ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

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Andreas Čap

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INFINITESIMAL AUTOMORPHISMS AND DEFORMATIONS OF PARABOLIC GEOMETRIES

ANDREAS ČAP

Abstract. We show that infinitesimal automorphisms and infinitesimal deformations of parabolic geometries can be nicely described in terms of the twisted de–Rham sequence associated to a certain linear connection on the adjoint tractor bundle. For regular normal geometries, this description can be related to the underlying geometric structure using the machinery of BGG sequences. In the locally flat case, this leads to a deformation complex, which generalizes the is well know complex for locally conformally flat manifolds.

Recently, a theory of subcomplexes in BGG sequences has been developed. This applies to certain types of torsion free parabolic geometries including, quaternionic structures, quaternionic contact structures and CR structures. We show that for these structures one of the subcomplexes in the adjoint BGG sequence leads (even in the curved case) to a complex governing deformations in the subcategory of torsion free geometries. For quaternionic structures, this deformation complex is elliptic.

1. Introduction

Given a smooth manifold M and a type of geometric structure, it is a natural idea to consider the moduli space, i.e. the space of isomorphism classes of structures of the given type on M. This moduli space can be viewed as the quotient of the space of all structures of the given type by the action of the diffeomorphism group of M, which acts by pulling back structures. In general, the moduli space is a highly complicated object. Trying to understand the moduli space locally, one is led to the study of deformations of geometric structures. Here deformations coming from the action of one–parameter groups of diffeomorphisms have to be considered as trivial. Reducing further to the formal infinitesimal level, one arrives at infinitesimal deformations. These describe

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the possible directions into which a given structure can be deformed. As before, there is the notion of a trivial infinitesimal deformation, and the quotient of the space of all infinitesimal deformations by the trivial ones is usually referred to as the *formal tangent space* of the moduli space at the given structure.

In this paper, we study infinitesimal deformations and the closely related infinitesimal automorphisms for parabolic geometries. These form a large class of geometric structures containing examples like conformal, quaternionic, hypersurface type CR, and certain higher codimension CR structures. For some of these structures, deformation theory has been developed quite far. Infinitesimal deformations are usually defined in an ad hoc manner as smooth sections of some bundle. Trivial infinitesimal deformations are those which lie in the image of some linear differential operator, whose kernel is the space of infinitesimal automorphisms. In particular, the formal tangent space is usually infinite dimensional.

It is a highly interesting problem to restrict the class of allowed deformations in such a way that one obtains a finite dimensional moduli space. This can be done by imposing integrability conditions on the geometric structure and looking only at deformations in the subclass of geometries defined in that way. For parabolic geometries, the simplest possible condition is local flatness, but in some cases much more subtle integrability conditions can be used, for example anti self duality for conformal structures in dimension four.

The unifying feature of parabolic geometries is that they can be viewed as Cartan geometries with homogeneous model a generalized flag manifold. Regular normal geometries of this type are then equivalent to underlying geometric structures including the examples listed above. For Cartan geometries, there are evident notions of infinitesimal deformations and infinitesimal automorphisms. These can be nicely formulated in terms of a certain linear connection (which surprisingly is different from the canonical normal tractor connection) on the adjoint tractor bundle, see Proposition 3.2. In particular, the relevant operators are part of the twisted de–Rham sequence associated to this linear connection.

The machinery of Bernstein–Gelfand–Gelfand sequences (or BGG–sequences), which was introduced in [12] and improved in [5], can be applied to this twisted de–Rham sequence to obtain a sequence of higher order operators acting on sections of bundles that can be easily interpreted in terms of the underlying structure. For regular normal geometries, the first operator in this sequence has the space of infinitesimal automorphisms as its kernel and the formal tangent space to the moduli space of normal geometries as its cokernel, see 3.4 and 3.6.

For locally flat parabolic geometries (which are automatically regular and normal, and locally isomorphic to the homogeneous model), the twisted de–Rham sequence is a complex. Thus also the corresponding BGG sequence is a complex which can be naturally interpreted as a deformation complex in the category of locally flat structures.

Finally we move to more subtle integrability conditions. The recent joint work [13] with V. Souček contains a theory of subcomplexes in BGG sequences. In that paper, we study several examples, in which there is an interesting notion of semi flatness which includes (and in most cases is equivalent to) torsion freeness. In particular, these include quaternionic structures and CR structures, but also quaternionic contact structures (torsion free ones in dimension 7) as introduced in [3, 4]. In section 4, we show that for all these geometries a certain subcomplex of the adjoint BGG sequence can be naturally interpreted as a deformation complex in the subcategory of semi flat geometries. For quaternionic structures, this deformation complex is elliptic.

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2. Some background

We very briefly review some background. Some more details can be found in [13] and much more information is available in [12, 10, 11].

2.1. **Parabolic geometries.** The basic data needed to define a parabolic geometry is a semisimple Lie algebra $\mathfrak{g}$ endowed with a $|k|$–grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ and a group G with Lie algebra $\mathfrak{g}$. The subgroup $P \subset G$ consisting of all elements $g \in G$ such that $\mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i$ for all i, where $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$, is a parabolic subgroup. We will also need the subgroup $G_0 \subset P$ of all elements whose adjoint action preserves the grading of $\mathfrak{g}$.

Parabolic geometries of type (G, P) are then defined as Cartan geometries of that type. Thus such a geometry on a smooth manifold M consists of a principal P–bundle $p : \mathcal{G} \to M$ and a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. The *homogeneous model* of parabolic geometries of type (G, P) is given by the canonical principal bundle $G \to G/P$ with the left Maurer–Cartan form as a Cartan connection. A morphism of parabolic geometries is a homomorphism of principal bundles which is compatible with the Cartan connections. In particular, any morphism is a local diffeomorphism.

The curvature of a Cartan connection ω can be viewed as $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by the structure equation

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

where ξ and η are vector fields on $\mathcal{G}$ and the bracket is in $\mathfrak{g}$. Since K is horizontal and equivariant, it can be interpreted as a two–form κ on M with values in the associated bundle $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$, see 3.1 for more details. The bundle $\mathcal{A}M$ is called the *adjoint tractor bundle*. The P–invariant filtration $\{\mathfrak{g}^i\}$ of $\mathfrak{g}$ gives rise to a filtration $\mathcal{A}M = \mathcal{A}^{-k}M \supset \cdots \supset \mathcal{A}^kM$ by smooth subbundles and the Lie bracket on $\mathfrak{g}$ gives rise to a tensorial bracket $\{\ ,\ \}$ on $\mathcal{A}M$, making it into a bundle of filtered Lie algebras modeled on $\mathfrak{g}$.

On the other hand, the Cartan connection ω induces an isomorphism between the tangent bundle TM and the associated bundle $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$. Hence there is a natural projection $\Pi : \mathcal{A}M \to \mathcal{T}M$ which induces an isomorphism $\mathcal{A}M/\mathcal{A}^0M \cong TM$. Via this isomorphism, the filtration of $\mathcal{A}M$ descends to a filtration $TM = T^{-k}M \supset \cdots \supset T^{-1}M$ of the tangent bundle by smooth subbundles.

Applying the projection Π to the values of κ, we obtain a TM–valued two–form κ_-, which is called the torsion of the Cartan connection ω. The geometry is called *torsion free* if this torsion vanishes.

Via the filtrations of TM and $\mathcal{A}M$, one has a natural notion of homogeneity for $\mathcal{A}M$–valued differential forms. In particular, we say that κ is homogeneous of degree $\geq \ell$, if $\kappa(T^iM, T^jM) \subset \mathcal{A}^{i+j+\ell}M$ for all $i, j = -k, \dots, -1$. A parabolic geometry is called *regular* if its curvature is homogeneous of degree ≥ 1. Note that torsion free parabolic geometries are automatically regular.

For parabolic geometries, there is a uniform normalization condition. This comes from the Kostant codifferential, which is the differential $\partial^* : \Lambda^k\mathfrak{p}_+ \otimes \mathfrak{g} \to \Lambda^{k-1}\mathfrak{p}_+ \otimes \mathfrak{g}$ in the standard complex computing Lie algebra homology of $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ with coefficients in the representation $\mathfrak{g}$. Now $\mathfrak{p}_+$ is dual to $\mathfrak{g}/\mathfrak{p}$ as a P–module via the Killing form, so $\mathcal{G} \times_P (\Lambda^k\mathfrak{p}_+ \otimes \mathfrak{g}) \cong \Lambda^kT^*M \otimes \mathcal{A}M$. Since ∂^* is P–equivariant it induces a bundle map $\Lambda^kT^*M \otimes \mathcal{A}M \to \Lambda^{k-1}T^*M \otimes \mathcal{A}M$ as well as a tensorial operator $\Omega^k(M, \mathcal{A}M) \to \Omega^{k-1}(M, \mathcal{A}M)$, which we all denote by ∂^*. A parabolic geometry is called *normal* if and only if $\partial^*(\kappa) = 0$.

Several important geometric structures like conformal structures, almost quaternionic structures, non–degenerate CR structures of hypersurface type, and quaternionic contact structures admit a unique regular normal Cartan connection of type (G, P) for an appropriate choice of (G, P). Usually, the underlying structure is easily encoded into a principal G_0–bundle endowed with certain partially defined differential forms. Using quite involved prolongations procedures (see [19, 18, 9]) one extends this bundle to a principal P–bundle and constructs a canonical regular normal Cartan connection. This leads to an equivalence of categories between regular normal parabolic geometries and the underlying structures. Thus parabolic geometries offer a powerful general machinery to study a variety of geometric structures.

2.2. **Bernstein–Gelfand–Gelfand sequences.** BGG sequences generalize the BGG resolutions of representation theory to sequences of invariant differential operators on parabolic geometries. They were introduced in [12] and the construction was improved in [5]. We will briefly sketch this improved construction for regular geometries in the special case of the adjoint tractor bundle, more details can be also found in [13, 6].

