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LOCATION OF THE ESSENTIAL SPECTRUM OF THE ENERGY OPERATORS OF THE QUANTUM SYSTEMS WITH NONINCREASING MAGNETIC FIELD

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§0. Introduction

In this article we continue the study of the essential spectrum of n-particle hamiltonians with magnetic field, which was begun in [1], [2]. As before we consider only magnetic fields, having the direction of x_3-axis and not depending on x_3. The main difficulty of the investigation of systems in such magnetic fields consists in the impossibility to separate the center-of-mass (c.m.) motion of the system in the x_1, x_2-plane: it is impossible even for a homogeneous magnetic field, if the particles are not identical [3]. To overcome this difficulty we considered in [1], [2] fields, which increase to $+\infty$ for all directions of x_1, x_2-plane and consequently do not permit the motion of the c.m. to infinity in this plane. For such fields the theorems on the localization of the essential spectrum were proved in [1], [2] for a wide class of quantum systems, including arbitrary atoms and molecules.

To obtain the localization theorem for nonincreasing (in x_1, x_2-plane) fields we suggested in [1] to use the $SO(2)$ symmetry of the system, that is to study the spectrum of hamiltonians on the subspaces of functions of fixed type m of $SO(2)$ symmetry. However, for such fields the results [1], [2] were established only under the condition, that the charges e_j of all particles of the system have the same sign (unfortunately by the authors fault this condition was omitted in [4], [5], where our theorems for nonincreasing fields were published without a proof). Later, for non-neutral system this condition was taken away, but only for homogeneous magnetic field and for systems without any neutral particles [6].

In this paper we are returning to the investigation of the energy operators of the quantum systems in nonincreasing magnetic fields for fixed type m of $SO(2)$ symmetry. We prove a general assertion on the localization of the essential spectrum of such systems and give a sufficient conditions for it application (Theorems 2.1,2.2) .These conditions for magnetic field can be understood in the sense, that

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magnetic vector-potential could not be strongly "irregular" near infinity and that the admissible "irregularity" depends on the ratio of the total charge of the system to the sum of the modulus of the charges of all particles of the system (see (2.5)).

The proved theorems detect the essential spectrum for the wide class of charged systems (the atomic and molecular ions, the systems, containing the neutral particles) in nonincreasing magnetic field, including homogeneous magnetic field. For the neutral systems our results can not be applied and the result of [3] for homogeneous field is the single one.

Our method is geometrical. Similar to [1] it is based on the partition of the configuration space into regions, which correspond to possible decompositions of the system. But here (in contrast to [1]) we apply different types of decompositions for different parts of the configuration space. From the beginning we make the decomposition in the direction of the third axis for large values of $\sum_i x_{i3}^2$ and after this — the decomposition in the x_1, x_2 plane (for bounded values of $\sum_i x_{i3}^2$). It is especially important, that the decomposition in the x_1, x_2 plane takes into account the possibility of motion of the c.m. of the system to infinity in this plane.

For simplicity in this article we do not take into account the permutational symmetry, since it can be taken into account as in [1].

§1. Definitions

§1.1. Let $Z_1 = (1, 2, \ldots, n)$ be a quantum system of n particles with the charges e_j, the masses k_j and coordinates $r_j = (\rho_j, x_{j3})$, $\rho_j = (x_{j1}, x_{j2})$, in an external magnetic field with magnetic vector-potential

$$\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_n), \quad \mathcal{B}_j = (\mathcal{B}_{j\perp}, 0), \quad \mathcal{B}_{j\perp} = B(|\rho_j|)(-x_{j2}, x_{j1}),$$

$$V_{st}(|r_{st}|) = V_{ts}(|r_{ts}|)$$

be the potentials of the interaction of particles s, t with each other, $r_{st} = r_t - r_s$,

$$M = \sum_{i=1}^{n} k_i, \quad r = (r_1, \ldots, r_n), \quad \rho = (\rho_1, \ldots, \rho_n), \quad X_3 = (x_{13}, \ldots, x_{n3}),$$

$$R^{3n} = \{r\}, \quad R^{2n} = \{\rho\},$$

$$R_{03} = \{r | r \in R^{3n}, \quad \rho_j = (0,0) \quad j = 1, 2, \ldots, n, \quad \sum_{i=1}^{n} k_i x_{i3} = 0\},$$

$$R_{0\perp} = \{r | r \in R^{3n}, \, x_{j3} = 0 \quad j = 1, \ldots, n\}, \quad R_0 = R_{0\perp} \oplus R_{03}.$$

R_0 is the space of the relative motion of the system Z_1 in the direction of the third axis.

We introduce in R^{3n} a new inner product and norm. Let

$$\tilde{r} = (\tilde{r}_1, \ldots, \tilde{r}_n), \quad (r, \tilde{r})_1 = \sum_{i=1}^{n} [(\rho_i, \tilde{\rho}_i)_{R^2} + k_i x_{i3} \tilde{x}_{i3} M^{\perp 1}], \quad |r|_1 = (r, r)_1^{1/2}$$

and P_0 be the projector[1] onto the subspace R_0. Then $q = (q_1, \dots, q_n) \equiv P_0 r$ are the coordinates in R_0, $q_j = (\rho_j, q_{j3})$, $q_{j3} = x_{j3} - M^{\perp 1}\sum_{s=1}^{n} k_s x_{s3}$, $r_{st} = q_t - q_s$.

Let

$$\nabla_{t\perp} = \left(\frac{\partial}{\partial x_{t1}}, \frac{\partial}{\partial x_{t2}}\right), \quad \nabla^{\circ}_{t3} = k_t^{\perp 1}\frac{\partial}{\partial q_{t3}} - M^{\perp 1}\sum_{s=1}^{n}\frac{\partial}{\partial q_{s3}}$$

$$T_t = \left(i\nabla_{t\perp} + e_t\mathcal{B}_{t\perp}\right)^2 k_t^{\perp 1}, \quad T^{\circ}_{t3} = -\nabla^{\circ 2}_{t3} k_t$$

$$A = \sum_{t=1}^{n} T_t, \quad T^{\circ}_{3} = \sum_{t=1}^{n} T^{\circ}_{t3}, \quad V(q) = \frac{1}{2}\sum_{s,t;s\neq t} V_{st}(|q_{st}|) \tag{1.1}$$

Then the energy operator of the system Z_1 (after the separation of the c.m. motion in the direction of x_3) can be written in the form

$$H_0 = A + T^{\circ}_3 + V(q) \tag{1.2}$$

where all operators depend only on q.

