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SUBCOMPLEXES IN CURVED BGG–SEQUENCES

ANDREAS ČAP AND VLADIMÍR SOUČEK

Abstract. BGG–sequences offer a uniform construction for invariant differential operators for a large class of geometric structures called parabolic geometries. For locally flat geometries, the resulting sequences are complexes, but in general the compositions of the operators in such a sequence are nonzero. In this paper, we show that under appropriate torsion freeness and/or semi–flatness assumptions certain parts of all BGG sequences are complexes.

Several examples of structures, including quaternionic structures, hypersurface type CR structures and quaternionic contact structures are discussed in detail. In the case of quaternionic structures we show that several families of complexes obtained in this way are elliptic.

1. Introduction

Parabolic geometries form a large class of geometric structures containing examples like conformal, quaternionic, hypersurface type CR, and certain higher codimension CR structures. Via the interpretation as Cartan geometries with homogeneous model a generalized flag manifold, these structures can be studied in a surprisingly uniform way. An important and difficult problem is the construction of invariant differential operators for such geometries, i.e. operators which are intrinsic to the structure.

For the homogeneous model G/P (and geometries locally isomorphic to the homogeneous model) this problem can be reformulated in terms of representation theory. Via the theory of homomorphisms of generalized Verma modules one obtains an almost complete answer. In particular, invariant differential operators between sections of bundles associated to irreducible representations show up in patterns which can be described combinatorially in terms of the Weyl group of the Lie algebra $\mathfrak{g}$ of G. The different patterns are indexed by certain weights for $\mathfrak{g}$. For dominant integral weights (which covers most of the cases

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of interest), the resulting pattern forms a resolution of the finite dimensional irreducible representation of $\mathfrak{g}$, the celebrated (generalized) Bernstein–Gelfand–Gelfand (or BGG) resolution, see [5, 20].

The representation theory arguments leading to the BGG resolutions are combinatorial in nature and finding an explicit interpretation in terms of differential operators is a highly nontrivial task. An independent construction of the BGG resolutions in terms of differential operators was given in [15] and improved in [9]. This construction has the advantage that it works without changes for curved geometries, thus providing a construction of a large number of invariant differential operators for arbitrary parabolic geometries. The resulting patterns of operators are referred to as BGG–sequences. The construction also relates sections of the bundles showing up in a BGG–sequence to differential forms with values in a so–called tractor bundle and the (higher order) operators in the sequence to a covariant exterior derivative on these forms.

A drawback of the curved BGG–sequences is that the operators in the sequence have nontrivial compositions in general, so usually one does not obtain complexes in this way. The machinery of [15, 9] however is strong enough to give explicit formulae for the compositions. Starting from these formulae, we prove a simple criterion (in terms of weights), which ensures that some of the compositions do vanish provided that the harmonic curvature of the geometry satisfies certain restrictions, see Theorem 3.1. The necessary restrictions on the curvature usually include torsion freeness, but in some cases one also needs assumptions like semi–flatness, the most prominent of those being (anti)self duality in four dimensional conformal geometry.

Using finite dimensional representation theory, the weight conditions are then systematically studied in several examples. We describe the form of the BGG patterns and identify in each case several sub–patterns, for which the weight conditions are always satisfied. Under the appropriate curvature restrictions, these sub–patterns therefore give rise to subcomplexes in each BGG sequence.

The first examples we discuss are almost Grassmannian and almost quaternionic structures, which are different real forms of the same complex geometry. In dimension four, the structures can be equivalently described as conformal structures in split signature respectively Riemannian signature. The curvature conditions amount to torsion freeness for higher dimensions and (anti) self duality in dimension four. Our results in this case vastly generalize the complexes found in [23] and [3] for quaternionic structures. We should also mention that many known applications of curved BGG–sequences actually use known special cases of these subcomplexes, see [8].

Next, we discuss Lagrangean contact structures and partially integrable almost CR structures of hypersurface type, which again are different real forms of the same complex geometry. The appropriate curvature restriction again is torsion freeness, which is equivalent to integrability in the CR case.

Finally, we discuss the case of quaternionic contact structures as introduced by O. Biquard, see [6, 7]. The curvature condition is torsion freeness in the case of dimension seven, while it is automatically satisfied in higher dimensions.

The complexes for quaternionic structures found in [23] (which involve first order operators only) are elliptic. In the last section we extend this result to other families of subcomplexes in the quaternionic case, which involve operators of arbitrarily high orders.

The BGG sequence associated to the adjoint representation is closely related to the theory of infinitesimal deformations of parabolic geometries. For the examples of structures discussed in this paper, one of the subcomplexes in the adjoint BGG sequence can be naturally interpreted as a deformation complex in the subcategory of structures satisfying the curvature restrictions. This is discussed in [10].

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2. Background

In this section, we briefly review some basic facts about parabolic geometries and BGG sequences, mainly to fix the notation used in the sequel. More detailed information can be found in [11, 14, 15].

2.1. **Parabolic geometries.** A type of parabolic geometries is determined by a parabolic subgroup P in a semisimple Lie group G. Parabolic subgroups can be nicely described in terms of $|k|$–gradings of the Lie algebra $\mathfrak{g}$ of G. Details about $|k|$–gradings can be found in [28].

A $|k|$–grading on a semisimple Lie algebra $\mathfrak{g}$ is a decomposition

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$$

such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. Defining $\mathfrak{g}^i := \mathfrak{g}_i \oplus \dots \oplus \mathfrak{g}_k$, we obtain a filtration of $\mathfrak{g}$ and $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$. In particular, $\mathfrak{p} := \mathfrak{g}^0$ is a Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_+ := \mathfrak{g}^1$ is a nilpotent ideal in $\mathfrak{p}$.

On the group level, we define $G_0 \subset P \subset G$ as the subgroups of elements whose adjoint action preserves each grading component $\mathfrak{g}_i$ respectively each filtration component $\mathfrak{g}^i$. It turns out that $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra and $P = N_G(\mathfrak{p})$ is the usual parabolic subgroup associated to $\mathfrak{p}$, while G_0 has Lie algebra $\mathfrak{g}_0$. Moreover, exponential map defines a diffeomorphism from $\mathfrak{p}_+$ onto a closed normal subgroup $P_+ \subset P$, and P is the semidirect product of G_0 and P_+. In the complex case, parabolic subalgebras (up to conjugacy) are in bijective correspondence with sets of simple roots. This leads to a description of $|k|$–gradings in terms of Dynkin diagrams with crosses, see [4]. In the real case, there is a similar description in terms of Satake diagrams, see [28].

Parabolic geometries are then defined as Cartan geometries of type (G, P). This means that a parabolic geometry of type (G, P) on M consists of a principal P–bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. The *homogeneous model* of the geometry is given by the natural bundle $p : G \to G/P$ endowed with the left Maurer-Cartan form as a Cartan connection. A *morphism* of parabolic geometries is a principal bundle map which is compatible with the Cartan connections. In particular, any morphism is a local diffeomorphism.

The curvature of a Cartan connection ω is defined as the $\mathfrak{g}$–valued two–form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by the structure equation

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

where ξ and η are vector fields on $\mathcal{G}$ and the bracket is in $\mathfrak{g}$. The form K is horizontal and equivariant, so it may be interpreted as a two–form κ on M with values in the associated bundle $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$, the *adjoint tractor bundle*. The P–invariant filtration $\{\mathfrak{g}^i\}$ of $\mathfrak{g}$ gives rise to a filtration $\mathcal{A}M = \mathcal{A}^{-k}M \supset \dots \supset \mathcal{A}^k M$ by smooth subbundles and the Lie bracket on $\mathfrak{g}$ gives rise to an algebraic bracket $\{\ ,\ \}$ on $\mathcal{A}M$ making it into a bundle of filtered Lie algebras modeled on $\mathfrak{g}$.

The Cartan connection ω induces an isomorphism $TM \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. Hence there is a natural projection $\Pi : \mathcal{A}M \to \mathcal{T}M$ which induces an isomorphism $\mathcal{A}M/\mathcal{A}^0 M \cong TM$. Via this isomorphism, the filtration of $\mathcal{A}M$ descends to a filtration $TM = T^{-k}M \supset \dots \supset T^{-1}M$ of the tangent bundle by smooth subbundles. Applying the projection Π to the values of κ we obtain a TM–valued two–form κ_-, which is called the *torsion* of the Cartan connection ω. The geometry is called *torsion free* if this torsion vanishes.

Via the filtrations of TM and $\mathcal{A}M$ one has a natural notion of homogeneity for $\mathcal{A}M$–valued differential forms. In particular, we say that κ is homogeneous of degree $\geq \ell$ if $\kappa(T^iM, T^jM) \subset \mathcal{A}^{i+j+\ell}M$ for all $i, j = -k, \dots, -1$. A parabolic geometry is called *regular* if its curvature is homogeneous of degree ≥ 1. Note that torsion free parabolic geometries are automatically regular.

2.2. **Lie algebra homology and normalization.** Parabolic geometries are mainly interesting as an equivalent conceptual description for a large number of (seemingly very diverse) examples of geometric structures. Usually, the given geometric structure can be easily encoded into what is called a regular infinitesimal flag structure, see [14]. This consists of a filtration $\{T^iM\}$ of TM and a principal G_0–bundle $\mathcal{G}_0 \to M$ endowed with certain partially defined differential forms. Under a cohomological condition, which is satisfied for all the structures considered in this paper, one can then apply involved prolongation procedures (see [27, 21, 13]). These extend $\mathcal{G}_0$ to a principal P–bundle $p : \mathcal{G} \to M$ endowed with a Cartan connection ω. In particular, the given filtration of TM coincides with the one obtained from $(\mathcal{G}, \omega)$ as in 2.1 and $\mathcal{G}/P_+ \cong \mathcal{G}_0$. The resulting parabolic geometry is uniquely determined (up to isomorphism) if one in addition requires the curvature of ω to satisfy a normalization condition to be discussed below. This leads to an equivalence of categories between regular normal parabolic geometries and the underlying structures.

By forming associated bundles to the Cartan bundle, natural vector bundles on parabolic geometries of type (G, P) are defined by representations of the Lie group P. A simple class of such representations are obtained by trivially extending representations of G_0. In particular, all irreducible representations of P are obtained in this way. Since the group G_0 turns out to be reductive, its representation theory is well understood. If $\mathbb{E}$ is such a representation, then $\mathcal{G} \times_P \mathbb{E} \cong \mathcal{G}_0 \times_{G_0} \mathbb{E}$. Therefore, the corresponding natural vector bundles admit a direct interpretation in terms of the underlying geometric structure.

A second source of representations of P is restrictions of representations of G. The corresponding natural vector bundles are called tractor bundles. While from a geometrical point of view these are rather unusual objects, they are in some respects easy to deal with. For example, the Cartan connection ω induces a linear connection on each tractor bundle. The general theory of tractor bundles is developed in [12].

An important link between the two classes of representations we have discussed is given by Lie algebra homology. Let $\mathbb{V}$ be a representation of G, viewed as a representation of P by restriction. Infinitesimally, we in particular get a representation of $\mathfrak{p}_+$ on $\mathbb{V}$. The standard complex for computing the Lie algebra homology $H_*(\mathfrak{p}_+,\mathbb{V})$ has the form

$$0\to\mathbb{V}\xrightarrow{\partial^*}\mathfrak{p}_+\otimes\mathbb{V}\to\cdots\to\Lambda^k\mathfrak{p}_+\otimes\mathbb{V}\xrightarrow{\partial^*}\Lambda^{k+1}\mathfrak{p}_+\otimes\mathbb{V}\to\dots$$

Following the literature on parabolic geometries, we denote the standard differential by ∂^* and call it the *Kostant codifferential*. Explicitly, it is given by

$$\partial^*(Z_0\wedge\cdots\wedge Z_n\otimes v)=\textstyle\sum_{i=0}^n(-1)^{i+1}Z_0\wedge\cdots\hat{i}\cdots\wedge Z_n\otimes Z_i\cdot v+$$
$$+\textstyle\sum_{i<j}(-1)^{i+j}[Z_i,Z_j]\wedge\cdots\hat{i}\cdots\hat{j}\cdots\wedge Z_n\otimes v$$

with hats denoting omission.

