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Maximally Homogeneous para–CR Manifolds

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MAXIMALLY HOMOGENEOUS PARA-CR MANIFOLDS

D.V. ALEKSEEVSKY, C. MEDORI AND A. TOMASSINI

Abstract. We define the notion of a (weak) almost para-CR structure on a manifold M as a distribution $HM \subset TM$ together with a field $K \in \Gamma(\mathrm{End}(HM))$ of involutive endomorphisms of HM. If K satisfies integrability conditions, then (HM, K) is called a (weak) para-CR structure. Under some regularity conditions, an almost para-CR structure can be identified with a Tanaka structure. The notion of maximally homogeneous almost para-CR structure of a semisimple type is defined. A classification of such maximally homogeneous almost para-CR structures is given in terms of appropriate gradations of real semisimple Lie algebras. All such maximally homogeneous structures of depth two (which correspond to depth two gradations) are listed and the integrability conditions are verified.

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Introduction

A *weak almost paracomplex structure* on a $2n$-dimensional manifold M is a field $K \in \Gamma(\mathrm{End(TM)})$ of endomorphisms with $K^2 = \mathrm{id}$. If the eigenspace distributions $T^{\pm}$ with eigenvalues ± 1 have the same rank, then K is called an *almost paracomplex structure* and if the Nijenhuis tensor K vanishes, then K is called a paracomplex structure.

Many notions and results of the complex geometry can be generalized to the para-complex case. For example, analogously to the Kähler structure one can define a *para-Kähler structure* on a manifold M as a pair (K, g), where K is a paracomplex structure on M, g is a pseudo-Riemannian metric such that

$$g(KX, KY) = -g(X, Y)$$

and the two form $\omega(X, Y) = g(X, KY)$ is closed.

A para-Kähler structure on M gives rise to a pair of involutive Lagrangian distributions T^-M and T^+M, such that $TM = T^+M \oplus T^-M$, called *bi-Lagrangian structure* (see [3]). Vice versa, starting from a symplectic manifold (M, ω), whose tangent bundle TM is the direct sum of two Lagrangian involutive distributions $T^{\pm}M$, one can define a para-Kähler structure on M, setting

$$K|_{T^{\pm}M} = \pm\mathrm{id} \quad \text{and} \quad g(X, Y) = \omega(KX, Y)\,.$$

For other references on paracomplex geometry see e.g. [1],[6],[7],[10],[17].

In this paper, we consider the analogue of a CR structure in the paracomplex context and we give a classification of maximally homogeneous para-CR manifolds of semisimple type.

A *weak almost para-CR structure* on a manifold M is a pair (HM, K) where HM is a distribution on M and $K \in \Gamma(\mathrm{End}(HM))$ is a field of endomorphisms on HM with $K^2 = \mathrm{id}$. It is called a weak para-CR structure if the eigenspace distributions $H^{\pm} \subset HM$ are involutive (see sections 1 and 2).

We associate with an almost para-CR structure and a point $x \in M$ a pair $(\mathfrak{m}(x), K_x)$ (called *para-CR algebra*) where $\mathfrak{m}(x) = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ is a negatively graded Lie algebra and K_x is an involutive endomorphism in $\mathfrak{m}^{-1}$. If the algebras $(\mathfrak{m}(x), K_x)$ are isomorphic to a fixed para-CR algebra

$(\mathfrak{m}, K_o)$, then the almost para-CR structure is called a *regular structure of type* $(\mathfrak{m}, K_o)$. A regular almost para-CR structure can be identified with a Tanaka structure (see section 3). In Section 4, we recall the basic results of the theory of prolongations of graded Lie algebras and Tanaka structures. We define a semisimple Tanaka structure of type $(\mathfrak{m}, G^0)$ as a Tanaka structure $(\mathfrak{m}, G^0)$ such that the full prolongation $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ of the associated non-positively graded Lie algebra

$$\mathfrak{m} + \mathfrak{g}^0 = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1} + \mathfrak{g}^0$$

is a finite dimensional semisimple Lie algebra and we give the notion of maximally homogeneous Tanaka structure (see section 5). We construct the standard model $F = G/P$ of a maximally homogeneous Tanaka structure of semisimple type $(\mathfrak{m}, G^0)$ and prove that any maximally homogeneous Tanaka structure of type $(\mathfrak{m}, G^0)$ is isomorphic to the standard one up to a covering. Applying this results to a weak almost para-CR structure, we reduce the classification of weak almost para-CR structures of semisimple type (up to a covering) to the classification of fundamental gradations $\mathfrak{g} = \mathfrak{g}^{-d} + \cdots + \mathfrak{g}^d$ of real semisimple Lie algebras with reducible $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$. We give a description of all such gradations in terms of Satake diagrams of the Lie algebra $\mathfrak{g}$ (see sections 6 and 7). In Section 8, we specialize these results to the case of gradations of depth 2. We describe explicitly all standard maximally homogeneous weak para-CR manifolds $F = G/P$.

1. Paracomplex structures

Let M be a $2n$-dimensional manifold. An *almost paracomplex structure* on M is a field of endomorphisms $K \in \mathrm{End}(TM)$ of the tangent bundle TM of M such that $K^2 = \mathrm{id}$ and its ± 1-eigenspace distributions $T^\pm M$ have the same rank (see e.g. [11], [7]).
An almost paracomplex structure K is called a *paracomplex structure*, if it is *integrable*, i.e.

$$S(X,Y) = [X,Y] + [KX,KY] - K[X,KY] - K[KX,Y] = 0$$

for any vector fields X, $Y \in \Gamma(TM)$.
This is equivalent to say that the distributions $T^\pm M$ are involutive.
A *para-Hermitian structure* on a $2n$-dimensional manifold M is a pair (K, g), where g is a pseudo-Riemannian metric and K is a g-skewsymmetric paracomplex structure.

Note that

$$g(KX, KY) = -g(X,Y)$$

for X, $Y \in \Gamma(TM)$ and that g has signature (n, n).
A *para-Kähler structure* on M is a para-Hermitian structure, with closed (non-degenerate) 2-form

$$\omega(X,Y) = g(KX,Y)\,.$$

The eigenspace distributions $T^{\pm}M$ define a bi-Lagrangian structure, i.e. a decomposition of TM into direct sum of two ω-Lagrangian distributions (see Bryant [3]).
Conversely, a bi-Lagrangian structure $TM = T^{+}M + T^{-}M$ on a symplectic manifold (M, ω) defines a para-Kähler structure (K, g) on M, where $K_{|_{T^{\pm}M}} = \pm\mathrm{id}$ and $g(X, Y) = \omega(KX, Y)$.

2. Para-CR manifolds

Recall that an *almost CR-structure* of codimension k on a $2n + k$-dimensional manifold M is a distribution $HM \subset TM$ of rank $2n$ together with a field of endomorphisms $J \in \mathrm{End}(HM)$ such that $J^2 = -\mathrm{id}$. An almost CR-structure is called *CR-structure*, if the $\pm i$-eigenspace subdistributions $H_{\pm}M$ of the complexified tangent bundle $T^{\mathbb{C}}M$ are involutive.
We define an almost para-CR structure in a similar way.

Definition 2.1. *A* weak almost para-CR structure *on a $2n+k$-dimensional manifold M is a pair (HM, K), where $HM \subset TM$ is a rank $2n$ distribution and $K \in \mathrm{End}(HM)$ is a field of endomorphisms such that $K^2 = \mathrm{id}$ and $K \neq \pm\mathrm{id}$.*
A weak almost para-CR structure is said to be a weak para-CR structure*, if it is* formally integrable*, i.e. if the following conditions hold:*

$$[KX, KY] + [X, Y] \in \Gamma(HM)\,, \tag{1}$$

$$S(X, Y) := [X, Y] + [KX, KY] - K([X, KY] + [KX, Y]) = 0 \tag{2}$$

for all X, $Y \in \Gamma(HM)$.

Let M be a manifold endowed with a weak almost para-CR structure (HM, K). Let

$$H_{\pm}M = \{X \pm KX \mid X \in \Gamma(M, HM)\}\,.$$

The structure (HM, K) is said to be an *almost para-CR structure*, if the distributions $H_{+}M$ and $H_{-}M$ have the same rank.
A straightforward computation shows that the integrability condition is equivalent to the involutivness of the distributions $H_{+}M$ and $H_{-}M$.
A manifold M, endowed with a (weak almost) para-CR structure, is called a (*weak almost*) *para-CR manifold.*
Note that a direct product of (weak almost) para-CR manifolds is a (weak almost) para-CR manifold.
We will show that a para-CR-structure can be considered as a Tanaka structure. First of all we recall the definition of a Tanaka structure.

3. Graded Lie algebras and Tanaka structures

3.1. **Gradations of a Lie algebra.** Recall that a *gradation* (more precisely $\mathbb{Z}$-gradation) of *depth* k of a Lie algebra $\mathfrak{g}$ is a direct sum decomposition

$$\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}^i = \mathfrak{g}^{-k} + \mathfrak{g}^{-k+1} + \cdots + \mathfrak{g}^0 + \cdots + \mathfrak{g}^j + \cdots \tag{3}$$

such that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, for any $i, j \in \mathbb{Z}$ and $\mathfrak{g}^{-k} \neq \{0\}$. Note that $\mathfrak{g}^0$ is a subalgebra and $\mathfrak{g}^i$ is a $\mathfrak{g}^0$-module.

We say that an element $x \in \mathfrak{g}^j$ has *degree* j and we write $d(x) = j$. The endomorphism δ of $\mathfrak{g}$ defined by

$$\delta_{|\mathfrak{g}_j} = j \cdot id$$

is a semisimple derivation of $\mathfrak{g}$ (with integer eigenvalues), whose eigenspaces determine the gradation. Conversely, any semisimple derivation δ of $\mathfrak{g}$ with integer eigenvalues defines a gradation where the grading space $\mathfrak{g}_j$ is the eigenspace of δ with eigenvalue j. If $\mathfrak{g}$ is a semisimple Lie algebra, then any derivation δ is inner, i.e. there exists $d \in \mathfrak{g}$ such that $\delta = \mathrm{ad}_d$. The element $d \in \mathfrak{g}$ is called the *grading element*.

