

**Pairs of Dual Gabor Frame Generators  
with Compact Support and Desired Frequency Localization**

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# Pairs of dual Gabor frame generators with compact support and desired frequency localization

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## Abstract

Let  $g \in L^2(\mathbb{R})$  be a compactly supported function, whose integer-translates  $\{T_k g\}_{k \in \mathbb{Z}}$  form a partition of unity. We prove that for certain translation- and modulation parameters, such a function  $g$  generates a Gabor frame, with a (non-canonical) dual generated by a finite linear combination  $h$  of the functions  $\{T_k g\}_{k \in \mathbb{Z}}$ ; the coefficients in the linear combination are given explicitly. Thus,  $h$  has compact support, and the decay in frequency is controlled by the decay of  $\hat{g}$ . In particular, the result allows the construction of dual pairs of Gabor frames, where both generators are given explicitly, have compact support, and decay fast in the Fourier domain. We further relate the construction to wavelet theory. Letting  $D$  denote the dilation operator and  $B_N$  be the  $N$ th order B-spline, our results imply that there exist dual Gabor frames with generators of the type  $g = \sum c_k D T_k B_N$  and  $h = \sum \tilde{c}_k D T_k B_N$ , where both sums are finite. It is known that for  $N > 1$ , such functions can not generate dual wavelet frames  $\{D^j T_k g\}_{j,k \in \mathbb{Z}}$ ,  $\{D^j T_k h\}_{j,k \in \mathbb{Z}}$ .

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# 1 Introduction

A sequence  $\{f_k\}_{k=1}^{\infty}$  of elements in a separable Hilbert space  $\mathcal{H}$  is a *frame* if there exist constants  $A, B > 0$  such that

$$A \|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$

If at least the upper frame condition is satisfied,  $\{f_k\}_{k=1}^{\infty}$  is called a *Bessel sequence*. For  $a, b \in \mathbb{R}$ , consider the translation operator  $(T_a f)(x) = f(x - a)$  and the modulation operator  $(E_b f)(x) = e^{2\pi i b x} f(x)$ , both acting on  $L^2(\mathbb{R})$ . A *Gabor frame* is a frame for  $L^2(\mathbb{R})$  of the form  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  for some  $g \in L^2(\mathbb{R})$  and  $a, b > 0$ . It is well-known that if  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame, then there exists a function  $h \in L^2(\mathbb{R})$  such that

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb} T_{na} h \rangle E_{mb} T_{na} g, \quad \forall f \in L^2(\mathbb{R}); \quad (1)$$

the classical choice of  $h$  is  $h = S^{-1}g$ , where  $S$  is the frame operator. The function  $h = S^{-1}g$  is called the *canonical dual generator*.

If  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  is a Gabor frame and  $ab < 1$ , then there exist infinitely many functions  $h \in L^2(\mathbb{R})$  which satisfy (1); any such function, with the additional property that  $\{E_{mb} T_{na} h\}_{m,n \in \mathbb{Z}}$  satisfies the upper frame condition, is called a *dual generator*. We will call  $(g, h)$  a *pair of dual frame generators*.

The aim of this paper is to provide a construction of a pair of dual frame generators  $(g, h)$  for which the functions  $g, h$  are given *explicitly* and have compact support. Here "explicitly" should be understood in a very strict sense: we want that it is possible to implement the functions  $g$  and  $h$  without any approximation, and this means that  $g$  and  $h$  have to be given in terms of (finite linear combinations of) elementary functions. Our concrete examples concern B-splines.

As a consequence of our results it is possible to find explicitly given dual generators  $(g, h)$ , with compact support, and fast decay in the Fourier domain.

We finally relate our results to recent constructions of dual pairs of wavelet frames. For  $c > 0$ , let  $(D_c f)(x) = \frac{1}{c^{1/2}} f(x/c)$ ; for convenience, we will write  $D := D_{1/2}$ . Letting  $B_N$  denote the  $N$ th B-spline,  $N \geq 2$ , it is known that two finite linear combinations

$$\psi = \sum c_k D T_k B_N, \quad \text{and} \quad \tilde{\psi} = \sum \tilde{c}_k D T_k B_N$$

can not generate a pair of dual wavelet frames  $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$  and  $\{D^j T_k \tilde{\psi}\}_{j,k \in \mathbb{Z}}$ . Our results imply that there exist dual pairs of Gabor frames with such generators.

