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Vienna, Preprint ESI 1669 (2005)

July 18, 2005

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

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There are two important notions of differentiability of functions in Banach spaces.

A function $F : X \to Y$ is said to be **Gâteaux differentiable** at a point x_0 in X if

(*) $$Tu = \lim_{t\to 0} \frac{F(x_0+tu)-F(x_0)}{t}$$

exists for every $u \in X$ and is a bounded linear operator in u. This T is often denoted by $D_F(x_0)$.

If F is Gâteaux differentiable at x_0 and the limit in (*) exists uniformly in u in the unit ball of X we say that F is **Fréchet differentiable** at x_0.

Alternatively, F is Fréchet differentiable at x_0 if there is a bounded linear operator T so that

$$F(x_0 + u) = F(x_0) + Tu + o(\|u\|) \ .$$

This T is necessarily $D_F(x_0)$. Thus F has a genuine linear approximation at a neighborhood of x_0. If X is finite-dimensional and F is a Lipschitz map both notions coincide.

In infinite-dimensional Banach spaces there exists no translation invariant probability measure. In such spaces the most commonly used measures are the **Gaussian measures**.

A probability measure μ is said to be a Gaussian measure if for every $x^* \in X^*$ the measure ν_{x^*} on R defined by

$$\nu_{x^*}(A) = \mu(x; < x^*, x > \in A)$$

is a Gaussian measure on R. We say that μ is **nondegenerate** if for every $x^* \neq 0$ the measure ν_{x^*} is a genuine Gaussian measure (i.e. not concentrated at one point).

In other words, the Gaussian measure μ is nondegenerate if it is not supported on any closed huperplane of X.

A (Borel) set A in X is said to be **Gauss null** if $\mu(A) = 0$ for evry nondegenerate Gaussian measure μ. In finite-dimensional spaces X, Gauss null sets are exactly the sets which have Lebesgue measure 0.

The natural class of maps for which one considers the existence of derivatives is the class of Lipschitz maps. Lebesgue's theorem that every Lipschitz map from R to itself is differentiable a.e. has a natural extension to Lipschitz maps between Banach spaces. We have however to be careful about the range space. The simple map from $[0,1]$ to $L_1(0,1)$ defined by $F(t) = \chi_{[0,t]}$ has no derivative at any point.

A Banach space has the **Radon Nikodym property (RNP)** if every Lipschitz map $F : R \to Y$ has points of differentiability (or equivalently is differentiable a.e.). We have just observed that $L_1(0,1)$ fails to have the RNP. On the other hand it is easy to check that separable conjugate spaces (in particular reflexive spaces) have the RNP.

There is a satisfactory general theorem on the existence of Gâteaux derivatives of Lipschitz maps.

THEOREM: Let X and Y be separable Banach spaces with Y having the RNP. Then any Lipschitz map $F : X \to Y$ has a Gâteaux derivative at any point outside a Gauss null set in X.

This theorem was proved independently by several mathematicians in the 1970's. Each one had a different notion of exceptional sets. The final touch was made by M. Csörnyei in 1999, who showed that the exceptional sets are the Gauss null sets (that is the hardest part in the proof).

The problem with this theorem is that in general for the usual applications of derivatives Gâteaux derivatives do not suffice and one really needs Fréchet derivatives which are genuine local linear approximations of the given map.

For example, if F is a Lipschitz equivalence between X and Y and the Gâteaux derivative $D_F(x_0)$ exists at some point x_0 then it has to be a linear isomorphism on X, but in general it will not map onto Y. In other words, $D_F(x_0)$ is a linear isomorphism from X onto a subspace of Y. On the other hand if the Fréchet derivative exists at x_0 then, as is easily checked, $D_F(x_0)$ is a linear isomorphism between X and Y. A similar situation exists in other contexts, when one is interested in linearizing Lipschitz maps.

Unfortunately there are only few general theorems that guarantee the existence of Fréchet derivatives and these are usually very hard to prove.

A class of sets which have an important role in the study of Fréchet derivatives is that of **porous sets**.

