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ON THE DYNAMICS OF NONCANONICALLY COUPLED OSCILLATORS AND ITS HIDDEN SUPERSTRUCTURE

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Abstract. The classical and quantum dynamics of noncanonically coupled oscillators is considered. It is shown that though the classical dynamics is well–defined for both harmonic and anharmonic oscillators, the quantum one is well–defined in a harmonic case, admits a hidden (super)hamiltonian formulation, and thus, preserves the initial operator relations, whereas a naïve quantization of the anharmonic case meets with principal difficulties.

I. INTRODUCTION

The classical and quantum dynamics of hamiltonian systems is often described by remarkable algebraic structures such as Lie algebras, their nonlinear generalizations and (quantum) deformations [1]. It seems that not less important objects govern a behaviour of the interacting hamiltonian systems and that they maybe unravelled in a certain way. There exist several forms of an interaction of hamiltonian systems: often it has a potential character, sometimes it is ruled by a deformation of the Poisson brackets; however, one of the most intriguing, physically important but mathematically less explored forms is a nonhamiltonian (noncanonical) interaction, which can not be described by deformations of the standard hamiltonian data (Poisson brackets and hamiltonians). Sometimes, such interaction may be realized by a dependence of the Poisson brackets of one hamiltonian system on the state of another. The pair of noncanonically coupled oscillators is one of the simplest and the most crucial examples of the nonhamiltonian interaction [2]; this paper is devoted to an investigation of the related classical and quantum dynamics. It is shown that though the classical dynamics is well–defined for both harmonic and anharmonic case, the quantum one is well-defined for noncanonically coupled harmonic oscillators, admits a hidden (super)hamiltonian formulation, and hence, preserves the initial operator relations (cf.[3]), whereas a naïve quantization of anharmonic oscillators meets with principal difficulties.

Key words and phrases. Classical and quantum dynamics, hamiltonian systems, nonhamiltonian interaction, Lie superalgebras.

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II. ISOTOPIC PAIR OF NONCANONICALLY COUPLED OSCILLATORS.

2.1. General algebraic definitions. Let's describe algebraic objects underlying the dynamics, which we are interested in.

Definition 1 [2]**.** The pair (V_1, V_2) of linear spaces is called *an isotopic pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X, Y) \mapsto [X, Y]_A = m_1(A, X, Y)$ $(X, Y \in V_1,\ A \in V_2)$ and $(A, B) \mapsto [A, B]_X = m_2(X, A, B)$ $(A, B \in V_2,\ X \in V_1)$ obey the Jacobi identity for all values of a subscript parameter (such operations will be called *isocommutators* and the subscript parameters will be called *isotopic elements*) and are compatible to each other, i.e. the identities

$$\begin{aligned}
[X,Y]_{[A,B]_Z} =& \tfrac{1}{2}([[X,Z]_A,Y]_B + [[X,Y]_A,Z]_B + [[Z,Y]_A,X]_B - \\
& [[X,Z]_B,Y]_A - [[X,Y]_B,Z]_A - [[Z,Y]_B,X]_A), \\
[A,B]_{[X,Y]_C} =& \tfrac{1}{2}([[A,C]_X,B]_Y + [[A,B]_X,C]_Y + [[C,B]_X,A]_Y - \\
& [[A,C]_Y,B]_X - [[A,B]_Y,C]_X - [[C,B]_Y,A]_X)
\end{aligned}$$

$(X, Y, Z \in V_1,\ A, B, C \in V_2)$ hold.

Let's discuss this defintion. First, it may be considered as a result of an axiomatization of the following trivial construction: let $\mathcal{A}$ be an associative algebra (f.e. any matrix one) and V_1, V_2 be two linear subspaces in it such that V_1 is closed under the isocommutators $(X, Y) \mapsto [X, Y]_A = XAY - YAX$ with isotopic elements A from V_2, whereas V_2 is closed under the isocommutators $(A, B) \mapsto [A, B]_X = AXB - BXA$ with isotopic elements X from V_1.

Second, one may compare def.1 with the definition of "anti–Jordan pairs" [4]. Namely,

Definition 2 (cf.[4])**.** The pair (V_1, V_2) of linear spaces is called *an anti–Jordan pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X, Y) \mapsto [X, Y]_A = m_1(A, X, Y)$ $(X, Y \in V_1,\ A \in V_2)$ and $(A, B) \mapsto [A, B]_X = m_2(X, A, B)$ $(A, B \in V_2,\ X \in V_1)$ are compatible to each other in the following manner

$$\begin{aligned}
[X,Y]_{[A,B]_Z} =& [[X,Z]_A,Y]_B + [[Z,Y]_A,X]_B - [[X,Y]_B,Z]_A, \\
[A,B]_{[X,Y]_C} =& [[A,C]_X,B]_Y + [[C,B]_X,A]_Y - [[A,B]_Y,C]_X
\end{aligned}$$

$(X, Y, Z \in V_1,\ A, B, C \in V_2)$ hold.