Evidently, all spaces in the standard complex are representations of P and ∂^* is P–equivariant. Hence each of the homology groups $H_k(\mathfrak{p}_+,\mathbb{V})$ naturally is a P–module. Moreover, one easily verifies that P_+ acts trivially on the homology, so the representations $H_k(\mathfrak{p}_+,\mathbb{V})$ come from the subgroup G_0.

In [18], B. Kostant gave an explicit algorithm to compute, for each k, the G_0–module $H^k(\mathfrak{p}_+,\mathbb{V})$, which is dual to $H_k(\mathfrak{p}_+,\mathbb{V}^*)$, in the case when $\mathfrak{g}$ is complex and simple and $\mathbb{V}$ is a complex irreducible representation. Using basic tricks of the trade in Lie algebra cohomology one may also deal with the real cases, so all the modules in question are explicitly computable.

This construction has a direct geometric counterpart. We have noted above that $TM\cong\mathcal{G}\times_P\mathfrak{g}/\mathfrak{p}$, with the action coming from the adjoint representation. Now $(\mathfrak{g}/\mathfrak{p})^*\cong\mathfrak{p}_+$ as P–modules via the Killing form of $\mathfrak{g}$. Hence $T^*M\cong\mathcal{G}\times_P\mathfrak{p}_+=\mathcal{A}^1M$, so T^*M naturally is a bundle of nilpotent filtered Lie algebras with the restriction of the bracket $\{\ ,\ \}$ from 2.1. Denoting the tractor bundle corresponding to the representation $\mathbb{V}$ by VM, we see that $\mathcal{G}\times_P(\Lambda^k\mathfrak{p}_+\otimes\mathbb{V})\cong\Lambda^kT^*M\otimes VM$, and the codifferential induces a natural bundle map

$$\partial^*:\Lambda^kT^*M\otimes VM\to\Lambda^{k-1}T^*M\otimes VM.$$

Of course, we have $\partial^*\circ\partial^*=0$ and the kernel and image of ∂^* are natural subbundles. Their quotient is by construction isomorphic to $\mathcal{G}\times_P H_k(\mathfrak{p}_+,\mathbb{V})$. We denote this bundle by $H_k(T^*M,VM)$ since it is obtained from taking the pointwise Lie algebra homology of T^*_xM with coefficients in the module V_xM. In particular, there is a natural bundle map $\pi_H:\ker(\partial^*)\to H_k(T^*M,VM)$ and we denote by the same symbol the induced tensorial map on sections.

The bundle map ∂^* also induces a tensorial operator on VM–valued differential forms, which we denote by the same symbol. In the special case $\mathbb{V}=\mathfrak{g}$ and $k=2$, we obtain $\partial^*:\Omega^2(M,\mathcal{A}M)\to\Omega^1(M,\mathcal{A}M)$.

A parabolic geometry is called *normal* if and only if $\partial^*(\kappa) = 0$, where κ denotes the Cartan curvature. This is the normalization condition referred to above. If this is satisfied then one defines the *harmonic curvature* $\kappa_H := \pi_H(\kappa) \in \Gamma(H_2(T^*M, \mathcal{A}M))$. This is much easier to interpret in terms of the underlying structure than the full curvature κ. In the regular normal case, κ_H is a complete obstruction to local flatness.

2.3. **Strongly invariant operators.** These form a class of invariant differential operators, which algebraically are of particularly simple nature. The starting point for this is that the first jet prolongation of a vector bundle associated to $\mathcal{G}$ can be identified with an associated bundle to $\mathcal{G}$: For an arbitrary representation $\mathbb{E}$ of P, one defines $J^1\mathbb{E} := \mathbb{E} \oplus L(\mathfrak{g}/\mathfrak{p}, \mathbb{E})$ endowed with a certain P–module structure. For each parabolic geometry $(p : \mathcal{G} \to M, \omega)$, one has the natural vector bundle $EM := \mathcal{G} \times_P \mathbb{E}$. One shows that the first jet prolongation J^1EM of this bundle is naturally isomorphic to $\mathcal{G} \times_P J^1\mathbb{E}$ for that P–module structure.

This does not extend to higher jets, but it does work for higher semi–holonomic jet prolongations. One can put a P–module structure on the space $\bar{J}^r\mathbb{E} = \oplus_{i=0}^r \otimes^i (\mathfrak{g}/\mathfrak{p})^* \otimes \mathbb{E}$, such that the corresponding associated bundle is naturally isomorphic to the rth semi–holonomic jet prolongation $\bar{J}^r EM$. The upshot of this is that any P–module homomorphisms $\Psi : \bar{J}^r\mathbb{E} \to \mathbb{F}$ gives rise to a vector bundle map $\bar{J}^r EM \to FM$, and thus to a natural rth order differential operator $\Gamma(EM) \to \Gamma(FM)$. Operators arising in this way are called strongly invariant. See [15, 25] for more information on these issues.

2.4. **BGG sequences.** We next sketch the geometric construction of the generalized BGG resolutions as introduced in [15] and improved in [9]. A nice explanation of the role of BGG sequences and some applications can be found in [17, 16, 8].

Let $\mathbb{V}$ be a representation of G and for a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ let VM be the corresponding tractor bundle. As we have noted in 2.2, the Cartan connection ω induces a linear connection $\nabla = \nabla^{\mathbb{V}}$, called the tractor connection, on VM. This extends to an operation

$$d^\nabla : \Omega^k(M, VM) \to \Omega^{k+1}(M, VM)$$

on VM–valued differential forms, called the covariant exterior derivative. This operation is defined by taking the usual formula for the exterior derivative and replacing the action of vector fields on functions by the covariant derivative.

The tractor bundle VM carries a natural filtration by smooth subbundles (see [15]) and correspondingly one has the notion of homogeneity for VM–valued differential forms. The codifferential ∂^* from 2.2 is compatible with homogeneities. For regular normal parabolic geometries, also d^∇ is compatible with homogeneities. Now one can view the composition $\partial^* \circ d^\nabla$ as an operator acting on sections of the bundle $\operatorname{im}(\partial^*) \subset \Lambda^k T^*M \otimes VM$. This preserves homogeneities and one verifies that its homogeneous component of degree zero is tensorial and invertible. (For this to have geometric meaning one has to view the homogeneous component of degree zero as acting on sections of the associated graded bundles.) This implies that $\partial^* \circ d^\nabla$ itself is invertible, and the inverse is a (by construction natural) differential operator Q acting on sections of $\operatorname{im}(\partial^*)$.

Now we define an operator $L : \Gamma(H_k(T^*M, VM)) \to \Omega^k(M, VM)$ as follows: For a section α of $H_k(T^*M, VM)$ choose a representative $\varphi \in \Omega^k(M, VM)$, i.e. $\partial^*(\varphi) = 0$ and $\pi_H(\varphi) = \alpha$, and put

$$L(\alpha) := \varphi - Q\partial^* d^\nabla \varphi.$$

Since different choices for φ differ by sections of the subbundle $\operatorname{im}(\partial^*)$ and the operator $Q\partial^* d^\nabla$ is the identity on such sections, this is a well defined invariant operator, called the *splitting operator*. Let us collect its main properties:

Theorem. *For any $\alpha \in \Gamma(H_k(T^*M, VM))$ we have $\partial^*(L(\alpha)) = 0$, $\pi_H(L(\alpha)) = \alpha$, and $\partial^*(d^\nabla L(\alpha)) = 0$. These three properties characterize $L(\alpha)$.*

Proof. Choosing a representative φ for α, we have $L(\alpha) = \varphi - Q\partial^* d^\nabla \varphi$. Since $\partial^*(\varphi) = 0$ and Q has values in $\operatorname{im}(\partial^*)$ we see that $\partial^*(L(\alpha)) = 0$ and $\pi_H(L(\alpha)) = \pi_H(\varphi) = \alpha$. Since Q is inverse to $\partial^* d^\nabla$ on $\operatorname{im}(\partial^*)$ we see that $\partial^* d^\nabla L(\alpha) = \partial^* d^\nabla \varphi - \partial^* d^\nabla \varphi = 0$.

Conversely, assume that $\psi \in \Omega^k(M, VM)$ satisfies $\partial^* \psi = 0$, $\pi_H(\psi) = \alpha$, and $\partial^* d^\nabla \psi = 0$. Then we can use ψ as a representative for α in the construction of L, and since $\partial^* d^\nabla \psi = 0$ we get $L(\alpha) = \psi$. $\square$

Since $\partial^* d^\nabla L(\alpha) = 0$, we obtain an invariant differential operator

$$D = D^{\mathbb{V}} := \pi_H \circ d^\nabla \circ L : \Gamma(H_k(T^*M, VM)) \to \Gamma(H_{k+1}(T^*M, VM)),$$

and these operators form the BGG sequence.

It is well known that $d^\nabla \circ d^\nabla$ is given by the action of the curvature of ∇. The curvature of a tractor connection is given by the action of the Cartan curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$, so we obtain $d^\nabla d^\nabla \varphi = \kappa \wedge \varphi$. This is the alternation of $(\xi_0, \dots, \xi_{k+1}) \mapsto \kappa(\xi_0, \xi_1) \bullet \varphi(\xi_2, \dots, \xi_{k+1})$ with the bundle map $\bullet : \mathcal{A}M \times VM \to VM$ induced by the action of $\mathfrak{g}$ on $\mathbb{V}$.

For locally flat geometries, we have $\kappa = 0$ and the twisted de-Rham sequence is a complex. This easily implies that also the BGG sequence is a complex and both complexes compute the same cohomology, see [15]. In the curved case, the compositions of the BGG operators are nontrivial in general.

3. Subcomplexes

3.1. **Compositions of BGG operators.** Given a representation $\mathbb{V}$ of G, the representations $H_k(\mathfrak{p}_+, \mathbb{V})$ of G_0 are always completely reducible. Hence they split into a direct sum of irreducible components and correspondingly the bundles $H_k(T^*M, VM)$ decompose into a direct sum of smooth subbundles. Doing this for k and $k+1$, the BGG operator splits into a family of operators acting between sections of the individual components. Likewise, the composition of two consecutive BGG operators splits into components acting between the irreducible pieces. Assuming restrictions on the Cartan curvature we can derive a purely algebraic criterion for vanishing of pieces of the composition.

Theorem. *Let $\mathbb{E}_0$ be a G_0–submodule of $H_2(\mathfrak{p}_+, \mathfrak{g})$ and let $\mathbb{F}_1$ and $\mathbb{F}_2$ be irreducible components of $H_k(\mathfrak{p}_+, \mathbb{V})$ and $H_{k+2}(\mathfrak{p}_+, \mathbb{V})$, respectively. Suppose further that $\mathbb{F}_2$ is not isomorphic to a G_0-submodule of $\oplus_{i\geq 0}(\otimes^i \mathfrak{p}_+ \otimes \mathbb{E}_0 \otimes \mathbb{F}_1)$.*

Then for any torsion free normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$ whose harmonic curvature κ_H is a section of $E_0M \subset H_2(T^*M, \mathcal{A}M)$, the component in $F_2M \subset H_{k+2}(T^*M, VM)$ of the restriction of $D \circ D$ to $F_1M \subset H_k(T^*M, VM)$ vanishes identically.

Proof. By torsion freeness, the covariant exterior derivative d^∇ coincides with the twisted exterior derivative $d^{\mathbb{V}}$ used in [15]. Hence the constructions of the splitting operators and BGG operators described in 2.4 coincides with the construction in [15], so in particular all the operators are strongly invariant.