We assume that the functions $V_{st}(|r_1|)$ and $B(|\rho_1|)$ are real,

$$V_{st}(|r_1|) \in \mathcal{L}_{2,loc}(R^3), \quad s,t = 1,\dots,n, \quad B(|\rho_1|) \in \mathcal{L}_{2,loc}(R^2) \tag{1.3}$$

and

$$\lim_{t\to+\infty} V_{sp}(t) = 0$$

§1.2. For an arbitrary subsystem $\mathcal{D} = (i_1, \dots, i_p) \subseteq Z_1$ we put

$$R[\mathcal{D}] = \{r | r \in R^{3n},\ r_i = (0,0,0)\, i\ \bar{\in}\ \mathcal{D}\}, \quad R_{0\perp}[\mathcal{D}] = \{r | r \in R[\mathcal{D}],\ x_{t3} = 0\, t \in \mathcal{D}\}$$

$$R_{03}[\mathcal{D}] = \{r | r \in R[\mathcal{D}],\ \rho_t = (0,0)\, t \in \mathcal{D}, \sum_{t\in\mathcal{D}} k_t x_{t3} = 0\},\ R_0[\mathcal{D}] = R_{0\perp}[\mathcal{D}] \oplus R_{03}[\mathcal{D}]$$

It is clear, that $R_0[\mathcal{D}]$ is the space of the relative motion of the cluster $\mathcal{D}$. Let $P_0[\mathcal{D}]$ be the projector in $R[\mathcal{D}]$ on the subspace $R_0[\mathcal{D}]$. Then

$$q[\mathcal{D}] \equiv P_0[\mathcal{D}]r = (q_1[\mathcal{D}], \dots, q_n[\mathcal{D}]), \quad q_t[\mathcal{D}] = (\rho_t, q_{t3}[\mathcal{D}]),$$

$$q_{t3}[\mathcal{D}] = x_{t3} - \sum_{j\in\mathcal{D}} k_j x_{j3} M[\mathcal{D}]^{\perp 1} \quad t \in \mathcal{D},$$

$$q_t[\mathcal{D}] = (0,0,0) \quad \text{for } t \notin \mathcal{D}$$

are the coordinates in $R_0[\mathcal{D}]$; here $M[\mathcal{D}] = \sum_{j\in\mathcal{D}} k_j$. Further, the operator of gradient on R_{03} has the components

$$\nabla^{\circ}_{t3}[\mathcal{D}] = k_t^{\perp 1}\frac{\partial}{\partial q_{t3}[\mathcal{D}]} - M[\mathcal{D}]^{\perp 1}\sum_{s\in\mathcal{D}}\frac{\partial}{\partial q_{s3}[\mathcal{D}]} \quad t \in \mathcal{D}$$

[1] all projectors in §§1.1–1.3 are projectors in the sense of $(\cdot,\cdot)_1$

and the energy operator $H_0[\mathcal{D}]$ of the relative motion of the cluster $\mathcal{D}$ (in the direction of the third axis) can be written in the form

$$H_0[\mathcal{D}] = \sum_{t\in\mathcal{D}} \Big(T_{t\perp} - k_t \overset{\circ}{\nabla}{}^2_{t3}[\mathcal{D}]\Big) + \frac{1}{2} \sum_{s,t\in\mathcal{D}, s\neq t} V_{st}(|r_{st}|)$$

§1.3. Let $Z = (C_1, \ldots, C_s)$ be an arbitrary decomposition of the initial system Z_1 into nonintersecting clusters C_j, $|Z| = s$,

$$V_Z = \sum_{j=1}^{s} \sum_{p,t\in C_j} V_{pt}, \quad I_Z = V - V_Z.$$

We define the spaces

$$R_{0\perp}(Z) = \sum_{j=1}^{s} \oplus R_{0\perp}[C_j] = R^{2n}, \quad R_{03}(Z) = \sum_{j=1}^{s} \oplus R_{03}[C_j],$$

$$R_0(Z) = R_{0\perp}(Z) \oplus R_{03}(Z), \quad R_c(Z) = R_{03} \ominus R_{03}(Z)$$

Obviously, $R_{03}(Z)$ and $R_c(Z)$ are respectively the spaces of the relative motion of s-clusters system Z and of the motion of the c.m. of clusters $C_1, \ldots, C_s$ in the direction of the third axis. It is clear $R_0 = R_0(Z) \oplus R_c(Z)$.

Let $P_{03}(Z), P_0(Z)$ and $P_c(Z)$ be the projectors onto $R_{03}(Z)$, $R_0(Z)$ and $R_c(Z)$. It is easy to see that

$$P_0(Z)r = \sum_{j=1}^{s} P_0[C_i]r, \quad P_c(Z)r = (\zeta_1(Z), \ldots, \zeta_n(Z))$$

where

$$\zeta_j(Z) = (0, 0, \eta_j), \quad \eta_j = \sum_{i\in C_t} k_i x_{i3} M[C_t]^{\perp 1} - M^{\perp 1} \sum_{i=1}^{n} k_i x_{i3} \text{ for } j \in C_t,$$

η_j is the third coordinate of the vector, which joins the center-of-masses Z_1 and the subsystem C_s, containing j. The operator of gradient on $R_c(Z)$ has the components

$$\nabla^c_{j3}(Z) = M[C_t]^{\perp 1} \nabla_{\eta_j} - M^{\perp 1} \sum_{\varrho=1}^{n} \nabla_{\eta_\varrho} \quad j \in C_t$$