It can be easily verified that isotopic pairs are always anti–Jordan pairs (to obtain it one should use the Jacobi identity linearized by subscript parameters), and that the anti–Jordan pairs with a multiplication, obeying Jacobi identity if a subscript parameter is fixed in any way, are just the isotopic pairs. So the isotopic pairs may be considered as a particular case of the anti–Jordan pairs. Note that there exist examples of anti–Jordan pairs, which are not isotopic ones [4].

Anti–Jordan pairs are closely related to the (polarized) anti–Lie triple systems and Lie superalgebras [4]. Namely,

Definition 3. The ternary algebra V with product $[xyz]$ is called *an anti–Lie triple system* if

$$\begin{aligned} &(1) \quad [xyz] = [xzy], \\ &(2) \quad [xyz] + [zxy] + [yzx] = 0, \\ &(3) \quad [[xyz]uv] = [[xuv]yz] + [x[yvu]z] + [xy[zuv]]. \end{aligned}$$

An anti–Lie triple system V is *polarized* iff $V = V_1 \oplus V_2$ and $[xyz] = 0$ for $y, z \in V_1$ or $y, z \in V_2$.

If V is an anti–Lie triple system let's put $R_{yz} \in \mathrm{End}(V) : R_{yz}x = [xyz]$. The operators R_{yz} are closed under commutators so that $\mathfrak{g}_0(V) = \mathrm{span}(R_{yz}; y, z \in V)$ is a Lie algebra. The space $\mathfrak{g}_0(V) \oplus V$ possesses a natural structure of a Lie superalgebra with the even part $\mathfrak{g}_0(V)$ and the odd part V [4]. It will be denoted by $\mathfrak{g}(V)$. Polarized anti–Lie triple systems $V = V_1 \oplus V_2$ produce polarized Lie superalgebras $\mathfrak{g}(V) = \mathfrak{g}_0(V) \oplus (V_1 \oplus V_2)$ such that $[V_i, V_i]_+ = 0$, $[\mathfrak{g}(V), V_i]_- \subseteq V_i$ (it should be marked that there is sometimes asserted that $V_2 \simeq V_1^*$ as $\mathfrak{g}_0(V)$–modules, however, we shall not do it in general).

An arbitrary anti–Jordan pair (so an isotopic pair, in particular) has a structure of a polarized anti–Lie triple system. Namely, one should put $[xyz] = [z, x]_y$ (iff z belongs to the same space V_i as x) and $[y, x]_z$ (iff y belongs to the same space V_i as x).

Remark. Let's summarize the relations between the concepts of "isotopic pair", "anti–Jordan pair", "polarized anti–Lie triple system" and "polarized Lie superalgebra" once more.

(1) Each isotopic pair is an anti–Jordan pair, though there exist anti–Jordan pairs, which are not isotopic. It means that isotopic pairs form a proper subclass of the class of anti–Jordan pairs.

(2) Categories of anti–Jordan pairs and polarized anti–Lie triple systems are equivalent. It means that each anti–Jordan pair defines a polarized anti–Lie triple system and vice versa.

(3) Categories of polarized anti–Lie triple systems and polarized Lie superalgebras are equivalent. On the other hand polarized anti–Lie triple systems and polarized Lie superalgebras are particular cases of anti–Lie triple systems and Lie superalgebras respectively and the marked equivalency of categories is a particular case of the equivalency of categories of anti–Lie triple systems and Lie superalgebras.

(4) As a consequence of (2) and (3) categories of anti–Jordan pairs and polarized Lie superalgebras are equivalent.

(5) As a consequence of (1) and (4) each isotopic pair defines a polarized Lie superalgebra but not vice versa. There exist complementary strong conditions, which extract a proper subclass of polarized Lie superalgebras such that the corresponding anti–Jordan pairs are isotopic pairs.

An illustrative example to the construction of a Lie superalgebra by an isotopic pair is convenient. *Example*: let H_1 and H_2 be two linear spaces, $(\mathrm{Hom}(H_1, H_2); \mathrm{Hom}(H_2, H_1))$ is an isotopic pair, the corresponding Lie superalgebra is isomorphic to $\mathfrak{gl}(n|m)$, $n = \dim H_1$, $m = \dim H_2$.

2.2. Nonlinear dynamical equations associated with isotopic pairs. Note that the isocommutators in an isotopic pair (V_1, V_2) define families of compatible Poisson brackets $\{\cdot,\cdot\}_A$ and $\{\cdot,\cdot\}_X$ ($A \in V_2$, $X \in V_1$) in the spaces $S^{\cdot}(V_1)$ and $S^{\cdot}(V_2)$, respectively. The compatibility means that a linear combination of any two Poisson brackets is also a Poisson bracket.

Definition 4 (cf.[2]). Let's consider two elements $\mathcal{H}_1$ and $\mathcal{H}_2$ (*"hamiltonians"*) in $S^{\cdot}(V_1)$ and $S^{\cdot}(V_2)$, respectively. The equations

$$\dot{X}_t = \{\mathcal{H}_1, X_t\}_{A_t}, \qquad \dot{A}_t = \{\mathcal{H}_2, A_t\}_{X_t},$$

where $X_t \in V_1$ and $A_t \in V_2$ are called *the (nonlinear) dynamical equations associated with the isotopic pair* (V_1, V_2) *and "hamiltonians"* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ (it should be marked that "hamiltonians" are not even integrals of motion in a general situation).