Definition 3.1. *A gradation* $\mathfrak{g} = \sum \mathfrak{g}^i$ *of a Lie algebra* (*or a graded Lie algebra* $\mathfrak{g}$) *is called*

1. fundamental*, if the negative part* $\mathfrak{m} = \sum_{i<0} \mathfrak{g}^i$ *is generated by* $\mathfrak{g}^{-1}$*;*
2. effective *or* transitive*, if the non-negative part*

   $$\mathfrak{g}^{\geq 0} = \mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \cdots$$

   has no non-trivial ideal of $\mathfrak{g}$*;*
3. non degenerate*, if*

   $$X \in \mathfrak{g}^{-1}\,,\ [X, \mathfrak{g}^{-1}] = 0 \implies X = 0\,.$$

3.2. **Fundamental algebra associated with a distribution.** We associate to a distribution $\mathcal{H}$ on a manifold M and to a point $x \in M$ a graded Lie algebra $\mathfrak{m}(x)$.

First of all, we consider a filtration of the Lie algebra $\mathcal{X}(M)$ of vector fields defined inductively by

$$\begin{aligned} \Gamma(\mathcal{H})_{-1} &= \Gamma(\mathcal{H})\,, \\ \Gamma(\mathcal{H})_{-i} &= \Gamma(\mathcal{H})_{-i+1} + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})_{-i+1}]\ , \text{for } i > 1. \end{aligned}$$

Evaluating vector fields at a point $x \in M$, we get a flag

$$T_xM \supset \mathcal{H}_{-d-1}(x) = \mathcal{H}_{-d}(x) \supsetneq \mathcal{H}_{-d+1}(x) \supset \cdots \supset \mathcal{H}_{-2}(x) \supset \mathcal{H}_{-1}(x) = \mathcal{H}_x$$

in T_xM, where

$$\mathcal{H}_{-i}(x) = \{X_{|x} \mid X \in \Gamma(\mathcal{H})_{-i}\}.$$

Let us assume that $\mathcal{H}_{-d}(x) = T_xM$.
The commutators of vector fields induce a structure of fundamental negatively graded Lie algebra on the associated graded space

$$\mathfrak{m}(x) = \mathrm{gr}(T_xM) = \mathfrak{m}^{-d}(x) + \mathfrak{m}^{-d+1}(x) + \cdots + \mathfrak{m}^{-1}(x)\,,$$

where $\mathfrak{m}^{-j}(x) = \mathcal{H}_{-j}(x)/\mathcal{H}_{-j+1}(x)$. Note that $\mathfrak{m}^{-1}(x) = \mathcal{H}_x$.
A distribution $\mathcal{H}$ is called a *regular distribution* of *depth* d and *type* $\mathfrak{m}$ if all graded Lie algebras $\mathfrak{m}(x)$ are isomorphic to a given graded fundamental Lie algebra

$$\mathfrak{m} = \mathfrak{m}^{-d} + \mathfrak{m}^{-d+1} + \cdots + \mathfrak{m}^{-1}\,.$$

In this case $\mathfrak{m}$ is called the *Lie algebra associated* with the distribution $\mathcal{H}$. A regular distribution $\mathcal{H}$ is called *non-degenerate* if the associated Lie algebra is non degenerate.

3.3. **Para-CR algebras and Tanaka structures.**

Definition 3.2. *Let $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ be a negatively graded Lie algebra generated by $\mathfrak{m}^{-1}$ and G^0 be a closed Lie subgroup of (grading preserving) automorphisms of $\mathfrak{m}$. A* Tanaka structure *of* type $(\mathfrak{m}, G^0)$ *on a manifold M is a regular distribution $\mathcal{H} \subset TM$ of type $\mathfrak{m}$ together with a principal G^0-bundle $\pi : Q \to M$ of adapted coframes of $\mathcal{H}$. A coframe $\varphi : \mathcal{H}_x \to \mathfrak{m}^{-1}$ is called* adapted *if it can be extended to an isomorphism $\varphi : \mathfrak{m}_x \to \mathfrak{m}$ of Lie algebra.*

Remark that the extension of φ is unique since $\mathfrak{m}$ is generated by $\mathfrak{m}^{-1}$. As an example, let $\mathcal{H}$ be a regular distribution of type $\mathfrak{m}$ and $J : x \mapsto J_x \in \mathrm{Ten}(\mathcal{H})_x$ a field of tensors in $\mathcal{H}$ which have a fixed canonical form $J_o \in \mathrm{Ten}(\mathfrak{m}^{-1})$. We denote by G^0 the subgroup of the group of automorphisms $\mathrm{Aut}(\mathfrak{m})$ which preserves J_o and by $\pi : Q \to M$ the G^0-principal bundle of admissible coframes $\varphi : \mathcal{H}_x \to \mathfrak{m}^{-1}$ which transform J_x into J_o. Then $(\mathcal{H}, \pi : Q \to M)$ is a Tanaka structure of type $(\mathfrak{m}^{-1}, G^0)$. Conversely, the geometric structure J on the distribution $\mathcal{H}$ can be reconstructed from the Tanaka structure $(\mathcal{H}, \pi)$.

Let (HM, K) be a (weak almost) para-CR structure on a manifold M. Assume that it is regular of type $\mathfrak{m}$, i.e. HM is a regular distribution of type $\mathfrak{m}$. Then the tensor field K induces an endomorphism K_0 of $\mathfrak{m}^{-1}$ such that $K_o^2 = \mathrm{id}$. Then the above construction shows that we can identify (HM, K) with a Tanaka structure $(HM, \pi : Q \to M)$ of type $(\mathfrak{m}^{-1}, G^0)$, where G^0 is the subgroup of automorphism group $\mathrm{Aut}(\mathfrak{m}^{-1})$ which preserves the eigenspace decomposition $\mathfrak{m}^{-1} = \mathfrak{m}^{-1}_{+} + \mathfrak{m}^{-1}_{-}$ of K_o and Q consists of admissible coframes which transform K_x into K_0.

In case of a para-CR structure (HM, K), the integrability conditions (1) and (2) imply that

$$[K_0X, K_0Y] + [X, Y] = 0\,, \tag{4}$$

for any X, $Y \in \mathfrak{m}^{-1}$, that is the ±1-eigenspaces $\mathfrak{m}^{-1}_{\pm}$ of K_0 are commutative subalgebras of $\mathfrak{m}$.

Definition 3.3. *A pair* $(\mathfrak{m}, K_o)$*, where* $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ *is a negatively graded fundamental Lie algebra and* K_o *is an involutive endomorphism of* $\mathfrak{m}^{-1}$*, is called a* para-CR algebra *of depth* d*. If, moreover, the* ± 1*-eigenspaces* $\mathfrak{m}^{-1}_{\pm}$ *of* K_o *on* $\mathfrak{m}^{-1}$ *are commutative subalgebras of* $\mathfrak{m}$ *then* $(\mathfrak{m}, K_o)$*, is called an* integrable para-CR structure*.*

Note that a regular (weak) almost para-CR structure of type $\mathfrak{m}$ defines a para-CR algebra $(\mathfrak{m}, K_0)$ and it is integrable if the para-CR structure is integrable.

We can reformulate the definition of a regular para-CR structure as follows.

Definition 3.4. *Let* $(\mathfrak{m}, K_o)$ *be a para-CR algebra of depth* d*. A weak almost para-CR structure* (HM, K) *on* M *is called* regular *of* type $(\mathfrak{m}, K_o)$ *and* depth d *if, for any* $x \in M$*, the pair* $(\mathfrak{m}(x), K_x)$ *is isomorphic to* $(\mathfrak{m}, K_o)$*. We say that the regular almost para-CR structure is non-degenerate if the graded algebra* $\mathfrak{m}$ *is non degenerate.*

In the sequel we will *identify a regular almost para-CR structure with the corresponding Tanaka structure.*

We will recall briefly the Tanaka construction of prolongations of graded Lie algebras and Tanaka structures in the next section.

4. Prolongations of graded Lie algebras and Tanaka structures

4.1. **Prolongations of negatively graded Lie algebras.** The *full prolongation* of a negatively graded fundamental Lie algebra $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ is defined as a maximal graded Lie algebra

$$\mathfrak{g}(\mathfrak{m}) = \mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) + \mathfrak{g}^{0}(\mathfrak{m}) + \mathfrak{g}^{1}(\mathfrak{m}) + \cdots$$

with the negative part

$$\mathfrak{g}^{-d}(\mathfrak{m}) + \cdots + \mathfrak{g}^{-1}(\mathfrak{m}) = \mathfrak{m}$$

such that the following transitivity condition holds:

$$\text{for any } k \geq 0,\ X \in \mathfrak{g}^{k}(\mathfrak{m})\,,\ [X, \mathfrak{g}^{-1}(\mathfrak{m})] = \{0\} \ \Rightarrow\ X = 0\,.$$

In [14], N. Tanaka proved that the full prolongation $\mathfrak{g}(\mathfrak{m})$ always exists and it is unique up to an isomorphism. Moreover, it can be defined inductively by

$$\mathfrak{g}^{i}(\mathfrak{m}) = \begin{cases} \mathfrak{m}^{i} & \text{if } i < 0\,, \\ \{A \in \mathrm{Der}(\mathfrak{m}, \mathfrak{m}) \,:\, A(\mathfrak{m}^{j}) \subset \mathfrak{m}^{j}\,, \forall j < 0\} & \text{if } i = 0\,, \\ \{A \in \mathrm{Der}(\mathfrak{m}, \sum_{h<i} \mathfrak{g}^{h}(\mathfrak{m})) \,:\, A(\mathfrak{m}^{j}) \subset \mathfrak{g}(\mathfrak{m})^{i+j}\,, \forall j < 0\} & \text{if } i > 0\,, \end{cases}$$

where $\mathrm{Der}(\mathfrak{m}, V)$ denotes the space of derivations of the Lie algebra $\mathfrak{m}$ with values in the $\mathfrak{m}$-module V.