In the rest of this introduction we relate our work to the Gabor theory known so far.

Despite the fact that Gabor frames have attracted much attention during the last 20 years, only few examples are known where the frame generator  $g$  and an appropriate dual generator  $h$  are given explicitly (in the above sense). Often, a Gabor frame with prescribed properties is constructed, and a corresponding dual is found numerically, see [16].

Due to work by Janssen [10, 11] there exist characterizations of all pairs of dual Gabor frame generators, see Lemma 2.1 below, but apparently they have not yet been used for concrete constructions. One of the most famous Gabor frames is discussed in the following example.

**Example 1.1** It is well-known [12, 14, 15], that the Gaussian  $g(x) = e^{-x^2}$  generates a Gabor frame  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  if  $ab < 1$ . The Gaussian has exponential decay in time and frequency, and the same holds for the canonical dual generator. However, the Gaussian does not have compact support, and the canonical dual generator is not given explicitly in the sense described above: it has the form

$$S^{-1}g = \sum_{m,n \in \mathbb{Z}} c_{m,n} E_{m/a} T_{n/b} g$$

for an *infinitely* supported sequence  $\{c_{m,n}\}$ , see [7].

One of the main problems in Gabor frame theory is that the canonical dual generator is given in terms of the inverse frame operator, and therefore in general is difficult to find. Under certain circumstances, an explicit expression for the canonical dual generator is known, see [1]:

**Lemma 1.2** *Let  $N \in \mathbb{N}$ , and let  $g \in L^2(\mathbb{R})$  be a function with support in  $[0, N]$ . Assume that  $b \leq 1/N$  and that there exist  $A, B > 0$  such that*

$$A \leq G(x) := \sum_{n \in \mathbb{Z}} |g(x - na)|^2 \leq B, \text{ a.e. } x.$$

*Then  $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$  is a frame for  $L^2(\mathbb{R})$ , and the canonical dual generator is given by*

$$S^{-1}g = \frac{b}{G}g. \tag{2}$$

The function  $S^{-1}g$  in (2) has compact support, but unless we can take  $A = B$ , it is not given explicitly (in the sense described above); further, the expression for  $S^{-1}g$  does not provide information about the decay properties of the Fourier transform of  $S^{-1}g$ , which makes the constructed frame less attractive from a signal processing perspective.

The special case  $A = B$  corresponds to  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  being a tight frame; this case is discussed already in [5]. Given any function  $g$  whose integer-translates form a partition of unity and numbers  $a > 0$ ,  $b \in ]0, 1/N]$ , Lemma 1.2 shows that the function

$$h(x) = \sqrt{bg(x/a)}$$

generates a tight Gabor frame with frame bound one. However, due to the square-root, the behaviour of  $h$  in the frequency domain is not apparent. Our constructions below allow us to use functions  $g$  of the above type as frame generators: we do not obtain tight frames, but we obtain explicitly given dual generators, with well-controlled behavior in the frequency domain.

## 2 Explicitly given dual frame generators

Our starting point is a result, which is due to Janssen [10].

**Lemma 2.1** *Two Bessel sequences  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_{na}h\}_{m,n \in \mathbb{Z}}$  form dual frames for  $L^2(\mathbb{R})$  if and only if*

$$\sum_{k \in \mathbb{Z}} \overline{g(x - n/b - ka)} h(x - ka) = b\delta_{n,0}, \text{ a.e. } x \in [0, a]. \quad (3)$$

We are now ready to state one version of our main result. For convenience, we take the translation parameter  $a = 1$ .

**Theorem 2.2** *Let  $N \in \mathbb{N}$ . Let  $g \in L^2(\mathbb{R})$  be a real-valued bounded function with  $\text{supp } g \subseteq [0, N]$ , for which*

$$\sum_{n \in \mathbb{Z}} g(x - n) = 1. \quad (4)$$

*Let  $b \in ]0, \frac{1}{2N-1}]$ . Then the function  $g$  and the function  $h$  defined by*

$$h(x) = bg(x) + 2b \sum_{n=1}^{N-1} g(x+n) \quad (5)$$

generate dual frames  $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$  for  $L^2(\mathbb{R})$ .