The Cartan connection ω induces a natural linear connection ∇, called the adjoint tractor connection, on the adjoint tractor bundle $\mathcal{A}M$. This in turn induced the covariant exterior derivative

$$d^\nabla : \Omega^k(M, \mathcal{A}M) \to \Omega^{k+1}(M, \mathcal{A}M).$$

The BGG machinery relates d^∇ to higher order operators acting on sections of certain subquotient bundles. Let $\partial^* : \Lambda^k T^*M \otimes \mathcal{A}M \to \Lambda^{k-1} T^*M \otimes \mathcal{A}M$ denote the bundle maps induced by the Kostant codifferential. The kernels and images of these bundle maps are natural subbundles, so we can look at the quotient bundles $\ker(\partial^*)/\operatorname{im}(\partial^*)$. By construction, they are associated to the representations $H_k(\mathfrak{p}_+, \mathfrak{g})$. It turns out that the latter representations are always completely reducible and they are algorithmically computable using Kostant's version of the Bott–Borel–Weil theorem. Since the associated bundles can be viewed as the fiber–wise homology groups of the bundle T^*M of Lie algebras with coefficients in the bundle $\mathcal{A}M$, we denote them by $H_k(T^*M, \mathcal{A}M)$.Note that by construction there is a natural bundle map $\pi_H : \ker(\partial^*) \to H_k(T^*M, \mathcal{A}M)$, and we will denote by the same symbol the induced tensorial operator on sections.

For a normal parabolic geometry, the Cartan curvature κ by definition is a section of $\ker(\partial^*)$, so we obtain the section $\kappa_H = \pi_H(\kappa)$ of the bundle $H_2(T^*M, \mathcal{A}M)$, which is called the *harmonic curvature*. This is a much simpler object than κ, but still a complete obstruction to local flatness. The components of κ_H (according to the decomposition of $H_2(\mathfrak{p}_+, \mathfrak{g})$ into irreducibles) are the fundamental invariants of a regular normal parabolic geometry.

We have observed in 2.1 that there is a natural notion of homogeneity for $\mathcal{A}M$–valued forms. The operators ∂^* preserve homogeneities, i.e. if $\varphi \in \Omega^k(M, \mathcal{A}M)$ is homogeneous of degree $\geq \ell$, then so is $\partial^*(\varphi) \in \Omega^{k-1}(M, \mathcal{A}M)$. For regular parabolic geometries, also d^∇ is compatible with homogeneities. Now the operator $\partial^* \circ d^\nabla$ evidently preserves the subspace $\Gamma(\mathrm{im}(\partial^*)) \subset \Omega^k(M, \mathcal{A}M)$. To get the machinery going, one only needs the fact that the lowest homogeneous component of the restriction of $\partial^* \circ d^\nabla$ to $\Gamma(\mathrm{im}(\partial^*))$ is tensorial and invertible. Using this one shows that the whole operator $\partial^* \circ d^\nabla$ is invertible on $\Gamma(\mathrm{im}(\partial^*))$ and the inverse is a (by construction natural) differential operator.

Using this inverse, one constructs a natural differential operator $L : \Gamma(H_k(T^*M, \mathcal{A}M)) \to \Omega^k(M, \mathcal{A}M)$ which is characterized by the properties that for $\alpha \in \Gamma(H_k(T^*M, \mathcal{A}M))$ one has $\partial^*(L(\alpha)) = 0$, $\pi_H(L(\alpha)) = \alpha$, and $\partial^*(d^\nabla L(\alpha)) = 0$. The first two properties say that L is a differential splitting of the tensorial projection $\pi_H : \Gamma(\ker(\partial^*)) \to \Gamma(H_k(T^*M, \mathcal{A}M))$. Therefore, the operators L are referred to as the *splitting operators*. The last property implies that we can define invariant differential operators by

$$D := \pi_H \circ d^\nabla \circ L : \Gamma(H_k(T^*M, \mathcal{A}M)) \to \Gamma(H_{k+1}(T^*M, \mathcal{A}M)),$$

and these operators form the adjoint BGG sequence. Each of the bundles $H_k(T^*M, \mathcal{A}M)$ splits into a direct sum of subbundles according to the splitting of the representation $H_*(\mathfrak{p}_+, \mathfrak{g})$ into irreducible components. Doing this in all degrees, one obtains a pattern of operators acting between the various components.

2.3. **Infinitesimal deformations of conformal structures.** For the convenience of the reader, we briefly review some basic results on infinitesimal deformations of conformal structures. Let M be a smooth manifold of dimension $n \geq 3$ and let $[g]$ be a conformal class of pseudo–Riemannian metrics on M. An infinitesimal deformation of a pseudo–Riemannian metric is simply a smooth section h of the bundle S^2T^*M. To obtain a deformation on the conformal class $[g]$ one first requires h to be trace free, and second one needs that rescaling the metric g in the conformal class, h has to rescale in the same way. This means that h has to be a section of the tensor product of $S^2_0T^*M$ with a certain density bundle. Using the notation and conventions of [8], the right bundle is $F_1 := S^2_0T^*M[2] = S^2_0T^*M \otimes \mathcal{E}[2]$.

Trivial deformations are those coming from pulling back the given structure along diffeomorphisms. Viewing the conformal class $[g]$ as a section $\mathbf{g}$ of F_1, this means that trivial infinitesimal deformations are those of the form $\mathcal{L}_\xi \mathbf{g}$, for some vector field ξ on M. Here $\mathcal{L}$ denotes the

Lie derivative. Hence the quotient of all infinitesimal deformations by the trivial ones can be interpreted as the cokernel of the (by construction invariant) linear differential operator $D_0 : \Gamma(TM) \to \Gamma(F_1)$ given by $D_0(\xi) = \mathcal{L}_\xi \mathbf{g}$. It is easy to verify that D_0 is the conformal Killing operator. In particular, its kernel is the space of conformal Killing fields, i.e. of infinitesimal conformal isometries of $(M, [g])$.

This is about how far one can get for general conformal structures. To proceed further one can impose some integrability condition on the conformal structure and look at deformations in the subclass of structures satisfying this condition. The simplest choice of such a condition is local conformal flatness. Since this is equivalent to vanishing of the Weyl curvature, it is natural to consider the bundle F_2, in which the Weyl curvature has its values, and the operator $D_1 : \Gamma(F_1) \to \Gamma(F_2)$, which computes the infinitesimal change of the Weyl curvature caused by an infinitesimal deformation of the conformal structure. If $(M, [g])$ is locally conformally flat, then sections in the kernel of D_1 correspond to infinitesimal deformations in the subcategory of locally conformally flat structures. Moreover, $D_1 \circ D_0 = 0$ in that case, so the quotient $\ker(D_1)/\operatorname{im}(D_0)$ is exactly the formal tangent space to the moduli space of locally conformally flat structures on M.

It turns out that, still in the locally conformally flat case, this extends to a fine resolution

$$0 \to \Gamma(TM) \xrightarrow{D_0} \Gamma(S^2_0 T^*M[2]) \xrightarrow{D_1} \Gamma(F_2) \xrightarrow{D_2} \dots \xrightarrow{D_n} \Gamma(F_n) \to 0$$

of the sheaf of conformal Killing fields on M. Constructing this resolution by hand is fairly involved, see the book [14].

In the case of four dimensional conformal structures, a weaker integrability condition is available. In this case, the bundle F_2 splits into the direct sum $F_2^+ \oplus F_2^-$ of self dual and anti self dual parts. Accordingly, the Weyl curvature splits as $W = W^+ + W^-$ and correspondingly $D_1 = D_1^+ + D_1^-$. Given an anti self dual conformal structure, i.e. one such that $W^+ = 0$, the kernel of the operator D_1^+ exactly consists of infinitesimal deformations in the subcategory of anti self dual conformal structures. It turns out that in this case

$$0 \to \Gamma(TM) \xrightarrow{D_0} \Gamma(S^2_0 T^*M[2]) \xrightarrow{D_1^+} \Gamma(F_2^+) \to 0$$

is a complex, which is elliptic for Riemannian signature. This is the basis of the deformation theory for anti self dual conformal Riemannian four manifolds, see [16] and [15].

3. Infinitesimal automorphisms and deformations

3.1. **The basic setup.** Fix a parabolic geometry $(p : \mathcal{G} \to M, \omega)$ of some type (G, P). By definition, the adjoint tractor bundle $\mathcal{A}M$ is the associated bundle $\mathcal{G} \times_P \mathfrak{g}$ corresponding to the restriction of the adjoint representation of G to P. Smooth sections of this bundle are in bijective correspondence with smooth functions $f : \mathcal{G} \to \mathfrak{g}$ such that $f(u{\cdot}g) = \mathrm{Ad}(g^{-1})(f(u))$ for all $u \in \mathcal{G}$ and $g \in P$. More generally, for $k = 1, \dots, \dim(M)$ the space $\Omega^k(M, \mathcal{A}M)$ can be identified with the space $\Omega^k_{\mathrm{hor}}(\mathcal{G}, \mathfrak{g})^P$ of P–equivariant, horizontal $\mathfrak{g}$–valued k–forms on $\mathcal{G}$. Here $\Phi \in \Omega^k(\mathcal{G}, \mathfrak{g})$ is horizontal, if it vanishes upon insertion of one fundamental vector field, and P–equivariant if $(r^g)^*\Phi = \mathrm{Ad}(g^{-1}) \circ \Phi$ for all $g \in P$.

Explicitly, this correspondence is given as follows. For vector fields $\xi_1, \dots, \xi_k \in \mathfrak{X}(M)$, there are P–invariant lifts $\tilde{\xi}_1, \dots, \tilde{\xi}_k \in \mathfrak{X}(\mathcal{G})$. For $\Phi \in \Omega^k(\mathcal{G}, \mathfrak{g})$ we consider the function $\Phi(\tilde{\xi}_1, \dots, \tilde{\xi}_k) : \mathcal{G} \to \mathfrak{g}$. If Φ is horizontal and equivariant, then this function is independent of the choice of the lifts and P–equivariant, so it defines a smooth section $\varphi(\xi_1, \dots, \xi_k)$ of $\mathcal{A}M$. One immediately verifies that this defines an element $\varphi \in \Omega^k(M, \mathcal{A}M)$. Note that this identification is independent of the Cartan connection ω.

This correspondence immediately leads to a geometric interpretation of $\Omega^1(M, \mathcal{A}M)$: Suppose that $\tilde{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a second Cartan connection on $\mathcal{G}$. Then the difference $\tilde{\omega} - \omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is by definition horizontal and P–equivariant, and thus corresponds to an element of $\Omega^1(M, \mathcal{A}M)$. There is an obvious notion of a deformation of the Cartan geometry $(\mathcal{G} \to M, \omega)$ as a smooth family ω_τ of Cartan connections on $\mathcal{G}$ parametrized by $\tau \in (-\epsilon, \epsilon) \subset \mathbb{R}$ such that $\omega_0 = \omega$. The initial direction of this deformation is the derivative $\frac{d}{d\tau}|_{\tau=0}\omega_\tau$ of this family at $\tau = 0$. By definition, this is the limit of $\frac{1}{\tau}(\omega_\tau - \omega_0)$, so it can be interpreted as $\varphi \in \Omega^1(M, \mathcal{A}M)$. On the other hand, if $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is horizontal and P–equivariant, then $\omega + \Phi$ is a Cartan connection provided that it restricts to a linear isomorphism on each tangent space. Since this is an open condition, we can view $\Omega^1(M, \mathcal{A}M)$ as the space of all directions of deformations of the Cartan connection ω, i.e. as the space of all infinitesimal deformations of ω.