The Bianchi identity for linear connection implies $d^\nabla \kappa = 0$, which together with $\partial^*(\kappa) = 0$ shows that $\kappa = L(\kappa_H)$. By [15, Theorem 2.5] the fact that $\kappa_H \in \Gamma(E_0M)$ implies that $L(\kappa_H)$ is a section $\mathcal{G} \times_P \mathbb{E} \subset \Lambda^2 T^*M \otimes \mathcal{A}M$, where $\mathbb{E} \subset \Lambda^2 \mathfrak{p}_+ \otimes \mathfrak{g}$ is the P–submodule generated by $\mathbb{E}_0 \subset \ker(\Box)$. In particular, $\mathbb{E}$ is a quotient of $\oplus_{i\geq 0}(\otimes^i \mathfrak{p}_+ \otimes \mathbb{E}_0)$. Likewise, for $\alpha \in \Gamma(F_1M)$, the section $L(\alpha)$ has values in a subbundle associated to a quotient of $\oplus_{i\geq 0}(\otimes^i \mathfrak{p}_+ \otimes \mathbb{F}_1)$. Therefore, $\kappa \wedge L(\alpha)$ has values in a subbundle induced by a quotient of $\oplus_{i\geq 0}(\otimes^i \mathfrak{p}_+ \otimes \mathbb{E}_0 \otimes \mathbb{F}_1)$.

From the point of view of G_0 there is no difference between submodules and quotients. Hence if we form some semi–holonomic jet of $\kappa \wedge$

$L(\alpha)$, it will be a section of a subbundle corresponding to a representation, which, as a G_0–module, is contained in $\oplus_{i\geq 0}(\otimes^i\mathfrak{p}_+\otimes\mathbb{E}_0\otimes\mathbb{F}_1)$. Since our assumptions imply that there is no nonzero G_0–homomorphism from any such submodule to $\mathbb{F}_2$, we can complete the proof by showing that $D^2(\alpha)$ is obtained by applying a strongly invariant operator to $\kappa\wedge L(\alpha)$.

The latter fact has been proved in [9], but for the sake of completeness we give the simple argument: By definition, we have $D(\alpha)=\pi_H(d^\nabla L(\alpha))$. Hence we may use $d^\nabla L(\alpha)$ as a lift of $D(\alpha)$, so

$$LD(\alpha)=d^\nabla L(\alpha)-Q\partial^*(\kappa\wedge L(\alpha)).$$

Applying $\pi_H d^\nabla$ we conclude that

$$D^2(\alpha)=\pi_H\circ(\mathrm{id}-d^\nabla Q\partial^*)(\kappa\wedge L(\alpha)).$$

□

3.2. **The Hasse graph and BGG diagrams in the complex case.** To apply the vanishing Theorem systematically, we need to use certain facts concerning the decomposition of $H_*(\mathfrak{p}_+,\mathbb{V})$ into irreducible components. If $\mathfrak{g}$ is complex, and $\mathbb{V}$ is a complex irreducible representation, then $H_*(\mathfrak{p}_+,\mathbb{V})$ was completely described as a $\mathfrak{g}_0$–representation in [18]. The answer is remarkably uniform and is expressed using the Hasse graph and BGG diagrams. Let us briefly describe the result.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}\subset\mathfrak{g}$ a Cartan subalgebra and $\Delta\subset\mathfrak{h}^*$ the corresponding set of roots. Then the real subspace $\mathfrak{h}_0\subset\mathfrak{h}$ on which all roots have real values is a real form of $\mathfrak{h}$ and the Killing form induces a positive definite inner product on $\mathfrak{h}_0$. Fix a choice $\Delta_+\subset\Delta$ of a positive subsystem and let Δ_0 be the corresponding set of simple roots. For $\alpha\in\Delta$ let $\sigma_\alpha:\mathfrak{h}_0\to\mathfrak{h}_0$ denote the reflection in the hyperplane $\ker(\alpha)$. The Weyl group W of $\mathfrak{g}$ is the finite subgroup of the orthogonal group $O(\mathfrak{h}_0)$ generated by these reflections. Then also the reflections σ_j corresponding to simple roots $\alpha_j\in\Delta_0$ generate the group W. For $w\in W$, the length $|w|$ is defined as the smallest positive integer n such that $w=\sigma_{j_1}\circ\ldots\circ\sigma_{j_n}$. Apart from the evident action of W on $\mathfrak{h}_0^*$, which we write by $\lambda\mapsto w(\lambda)$, there is also the *affine action*. This is defined by $w{\cdot}\lambda:=w(\lambda+\delta)-\delta$ for $\lambda\in\mathfrak{h}_0^*$, where δ denotes half the sum of all positive roots.

The choices of $\mathfrak{h}$ and Δ_+ give rise to the standard Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$, which is the sum of $\mathfrak{h}$ and all positive root spaces. The *Hasse graph* for $\mathfrak{b}$ (see [5]) is the directed graph with vertices the elements of W and labeled arrows defined by $w\xrightarrow{\alpha}w'$ if and only if $|w'|=|w|+1$ and $\alpha\in\Delta_+$ is such that $w'=\sigma_\alpha\circ w$.

Now if $\mathfrak{p} \subset \mathfrak{g}$ is obtained from some $|k|$–grading then one can always choose $\mathfrak{h}$ and Δ_+ in such a way that $\mathfrak{b} \subset \mathfrak{p}$. Moreover, $\mathfrak{p}$ then is the standard parabolic subalgebra corresponding to a subset $\Sigma \subset \Delta_0$. Explicitly, a simple root lies in Σ if and only if the corresponding root space is contained in $\mathfrak{g}_1$. The root spaces of the other simple roots then lie in $\mathfrak{g}_0$. We also split $\Delta_+ = \Delta_+(\mathfrak{g}_0) \sqcup \Delta_+(\mathfrak{p}_+)$ according to positive root spaces being contained in the indicated subalgebras.

Associating to every $w \in W$ the subset $\Phi_w := \{\alpha \in \Delta_+ : w^{-1}(\alpha) \in -\Delta_+\} \subset \Delta_+$, one obtains a bijection between W the set of all subsets of Δ_+ having a certain property. The Hasse graph of $\mathfrak{p}$ is then defined as the subgraph of the Hasse graph of $\mathfrak{b}$ consisting of all vertices $w \in W^{\mathfrak{p}} := \{w \in W : \Phi_w \subset \Delta_+(\mathfrak{p}_+)\}$ and all edges connecting these vertices. It turns out that only elements of $\Delta_+(\mathfrak{p}_+)$ can occur as labels for the remaining arrows.

There is an alternative characterization of $W^{\mathfrak{p}}$: We say that a weight $\lambda \in \mathfrak{h}_0^*$ is $\mathfrak{g}$–dominant (respectively $\mathfrak{p}$–dominant) if $\langle \lambda, \alpha \rangle \geq 0$ for all $\alpha \in \Delta_0$ (respectively all $\alpha \in \Delta_0 \setminus \Sigma$). Then $w \in W^{\mathfrak{p}}$ if and only if for one (or equivalently any) $\mathfrak{g}$–dominant weight λ the weight $w{\cdot}\lambda$ is $\mathfrak{p}$–dominant. Given a $\mathfrak{g}$–dominant weight λ, we define the *BGG diagram* associated to λ to be the graph obtained from the Hasse graph of $\mathfrak{p}$ by replacing the edge w by $w{\cdot}\lambda$.

Following the conventions in [4], will label representations by the highest weight of the dual rather than the highest weight of the given representation. Equivalently, $\mathbb{V} = \mathbb{V}_\lambda$ if $-\lambda$ is the lowest weight of $\mathbb{V}$. We will use this notation for representations of both G and G_0. With this convention, the vertices in the BGG diagram associated to $\mathfrak{g}$–dominant integral weight λ, are exactly the labels of the irreducible components (with respect to G_0) of $H_*(\mathfrak{p}_+, \mathbb{V}_\lambda)$. The irreducible components of $H_k(\mathfrak{p}_+, \mathbb{V}_\lambda)$ are exactly the vertices corresponding to elements $w \in W^{\mathfrak{p}}$ such that $|w| = k$. The paper [18] even gives an explicit description of a highest weight vector in each component, which is often helpful in determining how the components sit inside of $\Lambda^*\mathfrak{p}_+ \otimes \mathbb{V}$.

There are efficient methods to work out the form of the labeled Hasse diagrams for low gradings (for details see [19]). We shall quote the results in cases of interests below. Now we can formulate the condition in the vanishing theorem 3.1 in terms of weights as follows.

Lemma. *Let $\mathbb{E}_0$ be an irreducible component of $H_2(\mathfrak{p}_+, \mathfrak{g})$ and let $\mathbb{F}_1 = \mathbb{F}_{\lambda_1} \subset H_k(\mathfrak{p}_+, \mathbb{V})$ and $\mathbb{F}_2 = \mathbb{F}_{\lambda_2} \subset H_{k+2}(\mathfrak{p}_+, \mathbb{V})$ be irreducible components.*

If $\mathbb{F}_2$ is isomorphic to a G_0-submodule of $\oplus_{i\geq 0}\,(\otimes^i \mathfrak{p}_+ \otimes \mathbb{E}_0 \otimes \mathbb{F}_1)$, then $\lambda_2 - \lambda_1$ is a weight of $\oplus_{i\geq 0}\,(\otimes^i \mathfrak{g}_- \otimes \mathbb{E}_0^*)$.

Proof. First note that $\mathbb{F}_2$ is isomorphic to a G_0-submodule of

$$\left(\oplus_{i\geq 0} \otimes^i \mathfrak{p}_+\right) \otimes \mathbb{E}_0 \otimes \mathbb{F}_1,$$

if and only if $\mathbb{F}_2^*$ is isomorphic to a G_0-submodule of

$$\left(\oplus_{i\geq 0} \left(\otimes^i \mathfrak{g}_-\right) \otimes \mathbb{E}_0^* \otimes \mathbb{F}_1^*\right).$$

Now the result follows from the following fact which is well known in representation theory of semisimple and reductive Lie algebras: Let $\mathbb{V}$ be an irreducible representation of highest weight λ and $\mathbb{W}$ an arbitrary finite dimensional representation. Then the highest weight of any irreducible component of $\mathbb{V} \otimes \mathbb{W}$ can be written as the sum of λ and some weight of $\mathbb{W}$. □

3.3. **Hasse graph and BGG diagrams in the real case.** A $|k|$–grading on a real semisimple Lie algebra $\mathfrak{g}$ induces a $|k|$–grading on the complexification $\mathfrak{g}^{\mathbb{C}}$. The subalgebras $\mathfrak{p}_+ \subset \mathfrak{p} \subset \mathfrak{g}$ complexify to their counterparts obtained from the complex $|k|$–grading. Using this, we can deduce the decomposition of $H_k(\mathfrak{p}_+, \mathbb{V})$ from the complex case discussed in 3.2 above.

Let us review some basic facts about representations of real Lie algebras, see [22, 24] for details. For a real Lie algebra $\mathfrak{a}$, a complex representation can be simply viewed as a real representation $\mathbb{V}$ together with an $\mathfrak{a}$–invariant complex structure $J : \mathbb{V} \to \mathbb{V}$. In this case also $-J$ is an $\mathfrak{a}$–invariant almost complex structure and the resulting representation of $\mathfrak{a}$ is called the *conjugate* of $\mathbb{V}$ and denoted by $\bar{\mathbb{V}}$. A complex representation of $\mathfrak{a}$ uniquely extends to a representation of the complexification $\mathfrak{a}^{\mathbb{C}}$, but on the level of $\mathfrak{a}^{\mathbb{C}}$ the relation between the representations $\mathbb{V}$ and $\bar{\mathbb{V}}$ is more involved than on the level of $\mathfrak{a}$.

If $\mathbb{V}$ is a real irreducible representation of $\mathfrak{a}$, then one can form the complexification $\mathbb{V}^{\mathbb{C}}$. If $\mathbb{V}$ does not admit an $\mathfrak{g}$–invariant complex structure, then $\mathbb{V}^{\mathbb{C}}$ is again irreducible. However if $\mathbb{V}$ does admit an invariant complex structure, then $\mathbb{V}^{\mathbb{C}} \cong \mathbb{V} \oplus \bar{\mathbb{V}}$.