Note that

$$\text{(5)}\quad \mathfrak{g}^i(\mathfrak{m}) \;=\; \Big\{A \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{m}, \sum_{h<i} \mathfrak{g}^h) \mid A(\mathfrak{g}^h) \subset \mathfrak{g}^{h+i}\ \forall h<0\,,\ \text{such that}$$
$$[A(Y), Z] + [Y, A(Z)] = A([Y,Z])\ \forall Y, Z \in \mathfrak{m}\Big\}\,.$$

4.2. **Prolongations of non-positively graded Lie algebras.** Consider now a non-positively graded Lie algebra $\mathfrak{m}+\mathfrak{g}^0 = \mathfrak{m}^{-d}+\cdots+\mathfrak{m}^{-1}+\mathfrak{g}^0$. The *full prolongation* of $\mathfrak{m}+\mathfrak{g}^0$ is the subalgebra

$$(\mathfrak{m}+\mathfrak{g}^0)^\infty = \mathfrak{m}^{-d}+\cdots+\mathfrak{m}^{-1}+\mathfrak{g}^0+\mathfrak{g}^1+\mathfrak{g}^2+\cdots$$

of $\mathfrak{g}(\mathfrak{m})$, defined inductively by

$$\mathfrak{g}^i = \{X \in \mathfrak{g}(\mathfrak{m})^i \,:\, [X, \mathfrak{m}^{-1}] \subset \mathfrak{g}^{i-1}\}\,,\ \text{for any } i \geq 1\,.$$

Definition 4.1. *A graded Lie algebra* $\mathfrak{m}+\mathfrak{g}^0$ *is called of* finite type *if its full prolongation* $\mathfrak{g} = (\mathfrak{m}+\mathfrak{g}^0)^\infty$ *is a finite dimensional Lie algebra and it is called of* semisimple type *if* $\mathfrak{g}$ *is a finite dimensional semisimple Lie algebra.*

We have the following criterion

Lemma 4.2. *Let* $(\mathfrak{m} = \sum_{i<0}\mathfrak{m}^i, K_o)$ *be an integrable para-CR algebra and* $\mathfrak{g}^0$ *the subalgebras of* $\mathfrak{g}^0(\mathfrak{m})$ *consisting of any* $A \in \mathfrak{g}^0(\mathfrak{m})$ *such that* $A|_{\mathfrak{m}^{-1}}$ *commutes with* K_o*. Then the graded Lie algebra* $(\mathfrak{m}+\mathfrak{g}^0)$ *is of finite type if and only if* $\mathfrak{m}$ *is non-degenerate.*

Proof. Assume that $\mathfrak{m}$ is non-degenerate. By Corollary 1, Section 11, of [14], it is sufficient to show that

$$\mathfrak{h}^1 = \{X \in \mathfrak{g}^1(m) \mid [X, \mathfrak{g}^i(\mathfrak{m})] = 0 \quad \forall i < -1\} = 0.$$

Let $\xi \in \mathfrak{h}^1$ and $A : \mathfrak{g}^{-1}(\mathfrak{m}) \to \mathfrak{g}^0$ the corresponding $\mathbb{R}$-linear map. Then we have

$$\begin{cases} A(X)Y = A(Y)X \\ [A(X)Y, Z] + [Y, A(X)Z] = 0 \\ A(X)KY = KA(X)Y \end{cases}$$

for any $X, Y, Z \in \mathfrak{m}^{-1}$. Therefore, the above conditions and (4) imply that

$$\begin{aligned} 0 &= [A(KX)KY, Z] + [KY, A(KX)Z] \\ &= [KA(KX)Y, Z] - [Y, KA(KX)Z] \\ &= [KA(KX)Y, Z] + [KA(KX)Y, Z] \\ &= 2[A(Y)X, Z]\,. \end{aligned}$$

Hence, $A(Y)X = 0$ and consequently $A = 0$, which gives $\xi = 0$.
Conversely, let $\mathfrak{n} = \sum_{i<-1}\mathfrak{m}^i$ and let $\mathfrak{h}$ denote the graded Lie subalgebra of $\mathfrak{m}$ defined by

$$\mathfrak{h} = \{X \in \mathfrak{g}(\mathfrak{m}) \mid [X, \mathfrak{n}] = 0\}\,.$$

Assume that $\mathfrak{m}$ is degenerate, i.e. there is $0 \neq X \in \mathfrak{g}^{-1}(\mathfrak{m})$ such that $[X, Y] = 0$ for any $Y \in \mathfrak{g}^{-1}(\mathfrak{m})$. Now, we can assume that $X + KX$ is different from zero (otherwise we consider $X - KX$). We consider the one-dimensional K-invariant subspace $U = \mathbb{R}(X + KX) \subset \mathfrak{g}^{-1}(\mathfrak{m})$ and denote by W a K-invariant complement to U in $\mathfrak{g}^{-1}(\mathfrak{m})$. Then, we define $Y_0 \in \mathfrak{h}^0 = \mathfrak{h} \cap \mathfrak{g}^0 \subset \mathfrak{g}^0$ by

$$\begin{cases} [Y_0, Z] = 0 \quad \text{for} \quad Z \in W + \mathfrak{n} \\ [Y_0, Z] = Z \quad \text{for} \quad Z \in U\,. \end{cases}$$

Recurrently, we construct a sequence $\{Y_i\}_{i \geq 0}$, with $Y_i \in \mathfrak{h} \cap \mathfrak{g}^i(\mathfrak{m})$, by

$$\begin{cases} [Y_i, Z] = 0 \qquad \text{for } Z \in W + \mathfrak{n} \\ [Y_i, X + KX] = Y_{i-1}\,. \end{cases}$$

Since Y_i is a non zero element from $\mathfrak{g}^i(\mathfrak{m})$, this shows that $\mathfrak{g}(\mathfrak{m})$ is infinite dimensional. $\square$

The following result will be used in the last section.

Lemma 4.3. *Let $\mathfrak{g} = \sum_i \mathfrak{g}_i$ be a fundamental effective semisimple graded Lie algebra such that $\mathfrak{m} + \mathfrak{g}^0$ is of finite type. Then $\mathfrak{g}$ coincides with the full prolongation $(\mathfrak{m} + \mathfrak{g}^0)^\infty$ of $\mathfrak{m} + \mathfrak{g}^0$.*

For the proof, see e.g. [12], Theorem 3.21.

Lemma 4.4. *A fundamental effective graded semisimple Lie algebra $\mathfrak{g} = \sum \mathfrak{g}^i$ is degenerate if and only if it contains a graded (not necessary proper) ideal of depth one.*

Proof. It is clear that, if $\mathfrak{g}$ contains a graded ideal of depth one (which is automatically a direct summand of $\mathfrak{g}$), then $\mathfrak{g}$ is degenerate.
We prove the converse. Set

$$\mathfrak{g}_0^{\pm 1} = \{X \in \mathfrak{g}^{\pm 1} \mid [X, \mathfrak{g}^{\pm 1}] = 0\}.$$

Since $\mathfrak{g}^{-1}$ generates the negative part $\mathfrak{m}$ of $\mathfrak{g}$, it is in the center of $\mathfrak{m}$ and, obviously, it is $\mathrm{ad}_{\mathfrak{g}^0}$-invariant. Similarly, $\mathfrak{g}_0^1$ is an $\mathrm{ad}_{\mathfrak{g}^0}$-invariant subalgebra of the positive part of $\mathfrak{g}$. This implies that

$$\mathfrak{g}_0 = \mathfrak{g}_0^{-1} + [\mathfrak{g}_0^{-1}, \mathfrak{g}_0^1] + \mathfrak{g}_0^1$$

is a graded ideal of $\mathfrak{g}$ of depth one. $\square$

4.3. **Prolongations of Tanaka structures and Tanaka structures of semisimple type.** Let $(\mathcal{H}, \pi : Q \to M)$ be a Tanaka structure of type $(\mathfrak{m}, G^0)$. Then, the Lie algebra $\mathfrak{g}^0$ of G^0 is a subalgebra of the Lie algebra of derivations of $\mathfrak{m}$.
We say that the Tanaka structure of type $(\mathfrak{m}, G^0)$ is of *finite type* (respectively *semisimple type* $(\mathfrak{m}, G^0)$), if the graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$ is of finite type (respectively semisimple type).

Let P be a Lie subgroup of a connected Lie group G and $\mathfrak{p}$ (respectively, $\mathfrak{g}$) the Lie algebra of P (respectively, G).

Definition 4.5. *A Cartan connection of type* $\mathfrak{g}$ *on a* P*-principal bundle* $\pi : B \to M$ *is a* P*-equivariant* $\mathfrak{g}$*-valued 1-form* $\kappa : TB \to \mathfrak{g}$ *which extends the canonical vertical parallelism* $T_b^{\mathrm{vert}} B \approx \mathfrak{p}$ *and whose restriction* $\kappa : T_b B \to \mathfrak{g}$ *to the tangent space* $T_b B$, $b \in B$, *is an isomorphism of vector spaces.*

For a Tanaka structure $(\mathcal{H}, \pi : Q \to M)$ of semisimple type, Tanaka constructed the full prolongation which is a principal P-bundle $\beta : B \to M$ (where P is a Lie group with Lie algebra $\mathfrak{p} = \sum_{i \geq 0} \mathfrak{g}_i$), with a Cartan connection κ, such that the automorphism group $\mathrm{Aut}(\mathcal{H}, \pi) = \mathrm{Aut}(B, \kappa)$ (see [15], [5]). This implies the following (see e.g. ([14], [5], [2])

Theorem 4.6. *Let* $(\pi : Q \to M, \mathcal{H})$ *be a Tanaka structure on* M *of semisimple type* $(\mathfrak{m}, G^0)$. *Then the Tanaka prolongation of* $(\pi, \mathcal{H})$ *is a* P*-principal bundle* $\mathcal{G} \to M$, *with the parabolic structure group* P, *equipped with a Cartan connection* $\kappa : T\mathcal{G} \to \mathfrak{g}$, *where* $\mathfrak{g}$ *is the full prolongation of* $\mathfrak{m} + \mathfrak{g}^0$ *and* $\mathrm{Lie}P = \mathfrak{p} = \sum_{i \geq 0} \mathfrak{g}_i$. *Moreover,* $\mathrm{Aut}(\mathcal{H}, \pi)$ *is a Lie group and*

$$\dim \mathrm{Aut}(\mathcal{H}, \pi) \leq \dim \mathfrak{g} \,.$$

4.4. **Maximally homogeneous Tanaka structures.** Let $(\mathcal{H}, \pi : Q \to M)$ be a Tanaka structure of semisimple type $(\mathfrak{m}, G^0)$ and $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty = \mathfrak{m} + \mathfrak{p}$ the full prolongation of the non-positively graded Lie algebra $\mathfrak{m} + \mathfrak{g}^0$.

Definition 4.7. *A semisimple Tanaka structure* $(\mathcal{H}, \pi : Q \to M)$ *is called* maximally homogeneous *if the dimension of its automorphism group* $\mathrm{Aut}(\mathcal{H}, \pi)$ *is equal to* $\dim \mathfrak{g}$.