From 2.1 we know that the curvature of any Cartan connection on $\mathcal{G}$ is naturally interpreted as an element of $\Omega^2(M, \mathcal{A}M)$. In particular, for a deformation ω_τ of ω, the resulting infinitesimal change of the curvature can be viewed as an element of $\Omega^2(M, \mathcal{A}M)$.

To discuss $\Omega^0(M, \mathcal{A}M) = \Gamma(\mathcal{A}M)$ we need a second interpretation of $C^\infty(\mathcal{G}, \mathfrak{g})^P$. Since ω trivializes $T\mathcal{G}$, associating to a vector field ξ on $\mathcal{G}$

the function $\omega \circ \xi$ defines a bijection $\mathfrak{X}(\mathcal{G}) \to C^\infty(\mathcal{G}, \mathfrak{g})$. Equivariancy of ω immediately implies that $(\omega \circ \xi) \circ r^g = \mathrm{Ad}(g^{-1}) \circ (\omega \circ \xi)$ if and only if $(r^g)^*\xi = \xi$, so we obtain a bijection between $\Gamma(\mathcal{A}M)$ and the space $\mathfrak{X}(\mathcal{G})^P$ of P–invariant vector fields on $\mathcal{G}$. Notice that P–invariant vector fields are automatically projectable to vector fields on M, and this corresponds to the projection $\Pi : \mathcal{A}M \to TM$ from 2.1.

A vector field $\xi \in \mathfrak{X}(\mathcal{G})$ satisfies $(r^g)^*\xi = \xi$ if and only if its flow commutes with r^g, whenever the flow is defined. This is true for all $g \in P$ if and only if the local flows are principal bundle automorphisms. Thus we can view the space $\Gamma(\mathcal{A}M)$ as the space of infinitesimal principal bundle automorphisms of the Cartan bundle $\mathcal{G}$.

3.2. Given a section of $\mathcal{A}M$, we can look at the corresponding vector field on $\mathcal{G}$. The local flows of this vector field are principal bundle automorphisms, so we can use them to pull back the Cartan connection ω, which locally defines a deformation of ω. Deformations obtained in this way and also the corresponding infinitesimal deformations are called *trivial*. Note that while flows may be only locally defined the corresponding infinitesimal deformation is always defined globally.

An automorphism of the parabolic geometry $(\mathcal{G}, \omega)$ by definition is a principal bundle automorphism Φ of $\mathcal{G}$ such that $\Phi^*\omega = \omega$. Correspondingly, an infinitesimal automorphism is a P–invariant vector field ξ on $\mathcal{G}$ such that the induced infinitesimal deformation of the Cartan connection vanishes identically.

In studying the infinitesimal change of curvature caused by an infinitesimal deformation of the Cartan connection, there is an additional subtlety. For a deformation ω_τ of $\omega = \omega_0$, we may view the curvature κ_τ of ω_τ as an element of $\Omega^2(M, \mathcal{A}M)$, and we could simply differentiate this family of sections. However, the identification of $\Lambda^2T^*M \otimes \mathcal{A}M$ with the associated bundle $\mathcal{G} \times_P \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$, which is used to construct operators acting on the curvature, depends on the Cartan connection. The easiest way to take this into account is to first convert κ_τ into an equivariant function $\mathcal{G} \to \Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$ using ω_τ. Then one takes the derivative of this family of functions at $\tau = 0$ and converts it back to an element of $\Omega^2(M, \mathcal{A}M)$ using $\omega = \omega_0$.

Finally observe that using the projection $\Pi : \mathcal{A}M \to TM$, any section of $\mathcal{A}M$ has an underlying vector field on M. In particular, for $s \in \Gamma(\mathcal{A}M)$ we can insert $\Pi(s)$ into a (bundle valued) differential form on M, and we write i_s for the corresponding insertion operator. More generally, for $\varphi \in \Omega^\ell(M, \mathcal{A}M)$ and a vector bundle $V \to M$, we obtain an insertion operator $i_\varphi : \Omega^k(M, V) \to \Omega^{k+\ell-1}(M, V)$.

Proposition. *Let $(\mathcal{G} \to M, \omega)$ be a parabolic geometry with curvature $\kappa \in \Omega^2(M, \mathcal{A}M)$. Let ∇ be the adjoint tractor connection, and let $d^\nabla : \Omega^k(M, \mathcal{A}M) \to \Omega^{k+1}(M, \mathcal{A}M)$ be the corresponding covariant exterior derivative. Then we have:*
(1) For $s \in \Gamma(\mathcal{A}M)$, the infinitesimal deformation of ω induced by the corresponding invariant vector field is given by $\nabla s + i_s\kappa$. In particular, s is an infinitesimal automorphism if and only if $\nabla s = -i_s\kappa$.
(2) For an infinitesimal deformation $\varphi \in \Omega^1(M, \mathcal{A}M)$ of the Cartan connection ω, the induced infinitesimal change of the curvature is given by $d^\nabla\varphi - i_\varphi\kappa \in \Omega^2(M, \mathcal{A}M)$.

Proof. (1) The derivative at zero of the pullback of ω by the flow of ξ is the Lie derivative $\mathcal{L}_\xi\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Evaluating this on a vector field η, we obtain $\xi\cdot\omega(\eta) - \omega([\xi, \eta])$. If η is invariant and $t \in \Gamma(\mathcal{A}M)$ is the corresponding section, we can express this in terms of the operators on adjoint tractor fields introduced in [7, section 3]: The term $\xi\cdot\omega(\eta)$ corresponds exactly to the fundamental D–operator or fundamental derivative D_st, while the second term is computed in [7, 3.6]. Inserting this we see that $(\mathcal{L}_\xi\omega)(\eta)$ corresponds to $D_ts + \{t, s\} + \kappa(s, t)$, and by [7, 3.5] the first two terms add up to $\nabla_{\Pi(t)}s$, which implies the result.
(2) Let ω_τ be a deformation of ω, put $\Phi := \frac{d}{d\tau}|_{\tau=0}\omega_\tau \in \Omega^1_{\text{hor}}(\mathcal{G}, \mathfrak{g})^P$, and let $\varphi \in \Omega^1(M, \mathcal{A}M)$ be the corresponding element. Viewed as $K_\tau \in \Omega^2(\mathcal{G}, \mathfrak{g})$, the curvature of ω_τ is given by

$$K_\tau(\xi, \eta) = d\omega_\tau(\xi, \eta) + [\omega_\tau(\xi), \omega_\tau(\eta)].$$

The derivative of this expression with respect to τ at $\tau = 0$ is given by

$$d\Phi(\xi, \eta) + [\Phi(\xi), \omega(\eta)] + [\omega(\xi), \Phi(\eta)].$$

Choose ξ and η to be P–invariant and denote by s and t the corresponding sections of $\mathcal{A}M$. Inserting the definition of the exterior derivative, we can rewrite the above as

$$D_s(\varphi(\Pi(t))) - D_t(\varphi(\Pi(s))) - \varphi(\Pi([s, t])) - \{t, \varphi(\Pi(s))\} + \{s, \varphi(\Pi(t))\}.$$

As above, the first and last term adds up to $\nabla_{\Pi(s)}(\varphi(\Pi(t)))$ and similarly for the second and forth term. Since $\Pi([s, t]) = [\Pi(s), \Pi(t)]$, we see that $\frac{d}{d\tau}|_{\tau=0}K_\tau$ is represented by the covariant exterior derivative $d^\nabla\varphi$.

As discussed above, we should however first convert K_τ into a function using ω_τ, which means looking at $K_\tau(\omega_\tau^{-1}(X), \omega_\tau^{-1}(Y))$ for $X, Y \in \mathfrak{g}$, differentiate, and then convert the result back into a form using ω. Differentiating the equation $X = \omega_\tau(\omega_\tau^{-1}(X))$ we see that

$$\tfrac{d}{d\tau}|_{\tau=0}\omega_\tau^{-1}(X) = -\omega^{-1}(\Phi(\omega^{-1}(X))).$$

To get the expression for the change of the curvature, we thus have to add to $d^\nabla\varphi$ the terms

$$-K_0\Big(\omega^{-1}(\Phi(\omega^{-1}(X))),\omega^{-1}(Y)\Big)-K_0\Big(\omega^{-1}(X),\omega^{-1}(\Phi(\omega^{-1}(Y)))\Big),$$

which exactly represent $-i_\varphi\kappa$. □

Remark. We consider infinitesimal automorphisms and deformations on the level of the total space of the Cartan bundle here. As discussed in 2.1, regular normal parabolic geometries are equivalent to underlying structures. For several of these structures, notions of infinitesimal automorphisms and deformations are available in the literature, see 2.3 for a sketch of the conformal case.

For infinitesimal automorphisms, it is easy to see that the two concepts are equivalent: The construction of the canonical normal Cartan connection induces an equivalence of categories between regular normal parabolic geometries and underlying structures. An automorphism of the underlying structure uniquely lifts to an automorphism of the parabolic geometry, and conversely any automorphism of a parabolic geometry induces an automorphism of the underlying structure on the base. Applying this to local flows of vector fields, one immediately concludes that there is a bijective correspondence between infinitesimal automorphisms in the two senses. We shall see below that this correspondence is implemented by the machinery of BGG sequences.

In the case of infinitesimal deformations the question is a bit more subtle, but the concepts still coincide in all cases that I am aware of. The basic point here is the following: The underlying structures of parabolic geometries can all be encoded as infinitesimal flag structures, see [10]. These are principal G_0–bundles endowed with certain partially defined differential forms. A small deformation of the underlying structure cannot change the isomorphism type of the principal bundle, so it can be viewed as a deformation of the partially defined differential forms. Since the subgroup $P_+ \subset P$ is always contractible, the total space of the Cartan bundle must be a trivial P_+–principal bundles over the underlying G_0–bundle. Making choices, one can extend the partially defined differential forms from above to a Cartan connection of the principal P—bundle, and this transforms smooth families to smooth families. The canonical Cartan connection can then be constructed by a normalization process which again maps smooth families to smooth families. This construction will be described in detail in [11]. In this way, any deformation of the underlying structure gives rise to a deformation of the parabolic geometry, and since the converse direction is obvious, this establishes the equivalence of the two notions. We shall see below in examples that this correspondence is implemented by the BGG machinery.