Let us return to the question of decomposing $H_*(\mathfrak{p}_+, \mathbb{V})$ in the case of real $\mathfrak{g}$. If $\mathbb{V}$ is a complex representation of $\mathfrak{g}$, then $\Lambda^k \mathfrak{p}_+ \otimes \mathbb{V} \cong \Lambda^k \mathfrak{p}_+^{\mathbb{C}} \otimes_{\mathbb{C}} \mathbb{V}$ and this is compatible with the differentials. This easily implies that $H_*(\mathfrak{p}_+, \mathbb{V})$ is simply the restriction to $\mathfrak{g}_0 \subset \mathfrak{g}_0^{\mathbb{C}}$ of $H_*(\mathfrak{p}_+^{\mathbb{C}}, \mathbb{V})$. In particular, we obtain the same decomposition into irreducibles as in the complex case.

On the other hand, let us assume that the representation $\mathbb{V}$ does not admit a $\mathfrak{g}$–invariant complex structure. Then $\mathbb{V}^{\mathbb{C}}$ is irreducible, and one easily shows that $H_*(\mathfrak{p}_+^{\mathbb{C}}, \mathbb{V}^{\mathbb{C}})$ is the complexification of $H_*(\mathfrak{p}_+, \mathbb{V})$. For a $\mathfrak{p}$–dominant weight μ let $\bar{\mu}$ be the weight characterized by $\mathbb{E}_{\bar{\mu}} = \overline{\mathbb{E}_\mu}$.

Now for an irreducible component $\mathbb{E}_\mu \subset H_k(\mathfrak{p}_+^{\mathbb{C}}, \mathbb{V}^{\mathbb{C}})$ there are two possibilities. Either it is the complexification of a real irreducible component in $H_*(\mathfrak{p}_+, \mathbb{V})$. The other possibility is that also $\mathbb{E}_{\bar\mu} \subset H_k(\mathfrak{p}_+^{\mathbb{C}}, \mathbb{V}^{\mathbb{C}})$ and there is a complex irreducible component in $H_*(\mathfrak{p}_+, \mathbb{V})$ whose complexification is $\mathbb{E}_\mu \oplus \mathbb{E}_{\bar\mu}$. It is shown in [24] that the second possibility occurs if and only if $\mu \neq \bar\mu$.

Hence we obtain a complete description of the decomposition of $H_*(\mathfrak{p}_+, \mathbb{V})$ in the real case. If $\mathbb{V}$ is complex and has highest weight λ as a representation of $\mathfrak{g}^{\mathbb{C}}$ then the decomposition is given by the BGG diagram associated to λ. An entry μ in this diagram corresponds to the restriction of the complex representation $\mathbb{E}_\mu$ to $\mathfrak{g}_0 \subset \mathfrak{g}_0^{\mathbb{C}}$.

If $\mathbb{V}$ is real (i.e. does not admit an invariant complex structure) then let λ be the highest weight of the $\mathfrak{g}^{\mathbb{C}}$ representation $\mathbb{V}^{\mathbb{C}}$. Then the decomposition of $H_*(\mathfrak{p}_+, \mathbb{V})$ is obtained by identifying in the BGG diagram associated to λ each weight μ with $\bar\mu$. If $\mu = \bar\mu$ then the vertex corresponds to a real irreducible component, while vertices obtained by identifying μ with $\bar\mu \neq \mu$ correspond to complex irreducible components. In particular, we immediately obtain the following real version of lemma 3.2:

Lemma. *Let $\mathbb{V}$ be a real irreducible $\mathfrak{g}$-module. Let $\mathbb{E}_0$ be an irreducible component of $H_2(\mathfrak{p}_+, \mathfrak{g})$ and let $\mathbb{F}_1 \subset H_k(\mathfrak{p}_+, \mathbb{V})$ and $\mathbb{F}_2 \subset H_{k+2}(\mathfrak{p}_+, \mathbb{V})$ be complex irreducible components such that λ_i and $\bar\lambda_i$ are the labels of the irreducible components of their complexifications.*

If $\mathbb{F}_2$ is isomorphic to a G_0-submodule of $\oplus_{i\geq 0} \left(\otimes^i \mathfrak{p}_+ \otimes \mathbb{E}_0 \otimes \mathbb{F}_1\right)$, then at least one of $\lambda_2 - \lambda_1$, $\bar\lambda_2 - \lambda_1$, $\lambda_2 - \bar\lambda_1$, and $\bar\lambda_2 - \bar\lambda_1$ is a weight of $\oplus_{i\geq 0} \left(\otimes^i \mathfrak{g}_-^{\mathbb{C}} \otimes (\mathbb{E}_0^{\mathbb{C}})^*\right)$.

3.4. We next study the Hasse and BGG diagrams in the case relevant for quaternionic and Grassmannian geometries. Let us consider the algebra $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{C})$ with the grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ corresponding to the Dynkin diagram (with $n+1$ nodes)

$$\overset{\alpha_1}{\circ}\!\!-\!\!\overset{\alpha_2}{\times}\!\!-\!\!\overset{\alpha_3}{\circ}\!- \cdots -\!\overset{a_{n+1}}{\circ} \quad ,$$

where $\alpha_i = e_i - e_{i+1}$ are the simple roots in the standard notation for the A_n series. The other positive roots for $\mathfrak{g}$ are then given by $\beta^{ij} := \alpha_i + \ldots + \alpha_j = e_i - e_{j+1}$ with $1 \leq i < j \leq n+1$ and we put $\beta^{ii} = \alpha_i$.

The semisimple part of $\mathfrak{g}_0$ is $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$. We get $\Delta_+(\mathfrak{g}_0) = \{\beta^{11}\} \cup \{\beta^{ij} | 3 \leq i \leq j \leq n+1\}$ with the two sets corresponding to the two summands, and $\Delta_+(\mathfrak{p}_+) = \{\beta^{1j} | 2 \leq j \leq n+1\} \cup \{\beta^{2j} | 2 \leq j \leq n+1\}$.

The Hasse graph can be computed by the methods of the book [4], or, together with labels over the arrows, by the methods of [19]. It has a triangular shape, whose form is easily seen from the case $n = 4$ given below.

$$\begin{tikzcd}
w_{0,0} \arrow[dr] & & w_{1,1} \arrow[dr] & & w_{2,2} \arrow[dr] & & w_{3,3} \arrow[dr] & & w_{4,4} \\
& w_{0,1} \arrow[ur] \arrow[dr] & & w_{1,2} \arrow[ur] \arrow[dr] & & w_{2,3} \arrow[ur] \arrow[dr] & & w_{3,4} \arrow[ur] & \\
& & w_{0,2} \arrow[ur] \arrow[dr] & & w_{1,3} \arrow[ur] \arrow[dr] & & w_{2,4} \arrow[ur] & & \\
& & & w_{0,3} \arrow[ur] \arrow[dr] & & w_{1,4} \arrow[ur] & & & \\
& & & & w_{0,4} \arrow[ur] & & & &
\end{tikzcd}$$

In general, the elements of length k have the form $w_{i,j}$ with $i+j = k$ and $0 \leq i \leq j \leq n$. They can be computed explicitly, but we do not need the result. Concerning the labels of the arrows, the left edge of the diagram has the form

$$w_{0,0} \xrightarrow{\beta^{22}} w_{0,1} \xrightarrow{\beta^{23}} w_{0,2} \xrightarrow{\beta^{24}} \dots \xrightarrow{\beta^{2,n}} w_{0,n-1} \xrightarrow{\beta^{2,n+1}} w_{0,n},$$

while the right edge of the diagram looks like

$$w_{0,n} \xrightarrow{\beta^{12}} w_{1,n} \xrightarrow{\beta^{13}} w_{2,n} \xrightarrow{\beta^{14}} \dots \xrightarrow{\beta^{1,n}} w_{n-1,n} \xrightarrow{\beta^{1,n+1}} w_{n,n}.$$

In the rest of the diagram, parallel arrows have the same label.

For any representation $\mathbb{V}$ we will denote the corresponding splitting of the Lie algebra homology groups as $H_k(\mathfrak{p}_+, \mathbb{V}) = \oplus H_{i,j}(\mathfrak{p}_+, \mathbb{V})$. In particular, we have $H_2(\mathfrak{p}_+, \mathfrak{g}) = H_{0,2}(\mathfrak{p}_+, \mathfrak{g}) \oplus H_{1,1}(\mathfrak{p}_+, \mathfrak{g})$.

Proposition. *Put* $\mathbb{E}_0 := H_{1,1}(\mathfrak{p}_+, \mathfrak{g})$. *Then we have:*
(1) All weights of $\oplus_{i\geq 0} (\otimes^i \mathfrak{g}_- \otimes (\mathbb{E}_0)^*)$ *have the form*

$$-m_1\alpha_1 - m_2\alpha_2 + \textstyle\sum_{i=3}^{n+1} m_i\alpha_i$$

for some integers $m_1, \dots, m_{n+1}$ *such that* $0 < m_1 < m_2$.
(2) For all $i = 0, \dots, n-2$; $j = i, \dots, n-2$ *and any* $\mathfrak{g}$*-dominant integral weight* λ, *we have*

$$w_{i,j+2}\cdot\lambda - w_{i,j}\cdot\lambda = m_2\alpha_2 + \dots + m_{n+1}\alpha_{n+1}$$

for some integers $m_2, \dots, m_{n+1}$.
(3) For all $j = 2, \dots, n$; $i = 0, \dots, j-2$ *and any* $\mathfrak{g}$*-dominant integral weight* λ, *we have*

$$w_{i+2,j}\cdot\lambda - w_{i,j}\cdot\lambda = -m(\alpha_1 + \alpha_2) + m_3\alpha_3 + \dots + m_{n+1}\alpha_{n+1}$$

for some integers $m, m_3 \dots, m_{n+1}$.

Proof. (1) The standard recipes from [4] show that in terms of the fundamental weights λ_i for $\mathfrak{g}$ the highest weight of $\mathbb{E}_0^*$ is given by $-4\lambda_2 + 3\lambda_3 + \lambda_{n+1}$. In particular, the action of the semisimple part $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(n,\mathbb{C})$ of $\mathfrak{g}_0$ is only via the second summand. Therefore, any weight of $\mathbb{E}_0^*$ is obtained by subtracting roots of the form $\beta^{i,j}$ with $i \geq 3$ from the highest weight. In terms of simple roots, the highest weight of $\mathbb{E}_0^*$ reads as $-\alpha_1 - 2\alpha_2 + \alpha_3 + \cdots + \alpha_{n+1}$, and hence any weight of $\mathbb{E}_0^*$ has the form $-\alpha_1 - 2\alpha_2 + m_3\alpha_3 + \cdots + m_{n+1}\alpha_{n+1}$. On the other hand, the weights of $\mathfrak{g}_-$ are exactly the elements of $-\Delta_+(\mathfrak{p}_+)$. Since these are either of the form $-\alpha_1 - \alpha_2 - \cdots - \alpha_j$ or of the form $-\alpha_2 - \cdots - \alpha_j$, the claim follows.
(2) If we have a labeled arrow $w \xrightarrow{\alpha} w'$ in the Hasse diagram, then $w' = \sigma_\alpha(w)$, and hence for a $\mathfrak{g}$–dominant weight λ the difference $w'{\cdot}\lambda - w{\cdot}\lambda$ is an integer multiple of α. Now from above we see that the Hasse diagram contains $w_{i,j} \xrightarrow{\beta^{2,j+2}} w_{i,j+1} \xrightarrow{\beta^{2,j+3}} w_{i,j+2}$, and the claim follows since $\beta^{2,\ell} = \alpha_2 + \cdots + \alpha_{\ell+1}$.
(3) This is similar as in (2) taking into account that the Hasse diagram contains the part $w_{i,j} \xrightarrow{\beta^{1,i+2}} w_{i+1,j} \xrightarrow{\beta^{1,i+3}} w_{i+2,j}$ and that $\beta^{1,\ell} = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell$. □

3.5. **Grassmannian and quaternionic structures.** There are two real forms of the grading considered in 3.4 which lead to well known geometric structures. Since we are dealing with a $|1|$–grading here, an infinitesimal flag structure of type (G, P) on M (which is equivalent to a regular normal parabolic geometry, see 2.2) is simply a first order G_0–structure, i.e. a reduction of the frame bundle of M to the structure group G_0.