We construct maximal homogeneous Tanaka structures of semisimple type $(\mathfrak{m}, G^0)$ as follows. Let $G = \mathrm{Aut}(\mathfrak{g})$ be the Lie group of automorphisms of the Lie algebra $\mathfrak{g}$. Recall that G^0 is a closed subgroup of the automorphism group of the Lie algebra $\mathfrak{g}^- = \mathfrak{m}$. Since the Lie algebra $\mathfrak{g}$ is canonically associated with $\mathfrak{m}$, we can canonically extend the action of G^0 on $\mathfrak{m}$ to the action of G^0 on $\mathfrak{g}$ by automorphisms.
In other words, we have an embedding $G^0 \hookrightarrow \mathrm{Aut}(\mathfrak{g}) = G$ as a closed subgroup. We denote by G^+ the connected (closed) subgroup of G with Lie algebra $\mathfrak{g}_+ = \sum_{p>0} \mathfrak{g}^p$. Then $P = G^0 \cdot G^+ \subset G$ is a (closed) parabolic subgroup of G. Let $F = G/P$ be the corresponding flag manifold. We have a decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{p}$ and we identify $\mathfrak{m}$ with the tangent space $T_o F$. Then the natural action of G^0 on $\mathfrak{m}$ is the isotropy representation of G^0. We have a natural Tanaka structure $(\mathcal{H}, \pi : Q \to F)$ of type $(\mathfrak{m}^{-1}, G^0)$, where $\mathcal{H}$ is the G-invariant distribution defined by $\mathfrak{m}^{-1}$ and Q is the G^0-bundle of coframes on $\mathcal{H}$ which is generated by coframes

$$Q_{|_o} = G^0 \cdot \mathrm{id} \,, \quad \mathrm{id} : \mathfrak{m}^{-1} \to \mathfrak{m}^{-1} \,.$$

This Tanaka structure $(\mathcal{H}, \pi : Q \to F = G/P)$ is obviously maximally homogeneous and it is called the *standard (maximally homogeneous) Tanaka structure* of type $(\mathfrak{m}, G^0)$.

Theorem 4.8. *Let $(\mathcal{H}, \pi : Q \to F = G/P)$ be the standard maximally homogeneous Tanaka structure of a semisimple type $(\mathfrak{m}, G^0)$. Any maximally homogeneous Tanaka structure $(\mathcal{H}', \pi' : Q' \to M = G'/P')$ of a semisimple type $(\mathfrak{m}, G_0)$ is locally isomorphic to the standard maximally homogeneous Tanaka structure $(\mathcal{H}, \pi : Q \to F = G/P)$.*

Proof. Denote by $\mathfrak{g}'$ and $\mathfrak{p}'$ the Lie algebras of the groups G' and P', respectively. We have a natural filtration of $\mathfrak{g}'$ defined by

$$\mathfrak{g}^{(-1)} = \{X \in \mathfrak{g}' \mid \pi_* X \in \mathcal{H}'_{x_0}\},$$

where $\pi : G' \to M$ is the natural projection, and inductively

$$\mathfrak{g}^{(i)} = \begin{cases} [\mathfrak{g}^{(i+1)}, \mathfrak{g}^{(-1)}] + \mathfrak{g}^{(i+1)}, & \text{for } i < 0, \\ \{X \in \mathfrak{p}' \mid [X, \mathfrak{g}^{-1}] \subseteq \mathfrak{g}^{-1+i}\}, & \text{for } i \geq 0. \end{cases}$$

The graded Lie algebra associated with the filtered Lie algebra $\mathfrak{g}'$ is a subalgebra of the full prolongation $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$. By assumption, it has the same dimension as $\mathfrak{g}$. Hence, it coincides with $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, by Lemma 5.9 of [16], we obtain that $\mathfrak{g}'$ is isomorphic to $\mathfrak{g}$. Moreover, there is an isomorphism $\mathfrak{g}' \to \mathfrak{g}$ which maps $\mathfrak{p}'$ to $\mathfrak{p}$. This shows that the filtration of $\mathfrak{g}'$ is associated with a gradation and $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic as graded Lie algebras. This implies that the considered Tanaka structures are locally isomorphic. $\square$

5. Maximally homogeneous almost para-CR structures and graded Lie algebras

5.1. **Para-CR structures as Tanaka structures.** Let (HM, K) be a weak almost para-CR structure. Assume that it is *regular*, i.e. the distribution HM is a regular distribution of type $\mathfrak{m}$. Then, according to section 3, it can be considered as a Tanaka structure of type $(\mathfrak{m}, G^0)$, and

$$G^0 = \mathrm{Aut}(\mathfrak{m},\, K_o) \subset \mathrm{GL}_{m_+}(\mathbb{R}) \times \mathrm{GL}_{m_-}(\mathbb{R}),$$

and $m_\pm = \dim \mathfrak{m}_\pm^{-1}$ is the rank of $H_\pm M$.

A regular almost para-CR structure is of *finite type* or, respectively, of *semisimple type*), if the Lie algebra $(\mathfrak{m} + \mathfrak{g}^0)^\infty$ is finite-dimensional or, respectively, semisimple. Recall that $\mathfrak{g}^0 = Der(\mathfrak{m}, K_0)$ is the Lie algebra of Lie group $\mathrm{Aut}(\mathfrak{m}, K_0)$.

Let (HM, K) be a (regular) weak almost para-CR structure of type $(\mathfrak{m}, K_0)$. Assume that it has finite type, i.e. $m = \dim(\mathfrak{m} + \mathfrak{g}^0)^\infty < \infty$. According to the above definition, (HM, K) is *maximally homogeneous*, if it admits a (transitive) Lie group of automorphisms of dimension m.

By Theorem 4.8, a maximally homogeneous weak almost para-CR structure of semisimple type is locally equivalent to a standard structure associated with a gradation of a semisimple Lie algebra. In the following subsection we describe this correspondence in more details.

5.1.1. *Standard almost para-CR-structures and gradation of a semisimple Lie algebras.* Let $\mathfrak{g} = \sum_{-d}^{d} \mathfrak{g}^i = \mathfrak{g}^- + \mathfrak{g}^0 + \mathfrak{g}^+$ be an effective fundamental gradation of a semisimple Lie algebra $\mathfrak{g}$ with negative part $\mathfrak{m} = \mathfrak{g}^-$ and positive part $\mathfrak{g}^+$.
Denote by $F = G/P$ the real flag manifold associated with graded Lie algebra $\mathfrak{g}$ where $G = \mathrm{Aut}(\mathfrak{g})$ and $P = G^0 G^+$ is the connected subgroup with Lie algebra $\mathfrak{g}^0 + \mathfrak{g}^+$.
We will identify the tangent space $T_o F$ at the point $o = eP$ with the subspace

$$\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{m}\,.$$

Since the subspace $(\mathfrak{g}^{-1} + \mathfrak{p})/\mathfrak{p} \subset T_o F$ is invariant under the isotropy representation of P, it defines an invariant distribution $\mathcal{H}$ on F. Since the gradation is fundamental, one can easily check that, for any $x \in F$, the negatively graded Lie algebra $\mathfrak{m}(x)$ associated with $\mathcal{H}$ is isomorphic to the Lie algebra $\mathfrak{m}$.
Moreover, let

$$\mathfrak{g}^{-1} = \mathfrak{g}^{-1}_{+} + \mathfrak{g}^{-1}_{-} \tag{6}$$

be a decomposition of the G^0-module $\mathfrak{g}^{-1}$ into a sum of two submodules and K_0 the associated $\mathrm{ad}_{\mathfrak{g}^0}$-invariant endomorphism such that $\mathfrak{g}^{-1}_{\pm}$ are ± 1-eigenspaces of K_0.
The decomposition (6) defines two invariant complementary subdistributions $\mathcal{H}_{\pm}$ of the distribution $\mathcal{H} \subset TF$ associated with $\mathfrak{g}^{-1}$ and K_0 defines G-invariant weak almost para-CR structure (HF, K) on F. It is the standard almost para-CR structure associated with the graded Lie algebra $\mathfrak{g}$ and the decomposition (6). We get the following theorem.

Theorem 5.1. *Let $F = G/P$ be the flag manifold associated with a (real) semisimple effective fundamental graded Lie algebra $\mathfrak{g}$. A decomposition $\mathfrak{g}^{-1} = \mathfrak{g}^{-1}_{+} + \mathfrak{g}^{-1}_{-}$ of $\mathfrak{g}^{-1}$ into complementary G^0-submodules $\mathfrak{g}^{-1}_{\pm}$ determines an invariant weak almost para-CR structure (HM, K) such that ± 1-eigenspaces $H_{\pm}M$ of K are subdistributions of HM associated with $\mathfrak{g}^{-1}_{\pm}$. Conversely, any standard weak almost para-CR structure (HM, K) on F can be obtained in such a way.*
Moreover, (HM, K) is:

1. *an almost para-CR structure if $\mathfrak{g}^{-1}_{+}$ and $\mathfrak{g}^{-1}_{-}$ have the same dimensions,*
2. *a para-CR structure if and only if $\mathfrak{g}^{-1}_{+}$ and $\mathfrak{g}^{-1}_{-}$ are commutative subalgebras of $\mathfrak{g}$,*
3. *non degenerate if and only if $\mathfrak{g}$ has no graded ideals of depth one.*

Proof. We prove only the statement about the integrability, since the other statements are obvious.
We have to prove that the distributions $H_{\pm}M$, associated with $\mathfrak{g}^{-1}_{\pm}$, are involutive.

Remark that the subgroup $G^- \subset G$ generated by the subalgebra $\mathfrak{g}^-$ has the open orbit $G^- \cdot o$ in F and it acts on this orbit with discrete stabilizer. Due to this, we can locally identify G^- with F, via the map

$$G^- \ni g \mapsto x = g \cdot o \in F\,.$$

Then, the distributions $\mathcal{H}_\pm$ are identified with left invariant distributions

$$G^- \ni g \mapsto \mathcal{H}^\pm_g = (L_g)_*(\mathfrak{g}^{-1}_\pm)$$

associated with $\mathfrak{g}^{-1}_\pm \subset \mathfrak{g}^- = T_e G^-$.
The left invariant vector fields $X^*_\pm$ associated with $X_\pm \in \mathfrak{g}^{-1}_\pm$ generate the distribution $\mathcal{H}_\pm$. For $X_\pm$, $Y_\pm \in \mathfrak{g}^{-1}_\pm$, we have

$$[X^*_\pm, Y^*_\pm]_{|e} = -[X_\pm, Y_\pm]^*_{|e} = [X_\pm, Y_\pm] \in \mathfrak{g}^{-2}\,.$$

This shows that the distribution $\mathcal{H}_\pm$ is involutive if and only if

$$[\mathfrak{g}^{-1}_\pm, \mathfrak{g}^{-1}_\pm] = \mathfrak{g}^{-1}_\pm \cap \mathfrak{g}^{-2} = 0\,.$$

□

Definition 5.2. *A fundamental effective gradation of a semisimple Lie algebra* $\mathfrak{g}$ *is called* admissible *if the* $\mathfrak{g}^0$*-module* $\mathfrak{g}^{-1}$ *is reducible.*

By Theorem 5.1, the classification of maximally homogeneous weak almost para-CR structures of semisimple type, up to local isomorphisms (i.e. up to coverings), reduces to the description of all admissible gradation of semisimple Lie algebras and to the decomposition of the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ into irreducible submodules. We will give such a description for complex and real semisimple Lie algebras in the next two sections.