3.3. **A variant of the adjoint BGG sequence.** Proposition 3.2 suggests considering the linear connection $\tilde{\nabla}$ on the bundle $\mathcal{A}M$ which is defined by $\tilde{\nabla}s = \nabla s + i_s\kappa$:

Lemma. *(1) For $\varphi \in \Omega^k(M, \mathcal{A}M)$ we have $d^{\tilde{\nabla}}\varphi = d^{\nabla}\varphi + (-1)^k i_\varphi\kappa$.*
(2) The curvature $\tilde{R}$ of $\tilde{\nabla}$ is given by $\tilde{R}(\xi,\eta)(s) = (D_s\kappa)(\xi,\eta)$, where D_s denotes the fundamental derivative.

Proof. (1) is a straightforward computation using the standard formula

$$\begin{aligned}(d^{\tilde{\nabla}}\varphi)(\xi_0,\dots,\xi_k) &= \sum_i (-1)^i \tilde{\nabla}_{\xi_i}(\varphi(\xi_0,\dots,\hat{i},\dots,\xi_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\varphi([\xi_i,\xi_j],\xi_0,\dots,\hat{i},\dots,\hat{j},\dots,\xi_k)\end{aligned}$$

for the covariant exterior derivative.
(2) The action of $\tilde{R}$ on $s \in \Gamma(\mathcal{A}M)$ can be computed as $d^{\tilde{\nabla}}\tilde{\nabla}s$. Inserting the definition of $\tilde{\nabla}$ and using (1), this equals $d^{\nabla}\nabla s + d^{\nabla}(i_s\kappa) - i_{\tilde{\nabla}s}\kappa$. The first term gives the action $\kappa\bullet s$ of the curvature of ∇, i.e. $(\kappa\bullet s)(\xi,\eta) = \{\kappa(\xi,\eta), s\}$. Since κ is the curvature of ∇, the Bianchi identity for linear connections implies that $0 = d^{\nabla}\kappa$. Taking $t_1, t_2 \in \Gamma(\mathcal{A}M)$ and expanding $0 = d^{\nabla}\kappa(t_1, s, t_2)$ we obtain the formula

$$d^{\nabla}(i_s\kappa)(t_1,t_2) = \nabla_s(\kappa(t_1,t_2)) - \kappa([s,t_1],t_2) - \kappa(t_1,[s,t_2]).$$

By [7, Proposition 3.2] we get $\nabla_s(\kappa(t_1,t_2)) = D_s(\kappa(t_1,t_2)) + \{s, \kappa(\xi,\eta)\}$. On the other hand, [7, Proposition 3.6] reads as $[s,t_1] = D_st_1 - \tilde{\nabla}_{t_1}s$. Inserting all these facts into the above formula for $d^{\tilde{\nabla}}\tilde{\nabla}s$, the claim follows. □

Using part (1), we conclude from Proposition 3.2 that the infinitesimal change of curvature caused by an infinitesimal deformation of a Cartan connection is computed by $d^{\tilde{\nabla}}$.

Now suppose that we are dealing with a regular parabolic geometry $(p : \mathcal{G} \to M, \omega)$. By definition, this means that κ is homogeneous of degree ≥ 1, i.e. for $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$, we have $\kappa(\xi,\eta) \in \Gamma(\mathcal{A}^{i+j+1}M)$. If $\varphi \in \Omega^k(M,\mathcal{A}M)$ is homogeneous of degree $\geq \ell$, this immediately implies that $i_\varphi\kappa$ is homogeneous of degree $\geq \ell+1$. Therefore $d^{\tilde{\nabla}}\varphi$ is congruent to $d^{\nabla}\varphi$ modulo elements which are homogeneous of degree $\geq \ell+1$. Hence the lowest possibly nonzero homogeneous components of $d^{\nabla}\varphi$ and of $d^{\tilde{\nabla}}\varphi$ coincide. As pointed out in 2.2, this is all we need to apply the BGG machinery to the twisted de–Rham sequence induced by $\tilde{\nabla}$.

We write $\tilde{L} : \Gamma(H_k(T^*M, \mathcal{A}M)) \to \Omega^k(M, \mathcal{A}M)$ for the splitting operators obtained by this construction. Their values $\tilde{L}(\alpha)$ are characterized by $\partial^*(\tilde{L}(\alpha)) = 0$, $\pi_H(\tilde{L}(\alpha)) = \alpha$, and $\partial^*(d^{\tilde{\nabla}}\tilde{L}(\alpha)) = 0$. The induced BGG operators $\tilde{D}_k : \Gamma(H_k(T^*M, \mathcal{A}M)) \to \Gamma(H_{k+1}(T^*M, \mathcal{A}M)))$ are given by $\tilde{D}_k = \pi_H \circ d^{\tilde{\nabla}} \circ \tilde{L}$.

3.4. **Infinitesimal automorphisms.** It is easy to relate the BGG sequence obtained from $d^{\tilde{\nabla}}$ to infinitesimal automorphisms:

Theorem. *Let $(p : \mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, P) corresponding to a $|k|$–grading of $\mathfrak{g}$. Then the bundle $H_0(T^*M, \mathcal{A}M)$ equals $\mathcal{A}M/\mathcal{A}^{-k+1}M \cong TM/T^{-k+1}M$. The algebraic projection π_H and the differential operator $\tilde{L}$ restrict to inverse bijections between infinitesimal automorphisms of $(p : \mathcal{G} \to M, \omega)$ and smooth sections $\sigma \in \Gamma(TM/T^{-k+1}M)$ such that $\tilde{D}_0(\sigma) = 0$.*

Proof. The bundle $H_0(T^*M, \mathcal{A}M)$ corresponds to the representation $H_0(\mathfrak{p}_+, \mathfrak{g})$. By definition, this homology group is $\mathfrak{g}/[\mathfrak{p}_+, \mathfrak{g}]$, and it is well known that $[\mathfrak{p}_+, \mathfrak{g}] = \mathfrak{g}^{-k+1}$, so the statement about $H_0(T^*M, \mathcal{A}M)$ follows.

By part (1) of Proposition 3.2, a smooth section $s \in \Gamma(\mathcal{A}M)$ defines an infinitesimal automorphism if and only if $\tilde{\nabla}s = 0$. If this is the case, then in particular $\partial^*(\tilde{\nabla}s) = 0$, and since $\partial^*(s) = 0$ is automatically satisfied, this implies $s = \tilde{L}(\pi_H(s))$ and $\tilde{D}_0(\pi_H(s)) = 0$. Hence π_H restricts to an injection from infinitesimal automorphisms to $\ker(\tilde{D}_0)$.

Conversely, if $\sigma \in \Gamma(TM/T^{-k+1}M)$ satisfies $\tilde{D}_0(\sigma) = 0$, then put $s := \tilde{L}(\sigma)$. Then $\partial^*(\tilde{\nabla}s) = 0$ and $\tilde{D}_0(\sigma) = 0$ implies that $\pi_H(\tilde{\nabla}s) = 0$, so $\tilde{\nabla}s$ is a section of the subbundle $\mathrm{im}(\partial^*) \subset T^*M \otimes \mathcal{A}M$. By part (2) of Proposition 3.2, we get $d^{\tilde{\nabla}}\tilde{\nabla}s = D_s\kappa$ and by naturality of the fundamental derivative and normality we get $\partial^*(D_s\kappa) = D_s\partial^*(\kappa) = 0$. But from 2.2 we know that $\partial^* \circ d^{\tilde{\nabla}}$ is injective on sections of $\mathrm{im}(\partial^*)$, so $\tilde{\nabla}s = 0$ and s is an infinitesimal automorphism. □

3.5. To complete the discussion of infinitesimal automorphisms, it remains to compare the first operator $\tilde{D}_0$ in the BGG sequence associated to $\tilde{\nabla}$ with the first operator D_0 in the BGG sequence associated to ∇.

Theorem. *Let $(p : \mathcal{G} \to M, \omega)$ be a regular normal parabolic geometry of type (G, P), and let $\mathfrak{g}$ be the Lie algebra of G. Let L and $\tilde{L}$ be the splitting operators in degree zero and D_0 and $\tilde{D}_0$ the BGG operators obtained from ∇ and $\tilde{\nabla}$, respectively.*

(1) If $\mathfrak{g}$ is $|1|$–graded or $(p : \mathcal{G} \to M, \omega)$ is torsion free, then $L = \tilde{L} : \Gamma(TM/T^{-k+1}M) \to \Gamma(\mathcal{A}M)$ and $\tilde{D}_0(\sigma) = D_0(\sigma) + \pi_H(i_{L(\sigma)}\kappa)$.
(2) If $(p : \mathcal{G} \to M, \omega)$ is torsion free and $H_1(\mathfrak{p}_+, \mathfrak{g})$ is concentrated in non–positive homogeneous degrees then $\tilde{D}_0 = D_0$.

Proof. (1) We start by computing $\partial^*(i_\xi\kappa)$ for an arbitrary vector field $\xi \in \mathfrak{X}(M)$. Locally, we can write κ as a finite sum of terms of the form $\varphi \wedge \psi \otimes t$ for $\varphi, \psi \in \Omega^1(M)$ and $t \in \Gamma(\mathcal{A}M)$. By definition, $\partial^*(\kappa)$ is then the sum of the corresponding terms of the form

$$-\psi \otimes \{\varphi, t\} + \varphi \otimes \{\psi, t\} - \{\varphi, \psi\} \otimes t.$$

On the other hand, $i_\xi\kappa$ is the sum of the terms $\varphi(\xi)\psi \otimes t - \psi(\xi)\varphi \otimes t$. Thus $\partial^*(i_\xi\kappa)$ is the sum of the terms $\varphi(\xi)\{\psi, t\} - \psi(\xi)\{\varphi, t\}$, and we conclude that

$$\partial^*(i_\xi\kappa) = -i_\xi\Big(\partial^*(\kappa) - (\{\ ,\ \} \otimes \mathrm{id})(\kappa)\Big),$$

where in the last term we use $\{\ ,\ \} \otimes \mathrm{id} : \Lambda^2T^*M \otimes \mathcal{A}M \to T^*M \otimes \mathcal{A}M$. Since we are dealing with a normal parabolic geometry, we have $\partial^*(\kappa) = 0$. In the case of a $|1|$–grading the map $\{\ ,\ \} : \Lambda^2T^*M \to T^*M$ is identically zero, so we get $\partial^*(i_\xi\kappa) = 0$ in this case.