Now we can write the energy operator of the relative motion (respect to the third axis) of the compound system Z in the form

$$H_0(Z) = \sum_{t=1}^{s} H_0[C_t].$$

It is easy to see, that

$$H(Z) \equiv H_0 - I_Z = H_0(Z) + \sum_{t=1}^{n} \left(i\nabla_{t3}^{c}(Z)\right)^2$$

§1.4. Let m be the type of the irreducible representation of $SO(2)$ group by the operators $T_g : T_g\psi(r) = \psi(g^{\perp 1}r)$ $\{T_g\varphi(\rho) = \varphi(g^{\perp 1}\rho)\}$ $g \in SO(2)$ and $P^{(m)}$ be the projector onto the subspace of the functions, that transform under the action of the operators T_g in accordance with the representation of the type m,

$$H_0^{(m)} = P^{(m)}H_0. \quad H_0^{(m)}(Z) = P^{(m)}H_0(Z)$$

$$\mu^{(m)} = \min_{Z,|Z|=2} \inf H_0^{(m)}(Z) \tag{1.4}$$

§1.5. In the space R^{2n} we introduce the inner product

$$(\rho, \tilde{\rho})_2 = \sum_{j=1}^{n} k_j(\rho_j, \tilde{\rho}_j)_{R^2} M^{\perp 1}$$

and determine the projectors (in the sense of $(\cdot,\cdot)_2$) $P_{0\rho}$ and $P_{c\rho}$ onto the subspaces

$$R_{0\rho} = \{\rho | \rho \in R^{2n}, \quad \sum_{j=1}^{n} k_j\rho_j = 0\}, \quad R_{c\rho} = R^{2n} \ominus R_{0\rho}.$$

According to [7] (p. 125)

$$|P_{0\rho}\rho|_2^2 = \sum_{j=1}^{n} k_j|\rho_j - \rho_c|^2, \quad |P_{0c}\rho|_2^2 = M|\rho_c|^2 \tag{1.5}$$

where

$$\rho_c = \sum_{j=1}^{n} k_j\rho_j M^{\perp 1}.$$

Let $C \geq 0$, $\beta > 0$ be the arbitrary constants,

$$K(C;\beta) = \{\rho | \rho \in R^{2n}, \quad |\rho|_2 \geq C, \quad |P_{0\rho}\rho|_2 \leq \beta |P_{c\rho}\rho|_2\}$$

$$K(0;\infty) = R^{2n}. \quad \mathcal{D}(C;\beta) = \{\varphi(\rho) | \varphi(\rho) \in P^{(m)}C_0^2(K(C;\beta))\}.$$

We close the operator A (see §1.1) from $\mathcal{D}(C;\beta)$, denote the obtained operator by $A^{(m)}(C;\beta)$ and write

$$A_1^{(m)}(C;\beta) = \left(A^{(m)}(C,\beta) + I\right)^{\perp 1}, \; A^{(m)}(\beta) = A^{(m)}(0,\beta), \; A_1^{(m)}(\beta) = A_1^{(m)}(0,\beta)$$

$$A^{(m)} = A^{(m)}(+\infty), \quad A_1^{(m)} = A_1^{(m)}(+\infty).$$

Let S_d be the class of all self-adjoint operators, having only pure discrete spectrum with single limit point at $+\infty$. At last, we shall denote the essential spectrum of an arbitrary operator h by $\sum_{ess}(h)$.

§2. The results

§2.1. In this paragraph we formulate and talk over the results of the article.

Theorem 2.1.. *Let for some $\beta > 0$ the operator $A_1^{(m)}(\beta)$ be compact. Then*

$$\sum_{ess}(H_0^{(m)}) = [\mu^{(m)}, +\infty) \tag{2.1}$$

Remark.. *Compactness of the operator $A_1^{(m)}(\beta)$ is equivalent to the inclusion $A^{(m)}(\beta) \in S_d$.*

§2.2. Let us consider the physical sense of the condition of the theorem 2.1 and it difference from the conditions of the corresponding theorems [1],[2]. We know, that for each point λ of $\sum_{ess}(H_0^{(m)})$ there is such sequence (Weil sequence) $\psi_t(r) \in P^{(m)}C_0^2(R_0)$, $||\psi_t|| = 1$, that

$$\psi_t(r) \rightharpoonup 0 \text{ in } \mathcal{L}_2(R_0), \quad \lambda = \lim_{t\to\infty}(H_0\psi_t, \psi_t)$$

It is easy to see, that for this sequence

$$\lim_{t\to\infty}\int_\Omega |\psi_t|^2 d\Omega = 0 \tag{2.2}$$

for any compact region $\Omega \subset R_0$. It means, that $\psi_t(r)$ is such sequence of the states of our system, which describes the leaving of the system of any compact region of the configuration space R_0. Consequently $\psi_t(r)$ can describe one of two cases:

i) some decomposition of Z_1 into two or more clusters, may be together with motion of the c.m. to infinity

ii) the motion of the c.m. of the system to the infinity without any decomposition.

It is clear, that in the case i) it is naturally to expect, that

$$\lambda = \lim_{t\to\infty}(H_0^{(m)}\psi_t, \psi_t) \geq \mu^{(m)}, \tag{2.3}$$

since λ is the limit of the values of the energy for states $\psi_t(r)$, and $\mu^{(m)}$ is the minimal value of the energy over all decompositions of the system under condition of fixation of complete angular momentum m^2.

In the case ii) the inequality can be not valid and it means that (2.1) can be wrong. That is why the principal problem of the localization of the essential spectrum for the systems with magnetic field consists in the proof of the fact that Weil sequence can not realize case ii), that is it can not describe the motion of the c.m. of Z_1 to infinity without any kind decomposition.

§2.3. It is clear, that $\psi_t(r)$ can not correspond to motion of the c.m. to the infinity in the direction of the third axis, since in R_0 the third coordinate of the c.m. is fixed.

[2]the proof of the equality (2.3) for $\forall\lambda \in \sum_{ess}(H_0^{(m)})$ is the hard part of all geometrical proofs of the assertions on localization of the essential spectrum of many particle systems, i.e. theorems of the type HVZ-theorem.