Putting $G = SL(n+2,\mathbb{R})$, the subgroup P turns out to be the stabilizer of a plane and $G_0 \cong S(GL(2,\mathbb{R}) \times GL(n,\mathbb{R})) \subset GL(2n,\mathbb{R})$. Hence these geometries exist on manifolds of even dimension $2n$, and they are usually called *almost Grassmannian structures*, see for example [1, chapters 6 and 7] and [2] for the complex analog of these geometries. Essentially, such a structure is given by an isomorphism from the tangent bundle TM to the tensor product $E^* \otimes F$ for two auxiliary bundles E and F of rank 2 and n, respectively.

The other choice of interest is $G = PSL(n+1,\mathbb{H})$, so $\mathfrak{g}$ is a real form of $\mathfrak{sl}(2n+2,\mathbb{C})$. Then P turns out to be the stabilizer of a quaternionic line in $\mathbb{H}^{n+1}$ and $G_0 \cong S(GL(1,\mathbb{H})GL(n,\mathbb{H})) \subset GL(4n,\mathbb{R})$. The resulting geometry is called an *almost quaternionic structure* on a manifold M of dimension $4n$, see [23]. It is given by a rank 3subbundle $Q \subset L(TM,TM)$ which can be locally spanned by I, J, and IJ for two anti commuting almost complex structures I and J on M. Lifting the structure to a two fold covering of G_0 (which corresponds to replacing G by $SL(n+1,\mathbb{H})$ and locally is uniquely possible), this is equivalent to a tensor product decomposition of $TM\otimes\mathbb{C}$ into a factor of rank 2 and one of rank n, which shows the similarity to the almost Grassmannian case.

Assume that we have a manifold M equipped with one of these two types of structures, $\mathbb{V}$ is an irreducible representation of G and VM is the corresponding tractor bundle. Then we have the bundles $H_k(T^*M,VM)$ from 2.2, and they split according to the decomposition of $H_k(\mathfrak{p}_+,\mathbb{V})$. One verifies directly that the BGG diagram for $\mathfrak{g}^{\mathbb{C}}$ as described in 3.4 above may never contain two conjugate weights, so by 3.3 and 3.4 we always get $H_k(T^*M,VM)=\oplus H_{i,j}(T^*M,VM)$ with $i+j=k$ and $1\leq i\leq j\leq n$ (respectively $2n$ in the Grassmannian case), with the notation following 3.4. Restricting a BGG operator D to sections of one component $H_{i,j}(T^*M,VM)$ we obtain a splitting $D=D^{1,0}+D^{0,1}$ with the two components having values in sections of $H_{i+1,j}(T^*M,VM)$ and $H_{i,j+1}(T^*M,VM)$, respectively.

In particular, $H_2(T^*M,\mathcal{A}M)=H_{0,2}(T^*M,\mathcal{A}M)\oplus H_{1,1}(T^*M,\mathcal{A}M)$ and accordingly the harmonic curvature decomposes into two parts. The part with values in $H_{0,2}(T^*M,\mathcal{A}M)$ in both cases can be determined as a specific component of the torsion of an arbitrary linear connection on TM which is compatible with the G_0–structure. This component is independent of the choice of the connection and it is the only part of the torsion that cannot be eliminated by changing the connection. Hence it is exactly the obstruction to torsion freeness in the sense of first order structures, and its vanishing is also equivalent to torsion freeness of the corresponding regular normal parabolic geometry. Torsion free geometries are usually referred to as *Grassmannian* respectively *quaternionic* structures.

Theorem. *Let M be a smooth manifold of dimension $2n$ endowed with a Grassmannian structure or a quaternionic structure (which requires n to be even). Let $\mathbb{V}$ be an irreducible representation of G and let VM be the corresponding tractor bundle. For $1\leq i\leq j\leq n$ put $\mathcal{H}_{i,j}:=H_{i,j}(T^*M,VM)$. Then the BGG sequence associated to VM contains the subcomplexes*

$$\mathcal{H}_{j,j}\xrightarrow{D^{0,1}}\mathcal{H}_{j,j+1}\xrightarrow{D^{0,1}}\dots\xrightarrow{D^{0,1}}\mathcal{H}_{j,n}\qquad \text{for } j=0,\dots,n-2$$

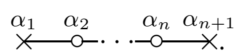
$$\mathcal{H}_{0,j}\xrightarrow{D^{1,0}}\mathcal{H}_{1,j}\xrightarrow{D^{1,0}}\dots\xrightarrow{D^{1,0}}\mathcal{H}_{j,j}\qquad \text{for } j=2,\dots,n$$

Proof. Since the assumptions of Theorem 3.1 are satisfied, and the BGG diagrams have the same form as for the complexification, we can use the weight condition from Proposition 3.4. For the compositions $D^{0,1} \circ D^{0,1}$, we see from part (2) of Proposition 3.4 that $\lambda_2 - \lambda_1$ is a linear combination of $\alpha_2, \dots, \alpha_{n+1}$ only. For the composition $D^{1,0} \circ D^{1,0}$ we see from part (3) of Proposition 3.4 that writing $\lambda_2 - \lambda_1$ as a linear combination of the α_i, the roots α_1 and α_2 have the same coefficient. Now the claim follows in both cases from part (1) of Proposition 3.4. □

Remark. (1) In the quaternionic case, this result vastly generalizes [23] and [3]. Indeed, the complexes in [23] are the $D^{0,1}$–complexes starting at $\mathcal{H}_{0,0}$ in the special case that $\mathbb{V}$ is a symmetric power of the dual of the standard representation. The paper [3] contains the $D^{1,0}$–complexes starting at $\mathcal{H}_{0,n}$ for arbitrary $\mathbb{V}$.

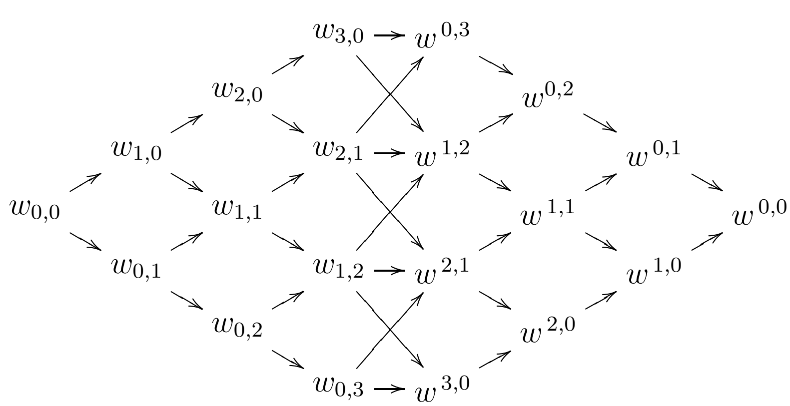
(2) The complexes constructed in the Theorem in general contain operators of arbitrarily high orders. For example in the $D^{0,1}$–complex starting at $\mathcal{H}_{0,0}$ the orders look as follows: Suppose that $\mathbb{V} = \mathbb{V}_\lambda$ and $\lambda = a_1\lambda_1 + \dots + a_{n+1}\lambda_{n+1}$ in terms of the fundamental weights. Then for each of the operators in the complex there is a unique i such that the order is $a_i + 1$. In particular, among these complexes the ones contained [23] are exactly those in which all operators are first order.

3.6. Next we study the case $\mathfrak{g} = \mathfrak{sl}(n+2, \mathbb{C})$ with $n \geq 2$, with the $|2|$–grading corresponding to the Dynkin diagram



We continue to use the notation from 3.4 for roots. The semisimple part of $\mathfrak{g}_0$ is $\mathfrak{sl}(n, \mathbb{C})$ and $\Delta_+(\mathfrak{g}_0) = \{\beta^{ij} | 2 \leq i \leq j \leq n\}$. On the other hand $\Delta_+(\mathfrak{p}_+)$ contains all $\beta^{1,j}$ and all $\beta^{i,n+1}$, and the root space of $\beta^{1,n+1}$ coincides with $\mathfrak{g}_2$.

The general shape of the Hasse diagram can be seen from the example $n = 3$:



Again the explicit form of the elements of $W^{\mathfrak{p}}$ is not important for our purposes. What we mainly need is that as in 3.4 parallel (or almost parallel) arrows have the same labels. In particular, we have sequences of $n+1$ vertices each, which always either go up or down. For the upward going sequences the labels over the arrows are (in the right order) $\beta^{1,1}, \beta^{1,2}, \dots, \beta^{1,n}$, while for the downward going ones they are $\beta^{n+1,n+1}, \beta^{n,n+1}, \dots, \beta^{2,n+1}$.

For any representation $\mathbb{V}$ of $\mathfrak{g}$ and $k \leq n$ we therefore obtain the splitting $H_k(\mathfrak{p}_+, \mathbb{V}) = \oplus H_{i,j}(\mathfrak{p}_+, \mathbb{V})$ with the sum over all $i, j \geq 0$ such that $i + j = k$. For $k > n$ we obtain $H_k(\mathfrak{p}_+, \mathbb{V}) = \oplus H^{i,j}(\mathfrak{p}_+, \mathbb{V})$ with the sum over all $i, j \geq 0$ such that $i + j = 2n + 1 - k$. Similarly to Proposition 3.4 one proves

Proposition. *Put $\mathbb{E}_0 := H_{1,1}(\mathfrak{p}_+, \mathfrak{g})$. Then we have:*
(1) All weights of $\oplus_{i \geq 0} \left(\otimes^i \mathfrak{g}_- \otimes (\mathbb{E}_0^{\mathbb{C}})^*\right)$ have the form

$$-m_1\alpha_1 - m_{n+1}\alpha_{n+1} + \sum_3^{n+1} m_i\alpha_i$$

for integers $m_1, \dots, m_{n+1}$ such that $m_1, m_{n+1} > 0$.
(2) Let μ_1 and μ_2 be two weights which are contained in an up going sequence in the BGG diagram of a $\mathfrak{g}$–dominant integral weight λ. Then $\mu_2 - \mu_1$ can be written as a linear combination of $\alpha_1, \dots, \alpha_n$.
(3) Let μ_1 and μ_2 be two weights which are contained in a down going sequence in the BGG diagram of a $\mathfrak{g}$–dominant integral weight λ. Then $\mu_2 - \mu_1$ can be written as a linear combination of $\alpha_2, \dots, \alpha_{n+1}$.

3.7. **Lagrangean contact structures.** There are various real forms of the situation discussed in 3.6 which are of interest in geometry. Putting $G := SL(n+2, \mathbb{R})$, one obtains Lagrangean (or Legendrean) contact structures, see [26] or [11]. Such a structure on a manifold M of dimension $2n+1$ is given by a codimension one subbundle $H \subset TM$ which defines a contact structure, and a fixed decomposition of $H = E \oplus F$ as the direct sum of two Legendrean subbundles. This means the the Lie bracket of two sections of E (or two sections of F) is a section of H.

Since we are dealing with a split real form here, all homology groups (and hence also the corresponding vector bundles) split according to the Hasse diagram discussed in 3.6. In particular, there are three components in the harmonic curvature. The $(0,2)$– and $(2,0)$–parts are torsions which are the obstructions to integrability of the subbundles $E, F \subset TM$. Vanishing of these two components is equivalent to torsion freeness of the corresponding parabolic geometry. Parallel to the proof of Theorem 3.5, Proposition 3.6 leads to

Theorem. *Let M a smooth manifold endowed with a torsion free Lagrangian contact structure. Then the BGG sequence associated to any finite dimensional irreducible representation of $\mathfrak{g}$ splits according to the Hasse diagram in 3.6 and any upgoing or downgoing subsequence is a complex.*

3.8. **CR structures.** The second class of interesting structures is obtained from $G := PSU(p+1, q+1)$ with $p \geq q$ and $p+q=n$. The resulting structures are partially integrable almost CR structures of hypersurface type which are non–degenerate of signature (p, q), see [13]. The analogy to Lagrangean contact structures can be seen by passing to the complexified tangent bundle. In particular, starting with a complex representation of G, the situation is completely parallel to the one discussed in 3.7.