In the torsion free case, we fist observe that the kernel of $[\ ,\ ] \otimes \mathrm{id}$ is a P–submodule in $\Lambda^2\mathfrak{p}_+ \otimes \mathfrak{g}$. For any normal parabolic geometry, the harmonic curvature $\kappa_H = \pi_H(\kappa)$ has values in $H_2(T^*M, \mathcal{A}M)$. By Kostant's version of the Bott–Borel–Weyl Theorem (see [17]) the corresponding subrepresentation has multiplicity one in $\Lambda^*\mathfrak{p}_+ \otimes \mathfrak{g}$. In particular, it has to be contained in the kernel of $[\ ,\ ] \otimes \mathrm{id}$. By [6, Theorem 3.2 (1)] the curvature of any torsion free parabolic geometry therefore has values in the kernel of $\{\ ,\ \} \otimes \mathrm{id}$, so we again conclude that $\partial^*(i_\xi\kappa) = 0$ for each ξ.

For a section σ of $TM/T^{-k+1}M$, consider $L(\sigma)$. By construction this satisfies $\partial^*(L(\sigma)) = 0$, $\pi_H(L(\sigma)) = \sigma$, and $\partial^*(\nabla L(\sigma)) = 0$. Since $\tilde{\nabla} L(\sigma) = \nabla L(\sigma) + i_{L(\sigma)}\kappa$, so we also have $\partial^*(\tilde{\nabla} L(\sigma)) = 0$. Hence $L(\sigma)$ satisfies the three properties which characterize $\tilde{L}(\sigma)$ and $\tilde{L} = L$ follows. Using this we obtain

$$\tilde{D}_0(\sigma) = \pi_H(\tilde{\nabla} L(\sigma)) = D_0(\sigma) + \pi_H(i_{L(\sigma)}\kappa).$$

(2) Since we are dealing with a torsion free geometry, we get $i_s\kappa \in \Omega^1(M, \mathcal{A}^0M) \subset \Omega^1(M, \mathcal{A}M)$ for each $s \in \Gamma(\mathcal{A}M)$. In particular, $i_s\kappa$ is always homogeneous of degree ≥ 1, so by the assumption on $H_1(\mathfrak{p}_+, \mathfrak{g})$ we get $\pi_H(i_{L(\sigma)}\kappa) = 0$ for any section σ of $TM/T^{-k+1}M$. $\square$

Corollary. *Suppose that* $(p : \mathcal{G} \to M, \omega)$ *is torsion free and* $H_1(\mathfrak{p}_+, \mathfrak{g})$ *is concentrated in non–positive homogeneous degrees, and that* $s \in \Gamma(\mathcal{A}M)$ *satisfies* $\nabla s = 0$. *Then* $i_s\kappa = 0$ *and in particular* s *is an infinitesimal automorphism.*

Proof. Since $\nabla s = 0$ we get $s = L(\pi_H(s))$ and $D_0(\pi_H(s)) = 0$. By the Theorem, we have $L = \tilde{L}$ and $D_0 = \tilde{D}_0$, and in the proof of Theorem 3.4, we have seen that $\tilde{D}_0(\pi_H(s)) = 0$ implies $\tilde{\nabla} s = 0$. □

Remark. (1) The condition that $H_1(\mathfrak{p}_+, \mathfrak{g})$ is concentrated in non–positive homogeneous degrees is easy to verify, see [20] or [9, Proposition 2.7]: The semisimple $|k|$–graded Lie algebra $\mathfrak{g}$ decomposes as a direct sum of $|k_i|$–graded simple ideals with $k_i \leq k$ for each i. The condition is equivalent to the fact that none of these simple ideals is of type A_ℓ or C_ℓ with the grading corresponding to the first simple root. If $\mathfrak{g}$ itself is simple, then this exactly excludes classical projective structures and a contact analog of these. Note that in the latter two cases regular normal parabolic geometries are automatically torsion free, so part (1) holds for all regular normal geometries in these cases.
(2) The statement of the corollary is rather surprising even in special cases like conformal structures. The identities responsible for its validity are contained in the proof of Lemma 3.3. From this proof one easily deduces $d^\nabla(i_s\kappa) = D_s\kappa - \kappa \bullet s + i_{\tilde{\nabla} s}\kappa$ for any $s \in \Gamma(\mathcal{A}M)$. If $\nabla s = 0$, then $0 = d^\nabla(\nabla s) = \kappa \bullet s$ and if the geometry is torsion free then this also implies that $\tilde{\nabla} s$ has values in $\mathcal{A}^0M$ and hence $i_{\tilde{\nabla} s}\kappa = 0$. Since $0 = D_s\partial^*(\kappa) = \partial^*(D_s\kappa)$ we obtain $\partial^* d^\nabla(i_s\kappa) = 0$, which under the assumptions of the Corollary implies $i_s\kappa = 0$.

3.6. **Infinitesimal deformations.** We next study infinitesimal deformations of parabolic geometries. Consider an infinitesimal deformation $\varphi \in \Omega^1(M, \mathcal{A}M)$ of a regular normal parabolic geometry $(p : \mathcal{G} \to M, \omega)$. Then φ is called normal, if the deformed curvature (infinitesimally) remains normal, so according to Propositions 3.2 and 3.3, this is the case if and only if $\partial^*(d^{\tilde{\nabla}}\varphi) = 0$.

The BGG machinery now easily implies that the operator $\tilde{D}_0$ whose kernel is the space of infinitesimal automorphisms, also has the formal tangent space to the moduli space of normal geometries as its cokernel:

Theorem. *Let* $(p : \mathcal{G} \to M, \omega)$ *be a regular normal parabolic geometry.*
(1) Any trivial infinitesimal deformation of ω *is normal.*
(2) The splitting operator $\tilde{L} : \Gamma(H_1(T^*M, \mathcal{A}M)) \to \Omega^1(M, \mathcal{A}M)$ *induces a bijection between* $\Gamma(H_1(T^*M, \mathcal{A}M))/\operatorname{im}(\tilde{D}_0)$ *and the formal*

tangent space at the given structure to the moduli space of all normal parabolic geometries on M.
(3) The BGG operator $\tilde{D}_1$ computes the infinitesimal change of the harmonic curvature caused by the infinitesimal deformation $\tilde{L}(\alpha)$ associated to $\alpha \in \Gamma(H_1(T^*M, \mathcal{A}M))$.

Proof. We have already observed in the proof of Theorem 3.4 that $d^{\tilde{\nabla}}\tilde{\nabla}s = D_s\kappa$ and that this has values in the kernel of ∂^*, so (1) follows.

For $\alpha \in \Gamma(H_1(T^*M, \mathcal{A}M))$ we put $\varphi := \tilde{L}(\alpha)$. Then by construction $\partial^*(d^{\tilde{\nabla}}\varphi) = 0$, so φ defines a normal infinitesimal deformation. By Proposition 3.2, $d^{\tilde{\nabla}}\varphi$ is the infinitesimal change of curvature caused by φ, and by definition $\tilde{D}_1(\alpha) = \pi_H(d^{\tilde{\nabla}}\varphi)$, which implies (3).

If $\alpha = \tilde{D}_0(\sigma)$, then put $s = \tilde{L}(\sigma)$, so $\alpha = \pi_H(\tilde{\nabla}s)$. Since $\partial^*(\tilde{\nabla}s) = 0$ and $\partial^*(d^{\tilde{\nabla}}\tilde{\nabla}s) = 0$ we conclude that $\tilde{\nabla}s = \tilde{L}(\alpha)$, so the resulting deformation is trivial. Thus $\tilde{L}$ induces a map from the quotient $\Gamma(H_1(T^*M, \mathcal{A}M))/\operatorname{im}(\tilde{D}_0)$ to normal infinitesimal deformations modulo trivial infinitesimal deformations.

Suppose that $\tilde{L}(\alpha) = \tilde{\nabla}s$. Then in particular $\partial^*(\tilde{\nabla}s) = 0$, so $s = \tilde{L}(\pi_H(s))$. Hence $\alpha = \tilde{D}_0(\pi_H(s))$ and our map is injective. To prove surjectivity, suppose that $\varphi \in \Omega^1(M, \mathcal{A}M)$ is any normal infinitesimal deformation. Put $s = -\tilde{Q}\partial^*(\varphi)$, where $\tilde{Q} : \Gamma(\operatorname{im}(\partial^*)) \to \Gamma(\operatorname{im}(\partial^*))$ is the inverse of $\partial^* \circ d^{\tilde{\nabla}}$, compare with 2.2. Replacing φ by the equivalent infinitesimal deformation $\psi = \varphi + \tilde{\nabla}s$, we see that $\partial^*(\psi) = 0$ and $\partial^*(d^{\tilde{\nabla}}\psi) = 0$, so $\psi = \tilde{L}(\pi_H(\psi))$ and surjectivity follows. □

The relation between the splitting operators and the BGG operators obtained from d^{∇} respectively $d^{\tilde{\nabla}}$ is much more complicated than for the first operator in the sequence. We just prove a simple general result here which is sufficient to deal with the cases discussed in this paper.

Lemma. *Let $(p : \mathcal{G} \to M, \omega)$ be a torsion free normal parabolic geometry. Suppose that $V \subset H_k(\mathfrak{p}_+, \mathfrak{g})$ and $W \subset H_{k+1}(\mathfrak{p}_+, \mathfrak{g})$ are irreducible components which are contained in homogeneity ℓ respectively $\ell + 1$. Then the components of the BGG operators $\tilde{D}_k$ and D_k, which map sections of $\mathcal{G} \times_P V$ to sections of $\mathcal{G} \times_P W$, coincide.*

Proof. Consider a section $\alpha \in \Gamma(\mathcal{G} \times_P V)$ and put $\varphi := L(\alpha) \in \Omega^k(M, \mathcal{A}M)$. Then φ is homogeneous of degree $\geq \ell$, $\partial^*(\varphi) = 0$ and $\pi_H(\varphi) = \alpha$. By part (1) of Lemma 3.3 we get $d^{\tilde{\nabla}}\varphi = d^{\nabla}\varphi + (-1)^k i_\varphi\kappa$ and therefore $\partial^*(d^{\tilde{\nabla}}\varphi) = (-1)^k\partial^*(i_\varphi\kappa)$. By torsion freeness κ is homogeneous of degree ≥ 2, so $i_\varphi\kappa$ is homogeneous of degree $\geq \ell + 2$.