A set $A \subset X$ is called **λ-porous**, $0 < \lambda < 1$, if for every $x \in A$ and $\epsilon > 0$ there is a $y \in X$ with $0 < \|y - x\| \leq \epsilon$ so that $A \cap B(y, \lambda\|y - x\|) = \emptyset$.

A set is called **porous** if it is λ-porous for some λ, $0 < \lambda < 1$.

A set is called **σ-porous** if it is a countable union of porous sets.

Porous sets are obviously nowhere dense and hence σ-porous sets are of the first category. If $\dim X < \infty$ then porous sets cannot have density points and hence by Lebesgue's density theorem they must be of measure 0 (this clearly implies that also σ-porous sets are of measure 0 in finite dimensions).

In infinite dimensions the measure situation is different. Preiss and Tiŝer proved in 1995 that every separable infinite-dimensional Banach space X can be represented as $U \cup V$ where U is a Gauss null set and V is σ-porous.

Here is an example which shows the connection between porous sets and differentiability. Let A be a porous set and $F(x) = d(x, A)$ which is clearly a Lipschitz function from X to R. F is not Fréchet defferentiable at any $x_0 \in A$. Indeed, the only possible derivative of the non-negative function F is 0. However arbitarily close to x_0 there is a point y with

$$B(y, \lambda\|x_0 - y\|) \cap A = \emptyset.$$

Clearly $d(y, A) \geq \lambda\|x_0 - y\|$ hence

$$\frac{F(y) - F(x_0)}{\|y - x_0\|} \geq \lambda$$

A useful variant of the notion of porous set is that of a directionally porous set. A set A is **directionally porous** if for every $x_0 \in A$ there is a unit vector u_0 so that the point y as in the definition of porous set is not only arbitrarily close to x_0 but is also on the ray $\{x_0 + \beta u_0; \beta > 0\}$. The little argument above shows that $d(x, A)$ is not even Gâteaux differentiable at x_0. Hence by the existence theorem for Gâteaux derivatives it follows that a directionally porous set has to be Gauss null.

We state now the three main positive results on existence of Fréchet derivatives. But first we mention that here we also have a restriction on the domain space. It can be shown that to get reasonable results in this context we have to assume that the domain X has a separable dual (besides the assumption that the range space Y has the RNP).

1. The first result is quite easy. It deals with convex continuous functions $f : X \to R$. Such functions are trivially locally Lipschitz. They are

much easier to handle than general Lipschitz functions $f : X \to R$. However, as we shall presently see, even the study of convex functions brings with it some surprises.

Asplund[1968], Preiss and Zajicek[1984]: Let X^* be separable and $f : X \to R$ be convex and continuous. Then f is Fréchet differentiable outside a σ-porous set in X.

2. **Preiss[1990], Lindenstrauss and Preiss[2000]: Let X^* be separable and $F : X \to R$ a Lipschitz function. Then F has points of Fréchet differentiability. Moreover there is a mean value theorem. If $F(u) - F(v) \geq m$ then there is a point x_0 where F is Fréchet differentiable and $< D_F(x_0), u - v > \geq m$.**

   The proofs are quite hard. The points of Fréchet differentiability are constructed by an iterative procedure and a limiting argument. We do not have here an a.e. result. And in fact it is still open if three Lipschitz functions $F_1 : \ell_2 \to R$, $F_2 : \ell_2 \to R$ and $F_3 : \ell_2 \to R$ have a common point of Fréchet differentiability (or alternatively if every Lipschitz map $F : \ell_2 \to R^3$ has a point of Fréchet differentiability). <For two functions from ℓ_2 to R there is some progress recently, but the situation is still not clear.>

   There is a (very complicated) example of a Lipschitz map $F : \ell_2 \to R^3$ which shows that the natural analogue of the mean value theorem in this situation may fail to hold.

3. Let X be a Banach space and denote by $\Gamma(X)$ the space of infinite-dimensional surfaces in X. By this we mean the set of all continuous maps $\gamma : T = [0,1]^{\aleph_0} \to X$ which have continuous first derivatives. We topologize $\Gamma(X)$ by the seminorms $\|\gamma\|_j = \sup_{0 \leq t \leq 1} \left| \frac{\partial \gamma(t)}{\partial t_j} \right|$. With these seminorms $\Gamma(X)$ becomes a Fréchet space (in particular complete metric space).