6. Fundamental gradations of a complex semisimple Lie algebra

We recall here the construction of a gradation of a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ and

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

the root decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We denote by

$$\Pi = \{\alpha_1, \dots, \alpha_\ell\} \subset R$$

a system of simple roots of the root system R and associate to each simple root α_i (or corresponding vertex of the Dynkin diagram) a non-negative integer d_i. Using the labelling vector $\vec{d} = (d_1, \dots, d_\ell)$, we define the *degree* of a root $\alpha = \sum_{i=1}^{\ell} k_i \alpha_i$ by

$$d(\alpha) = \sum_{i=1}^{\ell} k_i d_i\,.$$

This defines a gradation of $\mathfrak{g}$ by the conditions

$$d(\mathfrak{h}) = 0\,, \qquad d(\mathfrak{g}_\alpha) = d(\alpha), \quad \forall \alpha \in R\,,$$

which is called the *gradation associated with the label vector* $\vec{d}$.
We denote by $d \in \mathfrak{h}$ the corresponding grading element. Then $d(\alpha) = \alpha(d)$.
Any gradation of a complex semisimple Lie algebra $\mathfrak{g}$ is conjugated to a gradation of such a type (see [9]). In particular, it has the form

$$\mathfrak{g} = \mathfrak{g}^{-k} + \cdots + \mathfrak{g}^0 + \cdots + \mathfrak{g}^k\,,$$

where $\mathfrak{g}^0$ is a reductive subalgebra of $\mathfrak{g}$ and the grading spaces $\mathfrak{g}^{-i}$ and $\mathfrak{g}^i$ are dual with respect to the Killing form. It is clear now that any graded semisimple Lie algebra is a direct sum of graded simple Lie algebras. Hence, it is sufficient to describe gradations of simple Lie algebras.
We need the following (see [16])

Lemma 6.1. *The gradation of a complex semisimple Lie algebra $\mathfrak{g}$ associated with a label $\vec{d} = (d_1, \ldots, d_\ell)$ is fundamental if and only if all labels $d_i \in \{0, 1\}$.*

Let $\Pi^1 \subset \Pi$ be a set of simple roots. We denote by $\vec{d}_{\Pi^1}$ the labelling vector which associates label one to the roots from Π^1 and label zero to the other simple roots.
Now we describe the depth of a fundamental gradation.
Let μ be the maximal root with respect to the fundamental system Π. It can be written as a linear combination

$$\mu = m_1\alpha_1 + \cdots + m_\ell\alpha_\ell \tag{7}$$

of fundamental roots, where the coefficient m_i is a positive integer called the *Dynkin mark associated* with α_i.

Lemma 6.2. *Let $\Pi^1 = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\} \subset \Pi$ be a set of simple roots. Then the depth k of the fundamental gradation defined by the labelling vector $\vec{d}_{\Pi^1}$ is given by*

$$k = m_{i_1} + m_{i_2} + \cdots + m_{i_s}\,.$$

In particular, the gradation has depth two if and only if $\Pi^1 = \{\alpha_i\}$, where α_i is a simple root with Dynkin mark $m_i = 2$ or $\Pi^1 = \{\alpha_i, \alpha_j\}$, where α_i, α_j are simple roots with Dynkin mark $m_i = m_j = 1$.

Proof. The depth k of the gradation is equal to the maximal degree $d(\alpha)$, α being a root. If $\alpha = k_1\alpha_1 + \cdots + k_\ell\alpha_\ell$ is the decomposition of a root α with respect to simple roots, then

$$d(\alpha) = k_{i_1} + \cdots + k_{i_s} \leq d(\mu) = m_{i_1} + \cdots + m_{i_s} = k\,.$$

□

6.1. **Decomposition of a $\mathfrak{g}^0$-module $\mathfrak{g}^1$ into irreducible submodules.** Let $\mathfrak{g} = \sum \mathfrak{g}^i$ be a fundamental gradation of a complex semisimple Lie algebra $\mathfrak{g}$, defined by a label vector $\vec{d}$. Following [9], we describe the decomposition of a $\mathfrak{g}^0$-module into irreducible submodules. Set

$$R^i = \{\alpha \in R \mid d(\alpha) = i\} = \{\alpha \in R \mid \mathfrak{g}_\alpha \subset \mathfrak{g}^i\}$$

and

$$\Pi^i = \Pi \cap R^i = \{\alpha \in \Pi \mid d(\alpha) = i\}\,.$$

For any simple root $\gamma \in \Pi$, we put

$$R(\gamma) = (\{\gamma\} + R^0) \cap R = \{\alpha = \gamma + \alpha^0 \in R,\ \ \alpha^0 \in R^0\}\,.$$

We associate to any set of roots $Q \subset R$ a subspace

$$\mathfrak{g}(Q) = \sum_{\alpha \in Q} \mathfrak{g}_\alpha \subset \mathfrak{g}\,.$$

Proposition 6.3. ([9]) *The decomposition of a $\mathfrak{g}^0$-module $\mathfrak{g}^1$ into irreducible submodules is given by*

$$\mathfrak{g}^1 = \sum_{\gamma \in \Pi^1} \mathfrak{g}(R(\gamma))\,.$$

Moreover, γ is a lowest weight of the irreducible submodule $\mathfrak{g}(R(\gamma))$. In particular, the number of the irreducible components is equal to the number $\#\Pi^1$ of the simple roots of degree 1.

Since the $\mathfrak{g}^0$-modules $\mathfrak{g}^i$ and $\mathfrak{g}^{-i}$ are dual, Proposition 6.3 gives also the decomposition of $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ into irreducible submodules.

By applying this Proposition to depth two gradations described in Lemma 6.2, we get the following

Corollary 6.4. *Let $\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ be a fundamental depth two gradation of a simple Lie algebra defined by a labelling vector $\vec{d}$.*

i) *If $\vec{d} = \vec{d}_{\{\alpha_i\}}$ with $m_i = 2$, then the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ is irreducible.*

ii) *If $\vec{d} = \vec{d}_{\{\alpha_i, \alpha_j\}}$ with $m_i = m_j = 1$, then the $\mathfrak{g}^0$-module $\mathfrak{g}^{-1}$ has two irreducible components, with lowest weights α_i, α_j. Such gradations exist only for Lie algebras of type A_ℓ, D_ℓ and E_6.*

According to the previous Corollary, such a gradation of a complex simple Lie algebra is defined by a grading vector $\vec{d}_{\{\alpha_p, \alpha_q\}}$, where simple roots $\{\alpha_p, \alpha_q\}$ have Dynkin mark $m_p = m_q = 1$.
Note that, an admissible gradation of a real Lie algebra induces an admissible gradation on its complexification.

6.2. **Depth two admissible gradations and associated flag manifolds with almost para-CR structure.** We describe explicitly all admissible depth two fundamental gradations of complex simple Lie algebras $\mathfrak{g}$ and associated flag manifolds. We know that they are associated with a labelling vector $\vec{d}_{\{\alpha_p, \alpha_q\}}$, where simple roots α_p, α_q have Dynkin mark one.
We will use the numeration of simple roots from [9].
For a Lie algebra $\mathfrak{g}$ of type A_ℓ, any simple root has Dynkin mark one and we can take for α_p, α_q any two simple vectors.
For $\mathfrak{g}$ of type D_ℓ, we have three possibilities:

$$\{\alpha_1, \alpha_{\ell-1}\}, \quad \{\alpha_1, \alpha_\ell\}, \quad \{\alpha_{\ell-1}, \alpha_\ell\}\,.$$

Since pairs $\{\alpha_1, \alpha_{\ell-1}\}$ and $\{\alpha_1, \alpha_\ell\}$ are conjugated by an outer automorphism of the root system, it is sufficient to consider only gradation defined by $\{\alpha_1, \alpha_\ell\}$ and $\{\alpha_{\ell-1}, \alpha_\ell\}$.
For $\mathfrak{g}$ of type E_6, there exists only one pair $\{\alpha_1, \alpha_5\}$ of simple roots with Dynkin mark one. For the other simple Lie algebras there is no such pair.

Case $\mathbf{A}_\ell$ ($\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$). The root system is

$$R = \{\epsilon_i - \epsilon)_j \,|\, i, j \in \{1, \ldots, \ell+1\}\}$$

and the system of simple roots is

$$\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_\ell = \epsilon_{\ell-1} - \epsilon_\ell\}\,.$$

The gradation of A_ℓ associated with a pair $\{\alpha_p, \alpha_q\}, 1 \leq p < q \leq \ell$, is described as follows:

$$\begin{array}{rcl} \mathfrak{g}^0 & \simeq & \mathfrak{sl}_p(\mathbb{C}) \oplus \mathfrak{sl}_{q-p}(\mathbb{C}) \oplus \mathfrak{sl}_{\ell+1-q}(\mathbb{C}) \oplus \mathbb{C}^2\,, \\ \mathfrak{g}^{-1} & = & \mathfrak{g}^{-1}_+ \oplus \mathfrak{g}^{-1}_-\,, \\ \mathfrak{g}^{-2} & = & \mathbb{C}^{\ell+1-q} \otimes (\mathbb{C}^p)^*\,, \end{array}$$

where $\mathfrak{g}^{-1}_+ = \mathbb{C}^{q-p} \otimes (\mathbb{C}^p)^*$ is a $\mathfrak{g}^0$-module with trivial action of $\mathfrak{sl}_{\ell+1-q}(\mathbb{C})$ and $\mathfrak{g}^{-1}_- = \mathbb{C}^{\ell+1-q} \otimes (\mathbb{C}^{q-p})^*$ is a $\mathfrak{g}^0$-module with trivial action of $\mathfrak{sl}_p(\mathbb{C})$.
Note that

$$\begin{array}{rcl} \dim \mathfrak{g}^{-1}_- & = & p(q-p)\,, \\ \dim \mathfrak{g}^{-1}_+ & = & (q-p)(\ell+1-q)\,, \\ \dim \mathfrak{g}^{-2} & = & p(\ell+1-q)\,. \end{array}$$

If we decompose the vector space $V = \mathbb{C}^{\ell+1}$ into a direct sum $V_1 \oplus V_2 \oplus V_3$ of subspaces of dimension $p, q-p, \ell+1-q$ respectively, then, with respect to a basis consistent with this decomposition, the gradatation can be described in matrix form as follows:

$$\mathfrak{sl}_{\ell+1}(\mathbb{C}) = \mathfrak{g} = \begin{pmatrix} \mathfrak{g}^0_1 + \lambda_1 E_p & \mathfrak{g}^1 & \mathfrak{g}^2 \\ \mathfrak{g}^{-1} & \mathfrak{g}^0_2 + \lambda_2 E_{q-p} & \mathfrak{g}^1 \\ \mathfrak{g}^{-2} & \mathfrak{g}^{-1} & \mathfrak{g}^0_3 + \lambda_3 E_{\ell+1-q} \end{pmatrix}\,,$$

where E_m is the identity matrix of size m, and

$$p\lambda_1 + (q-p)\lambda_2 + (\ell+1-q)\lambda_3 = 0\,.$$

In particular,

$$\mathfrak{g}^0 = \mathrm{diag}(\mathfrak{g}^0_1 + \mathbb{C}E_p, \mathfrak{g}^0_2 + \mathbb{C}E_{q-p}, \mathfrak{g}^0_3 + \mathbb{C}E_{\ell+1-q})$$

consists of block diagonal matrices of $\mathfrak{sl}_{\ell+1}(\mathbb{C})$.