The possibility of the motion of the c.m. to the infinity in x_1, x_2-plane depends on the character of the interaction of magnetic field with the particles. This interaction determines the spectral properties of the operators $A^{(m)}(\beta)$, $A_1^{(m)}(\beta)$, $0 < \beta \leq +\infty$ and in its turn these properties can determine the main features of the behavior of the system in magnetic field. So the demand of the compactness of the operator $A_1^{(m)}$ for the validity of the theorem on the essential spectrum (which was introduced in [2], but in fact it has been used earlier in [1]) practically is equivalent to the prohibition of the motion of the c.m. to infinity in x_1, x_2-plane, because such motion results in the infinite increase of the energy. Really, if $\psi_t(r)$ describes this motion, than

$$\delta = \lim_{k\to\infty} \sup_t ||\psi_t||_{|\rho_c|\geq k} > 0$$

but in virtue of compactness of the operator $A_1^{(m)}$ the operator $A^{(m)} \in S_d$ and therefore if $\delta > 0$ we obtain the relation

$$\lim (A^{(m)}\psi_t, \psi_t) = +\infty$$

which contradicts to the inequality $\lambda < +\infty$.

§2.4. Unfortunately, a compactness of the operator $A_1^{(m)}$ prohibits to Weil sequence to correspond to every motion of the c.m. to infinity in x_1, x_2-plane, including such motion, which is accompanied by some kind of decomposition of the system and which does not hinder (2.3). The condition of compactness of the operator $A_1^{(m)}(\beta)$ in the theorem 2.1 has no this defect, since it forbids the motion of the c.m. to infinity only in cone $K(0, \beta)$. Out of this cone such motion is possible, but it is always accompanied with some decomposition, because out of this cone by (1.5)

$$\sum_{j=1}^{n} k_j |\rho_j - \rho_c|^2 \geq \beta^2 M |\rho_c|^2$$

and if $|\rho_c| \to \infty$, then $|\rho_j - \rho_c| \to \infty$ for some j, that is the system is decomposed.

§2.5. The sufficient conditions of compactness of operator $A_1^{(m)}(\beta)$ are given in the following theorem.

$$\text{Let } Q = \sum_{j=1}^{n} e_j, \quad Q_0 = \sum_{j=1}^{n} |e_j|$$

Theorem 2.2.. *Let $Q \neq 0$ and the function $B(t)$, determining the form of magnetic vector-potential (see §1.3), satisfies to the relation*

$$\lim_{t\to\infty} |B(t)t| = \infty \tag{2.4}$$

and at least one of the following conditions holds *or for some* $q > 0$, $C_0 > 0$, $\delta_0 > 0$

$$|B(t)B(s)^{\perp 1} - 1| \leq q < |Q|Q_0^{\perp 1} \tag{2.5}$$

when $|ts^{\perp 1} - 1| \leq \delta$, $\quad t, s \geq C_0$

or all charged[3] *particles have the charges of the same sign*

$$e_i e_j \geq 0 \quad \forall i, j. \tag{2.6}$$

Then there is such $\bar{\beta}$, *that for* $\beta < \bar{\beta}$ *the operator* $A_1^{(m)}(\beta)$ *is compact.*

Theorems 2.1, 2.2 can be applied to extensive class of magnetic vector-potentials with nonincreasing functions $B(t)$ and to wide class of nonneutral quantum systems. For example, by the theorem 2.2 the operator $A_1^{(m)}(\beta)$ is compact for small β (and consequently the assertion (2.1) holds), when

$$B(t) = at^{\alpha}(1 + \gamma \sin \omega t),$$

where α, γ are such real constants, that $-1 < \alpha \leq 0$, $\quad |\gamma| < |Q| \cdot (2Q_0 + |Q|)^{\perp 1}$, a, ω are arbitrary positive constants.

Let us note, that the case $\gamma = \alpha = 0$ was studied in [6] for the systems without neutral particles. As compared with [1], [2] our theorems permit to particles to have the charges of different signs and the charge zero (that was forbided in [1], [2]).

The proofs of the theorem 2.1, 2.2 are given respectively in §§3,4. Since the region $K(0, \beta) \backslash K(C, \beta)$ is compact, then the operators $A_1^{(m)}(C; \beta)$ and $A_1^{(m)}(\beta)$ are compact or noncompact simultaneously for any C. That is why further we shall consider the operator $A_1^{(m)}(C; \beta)$ for large $C > 0$ instead of $A_1^{(m)}(\beta)$.

§3. The proof of the theorem 2.1

§3.1. The proof of the theorem 2.1 consists of veryfying the inclusions

$$[\mu^{(m)}, +\infty) \subseteq \sum{}_{ess}(H_0^{(m)})$$

and

$$\sum{}_{ess}(H_0^{(m)}) \subseteq [\mu^{(m)}, +\infty) \tag{3.1}$$

The first inclusion is proved identically by the same manner as in [1] (§§2.7, 2.8). To prove the second inclusion it is sufficient to establish, that for any sequence

$$\{g_k(r)\}, \quad g_k(r) \in P^{(m)} C_0^2(R_0), \quad ||g_k|| = 1, \quad \sup_k |(H_0 g_k, g_k)| < +\infty,$$

$g_k(r) \rightharpoonup 0$ in $\mathcal{L}_2(R_0)$, we have

$$\underline{\lim} (H_0^{(m)} g_k, g_k) \geq \mu^{(m)} \tag{3.2}$$

[3]the system can contain some neutral particles, but all particles can not be neutral since $Q \neq 0$

The proof of (3.2) consists of two parts. In the first part we divide the configuration space R_0 into the regions, corresponding to the decompositions $Z_s = (C_1, \dots, C_s)$ of the system $Z_1 = (1, 2, \dots, n)$, arising as the result of the increasing of the distances between classes C_j only in the direction of the axis x_3, that is we make the decompositions only in the space R_{03}. Let us remark, that no one of the constructed regions corresponds to the motion of the c.m. to the infinity in the direction x_3, since in R_{03} the position of c.m. is fixed. This decomposition (3.5) is realized by functions (3.4), which make the partition of identity. Further we estimate the quadratic form of the operator $H_0^{(m)}$ over all obtained regions with the exception of the cylinder

$$\Omega = \{X_3|, \quad X_3 \in R_{03}, \quad |X_3|_1 \le b(1)\} \otimes R^{2n}$$

where $b(1)$ is some constant.