   With these notions at hand we can define a new class of null sets in X.

   A set A is Γ **null** if the set of all those $\gamma \in \Gamma(X)$ such that

$$\mu\{t : \gamma(t) \cap A\} = 0$$

   is residual in $\Gamma(X)$.

Here μ is the usual product measure on $T = [0,1]^{\aleph_0}$. Note that the definition of Γ null sets involves both the notions of measure and category.

This notion and the results which follow are taken from *Lindenstrauss and Preiss, Ann. of Math. 157 (2003).* Among the results proved in this paper are

(a) In finite-dimensional spaces, a set is Γ null if and only if its Lebesgue measure is 0.

(b) The result on Gâteaux differentiability also holds for Γ null sets: If X is separable and Y has the RNP then a Lipschitz map $f : X \to Y$ is Gâteaux differentiable outside a Γ null set.

(c) **Every convex continuous $f : X \to R$ where X^* is separable is Fréchet differentiable outside a Γ null set.**

In 1999 J. Matousek and E. Matouskova constructed an example of a convex continuous $F : \ell_2 \to R$ which is Fréchet differentiable only on a Gauss null set. Combining this with *(c)* above we see that we can decompose ℓ_2 as $A \cup B$ where A is Γ null and B is Gauss null. Thus the notions of Γ null and Gauss null are completely different.

The main result in the paper of Preiss and myself mentioned above is the following:

Let X^* be separable. Then every Lipschitz function $F : X \to R$ is Fréchet differentiable outside a Γ null set if and only if every porous set in X is Γ null. Moreover if X has this property any Lipschitz $FX \to Y$ (with Y having the RNP and with the space of bounded linear operators from X to Y separable) is Fréchet differentiable outside a Γ null set.

The class of spaces having this property is unfortunately rather restricted. It contains c_0 and the spaces $C(K)$ with K compact countable and also some reflexive spaces (like the Tsirelson space), but it does not contain any L_p, $1 < p < \infty$, and in particular not Hilbert space.

In view of the results on Γ null sets, Preiss,Tiŝer and myself asked ourselves under what conditions can one avoid a σ-porous set by finite-dimensional surfaces (in particular curves and 2-dimensional surfaces).

Let $k \geq 0$ be an integer and X a Banach space. By $\Gamma_k(X)$ we denote the space of k-dimensional surfaces in X. Any $\gamma \in \Gamma_k(X)$ is a continuous map from $[0,1]^k$ into X which has square integrable derivatives of order 1 in the sense of distributions. $\Gamma_k(X)$ is normed by

$$\|\gamma\| = \|\gamma\|_\infty + (\int_0^1 \|\gamma\,'(t)\|_2\,{}^2 dt)^{1/2}$$

. With this norm $\Gamma_k(X)$ is a Banach space.

A set $A \subset X$ is said to be Γ_k **null** if $\{\gamma; \mu_k\{t; \gamma(t) \cap A\} = 0\}$ is a residual set in $\Gamma_k(X)$ where μ_k is the Lebesgue measure on $[0,1]^k$.

I do not have time to discuss proofs and mention just the main results.

1. Suppose X^* is separable and A is porous. Then A is Γ_1 null.

2. Let $X = \ell_1$. Then there exists a σ-porous set with complement null on every C^1 curve.

3. Let $X = \ell_2$ and let A be porous. Then A is Γ_2 null.

4. Let $X = \ell_2$; there is a σ-porous set A in X which meets every 3-dimensional surface. In other words for every $\gamma \in \Gamma_3(X)$ the set $\{t; \gamma(t) \notin A, rank\ \gamma\,'(t) = 3\}$ is of measure 0.

5. In every separable space directionally porous sets are Γ_1 null and Γ_2 null.

6. There is a G_δ set A in R^2 of measure 0 which fails to be Γ_1 null.