The grading element $d \in \mathfrak{g}^0$ is given by

$$d = \mathrm{diag}(E_p, 0, -E_{\ell+1-p}) - \frac{2p-\ell-1}{\ell+1} E_{\ell+1}\,.$$

The parabolic subalgebra $\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ is the stabilizer of the standard (p, q)-flag $f_0 = (V_1 \subset V_1 + V_2)$. The associated flag manifold

$$F = G/P = SL_{\ell+1}(\mathbb{C})/P$$

is the manifold of (p, q)-flags in V.

Case $\mathbf{D}_\ell$ ($\mathfrak{g} = \mathfrak{so}_{2\ell}(\mathbb{C})$). The root system is

$$R = \{\pm\epsilon_i \pm \epsilon_j \mid i, j \in \{1, \dots, \ell\}, \, i \neq j\}$$

and the simple root system is

$$\Pi = \{\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_{\ell-1} = \epsilon_{\ell-1} - \epsilon_\ell, \alpha_\ell = \epsilon_{\ell-1} + \epsilon_\ell\} .$$

There are two cases.

1. Gradation defined by the labelling vector $\vec{d}_{\{\alpha_1, \alpha_\ell\}}$. We have

$$\Pi^0 = \{\alpha_2, \dots, \alpha_{\ell-1}\}, \quad \mathfrak{g}^0 = \mathbb{C}^2 + \mathfrak{sl}_{\ell-1}(\mathbb{C}) ,$$

$$R^1 = R^1_+ \cup R^1_- ,$$
$$R^1_+ = \{\epsilon_1 - \epsilon_i \mid \ i = 2, \dots, \ell\} , \quad R^1_- = \{\epsilon_i + \epsilon_j \mid i \neq j \in \{2, \dots, \ell\}\} ,$$
$$R^2 = \{\epsilon_1 - \epsilon_i \mid i = 2, \dots, \ell\} .$$

This formula follows from the description of roots R of type D_ℓ in terms of simple roots (see e.g. [9], table 1).

We fix a basis $e_{-1}, \dots, e_{-\ell}, e_1, \dots, e_\ell$ of the complex Euclidean vector space $V = (\mathbb{C}^{2\ell}, < \cdot, \cdot >)$ such that $< e_i, e_{-j} >= \delta_{ij}$ and all other products are zero. Consider the decomposition

$$V = \mathbb{C}e_{-1} \oplus U_- \oplus \mathbb{C}e_1 \oplus U_+ ,$$

where

$$U_- = \mathrm{Span}\{e_{-2}, \dots, e_{-\ell}\} , \quad U_+ = \mathrm{Span}\{e_2, \dots, e_\ell\} .$$

Then the gradation

$$\mathfrak{so}_{2\ell}(\mathbb{C}) = \mathfrak{so}(V) = \Lambda^2(V) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$$

is described as

$$\mathfrak{g}^0 = \mathbb{C}e_{-1} \wedge e_1 + U_- \wedge U_+ ,$$
$$\mathfrak{g}^1 = \mathfrak{g}^1_+ + \mathfrak{g}^1_- = \mathbb{C}e_1 \wedge U_- + U_+ \wedge U_+, \quad \mathfrak{g}^{-1} = \mathfrak{g}^{-1}_+ + \mathfrak{g}^{-1}_- = \mathbb{C}e_{-1} \wedge U_+ + U_- \wedge U_- ,$$
$$\mathfrak{g}^2 = \mathbb{C}e_1 \wedge U_+, \quad \mathfrak{g}^{-2} = \mathbb{C}e_{-1} \wedge U_- .$$

We have

$$\begin{aligned} \dim \mathfrak{g}^{-1}_- &= \frac{(\ell-1)(\ell-2)}{2} , \\ \dim \mathfrak{g}^{-1}_+ &= \ell - 1 , \\ \dim \mathfrak{g}^{-2} &= \ell - 1 . \end{aligned}$$

The corresponding grading element is given by

$$d = 3/2 e_1 \wedge e_{-1} + 1/2 \sum_{i=2}^{\ell} e_i \wedge e_{-i} .$$

The parabolic subalgebra $\mathfrak{p} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$ is the stabilizer of the isotropic $(1, \ell)$-flag

$$f_0 = (\mathbb{C}e_1 \subset \mathbb{C}e_1 + U_+)$$

and the associated flag manifold

$$F = G/P = SO^+_{2\ell}(\mathbb{C})/P$$

is the manifold of the isotropic $(1, \ell)$-flags in the complex Euclidean space V.

2. Gradation defined by the labelling vector $\vec{d}_{\alpha_{\ell-1},\alpha_\ell}$. We have:

$$\Pi^0 = \{\alpha_1, \dots, \alpha_{\ell-2}\}, \quad \mathfrak{g}^0 = \mathbb{C}^2 + \mathfrak{sl}_{\ell-1}(\mathbb{C}),$$

$$\begin{aligned}
R_0 &= \{\epsilon_i - \epsilon_j \mid i, j = 1, \dots, \ell - 1\}, \\
R^1 &= R^1_+ \cup R^1_-, \\
R^1_- &= \{\epsilon_i - \epsilon_\ell = \sum_{i \le k < \ell} \alpha_k \mid i = 2, \dots, \ell\}, \\
R^1_+ &= \{\epsilon_i + \epsilon_l = \sum_{i \le k < \ell-1} \alpha_k + \alpha_\ell \mid i \neq j \in \{2, \dots, \ell\}\}, \\
R^2 &= \{\epsilon_i + \epsilon_j \mid i < j \le \ell - 1\} = \\
&= \{\sum_{i \le k < j} \alpha_k + 2 \sum_{j \le k < \ell-1} \alpha_k + \alpha_{\ell-1} + \alpha_\ell \mid i < j < \ell - 1\} \cup \\
&\cup \ \{\sum_{i \le k < \ell-1} \alpha_k + \alpha_{\ell-1} + \alpha_\ell \mid i < j = \ell - 1\}.
\end{aligned}$$

We fix the same basis $e_{-1}, \dots, e_{-\ell}, e_1, \dots, e_\ell$ of the complex Euclidean space $V = (\mathbb{C}^{2\ell}, < \cdot, \cdot >)$ as above and we consider the decomposition

$$V = U_- + \mathbb{C}e_{-\ell} + U_+ + \mathbb{C}e_\ell,$$

where

$$U_\pm = \mathrm{span}\{e_{\pm 1}, \dots, e_{\pm(\ell-1)}\}.$$

Then the gradation

$$\mathfrak{so}_{2\ell}(\mathbb{C}) = \mathfrak{so}(V) = \Lambda^2(V) = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$$

is described as follows

$$\begin{aligned}
\mathfrak{g}^0 &= \mathbb{C}e_{-\ell} \wedge e_\ell + U_+ \wedge U_- \simeq \mathbb{C}^2 + \mathfrak{sl}(U_+), \\
\mathfrak{g}^{\pm 1} &= \mathfrak{g}^{\pm 1}_+ + \mathfrak{g}^{\pm 1}_- = \mathbb{C}e_\ell \wedge U_\pm + \mathbb{C}e_{-\ell} \wedge U_\pm, \\
\mathfrak{g}^{\pm 2} &= \Lambda^2(U_\pm).
\end{aligned}$$

We have

$$\begin{aligned} \dim \mathfrak{g}_-^{-1} &= \ell - 1\,, \\ \dim \mathfrak{g}_+^{-1} &= \ell - 1\,, \\ \dim \mathfrak{g}^{-2} &= \frac{(\ell-1)(\ell-2)}{2}\,. \end{aligned}$$

The corresponding grading element is given by

$$d = \sum_{i=1}^{\ell-1} e_i \wedge e_{-i}\,.$$

In other words, the gradation of $\mathfrak{so}(V)$ is induced by the gradation

$$V = V^{-1} + V^0 + V^1\,,$$

where

$$V^{\pm 1} = U_\pm\,, \quad V^0 = \mathrm{span}\{e_{-\ell}, e_\ell\}\,.$$

The parabolic subalgebra $\mathfrak{p} = \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$ is the stabilizer of the isotropic flag

$$f_0 = (U_+ \subset U_+ + \mathbb{C}e_\ell)\,.$$

Hence, the associated flag manifold

$$F = G/P = SO^+_{2\ell}(\mathbb{C})/P$$

is the manifold of the isotropic $(\ell-1,\ell)$-flags in the complex Euclidean space V.