There is a difference however, in the case of real representations. If $\mathbb{V}$ is a real representation and λ is the highest weight of $\mathbb{V}^{\mathbb{C}}$ then in the notation of 3.6 one has $\overline{(w_{i,j} \cdot \lambda)} = w_{j,i} \cdot \lambda$ and $\overline{(w^{i,j} \cdot \lambda)} = w^{j,i} \cdot \lambda$, see [24]. Hence the splitting of the real homologies $H_*(\mathfrak{p}_+, \mathbb{V})$ is obtained from the Hasse diagram in 3.6 by identifying the $w_{i,j}$ with $w_{j,i}$ as well as $w^{i,j}$ with $w^{j,i}$. Moreover, vertices with $i \neq j$ correspond to complex subrepresentations in $H_*(\mathfrak{p}_+, \mathbb{V})$ while vertices with $i = j$ correspond to real subrepresentations. The resulting picture for $n = 3$ looks as

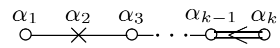
$$\begin{tikzcd}
 & & & w_{3,0} \arrow[r] \arrow[ddr] & w^{0,3} \arrow[dr] & & & \\
 & & w_{2,0} \arrow[ur] \arrow[dr] & & & w^{0,2} \arrow[dr] & & \\
 & w_{1,0} \arrow[ur] \arrow[dr] & & w_{2,1} \arrow[r] \arrow[uur] & w^{1,2} \arrow[ur] \arrow[dr] & & w^{0,1} \arrow[dr] & \\
w_{0,0} \arrow[ur] & & w_{1,1} \arrow[ur] & & & w^{1,1} \arrow[ur] & & w^{0,0}
\end{tikzcd}$$

In particular, since the adjoint representation is real, there are only two components in the harmonic curvature. There is just one torsion, which is represented by the Nijenhuis tensor, and hence is exactly the obstruction to integrability of the almost CR structure. Integrable structures are usually referred to as CR structures.

Theorem. *Let M be smooth manifold endowed with a non–degenerate CR structure of hypersurface type.*
(i) If $\mathbb{V}$ is an complex irreducible representation of G, then the associated BGG sequence splits according to the Hasse diagram from 3.6 and any upgoing or downgoing subsequence is a complex.
(ii) If $\mathbb{V}$ is a real irreducible representation of G, then the associated BGG according to the diagram above, and any upgoing or downgoing subsequence is a complex.

Proof. The complex case is done as before. For the real case, we use Lemma 3.3. The differences $\lambda_2 - \lambda_1$ and $\bar\lambda_2 - \bar\lambda_1$ can be handled as before. The differences $\bar\lambda_2 - \lambda_1$ and $\lambda_2 - \bar\lambda_1$ also cannot be among the weights described in part (1) of Proposition 3.6, since in this difference either α_1 or α_{n+1} must occur with a positive coefficient. $\square$

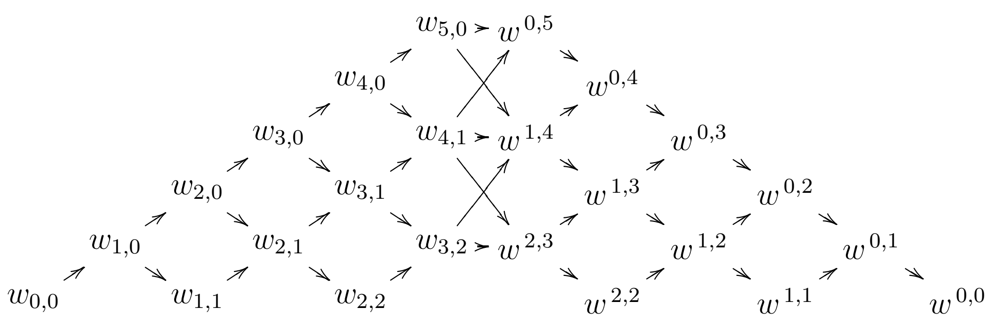
3.9. The last example we consider is $\mathfrak{g} = \mathfrak{sp}(2k, \mathbb{C})$ for $k \geq 3$ with the $|2|$–grading described by the Dynkin diagram



The positive roots are given by $\beta^{i,j} = \alpha_i + \ldots + \alpha_j$ for $1 \leq i \leq j \leq k$ and $\gamma^{i,j} = \beta^{i,k-1} + \beta^{jk}$ with $1 \leq i \leq k-1$ and $1 \leq j \leq k$. The semisimple part of $\mathfrak{g}_0$ is $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sp}(2(k-2), \mathbb{C})$. We have

$$\Delta_+(\mathfrak{p}_+) = \{\beta^{i,j} : i = 1, 2; 2 \leq j \leq k\} \cup \{\gamma^{i,j} : i = 1, 2; 1 \leq j \leq k-1\}.$$

The grading component $\mathfrak{g}_2$ consists of the root spaces of $\gamma^{1,1}$, $\gamma^{1,2}$, and $\gamma^{2,2}$. The general shape of the Hasse diagram can be seen from the example $k = 4$, which looks as



For general k, the left edge of the diagram, including the labels of the arrows, has the form

$$w_{0,0} \xrightarrow{\beta^{2,2}} \ldots \xrightarrow{\beta^{2,k-1}} w_{k-2,0} \xrightarrow{\gamma^{2,2}} w_{k-1,0} \xrightarrow{\beta^{2,k}} w_{k,0} \xrightarrow{\gamma^{2,k-1}} \ldots \xrightarrow{\gamma^{2,3}} w_{2k-3,0}.$$

The right edge of the diagram has the form

$$w^{0,2k-3} \xrightarrow{\beta^{1,2}} \ldots \xrightarrow{\beta^{1,k-1}} w^{0,k-1} \xrightarrow{\gamma^{1,1}} w^{0,k-2} \xrightarrow{\beta^{1,k}} w^{0,k-3} \xrightarrow{\gamma^{1,k-1}} \ldots \xrightarrow{\gamma^{1,3}} w^{0,0}.$$

In the rest of the diagram, parallel (or almost parallel) arrows have the same labels. As before, we will use the notation suggested by the diagram for the irreducible components of the homology groups $H_k(\mathfrak{p}_+, \mathbb{V})$.

Proposition. *Put $\mathbb{E}_0 := H_{1,1}(\mathfrak{p}_+, \mathfrak{g})$. Then we have:*
(1) All weights of $\oplus_{i \geq 0} (\otimes^i(\mathfrak{g}_-) \otimes \mathbb{E}_0^*)$ have the form

$$-m_1\alpha_1 - m_2\alpha_2 + \textstyle\sum_3^k m_i\alpha_i$$

for integers $m_1, \ldots, m_k$ such that $0 < m_1 < m_2$.

(2) Let μ_1 *and* μ_2 *be two weights contained in an upgoing subsequence of the BGG diagram associated to some* $\mathfrak{g}$*–dominant integral weight* λ*. Then the difference* $\mu_2 - \mu_1$ *can be written as a linear combination of* $\alpha_2, \dots, \alpha_k$*.*
(3) Let μ_1 *and* μ_2 *be two weights contained in an downgoing subsequence of the BGG diagram associated to some* $\mathfrak{g}$*–dominant integral weight* λ*. Then writing* $\mu_2 - \mu_1$ *as a linear combination of* $\alpha_1, \dots, \alpha_k$ *the roots* α_1 *and* α_2 *have the same coefficient.*

Proof. The algorithms of [4] show that in terms of fundamental weights the highest weight of $\mathbb{E}_0^*$ is $-5\lambda_2 + 4\lambda_3$, so in particular the $\mathfrak{sl}(2,\mathbb{C})$–factor in $\mathfrak{g}_0$ acts trivially on $\mathbb{E}_0^*$. In terms of simple roots we obtain $-5\lambda_2 + 4\lambda_3 = -\alpha_1 - 2\alpha_2 + 2(\alpha_3 + \dots + \alpha_{k-1}) + \alpha_k$. Now (1) follows as in the proof of Lemma 3.4. For (2) we only have to observe that in all roots which occur as labels in upgoing sequences, the coefficient of α_1 is trivial. Part (3) follows from the fact that in all roots occurring in downgoing sequences the coefficients of α_1 and α_2 are the same. □

3.10. **Quaternionic and split–quaternionic contact structures.** There are several real forms of the situation considered in 3.9 which are of interest in geometry. In all theses cases a, regular normal parabolic geometry is equivalent to a certain codimension three distribution $H \subset TM$ on a manifold M of dimension $4k-5$. Recall that given such a distribution and putting $Q := TM/H$, the Lie bracket of vector fields induces a tensorial map $H \times H \to Q$. For each $x \in M$ this makes $H_x \oplus Q_x$ into a nilpotent graded Lie algebra. The parabolic geometries then correspond to the case that for each x this is isomorphic to $\mathfrak{g}_-$.

For $G = PSp(p+1, q+1)$ with $p \geq q$ and $p+q = k-1$, one obtains for $\mathfrak{g}_-$ the quaternionic Heisenberg algebra given by a quaternionic Hermitian form of signature (p,q). In particular, for $q = 0$ the resulting geometries are exactly the quaternionic contact structures introduced by O. Biquard, see [6, 7]. The interest in these structures comes from the fact that they occur as conformal infinities of quaternionic Kähler manifolds. For the real form $G = PSp(2k, \mathbb{R})$, one obtains for $\mathfrak{g}_-$ the (uniquely determined) split quaternionic Heisenberg algebra. For $k = 3$ and hence $\dim(M) = 7$ the two types of rank 4 distributions obtained in this way are exactly the two generic types.

Concerning the structure of the harmonic curvature, the case $k = 3$ and hence $\dim(M) = 7$ is special. There are two independent harmonic curvature components, one of which is a torsion and one of which is a curvature. Hence torsion freeness is a nontrivial condition. It turns out that torsion freeness is also equivalent to existence of a twistor space.

On the other hand, for $k > 3$, the component $H_{2,0}(\mathfrak{p}_+, \mathfrak{g})$ consists of maps which are homogeneous of degree zero, and hence vanishing of the corresponding component of the harmonic curvature is a consequence of regularity.

It turns out that in all cases, the BGG diagrams of the complexification never contain conjugate weights. As for the other geometries we obtain.

Theorem. *Let M a smooth manifold of dimension $4k-5$, $k \geq 3$, endowed with a quaternionic contact structure or its split quaternionic analog. Assume further that this structure is torsion free if $k = 3$.*

Then for any irreducible representation $\mathbb{V}$ of G the associated BGG sequence splits according to the Hasse diagram from 3.9 and any upgoing or downgoing subsequence is a subcomplex.

4. Ellipticity

In this section, we want to show that many of complexes obtained in Theorem 3.5 are elliptic in the quaternionic case. To do this we first analyze their symbol sequences in the Grassmannian case.

4.1. **Symbol sequences.** As in 3.5, natural vector bundles on almost Grassmannian manifolds are associated to representations of the group P and in the case of irreducible representations one has to deal with $G_0 = S(GL(2,\mathbb{R}) \times GL(n,\mathbb{R}))$. The standard representations $\mathbb{E}$ and $\mathbb{F}$ of the two factors correspond to the bundles E and F, and $T^*M \cong E \otimes F^*$. If we have two representations $\mathbb{V}$ and $\mathbb{W}$, then the symbol of an rth order differential operator $D : \Gamma(VM) \to \Gamma(WM)$ between sections of the corresponding bundles is a bundle map $S^r T^*M \otimes VM \to WM$. In the case of an invariant differential operator, this bundle map is induced by a G_0–equivariant map $\sigma : S^r(\mathbb{E} \otimes \mathbb{F}^*) \otimes \mathbb{V} \to \mathbb{W}$. Determining all possible maps of this type is a sometimes tedious but standard task in representation theory. For $X \in \mathbb{E} \otimes \mathbb{F}^*$ we will write $\sigma_X : \mathbb{V} \to \mathbb{W}$ for the map $v \mapsto \sigma(X \vee \dots \vee X \otimes v)$.