The proof of this part has no difference from [1] (2.4, 2.5) and we give it very shortly.

In the second part of the proof we estimate the quadratic form of the operator $H_0^{(m)}$ in Ω. The approach of [1] can not be applied for this estimate, since now (in contrast to [1]) the operator $A_1^{(m)}$ is noncompact in R^{2n} for the systems, that contain the particles with the charges of the different signs or the neutral particles.

To estimate $(H_0^{(m)} g_k, g_k)_\Omega$ we make the special decomposition of Ω. The main features of this decomposition are the following:

i) it is generated by the decomposition of R^{2n}

ii) in all constructed regions (except $\Omega_{\perp 1}$) the motion of the c.m. of the system to infinity in x_1, x_2 plane is possible

iii) only in one region-Ω_0-such motion is theoretically possible without any the decomposition.

We realize this soecial decomposition with help the functions (3.8) and further estimate the quadratic form of $H_0^{(m)}$ over the obtained regions. Let us note, that compactness of the operator $A_1^{(m)}(\beta)$ is used for the estimate only in Ω_0: we prove, that the functions g_k describe the leaving of Ω_0 by the considered system.

§3.2. Thus from the beginning we follow to [1]. Let $Z_s = (C_1, \dots, C_s)$, $P_{03}(Z_s)$ and $P_c(Z_s)$ be the same, as in §1.3 for

$$Z = Z_s, \quad X_3 = (x_{13}, x_{23}, \dots, x_{n3}), \quad |X_3|_1^2 = \sum_{i=1}^n x_{i3}^2 k_i,$$

$$\tau_1 = |X_3|_1, \quad \tau_s = \tau(Z_s) = |P_0(Z_s)X_s| \cdot |P_c(Z_s)X_3|^{\perp 1} \tag{3.3}$$

We choose large numbers $a(1) < b(1)$ and small numbers $a(s) < b(s)$ $s \ge 2$ as in §2.4 [1] and define such real piece wise continuous differentiable functions $u_s(t)$, $v_s(t)$ that $0 \le u_s(t), v_s(t) \le 1$, $u_s(t) = 1$, for $t < a(s)$, $u_s(t) = 0$ for $t \ge b(s)$

$$u_s^2(t) + v_s^2(t) = 1 \tag{3.4}$$

Let $u_{Z_s} = u_s(\tau(Z_s)), \quad v_{Z_s} = v_s(\tau(Z_s)), \quad \psi(r) = g_k(r),$

$$\hat\psi_0 \equiv \psi, \quad \hat\psi_i = \hat\psi_{i\perp 1}(1 - \sum_{Z_i} u^2_{Z_i})^{1/2}, \quad \psi_{i\perp 1, Z_i} = \hat\psi_{i\perp 1} u_{Z_i}\, i = 1, \ldots, n \tag{3.5}$$

$$\psi_i = \sum_{Z_i} \psi_{i\perp 1, Z_i} \quad i = 1, 2, \ldots, n-1$$

Then according to relations (2.5), (2.6) [1] we obtain for $\forall\varepsilon > 0$ and large $a(1) = a(1, \varepsilon)$

$$(H_0 g_k, g_k) \geq \sum_{j=0}^{n\perp 1} \sum_{Z_{j+1}} (H(Z_{j+1})\psi_{j,Z_{j+1}}, \psi_{j,Z_{j+1}}) - 2\varepsilon ||g_k||^2 \tag{3.6}$$

$$(H(Z_{j+1})\psi_{j,Z_{j+1}}, \psi_{j,Z_{j+1}}) \geq \mu^{(m)} ||\psi_{j,Z_{j+1}}||^2 \quad j \neq 0 \tag{3.7}$$

§3.3. Now we shall estimate $(H(Z_1)\psi_{0,Z_1}, \psi_{0,Z_1})$. With the purpose we take into account, that supp$\psi_{0,Z_1} \subseteq \Omega$, where

$$\Omega = \{r | r = (r_1, \ldots, r_n), \quad r \in R_0, \quad |X_3|_1 \leq b(1)\},$$

and decompose Ω by the partition of R^{2n} into regions, which correspond to all possible variants of the decomposition of the system Z_1 in x_1, x_2 plane. But in contrast to situation of §3.2 our decomposition contains the regions, which correspond to possible motion of the c.m. Z_1 to the infinity in x_1, x_2 plane. Let us introduce in $R_\rho = R^{2n} = \{\rho\}$ a new inner product and a norm

$$(\rho, \tilde\rho)_2 = \sum_{i=1}^{n} k_i (\rho_i, \tilde\rho_i)_{R^2}, \quad |\rho|_2^2 = (\rho, \rho)_2, \quad \rho = (\rho_1, \ldots, \rho_n), \quad \tilde\rho = (\tilde\rho_1, \ldots, \tilde\rho_n)$$

For arbitrary decomposition $Z_s = (C_1, \ldots, C_s)$ we write:

$$R_{0\rho}(Z_s) = \{\rho | \rho \in R_\rho, \quad \sum_{j \in C_i} k_j \rho_j = 0 \quad i = 1, \ldots, s\},$$

$$R_{c\rho}(Z_1) = R_\rho \ominus R_{0\rho}(Z_1), \quad R_{c\rho}(Z_s) = R_{0\rho}(Z_1) \ominus R_{0\rho}(Z_s) \quad s \geq 2$$

Let $P_{\gamma\rho}(Z_s)$ be the projector (in the sense of $(\cdot,\cdot)_2$) onto subspace $R_{\gamma\rho}(Z_s), \gamma = 0, c,$
$\chi_0 = \chi(Z_0) = |\rho|_2,$[4] $\qquad \chi_s = \chi(Z_s) = |P_{0\rho}(Z_s)\rho|_2 \cdot |P_{c\rho}(Z_s)\rho|_2^{\perp 1} \quad s \geq 1$

We define the functions $\tilde u_s(t), \tilde v_s(t)$ similar to $v_s(t), v_s(t)$, but now $\tilde u_s(t) = 1$ for $0 \leq a(s+1), \quad \tilde u_s(t) = 0$ for $t \geq b(s+1)$. Further, we put

$$\tilde u_{Z_s} = \tilde u_s(\chi(Z_s)), \tilde v_{Z_s} = \tilde v_s(\chi(Z_s)) \quad s = 0, 1, \ldots, n \tag{3.8}$$

[4] the notation Z_0 is introduced only for the unity of the notations.