2.3. Isotopic pair of noncanonically coupled oscillators. Let's now consider the isotopic pairs of noncanonically coupled oscillators [2,5]. The space V_1 is spanned by the elements p, q and r and the space V_2 is spanned by the elements a, b and c. The isocommutators have the form

$$\begin{array}{lll} [p,q]_a = 2\varepsilon_1 q & [p,q]_b = 2\varepsilon_1 p & [p,q]_c = \varepsilon_3 r \\ [p,r]_a = \varepsilon_2 r & [p,r]_b = 0 & [p,r]_c = 0 \\ [q,r]_a = 0 & [q,r]_b = -\varepsilon_2 r & [q,r]_c = 0 \end{array}$$

$$\begin{array}{lll} [a,b]_p = 2\tilde{\varepsilon}_1 b & [a,b]_q = 2\tilde{\varepsilon}_1 a & [a,b]_r = \tilde{\varepsilon}_3 c \\ [a,c]_p = \tilde{\varepsilon}_2 c & [a,c]_q = 0 & [b,c]_r = 0 \\ [b,c]_p = 0 & [b,c]_q = -\tilde{\varepsilon}_2 c & [a,c]_r = 0 \end{array}$$

where

$$\varepsilon_1 + \tilde{\varepsilon}_1 = 0, \quad \varepsilon_2 - \tilde{\varepsilon}_2 = \varepsilon_1 - \tilde{\varepsilon}_1, \quad \varepsilon_3\tilde{\varepsilon}_3 - \varepsilon_2\tilde{\varepsilon}_2 = 0.$$

The corresponding Lie algebra $\mathfrak{g}_0(V_1 \oplus V_2)$ is spanned (for generic ε_i, $\tilde{\varepsilon}_i$) by 6 operators $R_{p,a}$, $R_{p,b}$, $R_{q,a}$, $R_{q,b}$, $R_{r,b} = \frac{\varepsilon_2}{\varepsilon_3} R_{p,c}$, $R_{r,a} = \frac{\varepsilon_2}{\varepsilon_3} R_{q,c}$, which have the form

$$R_{p,a} = \begin{pmatrix} 2\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \varepsilon_2 \end{pmatrix}, \quad R_{p,b} = \begin{pmatrix} 0 & 0 & 0 \\ 2\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{q,a} = \begin{pmatrix} 0 & -2\varepsilon_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R_{q,b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2\varepsilon_1 & 0 \\ 0 & 0 & -\varepsilon_2 \end{pmatrix}, \quad R_{p,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \varepsilon_3 & 0 & 0 \end{pmatrix}, \quad R_{q,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\varepsilon_3 & 0 \end{pmatrix}$$

in the basis (q, p, r) and the form

$$R_{p,a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2\tilde{\varepsilon}_1 & 0 \\ 0 & 0 & \tilde{\varepsilon}_2 \end{pmatrix}, \quad R_{p,b} = \begin{pmatrix} 0 & 0 & 0 \\ -2\tilde{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_{q,a} = \begin{pmatrix} 0 & 2\tilde{\varepsilon}_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$R_{q,b} = \begin{pmatrix} -2\tilde{\varepsilon}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\tilde{\varepsilon}_2 \end{pmatrix}, \quad R_{p,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\tilde{\varepsilon}_2 & 0 & 0 \end{pmatrix}, \quad R_{q,c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \tilde{\varepsilon}_2 & 0 \end{pmatrix}$$

in the basis (a,b,c).

The Lie superalgebra $\mathfrak{g}(V_1\oplus V_2)$ has a (super)dimension $(6|6)$ and is generated by $R_{p,a}$, $R_{p,b}$, $R_{q,a}$, $R_{q,b}$, $R_{p,c}$, $R_{q,c}$, p, q, r, a, b, c with (super)commutation relations

$$\begin{gathered}
[q,p]_+=[q,r]_+=[p,r]_+=[a,b]_+=[a,c]_+=[b,c]_+=[r,c]_+=0,\\
[p,a]_+=R_{p,a},\ [q,a]_+=R_{q,a},\ [p,b]_+=R_{p,b},\\
[q,b]_+=R_{q,b},\ [p,c]_+=R_{p,c},\ [q,c]_+=R_{q,c},\\
[r,a]_+=\tfrac{\varepsilon_2}{\varepsilon_3}R_{q,c},\ [r,b]_+=\tfrac{\varepsilon_2}{\varepsilon_3}R_{p,c};
\end{gathered}$$