As a preliminary step, we have to analyze the representations $\mathbb{V}^{k,\ell} := S^k\mathbb{E} \otimes \Lambda^\ell \mathbb{F}^*$. Then there is a unique (up to scale) G_0–homomorphism $\sigma : (\mathbb{E} \otimes \mathbb{F}^*) \otimes \mathbb{V}^{k,\ell} \to \mathbb{V}^{k+1,\ell+1}$ induced by the symmetric product in the first, and the wedge product in the second factor. Choosing a basis $\{e_1, e_2\}$ for $\mathbb{E}$, one may write any element $X \in \mathbb{E} \otimes \mathbb{F}^*$ as $e_1 \otimes \alpha_1 + e_2 \otimes \alpha_2$ for elements $\alpha_1, \alpha_2 \in \mathbb{F}^*$.

Lemma. *For each $k \geq 0$ the symbol sequence*

$$0 \to \mathbb{V}^{k,0} \overset{\sigma_X}{\to} \mathbb{V}^{k+1,1} \to \dots \overset{\sigma_X}{\to} \mathbb{V}^{k+n,n} \to 0$$

is exact for $X = e_1 \otimes \alpha_1 + e_2 \otimes \alpha_2$ provided that α_1 and α_2 are linearly independent.

Proof. Let us assume throughout the proof that α_1 and α_2 are linearly independent. First consider the sequence $\mathbb{V}^{0,j-1} \stackrel{\sigma_X}{\to} \mathbb{V}^{1,j} \stackrel{\sigma_X}{\to} \mathbb{V}^{2,j+1}$. We claim that this sequence is exact for all $j = 1, \dots, n-1$, the first map is injective for $j = 1$ and the last map is surjective for $j = n-1$.

Injectivity of the first map for $j = 1$ is obvious. An arbitrary element of $\mathbb{V}^{1,j}$ can be written as $e_1 \wedge \varphi_1 + e_2 \wedge \varphi_2$ for $\varphi_1, \varphi_2 \in \Lambda^j \mathbb{F}^*$. Applying σ_X, we obtain

$$e_1 \vee e_1 \otimes \alpha_1 \wedge \varphi_1 + e_1 \vee e_2 \otimes (\alpha_1 \wedge \varphi_2 - \alpha_2 \wedge \varphi_1) + e_2 \vee e_2 \otimes \alpha_2 \wedge \varphi_2.$$

From this, surjectivity of the last map for $j = n-1$ follows easily. Moreover, if this expression vanishes, then for $i = 1, 2$ vanishing of the coefficient of $e_i \vee e_i$ implies $\alpha_i \wedge \varphi_i = 0$, so $\varphi_i = \alpha_i \wedge \psi_i$. Vanishing of the coefficient of $e_1 \vee e_2$ leads to $\alpha_1 \wedge \alpha_2 \wedge (\psi_2 + \psi_1) = 0$. Since α_1 and α_2 are linearly independent, we obtain $\psi_1 + \psi_2 = \alpha_1 \wedge \rho_1 + \alpha_2 \wedge \rho_2$, and thus $\psi_1 - \alpha_1 \wedge \rho_1 = \psi_2 - \alpha_2 \wedge \rho_2 =: \beta$. By construction, $\alpha_i \wedge \beta = \alpha_i \wedge \psi_i = \varphi_i$ and hence $e_1 \otimes \varphi_1 + e_2 \otimes \varphi_2 = \sigma_X(\beta)$.

Let us inductively assume that $k > 1$ and we have proved that $\mathbb{V}^{i-1,j-1} \stackrel{\sigma_X}{\to} \mathbb{V}^{i,j} \stackrel{\sigma_X}{\to} \mathbb{V}^{i+1,j+1}$ is exact for all $i \leq k$ with the first map injective for $j = 1$ and the last map surjective for $j = n-1$. Consider the sequence

$$0 \to S^{\ell-1}\mathbb{E} \otimes \Lambda^2 \mathbb{E} \to S^\ell \mathbb{E} \otimes \mathbb{E} \to S^{\ell+1}\mathbb{E} \to 0,$$

where the maps are given by symmetrization in the first ℓ respectively in all factors. Clearly the composition of these two maps is trivial and looking at the dimensions one concludes that this is a short exact sequence. The two maps are evidently compatible with taking the symmetric product with some fixed element. Thus tensorizing with appropriate exterior powers of $\mathbb{F}^*$ we obtain the following commutative diagram with short exact columns, in which all horizontal maps are either σ_X or the tensor product of σ_X with an appropriate identity map:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{V}^{k-2,j-1} \arrow[r] \arrow[d] & \mathbb{V}^{k-1,j} \otimes \Lambda^2\mathbb{E} \arrow[r] \arrow[d] & \mathbb{V}^{k,j+1} \otimes \Lambda^2\mathbb{E} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{V}^{k-1,j-1} \otimes \mathbb{E} \arrow[r] \arrow[d] & \mathbb{V}^{k,j} \otimes \mathbb{E} \arrow[r] \arrow[d] & \mathbb{V}^{k+1,j+1} \otimes \mathbb{E} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathbb{V}^{k,j-1} \arrow[r] & \mathbb{V}^{k+1,j} \arrow[r] & \mathbb{V}^{k+2,j+1} \arrow[r] & \cdots
\end{tikzcd}$$

By induction, the two top rows are exact, so exactness of the bottom row (including the statements for $j = 1$ and $j = n - 1$) follows from the nine–lemma of category theory. □

We are interested in the BGG sequences associated to the representations $(S^k\mathbb{V}^* \otimes S^\ell\mathbb{V})_0$, where $\mathbb{V} = \mathbb{R}^{n+2}$ is the standard representation of $SL(n+2,\mathbb{R})$ and the subscript denotes the totally tracefree part. These are exactly those representations of G whose highest weight is a linear combination of the first and last fundamental weights. We consider the $D^{0,1}$–subcomplexes starting at $\mathcal{H}_{0,0}$ (i.e. the left edge of the diagram) from Theorem 3.5. Let $\mathbb{W}_j^{k,\ell}$ be the representation inducing $\mathcal{H}_{0,j}$ for the given choice. It will be convenient to put $\mathbb{W}_j^{k,\ell} = 0$ for $j < 0$ and $j > n$.

From the algorithms for determining Weyl orbits of weights in [4] and the shape of the Hasse diagram, one immediately concludes that $\mathbb{W}_j^{k,\ell}$ is the irreducible component of highest weight in $S^k\mathbb{E}^* \otimes \mathbb{W}_j^{0,\ell}$ for all $j = 0, \dots, n$. For $j < n$, one similarly concludes that $\mathbb{W}_j^{k,\ell}$ is the irreducible component of highest weight in $\mathbb{W}_j^{k,0} \otimes S^\ell\mathbb{F}$. Finally, one easily verifies directly that $W_j^{0,0} = S^j\mathbb{E} \otimes \Lambda^j\mathbb{F}^* \subset \Lambda^j(\mathbb{E} \otimes \mathbb{F})$ for all j. Thus we conclude that

$$\mathbb{W}_j^{k,\ell} = (S^j\mathbb{E} \otimes S^k\mathbb{E}^*)_0 \otimes (\Lambda^j\mathbb{F}^* \otimes S^\ell\mathbb{F})_0$$

for $j < n$.

Since $\mathbb{E}$ has dimension two, the wedge product induces an isomorphism $\mathbb{E}^* \cong \mathbb{E} \otimes \Lambda^2\mathbb{E}^*$. Following the usual conventions for conformal weights we indicate tensor product with the kth power of the line bundle $\Lambda^2\mathbb{E}^*$ by adding the symbol $[k]$. Likewise, adding $[-k]$ indicates a tensor product with the kth power of $\Lambda^2\mathbb{E}$. The the above isomorphism reads as $\mathbb{E}^* \cong \mathbb{E}[1]$. We also obtain an isomorphism $S^j\mathbb{E} \otimes S^k\mathbb{E}^* \cong (S^j\mathbb{E} \otimes S^k\mathbb{E})[k]$ under which the tracefree part corresponds to $S^{j+k}\mathbb{E}[k]$. Finally, one verifies directly that

$$\mathbb{W}_n^{k,\ell} = S^{k+n+\ell}\mathbb{E}[k] \otimes \Lambda^n\mathbb{F}^*.$$

The operators in our subcomplex are all of first order, except for the last one, which is of order $\ell + 1$. For $j < n - 1$ there evidently is a unique (up to scale) G_0–homomorphism $\mathbb{E} \otimes \mathbb{F}^* \otimes \mathbb{W}_j^{k,\ell} \to \mathbb{W}_{j+1}^{k,\ell}$ which is induced by taking the symmetric product in the $\mathbb{E}$ component and the wedge product in the $\mathbb{F}^*$–component. In the last step, the symbol should be a homomorphism

$$S^{\ell+1}(\mathbb{E} \otimes \mathbb{F}^*) \otimes S^{k+n-1}\mathbb{E} \otimes (\Lambda^{n-1}\mathbb{F}^* \otimes S^\ell\mathbb{F})_0 \to S^{k+n+\ell}\mathbb{E}[k] \otimes \Lambda^n\mathbb{F}^*.$$

Looking at the $\mathbb{E}$–components we see that such a homomorphism has to factorize through $S^{\ell+1}\mathbb{E}\otimes S^{\ell+1}\mathbb{F}^*\subset S^{\ell+1}(\mathbb{E}\otimes\mathbb{F}^*)$. But then there is again a unique (up to scale) G_0–homomorphism. This is induced by the symmetric product in the $\mathbb{E}$ component, while in the $\mathbb{F}^*$ component one has to take the unique contraction $S^{\ell+1}\mathbb{F}^*\otimes S^\ell\mathbb{F}\to\mathbb{F}^*$ followed by the wedge product. Hence we see that the symbols of the operators in the subcomplex are all uniquely determined up to scale by their invariance properties.

Theorem. *For all integers k and ℓ, the symbol sequence*

$$0\to\mathbb{W}_0^{k,\ell}\overset{\sigma_X}{\to}\mathbb{W}_1^{k,\ell}\to\dots\overset{\sigma_X}{\to}\mathbb{W}_n^{k,\ell}\to 0$$

is exact for $X=e_1\otimes\alpha_1+e_2\otimes\alpha_2$ provided that α_1 and α_2 are linearly independent.

Proof. We proceed by induction on ℓ. For $\ell=0$, we have $\mathbb{W}_j^{k,0}\cong\mathbb{V}^{k+j,j}[k]$, so the result follows directly from the Lemma.

Assuming that $\ell>0$, consider for $j=1,\dots,n-1$ the sequence

$$0\to(\Lambda^{j-1}\mathbb{F}^*\otimes S^{\ell-1}\mathbb{F})_0\to\Lambda^j\mathbb{F}^*\otimes S^\ell\mathbb{F}\to(\Lambda^j\mathbb{F}^*\otimes S^\ell\mathbb{F})_0\to 0,$$

in which the first map is given by tensorizing with the identity and then symmetrizing in the $\mathbb{E}$–part and alternating in the $\mathbb{F}$–part, and the second map is projection to the tracefree part. The dimensions of the representations can be easily computed using Weyl's formula, and this shows that the sequence is short exact.

Tensorizing this exact sequence with $S^{k+j}\mathbb{E}[k]$, we obtain, for each j, a short exact sequence

$$0\to\mathbb{W}_{j-1}^{k+1,\ell-1}[-1]\to\mathbb{W}_j^{k,0}\otimes S^\ell\mathbb{F}\to\mathbb{W}_j^{k,\ell}\to 0.$$

One easily verifies directly, that for $j<n-1$, we get a commutative diagram

$$\begin{tikzcd}
\mathbb{E}\otimes\mathbb{F}^*\otimes\mathbb{W}_{j-1}^{k+1,\ell-1}[-1] \arrow[r] \arrow[d] & \mathbb{W}_j^{k+1,\ell-1}[-1] \arrow[d] \\
\mathbb{E}\otimes\mathbb{F}^*\otimes\mathbb{W}_j^{k,0}\otimes S^\ell\mathbb{F} \arrow[r] & \mathbb{W}_{j+1}^{k,0}\otimes S^\ell\mathbb{F}
\end{tikzcd}$$

in which the vertical arrows come from the sequence above and the horizontal arrows are tensor products of the symbol homomorphism σ with appropriate identity maps. By exactness, these induce a homomorphism $\mathbb{E}\otimes\mathbb{F}^*\otimes\mathbb{W}_j^{k,\ell}\to\mathbb{W}_{j+1}^{k,\ell}$. From above, we know that this has to be a multiple of σ, and this multiple has to be nonzero, since $\sigma\otimes\mathrm{id}$ maps onto $\mathbb{W}_{j+1}^{k,0}\otimes S^\ell\mathbb{F}$ by irreducibility of $\mathbb{W}_{j+1}^{k,0}$.