Case E_6. We use the description of roots and fundamental weights of E_6 as in [9]. The root system is

$$R = \{\epsilon_i - \epsilon_j\,,\ \pm 2\epsilon,\ \epsilon_i + \epsilon_j + \epsilon_k \pm \epsilon\}\,,$$

where $i,j,k \in \{1,2,3,4,5,6\}$ are distinct and the vectors ϵ_i, ϵ satisfies the following conditions:

$$\begin{aligned} &\sum_{i=1}^{6} \epsilon_i = 0\,, \\ &\langle \epsilon_i, \epsilon_i \rangle = \frac{5}{6}\ \ , \quad \langle \epsilon_i, \epsilon_j \rangle = -\frac{1}{6}\ \ , \quad \text{for } i \neq j\,, \\ &\langle \epsilon, \epsilon_i \rangle = 0\ \ , \quad \langle \epsilon, \epsilon \rangle = \frac{1}{2}\,. \end{aligned}$$

The simple roots are

$$\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1}\,,\ \alpha_6 = \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon \mid i < 6\}\,.$$

The fundamental roots are

$$\pi_i = \epsilon_1 + \cdots + \epsilon_i + \min\{i, 6-i\} \cdot \epsilon\,,\ i < 6, \quad \pi_6 = 2\epsilon\,.$$

The gradation is defined by the labelling vector

$$\vec{d}_{\{\alpha_1, \alpha_5\}}\,.$$

Let $\alpha = k_1\alpha_1 + \cdots + k_6\alpha_6$ be the decomposition of a root α with respect to the simple roots. Then

$$k_i = \langle \pi_i, \alpha \rangle \, .$$

Hence,

$$\begin{array}{rcl} R^0 & = & \{\alpha \mid \langle \pi_1, \alpha \rangle = \langle \pi_5, \alpha \rangle = 0\} \, , \\ R^1_+ & = & \{\alpha \mid \langle \pi_1, \alpha \rangle = 1 \ , \ \langle \pi_5, \alpha \rangle = 0\} \, , \\ R^1_- & = & \{\alpha \mid \langle \pi_1, \alpha \rangle = 0 \ , \ \langle \pi_5, \alpha \rangle = 1\} \, , \\ R^2 & = & \{\alpha \mid \langle \pi_1, \alpha \rangle = 1 \ , \ \langle \pi_5, \alpha \rangle = 1\} \, . \end{array}$$

By computing this scalar products, we get

$$\begin{array}{rcl} R^0 & = & \{\epsilon_i - \epsilon_j \, , \ \epsilon_i + \epsilon_j + \epsilon_6 + \epsilon \, , \ \epsilon_1 + \epsilon_i + \epsilon_j - \epsilon\} \, , \\ R^1_+ & = & \{\epsilon_1 - \epsilon_i \, , \ \epsilon_1 + \epsilon_i + \epsilon_6 + \epsilon\} \, , \\ R^1_- & = & \{\epsilon_i - \epsilon_6 \, , \ \epsilon_i + \epsilon_j + \epsilon_k + \epsilon\} \, , \\ R^2 & = & \{\epsilon_1 - \epsilon_6 \, , \ \epsilon_1 + \epsilon_i + \epsilon_j + \epsilon \, , \ 2\epsilon\} \, , \end{array}$$

where $i, j, k \in \{2, 3, 4, 5\}$ are distinct. The Lie algebra $\mathfrak{g}^0$ is isomorphic to

$$\mathbb{C}^2 \oplus \mathfrak{so}_8(\mathbb{C})$$

and

$$\mathfrak{g}^1_+ = \mathfrak{g}(R^1_+) \, , \quad \mathfrak{g}^1_- = \mathfrak{g}(R^1_-) \, , \quad \mathfrak{g}^2 = \mathfrak{g}(R^2)$$

are three eight-dimensional irreducible representations, which can be transformed one into another by an outer automorphism of $\mathfrak{so}_8(\mathbb{C}) = D_4$ (triality principle).
The parabolic subalgebra associated with this gradation is

$$\mathfrak{p} = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2 = (\mathbb{C}^2 \oplus \mathfrak{so}_8(\mathbb{C})) + (\mathbb{C}^8 + \mathbb{C}^8) + \mathbb{C}^8 \, .$$

The corresponding flag manifold is

$$F = G/P = E_6/P \, ,$$

where P is the connected parabolic subgroup of E_6 generated by the subalgebra $\mathfrak{p}$.

7. Fundamental gradations of a real semisimple Lie algebra

7.1. **Real forms of a complex semisimple Lie algebra and Satake diagrams.** Now we recall the description of a real form of a complex semisimple Lie algebra in terms of Satake diagrams. It is sufficient to do this for complex simple Lie algebras.

Any real form of a complex semisimple Lie algebra $\mathfrak{g}$ is the fix point set $\mathfrak{g}^\sigma$ of an antilinear involution σ, that is, an antilinear map $\sigma : \mathfrak{g} \to \mathfrak{g}$, which is an automorphism of $\mathfrak{g}$ as a real algebra, such that $\sigma^2 = \mathrm{id}$. We fix a Cartan decomposition

$$\mathfrak{g}^\sigma = \mathfrak{k} + \mathfrak{m}$$

of the real form $\mathfrak{g}^\sigma$, where $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}^\sigma$ and $\mathfrak{m}$ is its orthogonal complement with respect to the Killing form B. Let

$$\mathfrak{h}^\sigma = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{m}$$

be a Cartan subalgebra of $\mathfrak{g}^\sigma$ which is consistent with this decomposition and such that $\mathfrak{h}_\mathfrak{m} = \mathfrak{h} \cap \mathfrak{m}$ has maximal dimension. Then the root decomposition of $\mathfrak{g}^\sigma$, with respect to the subalgebra $\mathfrak{h}^\sigma$, can be written as

$$\mathfrak{g}^\sigma = \mathfrak{h}^\sigma + \sum_{\lambda \in \Sigma} \mathfrak{g}^\sigma_\lambda ,$$

where $\Sigma \subset (\mathfrak{h}^\sigma)^*$ is a (non-reduced) root system. The number $m_\lambda = \dim \mathfrak{g}_\lambda$ is the *multiplicity* of a root $\lambda \in \Sigma$.

Denote by $\mathfrak{h} = (\mathfrak{h}^\sigma)^\mathbb{C}$ the complexification of $\mathfrak{h}^\sigma$ which is a σ-invariant Cartan subalgebra. We denote by σ^* the induced antilinear action of σ on $\mathfrak{h}^*$ given by

$$\sigma^* \alpha = \overline{\alpha \circ \sigma} , \quad \alpha \in \mathfrak{h}^* .$$

Consider the root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

of the Lie algebra $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. Note that σ^* preserves the root system R, i.e. $\sigma^* R = R$. Now we relate the root space decomposition of $\mathfrak{g}^\sigma$ and $\mathfrak{g}$. We define the subsystem of compact roots $R_\bullet$ by

$$R_\bullet = \{\alpha \in R \mid \sigma^* \alpha = -\alpha\} = \{\alpha \mid \alpha(\mathfrak{h}_\mathfrak{m}) = 0\}$$

and denote by $R' = R \setminus R_\bullet$ the complementary set of noncompact roots. We can choose a system Π of simple roots of R such that the corresponding system of positive roots R_+ satisfies the condition: $R'_+ = R' \cap R_+$ is σ-invariant. In this case, Π is called a *σ-fundamental system* of roots. We denote by $\Pi_\bullet = \Pi \cap R_\bullet$ the set of compact simple roots (which are also called black) and by $\Pi' = \Pi \setminus \Pi_\bullet$ the noncompact simple roots (called white). The action of σ^* on white roots satisfies the following property:

for any $\alpha \in \Pi'$ there exists a unique $\alpha' \in \Pi'$ such that $\sigma^* \alpha - \alpha'$ is a linear combination of black roots, i.e.

$$\sigma^* \alpha = \alpha' + \sum_{\beta \in \Pi_\bullet} k_\beta \beta, \quad k_\beta \in \mathbb{N} .$$

In this case, we say that the roots α, α' are *σ-equivalent* and we will write $\alpha \sim \alpha'$. The information about fundamental system ($\Pi = \Pi_\bullet \cup \Pi'$) together with the σ-equivalence can be visualized in terms of the *Satake diagram*, which is defined as follows.

On the Dynkin diagram of the system of simple roots Π, we paint the vertices which correspond to black roots into black and we join the vertices which correspond to σ-equivalent roots α, α' by a curved arrow.

Using slight abuse of notation, we will refer to the σ-fundamental system

$\Pi = \Pi_\bullet \cup \Pi'$, together with the σ-equivalence $\sim$, as the *Satake diagram.* This diagram is determined by the real form $\mathfrak{g}^\sigma$ of a complex simple Lie algebra $\mathfrak{g}$ and does not depend on the choice of a Cartan subalgebra and a σ-fundamental system. The list of Satake diagram of real simple Lie algebras is known (see e.g. [9])

Conversely, Satake diagram ($\Pi = \Pi_\bullet \cup \Pi', \sim$) allows to reconstruct the action of σ^* on Π, hence on $\mathfrak{h}^*$. This action can be canonically extended to the antilinear involution σ of the complex Lie algebra $\mathfrak{g}$. Hence, there is a natural $1-1$ correspondence between Satake diagrams subordinated to the Dynkin diagram of a complex semisimple Lie algebra $\mathfrak{g}$, up to isomorphisms, and real forms $\mathfrak{g}^\sigma$ of $\mathfrak{g}$, up to conjugations.

We will describe real forms $\mathfrak{g}^\sigma$ of a complex simple Lie algebra in terms of Satake diagrams ($\Pi = \Pi_\bullet \cup \Pi', \sim$).

7.2. **Description of gradations of a real semisimple Lie algebra.** Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{g}^\sigma$ a real form of $\mathfrak{g}$. As above, we consider a Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{m}$ of $\mathfrak{g}^\sigma$ and extend it to the Cartan subalgebra $\mathfrak{h} = (\mathfrak{h}^\sigma)^\mathbb{C}$. We denote by $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ a σ-fundamental system, which is a system of simple roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. We denote by $\Pi_\bullet$ and Π' the set of black and white roots respectively and by $\sim$ the σ-equivalence relation. Recall that black and white roots are represented by black and white vertices of the corresponding Satake diagram and the equivalence relation is indicated by curved arrows.

Let $\vec{d} = (d_1, \dots, d_\ell)$ be a label of the simple roots system Π and $\mathfrak{g} = \sum_{i\in\mathbb{Z}} \mathfrak{g}^i$ be the corresponding gradation of $\mathfrak{g}$, with the grading element $d \in \mathfrak{h} \subset \mathfrak{g}$.

The following theorem gives necessary and sufficient conditions in order that this gradation induces a gradation

$$\mathfrak{g}^\sigma = \sum_{i\in\mathbb{Z}} \mathfrak{g}^\sigma \cap \mathfrak{g}^i$$

of the real form $\mathfrak{g}^\sigma$. This means that the grading element d belongs to $\mathfrak{g}^\sigma$. We denote by $\Pi^0 \subset \Pi$ the set of simple roots with label zero.