$$\begin{gathered}
[R_{p,a},q]_-=2\varepsilon_1 q,\ [R_{p,a},p]_-=0,\ [R_{p,a},r]_-=\varepsilon_2 r,\\
[R_{q,a},q]_-=0,\ [R_{q,a},p]_-=-2\varepsilon_1 q,\ [R_{q,a},r]_-=0,\\
[R_{p,b},q]_-=2\varepsilon_1 p,\ [R_{p,b},p]_-=0,\ [R_{p,b},r]_-=0,\\
[R_{q,b},q]_-=0,\ [R_{q,b},p]_-=-2\varepsilon_1 p,\ [R_{q,b},r]_-=-\varepsilon_2 r,\\
[R_{p,c},q]_-=\varepsilon_3 r,\ [R_{p,c},p]_-=0,\ [R_{p,c},r]_-=0,\\
[R_{q,c},q]_-=0,\ [R_{q,c},p]_-=-\varepsilon_3 r,\ [R_{q,c},r]_-=0,\\
[R_{p,a},a]_-=0,\ [R_{p,a},b]_-=2\tilde{\varepsilon}_1 b,\ [R_{p,a},c]_-=\tilde{\varepsilon}_2 c,\\
[R_{q,a},a]_-=0,\ [R_{q,a},b]_-=2\tilde{\varepsilon}_1 a,\ [R_{q,a},c]_-=0,\\
[R_{p,b},a]_-=-2\tilde{\varepsilon}_1 b,\ [R_{p,b},b]_-=0,\ [R_{p,b},c]_-=0,\\
[R_{q,b},a]_-=-2\tilde{\varepsilon}_1 a,\ [R_{q,b},b]_-=0,\ [R_{q,b},c]_-=-\tilde{\varepsilon}_2 c,\\
[R_{p,c},a]_-=-\tilde{\varepsilon}_2 c,\ [R_{p,c},b]_-=0,\ [R_{p,c},c]_-=0,\\
[R_{q,c},a]_-=0,\ [R_{q,c},b]_-=\tilde{\varepsilon}_2 c,\ [R_{q,c},c]_-=0;
\end{gathered}$$

$$\begin{gathered}
[R_{p,a},R_{p,b}]_-=-2\varepsilon_1 R_{p,b},\ [R_{p,a},R_{q,a}]_-=2\varepsilon_1 R_{q,a},\ [R_{p,a},R_{p,b}]_-=0,\\
[R_{p,a},R_{p,c}]_-=\tilde{\varepsilon}_2 R_{p,c},\ [R_{p,a},R_{q,c}]_-=\varepsilon_2 R_{q,c},\ [R_{p,b},R_{q,a}]_-=2\varepsilon_1(R_{q,b}+R_{p,a}),\\
[R_{p,b},R_{q,b}]_-=2\varepsilon_1 R_{p,b},\ [R_{p,b},R_{p,c}]_-=0,\ [R_{p,b},R_{q,c}]_-=2\varepsilon_1 R_{p,c},\\
[R_{q,a},R_{q,b}]_-=-2\varepsilon_1 R_{q,a},\ [R_{q,a},R_{p,c}]_-=-2\varepsilon_1 R_{p,c},\ [R_{q,a},R_{q,c}]_-=0,\\
[R_{q,b},R_{p,c}]_-=-\varepsilon_2 R_{p,c},\ [R_{q,b},R_{q,c}]_-=-\tilde{\varepsilon}_2 R_{q,c},\ [R_{p,c},R_{q,c}]_-=0.
\end{gathered}$$

The even part of the Lie superalgebra $\mathfrak{g}(V_1\oplus V_2)$ is isomorphic to the semidirect sum of $\mathfrak{gl}(2,\mathbb{C})$ and $\mathbb{C}^2$. On the other hand $\mathfrak{g}(V_1\oplus V_2)$ may be considered as a semidirect product of the Lie superalgebra $\mathfrak{sl}(2|1,\mathbb{C})$ generated by $R_{p,a}$, $R_{p,b}$, $R_{q,a}$, $R_{q,b}$, p, q, a, b and the $(2|2)$–dimensional vector superspace $V^{2|2}$ generated by $R_{p,c}$, $R_{q,c}$, r, c.

III. CLASSICAL AND QUANTUM DYNAMICS OF NONCANONICALLY COUPLED OSCILLATORS.

3.1. Classical dynamics of noncanonically coupled harmonic oscillators. First of all, let's describe the classical dynamics of noncanonically coupled harmonic oscillators (the several misprints of signs of [3] are corrected). The dynamical equations with "hamiltonians" $\mathcal{H}_1=P^2+Q^2$ and $\mathcal{H}_2=A^2+B^2$ have the form

$$\begin{cases} \dot P = -4\varepsilon_1(Q^2A+PQB)-2\varepsilon_3 RQC \\ \dot Q = 4\varepsilon_1(PQA+P^2B)+2\varepsilon_3 RPC \\ \dot R = 2\varepsilon_2(PRA-QRB) \end{cases} \qquad \begin{cases} \dot A = -4\tilde\varepsilon_1(B^2P+ABQ)-2\tilde\varepsilon_3 CBR \\ \dot B = 4\tilde\varepsilon_1(ABP+A^2Q)+2\tilde\varepsilon_3 CAR \\ \dot C = 2\tilde\varepsilon_2(ACP-BCQ) \end{cases}$$