**Proof.** By assumption, the function  $g$  has compact support and is bounded; by the definition (5), the function  $h$  share these properties. It follows that  $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$  are Bessel sequences. In order to verify that these sequences form dual frames, we use Lemma 2.1: according to (3) we need to check that for  $x \in [0, 1]$ ,

$$\sum_{k \in \mathbb{Z}} g(x - n/b - k)h(x - k) = b\delta_{n,0}. \quad (6)$$

By assumption  $g$  has support in  $[0, N]$ , so by construction  $h$  has support in  $[-N + 1, N]$ ; thus (6) is satisfied for  $n \neq 0$  whenever  $1/b \geq 2N - 1$ , i.e., if  $b \in ]0, \frac{1}{2N-1}]$ . For  $n = 0$ , the condition means that

$$\sum_{k \in \mathbb{Z}} g(x - k)h(x - k) = b, \quad x \in [0, 1]; \quad (7)$$

due to the compact support of  $g$ , this is equivalent to

$$\sum_{k=0}^{N-1} g(x + k)h(x + k) = b, \quad x \in [0, 1]. \quad (8)$$

To check that (8) holds, we use that for  $x \in [0, 1]$ ,

$$1 = \sum_{k=0}^{N-1} g(x + k);$$

thus, again for  $x \in [0, 1]$ ,

$$\begin{aligned}
1 &= \left( \sum_{k=0}^{N-1} g(x+k) \right)^2 \\
&= (g(x) + g(x+1) + \cdots + g(x+N-1)) \times \\
&\quad (g(x) + g(x+1) + \cdots + g(x+N-1)) \\
&= g(x)[g(x) + 2g(x+1) + 2g(x+2) + \cdots + 2g(x+N-1)] \\
&\quad + g(x+1)[g(x+1) + 2g(x+2) + 2g(x+3) + \cdots + 2g(x+N-1)] \\
&\quad + g(x+2)[g(x+2) + 2g(x+3) + 2g(x+4) + \cdots + 2g(x+N-1)] \\
&\quad + \cdots \\
&\quad + \cdots \\
&\quad + g(x+N-2)[g(x+N-2) + 2g(x+N-1)] \\
&\quad + g(x+N-1)[g(x+N-1)] \\
&= \frac{1}{b} \sum_{k=0}^{N-1} g(x+k)h(x+k).
\end{aligned}$$

Thus the condition (8) is satisfied.  $\square$

Recall that the B-splines  $B_N$ ,  $N \in \mathbb{N}$ , are given inductively by

$$B_1 = \chi_{[0,1]}, \quad B_{N+1} = B_N * B_1.$$

The B-spline  $B_N$  has support on the interval  $[0, N]$ . Furthermore, it is well-known that the integer-translates of any B-spline form a partition of unity. Thus, we obtain the following immediate consequence of Theorem 2.2 (see Corollary 3.1 for a more general statement).

**Corollary 2.3** *For any  $N \in \mathbb{N}$  and  $b \in ]0, \frac{1}{2N-1}]$ , the functions  $B_N$  and*

$$h_N(x) := bB_N(x) + 2b \sum_{n=1}^{N-1} B_N(x+n) \quad (9)$$

*generate a pair of dual frames  $\{E_{mb}T_n B_N\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_n h_N\}_{m,n \in \mathbb{Z}}$ .*

Some of the important features of the dual pair of frame generators  $(B_N, h_N)$  in Corollary 2.3 are:

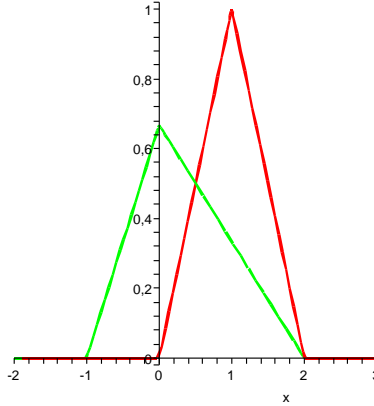
- The functions  $B_N$  and  $h_N$  are given explicitly in terms of elementary functions;
- $B_N$  and  $h_N$  have compact support, i.e., perfect time-localization;
- By choosing  $N$  sufficiently large, polynomial decay of  $\widehat{B}_N$  and  $\widehat{h}_N$  of any desired order can be obtained.