Denoting by $\tilde{Q}$ the operator used in the proof of the Theorem, we conclude that $\psi := (-1)^{k+1}\tilde{Q}\partial^*(i_\varphi\kappa)$ is homogeneous of degree $\geq \ell+2$. By construction $\partial^*(\varphi+\psi)=0$, $\pi_H(\varphi+\psi)=\alpha$, and $\partial^*(d^{\tilde{\nabla}}(\varphi+\psi))=0$, which implies $\tilde{L}(\alpha)=\varphi+\psi$. Now

$$d^{\tilde{\nabla}}(\varphi+\psi)=d^{\nabla}\varphi+(-1)^k i_\varphi\kappa+d^{\tilde{\nabla}}\psi,$$

and the last two terms are homogeneous of degree $\geq \ell+2$. By homogeneity, these terms cannot contribute to the component of the image under π_H that we are interested in. □

3.7. **On regularity.** To get a complete correspondence to underlying structures, one has to single out regular normal infinitesimal deformations among all normal ones. Here a normal infinitesimal deformation $\varphi \in \Omega^1(M,\mathcal{A}M)$ is called regular if and only if $d^{\tilde{\nabla}}\varphi \in \Omega^2(M,\mathcal{A}M)$ is homogeneous of degree ≥ 1. Notice that this condition is vacuous if the geometry corresponds to a $|1|$–grading, and Theorem 3.6 therefore gives a complete description of the formal tangent space to the moduli space of regular normal geometries.

In general, we can first show that trivial infinitesimal deformations of regular normal geometries are regular. Indeed, from part (2) of Lemma 3.3 we know that for $s \in \Gamma(\mathcal{A}M)$ we have $d^{\tilde{\nabla}}\tilde{\nabla}s = D_s\kappa$. If we start from a regular normal geometry, then κ is homogeneous of degree ≥ 1, and by naturality of the fundamental derivative the same is true for $D_s\kappa$. Theorem 3.6 now directly implies

Corollary. *Let $(p:\mathcal{G}\to M,\omega)$ be a regular normal parabolic geometry. Then the formal tangent space at the given structure to the moduli space of regular normal geometries is the quotient of the space of all $\alpha \in \Gamma(H_1(T^*M,\mathcal{A}M))$ such that $d^{\tilde{\nabla}}\tilde{L}(\alpha)\in\Omega^2(M,\mathcal{A}M)$ is homogeneous of degree ≥ 1 by the image of $\tilde{D}_0$.*

For any concrete choice of structure, the condition on the homogeneity of $d^{\tilde{\nabla}}\tilde{L}(\alpha)$ can be made more explicit by projecting out step by step the lowest possibly nonzero homogeneous components of $d^{\tilde{\nabla}}\circ\tilde{L}$. For structures correspondig to $|2|$–gradings, we can give a nicer description, which will be useful in the examples in section 4.

Proposition. *Suppose that $P\subset G$ corresponds to a $|2|$–grading of $\mathfrak{g}$. Then for any regular normal parabolic geometry $(p:\mathcal{G}\to M,\omega)$ of type (G,P) and any section $\alpha\in\Gamma(H_1(T^*M,\mathcal{A}M))$ the form $d^{\tilde{\nabla}}\tilde{L}(\alpha)\in\Omega^2(M,\mathcal{A}M)$ is homogeneous of degree ≥ 1 if and only $\tilde{D}_1(\alpha)$ is homogeneous of degree ≥ 1.*

Proof. By definition, we have $\tilde{D}_1(\alpha) = \pi_H(d^{\tilde{\nabla}}\tilde{L}(\alpha))$. If $\tilde{D}_1(\alpha)$ is homogeneous of degree ≥ 1, then so is $\tilde{L}(\tilde{D}_1(\alpha))$, which differs from $d^{\tilde{\nabla}}\tilde{L}(\alpha)$ by a section of $\mathrm{im}(\partial^*)$. Since we deal with a $|2|$–grading, any element of $\Lambda^3T^*M \otimes \mathcal{A}^M$ is homogeneous of degree ≥ 1, and the result follows since ∂^* preserves homogeneities. $\square$

Since any irreducible component of $H_2(\mathfrak{p}_+, \mathfrak{g})$ is contained in some homogeneous degree, the condition in the proposition simply means that all components of $\tilde{D}_1(\alpha)$ in bundles corresponding to irreducible pieces in homogeneity zero have to vanish.

3.8. **The locally flat case.** As a simple consequence of Theorem 3.6, we can deal with the case of locally flat geometries. The following result was first proved in [5].

Theorem. *Let $(p : \mathcal{G} \to M, \omega)$ be a locally flat parabolic geometry. Then the BGG sequence associated to the adjoint representation is a complex. It can be naturally viewed as a deformation complex, i.e. its homologies in degrees zero and one are the space of infinitesimal automorphisms respectively the formal tangent space to the moduli space of all locally flat parabolic geometries on M.*

Proof. By local flatness, $\nabla = \tilde{\nabla}$ and this connection is flat, so the twisted de-Rham sequence is a complex. This easily implies that $L \circ D = d^\nabla \circ L$, so the BGG sequence also is a complex. By Theorem 3.5, the cohomology of this complex in degree zero is isomorphic to the space of infinitesimal automorphisms. For $\alpha \in \Gamma(H_1(T^*M, \mathcal{A}M))$ with $D_1(\alpha) = 0$ we have $d^\nabla L(\alpha) = LD(\alpha) = 0$, so the infinitesimal deformation $L(\alpha)$ does not change the curvature infinitesimally. Since conversely $d^\nabla L(\alpha) = 0$ clearly implies $D_1(\alpha) = 0$, we see that the kernel of D_1 exactly corresponds to the infinitesimal deformations in the subcategory of locally flat geometries. Now the interpretation of the first cohomology follows from Theorem 3.6. $\square$

4. Deformation complexes for torsion free geometries

In the recent joint work [13] with V. Souček, we have developed a theory of subcomplexes in curved BGG sequences. This theory applies to torsion free geometries of certain types. To have interesting examples, one needs assumptions on the structure of the homology groups $H_2(\mathfrak{p}_+, \mathfrak{g})$, which form the degree two part of the adjoint BGG sequence and governs the structure of the harmonic curvature. The main examples of this situation are the ones discussed in [13].

Here we find for all these examples a certain subcomplex in the adjoint BGG sequence (obtained from d^∇). Using the results of section 3, we can show that the first two operators in this subcomplex coincide with their counterparts in the BGG sequence obtained from $d^{\tilde\nabla}$. This leads to an interpretation of the subcomplex as a deformation complex in the appropriate subcategory of torsion free geometries.

4.1. **Grassmannian structures.** An almost Grassmannian structure on a manifold M of dimension $2n$ is essentially given by two auxiliary bundles E and F over M of rank 2 respectively n, and an isomorphism $\Phi : E^* \otimes F \to TM$. The bundles E and F are the basic building blocks for bundles over M corresponding to irreducible representations of P.

The BGG sequences in this case have triangular shape, see [13, 3.4]. The bundle in degree k of the BGG sequence splits as a direct sum of irreducible subbundles $\mathcal{H}_{p,q}$ with $p+q=k$ and $0 \le p \le q \le n$. In particular, the second bundle splits as $\mathcal{H}_{0,2} \oplus \mathcal{H}_{1,1}$, and correspondingly there are two irreducible components in the harmonic curvature. Let us now restrict to the case $n > 2$, the case $n = 2$ will be discussed below. The harmonic curvature component in $\Gamma(\mathcal{H}_{0,2})$ is called the torsion of the almost Grassmannian structure. Vanishing of this torsion is equivalent to torsion freeness in the sense of G–structures, and the corresponding geometries are called Grassmannian rather than almost Grassmannian. The harmonic curvature component in $\mathcal{H}_{1,1}$ is a true curvature. It is shown in [13, Theorem 3.5] that in the case of Grassmannian structures for any $p = 0, \dots, n$ the parts $\mathcal{H}_{p,p} \to \cdots \to \mathcal{H}_{p,n}$ and for any $q = 0, \dots, n$ the parts $\mathcal{H}_{0,q} \to \cdots \to \mathcal{H}_{q,q}$ are subcomplexes in each BGG sequence.

The representations inducing the bundles in the adjoint BGG sequence are determined in [13, 4.1], where we have to take $k = \ell = 1$. For $j < n$, one obtains $\mathcal{H}_{0,j} = (S^jE \otimes E^*)_0 \otimes (\Lambda^j F^* \otimes F)_0$, where the subscript 0 denotes the tracefree part. In particular $\mathcal{H}_{0,0} = E^* \otimes F = TM$, which also follows from Theorem 3.4, and $\mathcal{H}_{0,1} = \mathfrak{sl}(E) \otimes \mathfrak{sl}(F)$. Evidently, $\mathcal{H}_{0,j}M \subset \Lambda^j(E \otimes F^*) \otimes (E^* \otimes F) = \Lambda^j T^*M \otimes TM$. Looking at homogeneities, this implies that the BGG operators $\Gamma(\mathcal{H}_{0,j-1}) \to \Gamma(\mathcal{H}_{0,j})$ are first order for all $j = 1, \dots, n-1$. Finally, $\mathcal{H}_{0,n} = (S^{n+1}E \otimes E^*)_0 \otimes \Lambda^n F^*$, and the last BGG operator $\Gamma(\mathcal{H}_{0,n-1}) \to \Gamma(\mathcal{H}_{0,n})$ is of second order.

Finally, we need the bundle $\mathcal{H}_{1,1}$ which turns out to be the highest weight part in $\Lambda^2 E \otimes S^2 F^* \otimes \mathfrak{sl}(F)$. This is contained in $\Lambda^2 T^*M \otimes L(TM, TM)$, so the BGG operator $\Gamma(\mathcal{H}_{0,1}) \to \Gamma(\mathcal{H}_{1,1})$ is a second order operator.

Theorem. *Let M be a Grassmannian manifold of dimension $2n \geq 6$. Then the subcomplex*

$$0 \to \Gamma(\mathcal{H}_{0,0}) \to \Gamma(\mathcal{H}_{0,1}) \to \cdots \to \Gamma(\mathcal{H}_{0,n}) \to 0$$

of the adjoint BGG sequence is a deformation complex in the subcategory of Grassmannian structures.

Proof. The first two operators in this sequence are just the first two operators in the full adjoint BGG sequence, and from Theorem 3.5 and Lemma 3.6 we conclude that they coincide with their counterparts constructed from $\tilde{\nabla}$ rather than ∇. The statement on the cohomology in degree zero then follows from Theorem 3.4.