Note that "hamiltonians" $\mathcal{H}_1=\mathcal{I}_1^2$ and $\mathcal{H}_2=\mathcal{I}_2^2$ are integrals of motion here, so it is rather convenient to put $P=\mathcal{I}_1\cos\varphi$, $Q=\mathcal{I}_1\sin\varphi$, $A=\mathcal{I}_2\cos\psi$, $B=\mathcal{I}_2\sin\psi$. Then

$$\begin{cases} \dot\varphi = 2\varepsilon_3 RC+4\varepsilon_1\mathcal{I}_1\mathcal{I}_2\sin(\varphi+\psi) \\ \dot\psi = 2\tilde\varepsilon_3 RC+4\tilde\varepsilon_1\mathcal{I}_1\mathcal{I}_2\sin(\varphi+\psi) \end{cases} \qquad \begin{cases} \dot R = 2\varepsilon_2\cos(\varphi+\psi)R \\ \dot C = 2\tilde\varepsilon_2\cos(\varphi+\psi)C \end{cases}$$

Let's introduce $\vartheta=\varphi+\psi$, $\chi=\varepsilon_3\psi-\tilde\varepsilon_3\varphi$ and mark that $\varepsilon_1+\tilde\varepsilon_1=0$, then

$$\begin{cases} \dot\vartheta = 2(\varepsilon_3+\tilde\varepsilon_3)RC \\ \dot\chi = -4\varepsilon_1\mathcal{I}_1\mathcal{I}_2(\varepsilon_3-\tilde\varepsilon_3)\sin\vartheta \end{cases}$$

Also $(RC)^{\cdot}=2(\varepsilon_2+\tilde\varepsilon_2)\cos\vartheta(RC)$, therefore,

$$(RC)'_\vartheta=\frac{\varepsilon_2+\tilde\varepsilon_2}{\varepsilon_3+\tilde\varepsilon_3}\cos\vartheta, \qquad \text{and} \qquad RC=\mathcal{L}+\frac{\varepsilon_2+\tilde\varepsilon_2}{\varepsilon_3+\tilde\varepsilon_3}\mathcal{I}_1\mathcal{I}_2\sin\vartheta,$$

whereas

$$\dot\vartheta=2\mathcal{L}(\varepsilon_3+\tilde\varepsilon_3)+2\mathcal{I}_1\mathcal{I}_2(\varepsilon_2+\tilde\varepsilon_2)\sin\vartheta.$$

Here $\mathcal{L}=RC-\frac{\varepsilon_2+\tilde\varepsilon_2}{\varepsilon_3+\tilde\varepsilon_3}(QA+PB)$ is an integral of motion. Note that $(R^{\tilde\varepsilon_2}C^{-\varepsilon_2})^{\cdot}=0$, so it is convenient to put $\Lambda=R^{\frac{\tilde\varepsilon_2}{\varepsilon_2+\tilde\varepsilon_2}}C^{\frac{\varepsilon_2}{\varepsilon_2+\tilde\varepsilon_2}}$.

Then

$$\begin{cases} R = \Lambda\left(\mathcal{L}-\frac{\varepsilon_2+\tilde\varepsilon_2}{\varepsilon_3+\tilde\varepsilon_3}\mathcal{I}_1\mathcal{I}_2\sin\vartheta\right)^{\frac{\varepsilon_2}{\varepsilon_2+\tilde\varepsilon_2}} \\ C = \frac{1}{\Lambda}\left(\mathcal{L}-\frac{\varepsilon_2+\tilde\varepsilon_2}{\varepsilon_3+\tilde\varepsilon_3}\mathcal{I}_1\mathcal{I}_2\sin\vartheta\right)^{\frac{\tilde\varepsilon_2}{\varepsilon_2+\tilde\varepsilon_2}} \end{cases}$$

$\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{L}$ and Λ form a complete set of integrals of motion for generic values of ε_i, $\tilde\varepsilon_i$.

Let's also denote

$$\begin{aligned}\xi &= (\varepsilon_2+\tilde\varepsilon_2)\chi+2\varepsilon_1(\varepsilon_3-\tilde\varepsilon_3)\vartheta \\ &= [(\varepsilon_2+\tilde\varepsilon_2)\varepsilon_3+2\varepsilon_1(\varepsilon_3-\tilde\varepsilon_3)]\psi-[(\varepsilon_2+\tilde\varepsilon_2)\tilde\varepsilon_3+2\tilde\varepsilon_1(\varepsilon_3-\tilde\varepsilon_3)]\varphi,\end{aligned}$$

then

$$\xi=4\mathcal{L}(\varepsilon_3^2-\tilde\varepsilon_3^2)\varepsilon_1 t+\xi_0.$$

3.2. Classical dynamics of noncanonically coupled anharmonic oscillators. The dynamical equations with anharmonic "hamiltonians" $\mathcal{H}_1 = P^2 + V(Q^2)$ and $\mathcal{H}_2 = A^2 + V(B^2)$ have the form

$$\begin{cases} \dot{P} = -V'(Q^2)(4\varepsilon_1(Q^2A + PQB) + 2\varepsilon_3 RQC) \\ \dot{Q} = 4\varepsilon_1(PQA + P^2B) + 2\varepsilon_3 RPC \\ \dot{R} = 2\varepsilon_2(PRA - QRBV'(Q^2)) \end{cases}$$

$$\begin{cases} \dot{A} = -V'(B^2)(4\tilde{\varepsilon}_1(B^2P + ABQ) + 2\tilde{\varepsilon}_3 CBR) \\ \dot{B} = 4\tilde{\varepsilon}_1(ABP + A^2Q) + 2\tilde{\varepsilon}_3 CAR \\ \dot{C} = 2\tilde{\varepsilon}_2(ACP - BCQV'(B^2)) \end{cases}$$