Hence for $j = 0, \dots, n-2$ we obtain a commutative diagram with short exact columns in which the horizontal arrows are (nonzero multiples of) σ_X or the tensor product of σ_X with an appropriate identity map (recall that $\mathbb{W}^{r,s}_j = 0$ for $j < 0$):

$$\begin{tikzcd}
\mathbb{W}^{k+1,\ell-1}_{j-2}[-1] \arrow[r] \arrow[d] & \mathbb{W}^{k+1,\ell-1}_{j-1}[-1] \arrow[r] \arrow[d] & \mathbb{W}^{k+1,\ell-1}_{j}[-1] \arrow[d] \\
\mathbb{W}^{k,0}_{j-1} \otimes \mathbb{S}^\ell F \arrow[r] \arrow[d] & \mathbb{W}^{k,0}_{j} \otimes \mathbb{S}^\ell F \arrow[r] \arrow[d] & \mathbb{W}^{k,0}_{j+1} \otimes \mathbb{S}^\ell F \arrow[d] \\
\mathbb{W}^{k,\ell}_{j-1} \arrow[r] & \mathbb{W}^{k,\ell}_{j} \arrow[r] & \mathbb{W}^{k,\ell}_{j+1}.
\end{tikzcd}$$

By induction the two top rows are exact, so exactness of our symbol sequence at $\mathbb{W}^{k,\ell}_j$ for $j = 0, \dots, n-2$ follows from the nine lemma.

For the last part, we first observe that $\mathbb{W}^{k+1,\ell-1}_n[-1] \cong \mathbb{W}^{k,\ell}_n$. On the other hand, for $j = n$, the above short exact sequences degenerate to isomorphisms $(\Lambda^{n-1}\mathbb{F}^* \otimes S^{\ell-1}\mathbb{F})_0 \cong \Lambda^n\mathbb{F}^* \otimes S^\ell\mathbb{F}$ respectively $\mathbb{W}^{k+1,\ell-1}_{n-1}[-1] \cong \mathbb{W}^{k,0}_n \otimes S^\ell\mathbb{F}$.

Hence we obtain the following commutative diagram in which the first two columns are short exact, and the horizontal arrows are nonzero multiples of σ_X or a tensor product of σ_X with an appropriate identity map:

$$\begin{tikzcd}
\mathbb{W}^{k+1,\ell-1}_{n-3}[-1] \arrow[r] \arrow[d] & \mathbb{W}^{k+1,\ell-1}_{n-2}[-1] \arrow[r] \arrow[d] & \mathbb{W}^{k+1,\ell-1}_{n-1}[-1] \arrow[r] \arrow[d, "\cong"'] & \mathbb{W}^{k+1,\ell-1}_{n}[-1] \arrow[r] & 0 \\
\mathbb{W}^{k,0}_{n-2} \otimes S^\ell\mathbb{F} \arrow[r] \arrow[d] & \mathbb{W}^{k,0}_{n-1} \otimes S^\ell\mathbb{F} \arrow[r] \arrow[d] & \mathbb{W}^{k,0}_{n} \otimes S^\ell\mathbb{F} \arrow[rr] & & 0 \\
\mathbb{W}^{k,\ell}_{n-2} \arrow[r] & \mathbb{W}^{k,\ell}_{n-1} & & &
\end{tikzcd}$$

By induction, the two top rows are exact. We can define a map $\mathbb{W}^{k,\ell}_{n-1} \to \mathbb{W}^{k+1,\ell-1}_n[-1] \cong \mathbb{W}^{k,\ell}_n$ as follows: Choose a preimage in $\mathbb{W}^{k,0}_{n-1} \otimes S^\ell\mathbb{F}$, map it to $\mathbb{W}^{k,0}_n \otimes S^\ell\mathbb{F}$ go up via the isomorphism, and map to $\mathbb{W}^{k+1,\ell-1}_n[-1]$. Diagram chasing shows that this is well defined and inserting it as the last map in the sequence we get exactness at $\mathbb{W}^{k,\ell}_{n-1}$ and $\mathbb{W}^{k,\ell}_n$. To complete the proof, it thus remains to show that this map is a nonzero multiple of σ_X.

The inclusion of the tracefree part into $\Lambda^{n-1}\mathbb{F}^* \otimes S^\ell\mathbb{F}$ induces a G_0–homomorphism $\mathbb{W}^{k,\ell}_{n-1} \to \mathbb{W}^{k,0}_{n-1} \otimes S^\ell\mathbb{F}$. Tensorizing with the identity on $\mathbb{E} \otimes \mathbb{F}^*$ and composing, we get a homomorphism

$$\mathbb{E} \otimes \mathbb{F}^* \otimes \mathbb{W}^{k,\ell}_{n-1} \to \mathbb{E} \otimes \mathbb{F}^* \otimes \mathbb{W}^{k,0}_{n-1} \otimes S^\ell\mathbb{F} \xrightarrow{\sigma\otimes\mathrm{id}} \mathbb{W}^{k,0}_n \otimes S^\ell\mathbb{F}.$$

From above we know that the target of this homomorphism is isomorphic to $\mathbb{W}^{k+1,\ell-1}_{n-1}[-1]$ and hence in particular irreducible. A moment of thought shows that the composition is nonzero and hence surjective by irreducibility. Hence we have obtained a surjective G_0–homomorphism $\mathbb{E}\otimes\mathbb{F}^*\otimes\mathbb{W}^{k,\ell}_{n-1}\to\mathbb{W}^{k+1,\ell-1}_{n-1}[-1]$. Tensorize this with the identity on $S^\ell(\mathbb{E}\otimes\mathbb{F}^*)$ and consider the composition

$$S^\ell(\mathbb{E}\otimes\mathbb{F}^*)\otimes\mathbb{E}\otimes\mathbb{F}^*\otimes\mathbb{W}^{k,\ell}_{n-1}\to S^\ell(\mathbb{E}\otimes\mathbb{F}^*)\otimes\mathbb{W}^{k+1,\ell-1}_{n-1}[-1]\xrightarrow{\sigma}\mathbb{W}^{k,\ell}_n.$$

This is surjective by induction, and looking at the explicit form of the representation $\mathbb{W}^{k,\ell}_n$ one immediately sees that it has to factor through $S^{\ell+1}\mathbb{E}\otimes S^{\ell+1}\mathbb{F}^*\subset S^\ell(\mathbb{E}\otimes\mathbb{F}^*)\otimes\mathbb{E}\otimes\mathbb{F}^*$. Therefore, it restricts to a nonzero multiple of the symbol map on that part, and inserting copies of X, the claim follows. □

4.2. **Dual sequences.** By duality, we can prove ellipticity for the $D^{1,0}$–subcomplexes starting at $\mathcal{H}_{0,n}$ in the BGG sequences considered in 4.1. Let us denote by $\tilde{\mathbb{W}}^{k,\ell}_j$ the G_0–representation corresponding to the bundle $\mathcal{H}_{j,n}$ in the BGG sequence associated to $(S^k\mathbb{V}^*\otimes S^\ell\mathbb{V})_0$. The crucial point here is that $\tilde{\mathbb{W}}^{k,\ell}_j\cong(\mathbb{W}^{\ell,k}_{n-j})^*\otimes\Lambda^{2n}\mathfrak{g}^*_-$.

This isomorphism comes from the bilinear map

$$\Lambda^{n+j}\mathfrak{g}^*_-\otimes(S^k\mathbb{V}^*\otimes S^\ell\mathbb{V})_0\times\Lambda^{n-j}\mathfrak{g}^*_-\otimes(S^k\mathbb{V}\otimes S^\ell\mathbb{V}^*)_0\to\Lambda^{2n}\mathfrak{g}^*_-$$

given by the wedge product and the paring between dual representations. Note that $\dim(\mathfrak{g}_-)=2n$, so $\Lambda^{2n}\mathfrak{g}^*_-$ is one–dimensional.

In particular, this implies that dualizing and tensorizing with the identity of this one–dimensional representation induces an isomorphism

$$L(\tilde{\mathbb{W}}^{k,\ell}_j,\tilde{\mathbb{W}}^{k,\ell}_{j+1})\cong L(\mathbb{W}^{\ell,k}_{n-j-1},\mathbb{W}^{\ell,k}_{n-j}).$$

In this case, the first operator of the sequence is of order $k+1$, while all others are of first order. Now the symbol of an rth order natural differential operator can equivalently be interpreted as a G_0–equivariant map from $S^r(\mathbb{E}\otimes\mathbb{F}^*)$ to the module of linear maps between the representations inducing the bundles.

Thus, the results of 4.1 immediately imply that the symbols in our sequence are uniquely determined up to scale by G_0–equivariancy. Moreover, for any element $X\in\mathbb{E}\otimes\mathbb{F}^*$, and each j, the symbol map $\sigma_X:\tilde{\mathbb{W}}^{k,\ell}_j\to\tilde{\mathbb{W}}^{k,\ell}_{j+1}$ is the dual of the symbol map $\sigma_X:\mathbb{W}^{\ell,k}_{n-j-1}\to\mathbb{W}^{\ell,k}_{n-j}$. Since the dual of an exact sequence is exact, we obtain

Theorem. *The symbol sequence*

$$0\to\tilde{\mathbb{W}}^{k,\ell}_0\xrightarrow{\sigma_X}\tilde{\mathbb{W}}^{k,\ell}_1\to\dots\xrightarrow{\sigma_X}\tilde{\mathbb{W}}^{k,\ell}_n\to 0$$

is exact for $X=e_1\otimes\alpha_1+e_2\otimes\alpha_2$ if α_1 and α_2 are linearly independent.

4.3. **Elliptic complexes for quaternionic structures.** The results on symbol sequences in the Grassmannian case derived above have immediate consequences for quaternionic structures:

Theorem. *Let M be a quaternionic manifold of dimension $4n \geq 8$. Let $\mathbb{V} = \mathbb{H}^{n+1}$ be the standard representation of $SL(n+1, \mathbb{H})$.*

Then for all integers $k, \ell \geq 0$, the subcomplexes

$$0 \to \Gamma(\mathcal{H}_{0,0}M) \overset{D^{0,1}}{\to} \Gamma(\mathcal{H}_{0,1}M) \overset{D^{0,1}}{\to} \dots \overset{D^{0,1}}{\to} \Gamma(\mathcal{H}_{0,n}M) \to 0$$

$$0 \to \Gamma(\mathcal{H}_{0,n}M) \overset{D^{1,0}}{\to} \Gamma(\mathcal{H}_{1,n}M) \overset{D^{1,0}}{\to} \dots \overset{D^{1,0}}{\to} \Gamma(\mathcal{H}_{n,n}M) \to 0$$

of the BGG sequence associated to the representation $S^k\mathbb{V}^* \otimes S^\ell\mathbb{V}$ are elliptic. In particular, this applies to the deformation complex for quaternionic structures, see [10]*, which is the $D^{0,1}$–complex for $k = \ell = 1$.*

Proof. The symbol sequences have the same complexifications as the ones in Theorems 4.1 and 4.2. Since a nonzero quaternionic linear map $\mathbb{H} \to \mathbb{H}^n$ always has complex rank two, the condition for exactness of the symbol sequence in these Theorems is always satisfied. □

The $D^{0,1}$–complexes for $\ell = 0$ are all the elliptic complexes found in [23], except the one for $r = -1$ in [23, Theorem 5.5], which belongs to singular infinitesimal character.

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