**Example 2.4** For the B-spline

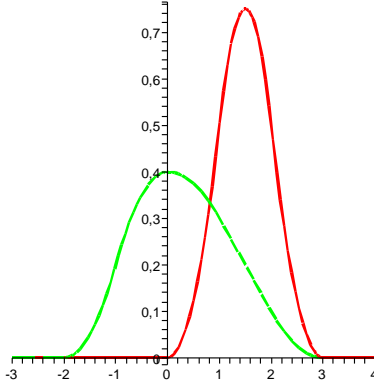
$$B_2(x) = \begin{cases} x & x \in [0, 1[ \\ 2 - x & x \in [1, 2[ \\ 0 & x \notin [0, 2[, \end{cases}$$

we can use Theorem 2.2 for  $b \in ]0, 1/3]$ . For  $b = 1/3$  we obtain the dual

$$\begin{aligned} h_2(x) &= \frac{1}{3} \begin{cases} x & x \in [0, 1[ \\ 2 - x & x \in [1, 2[ \\ 0 & x \notin [0, 2[, \end{cases} + \frac{2}{3} \begin{cases} x + 1 & x \in [-1, 0[ \\ 1 - x & x \in [0, 1[ \\ 0 & x \notin [-1, 1[ \end{cases} \\ &= \begin{cases} 2/3(x + 1) & x \in [-1, 0[ \\ 1/3(2 - x) & x \in [0, 2[ \\ 0 & x \notin [-1, 2[. \end{cases} \end{aligned} \quad (10)$$



**Figure 2.5** The B-spline  $B_2$  and the dual generator  $h_2$  in (10).



**Figure 2.6** The B-spline  $B_3$  and the dual generator  $h_3$  with  $b = 1/5$ .

We note that if we do not require  $g$  to be a spline, Theorem 2.2 even makes it possible to construct dual generators with exponential decay in the Fourier domain: we simply choose the function  $g$  such that  $\hat{g}$  decays exponentially.

We now aim at a version of Theorem 2.2 which is valid for any translation parameter  $a > 0$ . For this purpose we need the following lemma.

**Lemma 2.7** Let  $\{f_k\}$  and  $\{g_k\}$  be dual frames for a Hilbert space  $\mathcal{H}$ , and  $U : \mathcal{H} \rightarrow \mathcal{H}$  a unitary operator. Then  $\{Uf_k\}$  and  $\{Ug_k\}$  also form a pair of dual frames for  $\mathcal{H}$ .

**Theorem 2.8** Let  $N \in \mathbb{N}$ . Let  $g \in L^2(\mathbb{R})$  be a real-valued bounded function with  $\text{supp } g \subseteq [0, N]$ , for which

$$\sum_{n \in \mathbb{Z}} g(x - n) = 1.$$

Let  $a, b > 0$  be given such that  $ab \in ]0, \frac{1}{2N-1}]$ , and let

$$h(x) = abg(x) + 2ab \sum_{n=1}^{N-1} g(x + n).$$

Then the function  $D_a g$  and the function  $D_a h$  generate dual Gabor frames  $\{E_{mb}T_{na}D_a g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_{na}D_a h\}_{m,n \in \mathbb{Z}}$  for  $L^2(\mathbb{R})$ .

**Proof.** By assumptions and Theorem 2.2, the Gabor systems  $\{E_{mab}T_n g\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mab}T_n h\}_{m,n \in \mathbb{Z}}$  form dual frames; since  $D_a E_{mab}T_n = E_{mb}T_{na}D_a$ , the result follows from  $D_a$  being unitary and Lemma 2.7.  $\square$

We note that functions  $g$  of the type discussed in Theorem 2.8 were also considered in [9] by Gröchenig et al.: there it was proved that a continuous compactly supported function  $g$  whose integer-translates form a partition of unity does not generate a Gabor frame  $\{E_{mb}T_{na}g\}_{m,n \in \mathbb{Z}}$  for any  $a > 0$  when  $b = 2, 3, 4, \dots$ . Since the frame generator in Theorem 2.8 is the function  $D_a g$  and not  $g$  itself, there is no contradiction between [9] and our results.