Let's put $S = RC$, $T = AQ + BP$, then

$$\begin{cases} \dot{S} = 2S((\varepsilon_2 + \tilde{\varepsilon}_2)PA - QB(\varepsilon_2 V'(Q^2) + \tilde{\varepsilon}_2 V'(B^2))) \\ \dot{T} = 2S((\varepsilon_3 + \tilde{\varepsilon}_3)PA - QB(\varepsilon_3 V'(Q^2) + \tilde{\varepsilon}_3 V'(B^2))) \end{cases}$$

At the same time

$$\begin{cases} \dot{P} = -QV'(Q^2)(4\varepsilon_1 T + 2\varepsilon_3 S) \\ \dot{Q} = 4\varepsilon_1 PT + 2\varepsilon_3 PS \end{cases} \qquad \begin{cases} \dot{A} = -BV'(B^2)(4\tilde{\varepsilon}_1 T + 2\tilde{\varepsilon}_3 S) \\ \dot{B} = 4\tilde{\varepsilon}_1 AT + 2\tilde{\varepsilon}_3 AS \end{cases}$$

Let's consider the case $\varepsilon_3 = \varepsilon_2$ (and, hence, $\tilde{\varepsilon}_3 = \tilde{\varepsilon}_2$). Then $\mathcal{L} = S - T$ is an integral of motion. Therefore,

$$\begin{cases} \dot{Q} = P(2\varepsilon_3\mathcal{L} + (2\varepsilon_3 + 4\varepsilon_1)T) \\ \dot{P} = -QV'(Q^2)(4\varepsilon_3\mathcal{L} + (2\varepsilon_3 + 4\varepsilon_1)T) \end{cases} \begin{cases} \dot{B} = A(2\tilde{\varepsilon}_3\mathcal{L} + (2\tilde{\varepsilon}_3 + 4\tilde{\varepsilon}_1)T) \\ \dot{A} = -BV'(B^2)(4\tilde{\varepsilon}_3\mathcal{L} + (2\tilde{\varepsilon}_3 + 4\tilde{\varepsilon}_1)T) \end{cases}$$

Note that the "hamiltonians" $\mathcal{H}_i = \mathcal{I}_i^2$ are integrals of motion so put $P = \sqrt{\mathcal{I}_1^2 - V(Q^2)}$, $A = \sqrt{\mathcal{I}_2^2 - V(B^2)}$ and, hence

$$\begin{cases} \dot{Q} = \sqrt{\mathcal{I}_1^2 - V(Q^2)}(4\varepsilon_3\mathcal{L} + (2\varepsilon_3 + 4\varepsilon_1)T) \\ \dot{B} = \sqrt{\mathcal{I}_2^2 - V(B^2)}(4\tilde{\varepsilon}_3\mathcal{L} + (2\tilde{\varepsilon}_3 + 4\tilde{\varepsilon}_1)T) \end{cases}$$

where $T = \sqrt{\mathcal{I}_2^2 - V(B^2)}Q + \sqrt{\mathcal{I}_1^2 - V(B^2)}B$. Put $F_i(x) = \int \frac{dx}{\sqrt{\mathcal{I}_i^2 - V(x)}}$, then

$$\begin{cases} \dot{F}_1(Q) = 4\varepsilon_3\mathcal{L} + (2\varepsilon_3 + 4\varepsilon_1)(\dfrac{Q}{F_2'(B)} + \dfrac{B}{F_1'(Q)}) \\ \dot{F}_2(B) = 4\tilde{\varepsilon}_3\mathcal{L} + (2\tilde{\varepsilon}_3 + 4\tilde{\varepsilon}_1)(\dfrac{Q}{F_2'(B)} + \dfrac{B}{F_1'(Q)}) \end{cases}$$

Let's denote $\Theta = F_1(Q)$, $\Xi = F_2(B)$ and put $G_i = F_i^{-1}$, then $\Xi = \alpha\Theta + \beta t + \gamma$, where α, β are constants, which may be easily expressed via ε_i, $\tilde{\varepsilon}_i$ and $\mathcal{L}$, γ is an arbitrary number, determined by the initial conditions.

Put $G(x,t) = G_1(x)G_2(\alpha x + \beta t + \gamma)$, then Θ obeys the following differential equation

$$\dot{\Theta} = 4\varepsilon_3\mathcal{L} + (2\varepsilon_3 + 3\varepsilon_1)\frac{\partial G(\Theta, t)}{\partial \Theta}.$$

3.3. Representations of the isotopic pairs of noncanonically coupled oscillators. To quantize the classical dynamics one needs in representations of algebraic objects underlying it.