$\varphi = \psi_{0,Z_1}$, $\hat{\varphi}_{\perp 1} = \varphi$, $\hat{\varphi}_i = \hat{\varphi}_{i\perp 1}(1 - \sum_{Z_i} \tilde{u}^2_{Z_i})^{1/2}$, $\varphi_{i\perp 1, Z_i} = \hat{\varphi}_{i\perp 1}\tilde{u}_{Z_i}$ $i = 0, 1, \ldots, n$.

Let us remark, that the function $\tilde{u}_{Z_0} = \tilde{u}_0(|\rho|_2)$ describes the situation, when all particles are in compact region in R^{2n} (this function works similar to function $u_{Z_1}(\tau_1)$ in R_{03}) and the function $\tilde{u}_{Z_1}$ corresponds to possible motion of the c.m. to the infinity in the cone $K(0; b(2))$.

By the choice of numbers $a(s), b(s)$ we have that

$$\mathrm{supp}\varphi_{i,Z_{i+1}} \cap \mathrm{supp}\varphi_{i,Z'_{i+1}} = \emptyset \text{ if } Z_{i+1} \neq Z'_{i+1}$$

(see Corollary 1 of lemma 3.5 [8])

§3.4. Similar to (2.3) [1] we can obtain for large $b(1)$

(3.9)

$$(H_0\varphi, \varphi) \geq \sum_{i=\perp 1}^{n\perp 1} \sum_{Z_{i+1}} (H_0\varphi_{i,Z_{i+1}}, \varphi_{i,Z_{i+1}}) - C||\hat{\varphi}_0|\rho|^{\perp 1}||^2 - \varepsilon||\varphi_{\perp 1, Z_0}(1 + |\rho|_2)^{\perp 1}||^2$$

where ε is arbitrary small number, C and $b(1)$ depend on ε. If $r \in \mathrm{supp}\varphi_{i,Z_{i+1}}$, then for j, t from the different clusters of the Z_{i+1}, $i \geq 1$

$$|\rho_j - \rho_t| \geq a(1) const$$

(see lemma 3.7 and the relations (2.18) of [8]).

Therefore, for $i \neq -1, 0$,arbitrary $\varepsilon_1 > 0$ and large $a(1)$

$$(H_0\varphi_{i,Z_{i+1}}, \varphi_{i,Z_{i+1}}) \geq (H(Z_{i+1})\varphi_{i,Z_{i+1}}, \varphi_{i,Z_{i+1}}) - \varepsilon_1||\varphi_{i,Z_{i+1}}||^2 \tag{3.10}$$

and similar to (3.7)

$$(H(Z_{i+1})\varphi_{i,Z_{i+1}}, \varphi_{i,Z_{i+1}}) \geq \mu^{(m)}||\varphi_{i,Z_{i+1}}||^2 \tag{3.11}$$

§3.5. Now we shall estimate $(H_0\varphi_{i,Z_{i+1}}, \varphi_{i,Z_{i+1}})$ for $i = -1, 0$.By the construction

$$\varphi_{i,Z_{i+1}}(r) = g_k(r)\omega_i(r)$$

where $\omega_i(r) \in C^2$,

$$\sup_{r,j,k,p,l} \left\{ |\omega_i(r)| + |\frac{\partial \omega_i}{\partial x_{jk}}| + \frac{\partial^2 \omega_i}{\partial x_{jk}\partial x_{pl}}| \right\} < +\infty \quad i = -1, 0$$

$$\Omega_{\perp 1} \equiv \mathrm{supp}\{\omega_{\perp 1}(r)\} = \{r | r \in R_0,\ |X_3|_1 \leq b(1),\ |\rho|_2 \leq b(1)\}$$

$$\Omega_0 \equiv \mathrm{supp}\{\omega_0(r)\} = \{r | r \in R_0,\ |X_3|_1 \leq b(1),\ \rho \in K(b(1); b(2))\}$$

$$\omega_i(r) = 0 \quad r \in \partial\Omega_i.$$

We denote by H_{0i} the closure of the operator H_0 in $\mathcal{L}_2(\Omega_i)$ from the domain $P^{(m)}C^2_0(\Omega_i)$.

Let us demonstrate, that the operators H_{0i} belong to S_d (the definition S_d see in §1.5). For $H_{0,\perp 1}$ it is obviously, since $\Omega_{\perp 1}$ is compact. For the operator $H_{0,0}$ we have similar to lemma 2.1 [1]

$$H_{0,0}^{(m)} \geq h \equiv (\delta_1 A(b(2)) - \delta_2 \nabla_{03}^2 - \delta_3) P^{(m)}$$

where δ_i are some positive constants. Since in Ω_0 $|X_3|_1 \leq b(1)$ and $\rho \in K(b(1); b(2))$ then for $b(2) < \beta$ it follows from the separation of variables that $h \in S_d$. Consequently, $H_{0,0}^{(m)} \in S_d$ also.