By part (2) of Theorem 3.6 and since regularity is automatic for $|1|$–gradings, the quotient $\Gamma(\mathcal{H}_{0,1})/\operatorname{im}(D_0)$ is isomorphic to infinitesimal deformations of M in the category of almost Grassmannian structures modulo trivial infinitesimal deformations. On the other hand, part (3) of Theorem 3.6 implies that the kernel of $\Gamma(\mathcal{H}_{0,1}M) \to \Gamma(\mathcal{H}_{0,2}M)$ corresponds exactly to those deformations for which the infinitesimal change of torsion is trivial, so these are exactly the infinitesimal deformations in the category of Grassmannian structures. □

Remark. (1) For almost Grassmannian structures, the right definition of an infinitesimal deformation is not immediately evident. Is is a nice feature of the approach via parabolic geometries and the BGG machinery, that it shows that infinitesimal deformations are smooth sections of the bundle $\mathfrak{sl}(E) \otimes \mathfrak{sl}(F)$. This can be seen directly as follows.

The only part of an almost Grassmannian structure that can be deformed nontrivially is the isomorphism $\Phi : E^* \otimes F \to TM$. Infinitesimally, deformations of this isomorphisms are linear maps $E^* \otimes F \to TM$ modulo those, which are compatible with Φ. Using Φ to convert the target of such a map back to $E^* \otimes F$, these are exactly endomorphisms of $E^* \otimes F$ modulo those which are of the form $\varphi \otimes \mathrm{id}_F + \mathrm{id}_E \otimes \psi$.
(2) By Theorem 3.6, the splitting operator $\tilde{L} : \Gamma(\mathcal{H}_{0,1}) \to \Omega^1(M, \mathcal{A}M)$ computes the infinitesimal deformation of the canonical Cartan connection caused by an infinitesimal deformation of the underlying structure.

4.2. **The case $n = 2$.** In this case, $\dim(M) = 4$ and an almost Grassmannian structure is equivalent to a conformal spin structure with split signature $(2, 2)$. Basically, this is due to the fact that $SL(4, \mathbb{R})$ naturally is a two fold covering of $SO(3, 3)$. Here the situation is more symmetric than for general Grassmannian structures and the two components of the harmonic curvature are the self dual and the anti self dual parts of the Weyl curvature. Theorem 3.6 directly leads to a complex

$$0 \to \Gamma(\mathcal{H}_{0,0}) \to \Gamma(\mathcal{H}_{0,1}) \to \Gamma(\mathcal{H}_{0,2}) \to 0$$

inside the BGG sequence obtained from $d^{\tilde{\nabla}}$, and, for anti self dual structures, an interpretation as a deformation complex in the category of anti self dual conformal structures. This is exactly the split signature version of the complex discussed in 2.3. However, in this case the second operator (which has order two) differs (tensorially) from its counterpart in the standard adjoint BGG sequence.

4.3. **Quaternionic structures.** An almost quaternionic structure on a smooth manifold of dimension $4n$ is given by a rank 3 subbundle $Q \subset L(TM, TM)$ which is locally spanned by three almost complex structures I, J, and $K = IJ = -JI$. However, these local almost complex structures are an additional choice and not an ingredient of the structure. Equivalently, one can view an almost quaternionic structure are a reduction of the structure group of the linear frame bundle to the subgroup $S(GL(1,\mathbb{H})GL(n,\mathbb{H})) \subset GL(4n,\mathbb{R})$. Replacing $S(GL(1,\mathbb{H})GL(n,\mathbb{H}))$ by the two–fold covering $S(GL(1,\mathbb{H}) \times GL(n,\mathbb{H}))$ one has an equivalent description as an identification of the complexified tangent bundle $TM \otimes \mathbb{C}$ into the tensor product $E \otimes F$, where E has complex rank two and F has complex rank $2n$. Hence after complexification we are in the same situation as for almost Grassmannian structures with even dimensional F.

In particular, the BGG sequences have the same shape as in the almost Grassmannian case, and the operators have the same orders. Moreover, after complexification the bundles showing up in each BGG sequence are the same as in the almost Grassmannian case. In particular, there are again two harmonic curvature components and for $n > 1$ (the case $n = 1$ will be discussed below) one of them is a torsion and the other is a true curvature. Vanishing of the torsion is again equivalent to torsion freeness in the sense of G–structures and the corresponding geometries are referred to as quaternionic rather than almost quaternionic. For quaternionic structures one obtains subcomplexes in all BGG sequences which have the same form as in the Grassmannian case.

Theorem. *Let M be a quaternionic manifold of dimension $4n \geq 8$. Then the subcomplex*

$$0 \to \Gamma(\mathcal{H}_{0,0}) \to \Gamma(\mathcal{H}_{0,1}) \to \cdots \to \Gamma(\mathcal{H}_{0,n}) \to 0$$

of the adjoint BGG sequence is an elliptic complex, which can be naturally interpreted as a deformation complex in the category of quaternionic structures.

Proof. The interpretation as a deformation complex works exactly as in the Grassmannian case. Exactness of the symbol sequence is the special case $k = \ell = 1$ of [13, Theorem 4.3]. □

Describing the bundles which show up in the deformation complex is straightforward, but we do not need this description here. Let us just note the description of the bundle $\mathcal{H}_{0,1}$, whose sections are the infinitesimal deformations of an almost quaternionic structure. For $Q \subset L(TM, TM)$ let $L_Q(TM, TM) \subset L(TM, TM)$ denote the subbundle of those endomorphisms which commute with any element of Q. Then it turns out that $L(TM, TM) \cong Q \oplus L_Q(TM, TM) \oplus \mathcal{H}_{0,1}$ and that $\mathcal{H}_{0,1}$ is isomorphic to the tensor product of Q with the space of tracefree elements of $L_Q(TM, TM)$.

In the special case $n = 1$, an almost quaternionic structure on a four manifold is equivalent to a conformal Riemannian spin structure. As in 4.2, we obtain the deformation complex discussed in 2.3 directly from Theorem 3.6. Again, the second operator in the sequence differs from the one in the standard adjoint BGG sequence. Ellipticity can be easily verified directly.

4.4. **Lagrangean contact structures.** A Lagrangean contact structure on a smooth manifold M of dimension $2n + 1$ is given by a codimension one subbundle $H \subset TM$, which defines a contact structure on M, together with a decomposition $H = E \oplus F$ as a direct sum of two Lagrangean (or Legendrean) subbundles. This means that the Lie bracket of two sections of E (respectively F) is a section of H. Since H defines a contact structure, this forces E and F to be of rank at most (and hence equal to) n. We will assume $n \geq 2$ throughout.

The form of the BGG sequences is described in [13, 3.6]. For $k \leq n$, the bundle in degree k of each BGG sequence splits as $\oplus_{p,q}\mathcal{H}_{p,q}$ with $p + q = k$ and $0 \leq p, q$. The decomposition of the bundle in degree $n + k$ has the same form as for degree $n - k + 1$. In particular, there are three components in the harmonic curvature. The components in $\Gamma(\mathcal{H}_{2,0})$ and $\Gamma(\mathcal{H}_{0,2})$ are torsions, which are exactly the obstructions to integrability of the subbundles E and F of TM. The component in $\Gamma(\mathcal{H}_{1,1})$ is a true curvature. For torsion free geometries, the bundles E and F are integrable, so M locally admits two transversal fibrations onto manifolds of dimension $n + 1$ such that the two vertical bundles span a contact distribution on M and both are Legendrean. In the torsion free case, there many subcomplexes in each BGG sequence (see [13, 3.7]), in particular the bundles $\mathcal{H}_{0,q}$ for $q = 0, \dots, n$ and $\mathcal{H}_{p,0}$ for $p = 0, \dots, n$ both form subcomplexes.

Specializing to the adjoint BGG sequence, we know from 3.4 that the first bundle $\mathcal{H}_{0,0}$ is the quotient $Q := TM/H$ of the tangent bundle by the contact subbundle. For $0 < j \leq n$ one easily verifies that the bundle $\mathcal{H}_{0,j}$ can be described as follows. Since E and F are Legendrean, the contact structure defines isomorphisms $F \cong E^* \otimes Q$. Thus $\Lambda^j E^* \otimes F \cong \Lambda^j E^* \otimes E^* \otimes Q$, and $\mathcal{H}_{0,j} \subset \Lambda^j E^* \otimes F$ corresponds to the kernel of the alternation $\Lambda^j E^* \otimes E^* \otimes Q \to \Lambda^{j+1} E^* \otimes Q$. In particular, the operator mapping sections of $\mathcal{H}_{0,0}$ to sections of $\mathcal{H}_{0,1}$ must be of second order, while for $1 \leq j < n$ the operator $\Gamma(\mathcal{H}_{0,j}) \to \Gamma(\mathcal{H}_{0,j+1})$ is first order. In the same way, the bundles $\mathcal{H}_{j,0}$ for $1 \leq j \leq n$ can be described as subbundles in $\Lambda^j F^* \otimes E$ and one gets the analogous results for the orders of the operators.

Theorem. *Let (M, E, F) be a torsion free Lagrangean contact structure. Then the subcomplex*

$$0 \to \Gamma(\mathcal{H}_{0,0}) \to \begin{matrix} \Gamma(\mathcal{H}_{0,1}) \\ \oplus \\ \Gamma(\mathcal{H}_{1,0}) \end{matrix} \to \cdots \to \begin{matrix} \Gamma(\mathcal{H}_{0,n}) \\ \oplus \\ \Gamma(\mathcal{H}_{n,0}) \end{matrix} \to 0$$

in the adjoint BGG sequence can be naturally viewed as a deformation complex in the category of torsion free Lagrangean contact structures.

Proof. From Theorem 3.5 and Lemma 3.6 we see that the first two operators in this complex coincide with their counterparts in the BGG sequence obtained from $d^{\tilde{\nabla}}$. Since the bundles $\mathcal{H}_{2,0}$, $\mathcal{H}_{1,1}$, and $\mathcal{H}_{0,2}$ are all contained in positive homogeneous degrees, normal infinitesimal deformations are automatically regular by Proposition 3.7. Now the interpretation as a deformation complex works as for Grassmannian structures. □

We can again see directly that sections of the bundle $\mathcal{H}_{0,1} \oplus \mathcal{H}_{1,0}$ are the right notion for infinitesimal deformations of a Lagrangean contact structure. Since contact structures are rigid, the only way to deform such a structure is deforming the decomposition $H = E \oplus F$. Infinitesimally, a deformation of the subbundle $E \subset H$ is given by a linear map $E \to H/E \cong F$. Such a deformation is in the direction of a Legendrean subbundle if and only if the corresponding map $E \times E \to Q$ has trivial alternation. Likewise, $\Gamma(\mathcal{H}_{1,0})$ describes infinitesimal Lagrangean deformations of $F \subset H$. The splitting operator $\tilde{L}_1$ again computes the infinitesimal deformation of the normal Cartan connection caused by an infinitesimal deformation of a Lagrangean contact structure.