Definition 5 [5]. *A representation of the isotopic pair* (V_1, V_2) *in the linear space* W is a pair (T_1, T_2) of mappings $T_i : V_i \mapsto \mathrm{End}(W)$ such that

$$\begin{aligned} T_1([X,Y]_A) =& T_1(X)T_2(A)T_1(Y) - T_1(Y)T_2(A)T_1(X), \\ T_2([A,B]_X) =& T_2(A)T_1(X)T_2(B) - T_2(B)T_1(X)T_2(A), \end{aligned}$$

where $X, Y \in V_1$, $A, B \in V_2$. A representation of the isotopic pair (V_1, V_2) in the linear space W is called *split* iff $W = W_1 \oplus W_2$ and

$$\begin{cases} (\forall X \in V_1)\ T_1(X)|_{W_2} = 0,\ T_1(X) : W_1 \mapsto W_2, \\ (\forall A \in V_2)\ T_2(A)|_{W_1} = 0,\ T_2(A) : W_2 \mapsto W_1. \end{cases}$$

Otherwords, operators $T(X)$ and $T(A)$ have the form $\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, respectively.

Not that a split representation of an isotopic pair (V_1, V_2) defines a representation T of the corresponding anti–Lie triple system and Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$ (or its central extension $\hat{\mathfrak{g}}(V_1 \oplus V_2)$). The resulted representation of the Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$ always have a special "polarized" form: $W = W_1 \oplus W_2$, $T(V_1) : W_1 \mapsto W_2$, $T(V_2) : W_2 \mapsto W_1$, $\mathfrak{g}_0(V_1 \oplus V_2) : W_i \mapsto W_i$. Note that each representation (T_1, T_2) of the isotopic pair (V_1, V_2) in the space W defines a split representation (T_1^s, T_2^s) of the same pair in the space $W_1 \oplus W_2$ $(W_i \simeq W)$:

$$(\forall X \in V_1)\ T_1^s(X) = \begin{pmatrix} 0 & 0 \\ T_1(X) & 0 \end{pmatrix}, \quad (\forall A \in V_2)\ T_2^s(A) = \begin{pmatrix} 0 & T_2(A) \\ 0 & 0 \end{pmatrix}.$$

3.4. Quantum dynamics of noncanonically coupled harmonic oscillators. The formal quantum dynamical equations have the form

$$\begin{cases} \frac{d}{dt}\hat{P}_t = -2\varepsilon_1(\hat{P}_t\hat{B}_t\hat{Q}_t + \hat{Q}_t\hat{B}_t\hat{P}_t + 2\hat{Q}_t\hat{A}_t\hat{Q}_t) - \varepsilon_3(\hat{R}_t\hat{C}_t\hat{Q}_t + \hat{Q}_t\hat{C}_t\hat{R}_t) \\ \frac{d}{dt}\hat{Q}_t = 2\varepsilon_1(\hat{P}_t\hat{A}_t\hat{Q}_t + \hat{Q}_t\hat{A}_t\hat{P}_t + 2\hat{P}_t\hat{B}_t\hat{P}_t) + \varepsilon_3(\hat{R}_t\hat{C}_t\hat{P}_t + \hat{P}_t\hat{C}_t\hat{R}_t) \\ \frac{d}{dt}\hat{R}_t = \varepsilon_2(\hat{P}_t\hat{A}_t\hat{R}_t + \hat{R}_t\hat{A}_t\hat{P}_t - \hat{Q}_t\hat{B}_t\hat{R}_t - \hat{R}_t\hat{B}_t\hat{Q}_t) \end{cases}$$

$$\begin{cases} \frac{d}{dt}\hat{A}_t = -2\tilde{\varepsilon}_1(\hat{A}_t\hat{Q}_t\hat{B}_t + \hat{B}_t\hat{Q}_t\hat{A}_t + 2\hat{B}_t\hat{P}_t\hat{B}_t) - \tilde{\varepsilon}_3(\hat{C}_t\hat{R}_t\hat{B}_t + \hat{B}_t\hat{R}_t\hat{C}_t) \\ \frac{d}{dt}\hat{B}_t = 2\tilde{\varepsilon}_1(\hat{A}_t\hat{P}_t\hat{B}_t + \hat{B}_t\hat{P}_t\hat{A}_t + 2\hat{A}_t\hat{Q}_t\hat{A}_t) + \tilde{\varepsilon}_3(\hat{C}_t\hat{R}_t\hat{A}_t + \hat{A}_t\hat{R}_t\hat{C}_t) \\ \frac{d}{dt}\hat{C}_t = \tilde{\varepsilon}_2(\hat{A}_t\hat{P}_t\hat{C}_t + \hat{C}_t\hat{P}_t\hat{A}_t - \hat{B}_t\hat{Q}_t\hat{C}_t - \hat{C}_t\hat{Q}_t\hat{B}_t) \end{cases}$$

The dynamics is considered in arbitrary representation of the isotopic pair of noncanonically coupled oscillators. Let's consider such dynamics in the corresponding split representation. First of all renormalize c and r so that $R_{p,c} = R_{b,r}$ and $R_{q,c} = R_{a,r}$.