Further, in virtue of §3.1 and of the inequality (3.6) the sequence $g_k(r)\omega_i(r)$ possesses with the needed properties for the application of lemma 3.1 of §3.7. By this lemma for $i = -1$ and $i = 0$

$$\underline{\lim}_{k\to\infty} (H_0 \varphi_{i,Z_{i+1}}, \varphi_{i,Z_{i+1}}) = \underline{\lim}_{k\to\infty} (H_0 g_k\omega_i, g_k\omega_i) \geq 0 \tag{3.12}$$

$$\lim_{k\to\infty} ||\varphi_{i,Z_{i+1}}|| = \lim_{k\to\infty} ||g_k\omega_i|| = 0 \tag{3.13}$$

§3.6. By the construction

$$||\psi_{0,Z_1}||^2 = ||\varphi||^2 = \sum_{i=\perp 1}^{n\perp 1} \sum_{Z_{i+1}} ||\varphi_{i,Z_{i+1}}||^2 \tag{3.14}$$

In virtue of (3.9)–(3.11)

$$(H_0\psi_{0,Z_1}, \psi_{0,Z_1}) \geq (\mu^{(m)} - \varepsilon_2)||\psi_{0,Z_1}||^2 - \mu^{(m)}||g_k\omega_1||^2 - \mu^{(m)}||g_k\omega_2||^2 + \gamma_k \tag{3.15}$$

where $\gamma_k = (H_0 g_k\omega_{\perp 1}, g_k\omega_{\perp 1}) + (H_0 g_k\omega_0, g_k\omega_0)$, ε_2 is small if $a(1)$ is large.

By (3.5)–(3.7),(3.15) and since

$$\text{supp}\psi_{j,Z_{j+1}} \cap \text{supp}\psi_{j,Z'_{j+1}} = \emptyset \text{ if } Z_{j+1} \neq Z'_{j+1}$$

(see corollary od lemma 3.5 [8]) we obtain, that

$$(H_0 g_k, g_k) \geq (\mu^{(m)} - \varepsilon_3)||g_k||^2 - \mu^{(m)}(||g_k\omega_{\perp 1}||^2 + ||g_k\omega_0||^2) + \gamma_k \tag{3.16}$$

where ε_3 is small.

In virtue (3.12),(3.13)

$$\underline{\lim}_{k\to\infty} \gamma_k \geq 0, \quad \lim_{k\to\infty} ||g_k\omega_i|| = 0 \quad i = -1, 0.$$

And the inequality (3.2) follows from (3.16).

§3.7. Now we shall prove the assertion, which was used above.

Lemma 3.1. *Let T be some self-adjoint operator in $\mathcal{L}_2(R_0)$ with the domain D_T and $T \in S_d$. Then for any sequence $f_k(r) \in D_T$, $||f_k|| \leq 1$, $f_k \rightharpoonup 0$ in $\mathcal{L}_2(R_0)$ for which*

$$\overline{\lim}(Tf_k, f_k) < +\infty$$

the following relations hold:

$$\lim(Tf_k, f_k) \geq 0, \quad \lim_{k\to\infty} ||f_k|| = 0$$

The proof is trivial. Really, let $g_s(r)$ be the orthogonal eigenfunctions of the operator T and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s$ be the corresponding eigenvalues. Then

$$f_k(r) = \sum_{s=1}^{\infty}(f_k, g_s)g_s(r), \quad ||f_k||^2 = \sum_{s=1}^{\infty}|(f_k, g_s)|^2,$$

$$(Tf_k, f_k) = \sum_{s=1}^{\infty}\lambda_s|(f_k, g_s)|^2.$$

By the condition of the lemma $(f_k, g_s) \to 0$ if $k \to \infty$, then for any fixed N

$$\underline{\lim}(Tf_k, f_k) \geq \lambda_N \sum_{s=N}^{\infty}|(f_k, g_s)|^2 = \lambda_N \lim_{k\to\infty} ||f_k||^2.$$

Since $\lambda_N \to \infty$ for $N \to \infty$, then $\lim_{k\to\infty} ||f_k|| = 0$, and lemma is proved.

§4. The proof of the theorem 2.2

§4.1. For the validity of theorem 2.2 it is sufficient to prove that

$$\lim(A\varphi_t, \varphi_t) = +\infty \tag{4.1}$$

for arbitrary sequence $\varphi_t(\rho) \in \mathcal{D}(C; \beta)$, $||\varphi||_{R_\rho} = 1$, for which

$$\lim ||\varphi_t(\rho)||_\Omega = 0 \tag{4.2}$$

for every compact region $\Omega \subset R_\rho$.

Since $\varphi_t(\rho) = P^{(m)}\varphi_t(\rho)$ we can expend the function $\varphi_t(\rho)$ in the eigenfunctions of the operator of complete angular momentum which correspond to its eigenvalue m. Let $\varphi(\rho) \equiv \varphi_t(\rho)$. We have

$$\varphi(\rho) = \sum(m)\varphi_{m_1,\ldots,m_n}(\rho) \tag{4.3}$$

where $\sum(m)$ is the sum over all integer $m_1, \ldots, m_n$, $m_1 + \cdots + m_n = m$,

$$\varphi_{m_1,\ldots,m_n}(\rho) = e^{\perp i(m_1\gamma_1+\cdots+m_n\gamma_n)} f_{m_1,\ldots,m_n}(\varrho), \tag{4.4}$$

$$f_{m_1,\ldots,m_n}(\varrho) = (2\pi)^{\perp n} \int_0^{2\pi} \cdots \int_0^{2\pi} \varphi(\rho) e^{i(m_1\gamma_1+\cdots+m_n\gamma_n)} d\gamma_1 \ldots d\gamma_n \tag{4.5}$$

$\varrho = (\varrho_1, \dots, \varrho_n)$, ϱ_j, γ_j are the polar coordinates in the plane x_1, x_2:

$$x_{j1} = \varrho_j \cos\gamma_j, \quad x_{j2} = \varrho_j \sin\gamma_j.$$

It is evidently, that

$$(4.6) \qquad (A\varphi, \varphi) = \sum(m)(A\varphi_{m_1,\dots,m_n}, \varphi_{m_1,\dots,m_n})$$

and

$$(4.7) \qquad (A\varphi_{m_1,\dots,m_n}, \varphi_{m_1,\dots,m_n}) = (2\pi)^n (F_{m_1,\dots,m_n}(\varrho) f_{m_1,\dots,m_n}, f_{m_1,\dots,m_n})_{R^n}$$

where

$$(4.8) \qquad F_{m_1,\dots,m_n}(\varrho) = \sum_{j=1}^{n} k_j^{\perp 1}(m_j^2 \varrho_j^{\perp 2} + 2m_j e_j B_j + e_j^2 B_j^2 \varrho_j^2),$$

$$B_j = B(\varrho_j), \quad R^n = \{\varrho\}$$

We consider $m_1, \dots, m_n$ as continuous parameters and minimize $F_{m_1,\dots,m_n}(\varrho)$ for fixed ϱ over $m_1, \dots, m_n$ under condition $m_1 + m_2 + \dots + m_n = m$. Then we obtain the following inequality

$$(4.9) \qquad F_{m_1,\dots,m_n}(\varrho) \geq F_m(\varrho) \equiv (m + \sum_{j=1}^{n} e_j B_j \varrho_j^2)^2 (\sum_{j=1}^{n} k_j \varrho_j^2)^{\perp 1}$$