### 3 Relationship to wavelet frames

Since the fundamental paper [4] by Daubechies (see also [5]) it has been known that there are many parallels between Gabor frame theory and wavelet frame theory. The purpose of this section is to relate our results to wavelet theory; in particular, we prove that certain functions which are known *not* to generate dual wavelet frames, in fact generate dual Gabor frames for certain parameters.

In order to make the relationship to wavelet theory clear, we change the notation slightly so it fits the standard notation in wavelet analysis.

Recall that a function  $\phi \in L^2(\mathbb{R})$  is called a scaling function if there exists a bounded 1-periodic function  $H$  such that

$$\hat{\phi}(2\gamma) = H(\gamma)\hat{\phi}(\gamma), \text{ a.e. } \gamma.$$

It is well-known that any scaling function has the partition of unity property in Theorem 2.2 (see Cor. 7.54 in [17], or any other standard text on wavelets). Thus, we obtain:

**Corollary 3.1** *Let  $N \in \mathbb{N}$  and  $b \in ]0, \frac{1}{2N-1}]$ . Any bounded scaling function  $\phi$  with support in  $[0, N]$  generates a Gabor frame  $\{E_{mb}T_n \phi\}_{m,n \in \mathbb{Z}}$ , with a dual generator given by*

$$\tilde{\phi}(x) = b\phi(x) + 2b \sum_{n=1}^{N-1} \phi(x+n).$$

It is known [3] that if  $\phi$  is a scaling function for which  $H(-1/4) \neq \frac{1}{\sqrt{2}}$  and  $\{T_k\phi\}_{k \in \mathbb{Z}}$  is a Riesz sequence, then two finite linear combinations

$$\psi = \sum c_k DT_k\phi \quad \text{and} \quad \tilde{\psi} = \sum \tilde{c}_k DT_k\phi \quad (11)$$

can not generate a pair of dual wavelet frames  $\{D^j T_k \psi\}_{j,k \in \mathbb{Z}}$  and  $\{D^j T_k \tilde{\psi}\}_{j,k \in \mathbb{Z}}$ . In particular, this result excludes the choice  $\phi = B_N$  for  $N > 1$ . As a consequence of Theorem 2.8 we can prove that functions of the type (11) very well can generate dual Gabor frames, in fact, with  $\psi = D\phi$ :

**Theorem 3.2** *Let  $b \in ]0, \frac{2}{2^N-1}]$  for some  $N \in \mathbb{N}$ . For any bounded scaling function  $\phi$  with support in  $[0, N]$ , the functions*

$$\psi = D\phi \quad \text{and} \quad \tilde{\psi} = \sum_{k=-2N}^0 \tilde{c}_k DT_k\phi = \frac{b}{2}D\phi + b \sum_{n=1}^N DT_{-2n}\phi$$

generate dual Gabor frames  $\{E_{mb}T_{n/2}\psi\}_{m,n \in \mathbb{Z}}$  and  $\{E_{mb}T_{n/2}\tilde{\psi}\}_{m,n \in \mathbb{Z}}$  for  $L^2(\mathbb{R})$ .

**Proof.** The choice  $a = 1/2$  in Theorem 2.8 leads to the frame generator  $\psi = D\phi$  and the dual generator

$$\begin{aligned} \tilde{\psi}(x) &= Dh(x) \\ &= \frac{b}{2}D\phi(x) + b \sum_{n=1}^N (D\phi)(x+n) \\ &= \frac{b}{2}D\phi(x) + b \sum_{n=1}^N (T_{-n}D\phi)(x) \\ &= \frac{b}{2}D\phi(x) + b \sum_{n=1}^N (DT_{-2n}\phi)(x). \quad \square \end{aligned}$$

In particular, Theorem 3.2 applies for  $\phi = B_N$  for any  $N \in \mathbb{N}$ .

For functions  $\phi$  with large support, the conditions in Theorem 3.2 forces the modulation parameter  $b$  to be small. It would be interesting to know whether similar frame constructions are possible for larger values of the product  $ab$ .

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