Proposition. *Equations of quantum dynamics of noncanonically coupled oscillators are a reduction of formal super Heisenberg equations*

$$\tfrac{d}{dt}\hat{F}_t = [\hat{H}_{\text{hidden}}, \hat{F}_t]$$

in $\mathcal{U}(\mathfrak{g}(V_1 \oplus V_2))$ *with quadratic quantum hamiltonian*

$$\hat{H}_{\text{hidden}} = \hat{R}^2_{q,a} + \hat{R}^2_{p,b} + \hat{R}^2_{q,b} + \hat{R}^2_{p,a} + \hat{R}^2_{p,c} + \hat{R}^2_{q,c}$$

Proof. The statement of the proposition is verified by straightforward explicit computation.

So quantum dynamics of noncanonically coupled oscillators admits a hidden super–hamiltonian formulation in terms of Lie superalgebra $\mathfrak{g}(V_1 \oplus V_2)$. It leads to a very important consequence.

Corollary. *The quantum dynamics preserves the initial operator relations:*

$$\begin{gathered}\hat{P}_t\hat{A}_t\hat{Q}_t - \hat{Q}_t\hat{A}_t\hat{P}_t = 2\varepsilon_1\hat{Q}_t,\ \hat{P}_t\hat{A}_t\hat{R}_t - \hat{R}_t\hat{A}_t\hat{P}_t = \varepsilon_2\hat{R}_t,\ \hat{Q}_t\hat{A}_t\hat{R}_t - \hat{R}_t\hat{A}_t\hat{Q}_t = 0,\\ \hat{P}_t\hat{B}_t\hat{Q}_t - \hat{Q}_t\hat{B}_t\hat{P}_t = 2\varepsilon_1\hat{P}_t,\ \hat{P}_t\hat{B}_t\hat{R}_t - \hat{R}_t\hat{B}_t\hat{P}_t = 0,\ \hat{Q}_t\hat{B}_t\hat{R}_t - \hat{R}_t\hat{B}_t\hat{Q}_t = -2\varepsilon_2\hat{R}_t,\\ \hat{P}_t\hat{C}_t\hat{Q}_t - \hat{Q}_t\hat{C}_t\hat{P}_t = \varepsilon_3\hat{R}_t,\ \hat{P}_t\hat{C}_t\hat{R}_t - \hat{R}_t\hat{C}_t\hat{P}_t = 0,\ \hat{Q}_t\hat{C}_t\hat{R}_t - \hat{R}_t\hat{C}_t\hat{Q}_t = 0,\end{gathered}$$

$$\begin{gathered}\hat{A}_t\hat{P}_t\hat{B}_t - \hat{B}_t\hat{P}_t\hat{A}_t = 2\tilde{\varepsilon}_1\hat{B}_t,\ \hat{A}_t\hat{P}_t\hat{C}_t - \hat{C}_t\hat{P}_t\hat{A}_t = \tilde{\varepsilon}_2\hat{C}_t,\ \hat{B}_t\hat{P}_t\hat{C}_t - \hat{C}_t\hat{P}_t\hat{B}_t = 0,\\ \hat{A}_t\hat{Q}_t\hat{B}_t - \hat{B}_t\hat{Q}_t\hat{A}_t = 2\tilde{\varepsilon}_1\hat{A}_t,\ \hat{A}_t\hat{Q}_t\hat{C}_t - \hat{C}_t\hat{Q}_t\hat{A}_t = 0,\ \hat{B}_t\hat{Q}_t\hat{C}_t - \hat{C}_t\hat{Q}_t\hat{B}_t = -\tilde{\varepsilon}_2\hat{C}_t,\\ \hat{A}_t\hat{R}_t\hat{B}_t - \hat{B}_t\hat{R}_t\hat{A}_t = \tilde{\varepsilon}_3\hat{C}_t,\ \hat{B}_t\hat{R}_t\hat{C}_t - \hat{C}_t\hat{R}_t\hat{B}_t = 0,\ \hat{A}_t\hat{R}_t\hat{C}_t - \hat{C}_t\hat{R}_t\hat{A}_t = 0.\end{gathered}$$

3.5. Remark on the quantum dynamics of noncanonically coupled anharmonic oscillators. It should be marked that the quantization of anharmonic oscillators meets with a principal difficulty. Namely, each representation of an isotopic pair may be splitted. After such splitting the elements of $V_1 \oplus V_2$ become odd, and therefore, nilpotent. It provides that *all terms in the classical equations of motion related to the higher (non–quadratic) terms of a hamiltonian are killed by the quantization.* Such effect is not realistic. Of course, one may consider well–defined quantum hamiltonians by use of the hidden Lie superalgebraic structure in a way analogous to the proposition above. The quantum dynamics will preserve the initial operator relations for such hamiltonians. However, there is no any *a priori* relation between it and classical one.

IV. CONCLUSIONS

So the classical and quantum dynamics of noncanonically coupled oscillators was investigated. It appears that though the classical one is well-defined for both harmonic and anharmonic oscillators, the quantum one is well-defined for the noncanonically coupled harmonic oscillators, admits a hidden (super)hamiltonian formulation, and thus, preserves the initial operator relations, but a naïve quantization of the anharmonic oscillators meets with principal difficulties.

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