§4.2. Now let us estimate $F_m(\varrho)$ for $\varrho \in \mathrm{supp} f_{m_1,\dots,m_n}(\varrho)$ With the purpose we describe $\mathrm{supp} f_{m_1,\dots,m_n}(\varrho)$. In virtue of (1.5)

$$(4.10) \qquad |\rho|_2^2 = |P_{0\rho}\rho|_2^2 + |P_{c\rho}|_2^2$$

If $\rho \in K(C; \beta)$, then by (1.5)

$$(4.11) \qquad |\varrho_i |\rho_c|^{\perp 1} - 1| \leq \beta_1, \quad |\rho_c| \geq |\rho|_2 (1 + \beta_1^2)^{\perp 1/2} \geq N_1$$

where $\beta_1 = \beta M^{1/2} k_0^{\perp 1/2}$, $N_1 = C(1 + \beta_1^2)^{\perp 1/2} M^{\perp 1/2}$, $k_0 = \min_j k_j$

In virtue (4.11) for $\rho \in K(C; \beta)$

$$(4.12) \qquad |\varrho_i \varrho_j^{\perp 1} - 1| \leq \beta_2, \quad \varrho_j \geq (1 - \beta_1)|\rho_c| \geq N_2 \quad i, j = 1, 2, \dots, n$$

where $\beta_2 = 2\beta_1(1 - \beta_1)^{\perp 1}$, $N_2 = (1 - \beta_1)N_1$.

Since $\mathrm{supp}\varphi(\rho) \subset K(C, \beta)$ it is clear, that $\varphi(\rho_1, \dots, \rho_n) \equiv 0$ for all angls $\gamma_1, \dots, \gamma_n$ if for some i, j, t

$$(4.13) \qquad |\varrho_i \varrho_j^{\perp 1} - 1| > \beta_2 \text{ or } \varrho_t < N_2$$

By (4.5) $f_{m_1,\dots,m_n}(\varrho) \equiv 0$ if (4.13) holds, that is for each $\varrho \in \operatorname{supp} f_{m_1,\dots,m_n}(\varrho)$ the inequalities (4.12) are fulfilled. Further we choose β so small and C so large , that $\beta_2 < \delta_0$ and $N_2 > C_0$.

§4.3. Let us consider the case, when there are particles with the charges of different signs in the system. For $\forall s$ and $\varrho \in \operatorname{supp} f_{m_1,\dots,m_n}(\varrho)$ in virtue (4.12)

$$\varrho_j \varrho_s^{-1} = 1 + \beta_{js}, \quad B_j B_s^{-1} = 1 + q_{js} \tag{4.14}$$

where β_{js} and q_{js} are some functions, $|\beta_{js}| < \beta_2 \leq 1$, $|q_{js}| \leq q$.

Using (4.14) we have for the situation (2.5):

$$|\sum_{j=1}^{n} e_j B_j \varrho_j^2 + m| = |B_s| \varrho_s^2 | \sum{}' e_j (1 + q_{js})(1 + \beta_{js})^2 + m \varrho_s^{-2} B_s^{-1}| \geq$$

$$\geq |B_s| \varrho_s^2 [|Q| - Q_0 (q + 3\beta_2 + 3\beta_2 q + m \varrho_s^{-2} |B_s|^{-1} Q_0^{-1})] \tag{4.15}$$

where $\sum'$ is the sum over all j, for which $e_j \neq 0$, and s is one of such j.

In virtue (2.4)(2.5) and (4.12) for small β_2, large N_2 and some q

$$0 < q + 3\beta_2 + 3\beta_2 q + m \varrho_s^{-2} B_s^{-1} Q_0^{-1} \leq q_1 < |Q| Q_0^{-1}.$$

Consequently, for some positive ω_1

$$|\sum{}' e_j B_j \varrho_j^2 + m| \geq \omega_1 |B_s| \varrho_s^2. \tag{4.16}$$

For the case, when all charged particles have the charges of the same sign, we obtain at once

$$|\sum{}' e_j B_j \varrho_j^2 + m| \geq \omega_2 |B_s|^2 \varrho_s^2 \tag{4.17}$$

where $\omega_2 = |e_s| 2^{-1}, |e_s| = \min_j |e_j|, e_j \neq 0$, and N_2 is large.

Further

$$\sum_{j=1}^{n} k_j \rho_j^2 = \rho_s^2 \sum_{j=1}^{n} k_j \rho_j^2 \rho_s^{-2} \leq 2M \rho_s^2 \tag{4.18}$$

Using (4.16)-(4.17) in (4.9) we have

$$F_{m_1,\dots,m_n}(\varrho) \geq \omega^2 |B_s|^2 \varrho_s^2$$

where $\omega^2 = \min\{\omega_1^2, \omega_2^2\}(2M)^{-1}$.

By (4.6),(4.7)

$$(A\varphi, \varphi) \geq \sum(m) \omega^2 (|B_s|^2 \varrho_s^2 f_{m_1,\dots,m_n}, f_{m_1,\dots,m_n})(2\pi)^n = \omega^2 (|B_s|^2 \varrho_s^2 \varphi, \varphi) \tag{4.19}$$

In virtue (4.12), (2.5) $|B_s| \varrho_s \to \infty$ if $|\varrho| = |\rho| \to \infty$. Since $\varphi(\rho) = \varphi_t(\rho)$ and by (4.2) the relation (4.1) holds.

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