Note that the deformation complex cannot be elliptic or subelliptic, since Q is a real line bundle, while all other bundles showing up in the subcomplex have even rank.

4.5. **CR structures.** This case is closely parallel to the case of Lagrangean contact structures. The geometries in question are non–degenerate almost CR structures of hypersurface type, which satisfy partial integrability, a weakening of the usual integrability condition for CR structures, see [9, 4.15]. Compared the the case of Lagrangean contact structures, one only replaces the decomposition of the contact subbundle (which can be interpreted as an almost product structure) by an almost complex structure J on the contact subbundle H. The condition that the two subbundles are Legendrean corresponds to the partial integrability condition for almost CR structures. The complexification $H \otimes \mathbb{C}$ splits as $H^{1,0} \oplus H^{0,1}$ into holomorphic and anti holomorphic part, so an this level the picture is parallel to the Lagrangean contact case.

In particular, complex BGG sequences have exactly the same form as for Lagrangean contact structures. However, BGG sequences corresponding to real representations without an invariant complex structure (like the adjoint representation) are different. They are obtained by "folding" a complex BGG pattern, see [13, 3.8]. In particular, there are only two irreducible components in the harmonic curvature. One of these components is a torsion (corresponding to the two torsions for Lagrangean contact structures), while the other is a curvature. The torsion is a multiple of the Nijenhuis tensor, so the torsion free geometries are exactly CR structures, see [9, 4.16]. For CR structures, one obtains many subcomplexes in BGG sequences, see [13, Theorem 3.8].

Using the notation of [13, 3.8], there is a subcomplex in the adjoint BGG sequence which starts at $\mathcal{H}_{0,0}$. This has the form

$$0 \to \Gamma(\mathcal{H}_{0,0}) \to \Gamma(\mathcal{H}_{1,0}) \to \cdots \to \Gamma(\mathcal{H}_{n,0}) \to 0.$$

and apart from $\mathcal{H}_{0,0} = Q := TM/H$, all the bundles $\mathcal{H}_{j,0}$ in the sequence are complex vector bundles. To identify them, we just have to observe that their complexification splits into a direct sum of two complex vector bundles which exactly correspond to the two bundles in the Lagrangean contact case with E and F replaced by $H^{1,0}$ and $H^{0,1}$. In particular we see that $\mathcal{H}_{j,0} \otimes \mathbb{C} \subset L_{\mathbb{C}}(\Lambda^j(H \otimes \mathbb{C})^*, H \otimes \mathbb{C})$ and the components are singled out by their complex (anti–)linearity properties. For example, $\mathcal{H}_{0,1} \otimes \mathbb{C}$ is contained in $L(H^{1,0}, H^{0,1}) \oplus L(H^{0,1}, H^{1,0})$, which exactly means that $\mathcal{H}_{0,1}$ consists of conjugate linear maps $H \to H$. A conjugate linear map φ lies in $\mathcal{H}_{0,1}$ if and only if the corresponding bilinear map $H \times H \to Q$ is symmetric. In particular, conjugate linear maps are exactly infinitesimal deformations of the almost complex structure J (which is the only deformable ingredient in the structure) and the symmetry condition takes care about partial integrability. As before we deduce:

Theorem. *Let (M, H, J) be a non–degenerate CR structure of hypersurface type. Then the subcomplex*

$$0 \to \Gamma(\mathcal{H}_{0,0}) \to \Gamma(\mathcal{H}_{1,0}) \to \cdots \to \Gamma(\mathcal{H}_{n,0}) \to 0$$

in the adjoint BGG sequence can be naturally interpreted as a deformation complex in the category of CR structures.

Remark. This deformation complex has been found (by ad hoc methods) and successfully applied to the deformation theory of strictly pseudoconvex compact CR manifolds earlier. The part starting from $\mathcal{H}_{1,0}M$ is used in the work of T. Akahori in the case $n \geq 3$, see e.g. [1]. The full complex was constructed in [2] for $n = 2$. Since the first bundle in the complex is a real line bundle while all others a complex vector bundles, there is again no hope for the whole complex to be elliptic or subelliptic. Nonetheless, for some of the operators in the complex one can prove subelliptic estimates (in the strictly pseudoconvex case), which play a crucial role in the applications to deformation theory.

4.6. **Quaternionic contact structures.** These geometries are given by certain codimension three subbundles in the tangent bundles of manifolds of dimension $4n + 3$. Recall first that for $p + q = n$, there is (up to isomorphism) a unique quaternionic Hermitian form of signature (p, q) on $\mathbb{H}^n$. The imaginary part of this form is a skew symmetric bilinear map $\mathbb{H}^n \times \mathbb{H}^n \to \mathrm{im}(\mathbb{H})$. Putting $\mathfrak{g}_1 := \mathbb{H}^n$ and $\mathfrak{g}_2 := \mathrm{im}(\mathbb{H})$ this imaginary part makes $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ into a nilpotent graded Lie algebra, called the quaternionic Heisenberg algebra of signature (p, q). Since the forms of signature (p, q) and (q, p) differ only by sign, we may assume $p \geq q$. Similarly, one may look at the algebra of split quaternions, for which there is a unique Hermitian form in each dimension. Correspondingly, one obtains a unique split quaternionic Heisenberg algebra of dimension $4n + 3$ for each $n \geq 1$.

Recall that if $H \subset TM$ is a subbundle in the tangent bundle of a smooth manifold M, then the Lie bracket of vector fields induces a tensorial map $\mathcal{L} : H \times H \to TM/H =: Q$. For any $x \in M$ we put $\mathrm{gr}_1(T_xM) = H_x$ and $\mathrm{gr}_2(T_xM) := Q_x$. Then we can view $\mathcal{L}$ as defining on each of the spaces $\mathrm{gr}(T_xM) = \mathrm{gr}_1(T_xM) \oplus \mathrm{gr}_2(T_xM)$ the structure of a nilpotent graded Lie algebra. A *quaternionic contact structure* of signature (p,q) on a smooth manifold M of dimension $4(p+q)+3$ is a smooth subbundle $H \subset TM$ of corank 3 such that for each $x \in M$ the nilpotent graded Lie algebra $\mathrm{gr}(T_xM)$ is isomorphic to the quaternionic Heisenberg algebra of signature (p,q). A *split quaternionic contact structure* on a smooth manifold of dimension $4n+3$ is defined similarly using the split quaternionic Heisenberg algebra.

For $n=1$ we have $\dim(M)=7$ and there is only one possible signature. It turns out that both the quaternionic and the split quaternionic Heisenberg algebra are rigid in this case. Moreover, corank three distributions defining quaternionic and split quaternionic contact structures the two generic types of rank 4 distributions in dimension 7. In particular, a generic real hypersurface in a two–dimensional (split) quaternionic vector space carries a (split) quaternionic contact structure.

For $n>1$, there are no generic distributions of rank $4n$ in manifolds of dimension $4n+3$, but it is known from the works of O. Biquard, see [3, 4], that there are many examples of quaternionic contact structures of signature $(n,0)$.

For all these structures, the BGG sequences have the same form, see [13, 3.9]. For $k=0,\dots,2n+1$ the bundle in degree k splits into a direct sum of bundles $\mathcal{H}_{p,q}$ with $p+q=k$ and $p \geq q$, and for the bundle in degree $2n+1+k$ decomposes in the same way as the one in degree $2n+2-k$. In particular, in degree two we obtain two irreducible components $\mathcal{H}_{2,0}$ and $\mathcal{H}_{1,1}$. The harmonic curvature component having values in the bundle $\mathcal{H}_{2,0}$ of the adjoint BGG sequence is a torsion, while the one having values in $\mathcal{H}_{1,1}$ is a curvature. For $n=1$, one obtains a subcategory of torsion free (split) quaternionic contact structures. However, for $n>1$, bundle $\mathcal{H}_{2,0}$ is contained in homogeneity zero, so vanishing of the corresponding harmonic curvature component is forced by regularity, and any (split) quaternionic contact structure is automatically torsion free.

By [13, Theorem 3.10] there is a number of subcomplexes in each BGG sequence for a manifold endowed with a torsion free (split) quaternionic contact structure. In particular, the bundles $\mathcal{H}_{p,0}$ with $p=0,\dots,2n+1$ form a subcomplex. For the adjoint BGG sequence, one verifies directly that the operator $\Gamma(\mathcal{H}_{n,0}) \to \Gamma(\mathcal{H}_{n+1,0})$ is of second order, while all other operators in the subcomplex are of first order.

Theorem. *Let M be a smooth manifold of dimension $4n+3 \geq 11$ endowed with a quaternionic contact structure or split quaternionic contact structure. Then the subcomplex*

$$0 \to \Gamma(\mathcal{H}_{0,0}) \to \Gamma(\mathcal{H}_{1,0}) \to \cdots \to \Gamma(\mathcal{H}_{2n+1,0}) \to 0$$

of the adjoint BGG sequence can be naturally interpreted as a deformation complex in the category of torsion free (split) quaternionic contact structures.

Proof. By Theorem 3.5 and Lemma 3.6 the first two operators in this sequence coincide with their counterparts obtained from $d^{\tilde{\nabla}}$. Using Proposition 3.7 we conclude that the kernel of the operator $\mathcal{H}_{1,0} \to \mathcal{H}_{2,0}$ exactly corresponds to regular normal deformations. The interpretation as a deformation complex then works as before. $\square$

Remark. The situation in the seven–dimensional case is not completely clear. The problem here is that the operator $\mathcal{H}_{1,0} \to \mathcal{H}_{2,0}$ in the adjoint BGG sequence is of second order. It seems that the two operators obtained from d^{∇} respectively from $d^{\tilde{\nabla}}$ differ (tensorially) from each other. Therefore, there seems to be no direct way to relate the BGG sequence based on d^{∇} (for which we can prove the existence of the relevant subcomplex) to the one based on $d^{\tilde{\nabla}}$ (for which we have the interpretation in terms of infinitesimal deformations).

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Institut für Mathematik, Universität Wien, Nordbergstrsse 15, A–1090 Wien, Austria and International Erwin Schrödinger Institute for Mathematical Physics, Boltzmanngasse 9, A–1090 Wien, Austria

E-mail address: `Andreas.Cap@esi.ac.at`