Theorem 7.1. ([8]) *A gradation of a complex semisimple Lie algebra $\mathfrak{g}$, associated with a label vector $\vec{d} = (d_1, \dots, d_\ell)$, induces a gradation of the real form $\mathfrak{g}^\sigma$, which corresponds to a Satake diagram ($\Pi = \Pi_\bullet \cup \Pi', \sim$) if and only if the following two conditions hold:*

i) $\Pi_\bullet \subset \Pi^0$*, i.e. any black vertex of the Satake diagram has label zero;*
ii) *if $\alpha \sim \alpha'$ for α, $\alpha' \in \Pi \setminus \Pi_\bullet$, then $d(\alpha) = d(\alpha')$, i.e. white vertices of the Satake diagram which are joint by a curved arrow have the same label.*

Proof. Since $\mathfrak{h} = (\mathfrak{h}_\mathfrak{k})^\mathbb{C} + (\mathfrak{h}_\mathfrak{m})^\mathbb{C}$ and ad_h has real eigenvalues for $h \in \mathfrak{h}_\mathfrak{m}$ and purely imaginary eigenvalues for $h \in \mathfrak{h}_\mathfrak{k}$, the grading element d belongs to the space $i\mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{m}$. It belongs to $\mathfrak{h}^\sigma = \mathfrak{h}_\mathfrak{k} + \mathfrak{h}_\mathfrak{m}$ if and only if its projection

d' on $i\mathfrak{h}_{\mathfrak{k}}$ vanishes. We can write $d = d' + d''$, where d'' belongs to $\mathfrak{h}_{\mathfrak{m}}$. Then

$$\sigma(d) = -d' + d'' = d - 2d'.$$

For any simple root $\alpha \in \Pi$, we have

$$(\sigma^*\alpha)(\sigma(d)) = \overline{\alpha(d)} = \alpha(d)\,, \tag{8}$$

since $\alpha(d) \in \mathbb{R}$. On the other hand, we have

$$(\sigma^*\alpha)(\sigma(d)) = (\sigma^*\alpha)(d - 2d') = (\sigma^*\alpha)(d) - 2(\sigma^*\alpha)(d'). \tag{9}$$

Assume now that conditions i) and ii) hold. Then, $d(\beta) = 0$, for any black root β. Since $\sigma^*\alpha = \alpha' + \sum_{\beta\in\Pi_\bullet} k_\beta\beta$, we have

$$(\sigma^*\alpha)(d) = d(\sigma^*\alpha) = d(\alpha' + \sum_{\beta\in\Pi_\bullet} k_\beta d(\beta)) = d(\alpha')$$

and equalities (8), (9), together with condition ii), give

$$0 = \alpha(d) - \alpha(d') = 2(\sigma^*\alpha)(d')\,,$$

for any $\alpha \in \Pi$. Since Π forms a basis of $\mathfrak{h}$, we have that $d' = 0$.
Conversely, if $d' = 0$, then (9) implies conditions i) and ii). $\square$

A labelling $\vec{d} = (d_1, \ldots, d_\ell)$ of a Satake diagram $(\Pi = \{\alpha_1, \ldots, \alpha_\ell\} = \Pi_\bullet \cup \Pi', \sim)$ (and the corresponding gradation of $\mathfrak{g}$) is called of *real type* if it satisfies conditions i) and ii) of the theorem, that is black vertices have label zero and vertices related by a curved arrow have the same label. Using this definition, we can state Theorem 7.1 as follows

Corollary 7.2. *There exists a natural $1-1$ correspondence between labellings $\vec{d}$ of real type of a Satake diagram of a real semisimple Lie algebra $\mathfrak{g}^\sigma$ and gradations of $\mathfrak{g}^\sigma$. The gradation of $\mathfrak{g}^\sigma$ is fundamental if and only if the corresponding gradation of $\mathfrak{g}$ is fundamental, i.e. $\vec{d} = \vec{d_{\Pi^1}}$.*

Decomposition of $\mathfrak{g}^0$-module $\mathfrak{g}^1$ into irreducible submodules. Let $\mathfrak{g} = \sum \mathfrak{g}^i$ be a gradation of a complex semisimple Lie algebra $\mathfrak{g}$ with grading element d and $\mathfrak{g}^\sigma = \sum(\mathfrak{g}^\sigma)^i = \sum \mathfrak{g}^i \cap \mathfrak{g}^\sigma$ be a real form of $\mathfrak{g}$, consistent with this gradation. We denote by $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ the Satake diagram of $\mathfrak{g}^\sigma$. Recall that the decomposition of $\mathfrak{g}^1$ into irreducible $\mathfrak{g}^0$-submodules is given by

$$\mathfrak{g}^1 = \sum_{\gamma\in\Pi^1} \mathfrak{g}(R(\gamma))\,,$$

where Π^1 is the set of simple roots of degree one. The following obvious proposition describes the decomposition of $(\mathfrak{g}^\sigma)^0$-module $(\mathfrak{g}^\sigma)^1$ into irreducible submodules.

Proposition 7.3. *For any simple root $\gamma \in \Pi^1$ of degree one, there are two possibilities:*

i) $\sigma^*\gamma = \gamma + \sum_{\beta\in\Pi_\bullet} k_\beta\beta$. *Then $\sigma^*\gamma \in R(\gamma)$ and the $\mathfrak{g}^0$-module $\mathfrak{g}(R(\gamma))$ is σ-invariant;*

ii) $\sigma^*\gamma = \gamma' + \sum_{\beta\in\Pi_\bullet} k_\beta\beta$, *where* $\gamma \neq \gamma' \in \Pi^1$. *Then,* $\sigma^* R(\gamma) = R(\gamma')$ *and the two irreducible* $\mathfrak{g}^0$*-modules* $\mathfrak{g}(R(\gamma))$ *and* $\mathfrak{g}(R(\gamma'))$ *determine one irreducible submodule*
$$\mathfrak{g}^\sigma \cap (\mathfrak{g}(R(\gamma)) + \mathfrak{g}(R(\gamma')))$$
of $\mathfrak{g}^\sigma$.

The following theorems, which are reformulations of the above proposition in terms of Satake diagrams $(\Pi = \Pi_\bullet \cup \Pi', \sim)$ of a real semisimple Lie algebra $\mathfrak{g}^\sigma$, give a description of all admissible gradation of a real semisimple Lie algebra.

Theorem 7.4. *Let* $\mathfrak{g}^\sigma = \sum (\mathfrak{g}^\sigma)^i$ *be the gradation of a real semisimple Lie algebra* $\mathfrak{g}^\sigma$*, associated with a labelling vector* $\vec{d}$ *of real type. Then irreducible submodules of the* $(\mathfrak{g}^\sigma)^0$*-module* $(\mathfrak{g}^\sigma)^{-1}$ *correspond to vertices* γ *with label one without curved arrow and to pairs* (γ, γ') *of vertices with label one related by a curved arrow. In particular, if the gradation is fundamental, i.e.* $\vec{d} = \vec{d}_{\Pi^1}$, *then it is admissible if and only if there are two vertices from* Π^1 *which are not connected by a curved arrow.*

Corollary 7.5. *Let* $\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma)^{-2} + \cdots + (\mathfrak{g}^\sigma)^2$ *be a depth two gradation of a real semisimple Lie algebra* $\mathfrak{g}^\sigma$*, associated with a Satake diagram with a labelling vector of the (real) type* $\vec{d}_{\gamma,\gamma'}$. *If the vertices* γ, γ' *of the Satake diagram are related by a curved arrow, then the* $(\mathfrak{g}^\sigma)^0$*-module* $(\mathfrak{g}^\sigma)^{-1}$ *is irreducible. In the opposite case, the module* $(\mathfrak{g}^\sigma)^1$ *has two irreducible submodules* $(\mathfrak{g}^\sigma)^{-1}_\pm = (\mathfrak{g}^\sigma)^{-1} \cap \mathfrak{g}^{-1}_\pm$.

In particular, admissible gradations of a real simple Lie algebra $\mathfrak{g}^\sigma$ *correspond bijectively to labelling vectors of the form* $\vec{d}_{\{\alpha_p,\,\alpha_q\}}$, *where* $\{\alpha_p, \alpha_q\}$ *are white vertices of the Satake diagram which are not connected by a curved arrow.*

8. Maximal homogeneous para-CR manifolds of depth 2

By using Corollary 7.5 and the list of Satake diagrams of type A_ℓ, D_ℓ and E_6, we can describe admissible gradations (see Definition 5.2) in terms of Satake diagrams with an appropriate labelling.

Let $\mathfrak{g}$ be a complex simple Lie algebra. We have proved that all admissible gradations of complex simple Lie algebras are exhausted by the following cases:

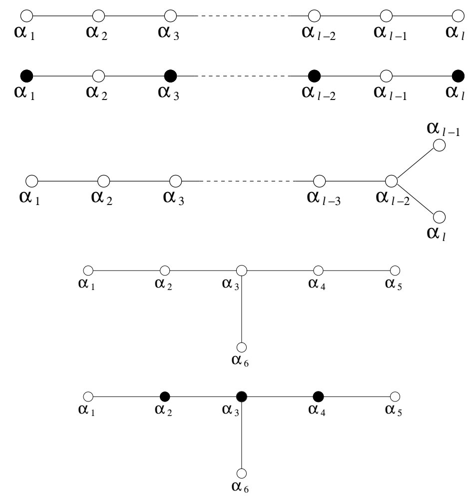
- gradations of $\mathfrak{g}$ of type A_ℓ defined by the labelling vector $\vec{d}_{\{\alpha_p,\,\alpha_q\}}$, where α_p, α_q are any simple roots;
- gradations of $\mathfrak{g}$ of type D_ℓ defined by the labelling vector $\vec{d}_{\{\alpha_1,\,\alpha_\ell\}}$ or $\vec{d}_{\{\alpha_{\ell-1},\alpha_\ell\}}$;
- gradations of $\mathfrak{g}$ of type E_6 defined by the labelling vector $\vec{d}_{\{\alpha_1,\alpha_5\}}$.

Any such a gradation defines an admissible gradation of the real form $\mathfrak{g}^\sigma$, if and only if the two vertices on the Satake diagram with label one are white and not connected by a curved arrow. By examining the Satake diagrams of type A_ℓ, D_ℓ, E_6, we get the following result:

Proposition 8.1. *Admissible graded real simple Lie algebras*

$$\mathfrak{g}^\sigma = (\mathfrak{g}^\sigma)^{-2} + \cdots + (\mathfrak{g}^\sigma)^2$$

of depth two are described as follows:
Type 1. $\mathfrak{g}$ *is a complex Lie algebra (considered as a real Lie algebra) of type A_ℓ, D_ℓ, E_6, with an admissible gradation;*
Type 2. $\mathfrak{g}$ *is a simple real Lie algebra without complex structure, described by one of the following Satake diagram*



where only two vertices on the Satake diagram have label one. They are white and non connected by a curved arrow.

Theorem 8.2. *Let M be a non degenerate maximally homogeneous weak para-CR manifold of semisimple type $(\mathfrak{m}, K_0)$ and depth 2. Then, up to coverings, M is isomorphic to a direct product of the following flag manifolds $F = G/P$ of a simple Lie group G associated with a graded Lie algebra $\mathfrak{g} = (\mathfrak{m} + \mathfrak{g}^0)^\infty$ equipped with an invariant para-CR structure:*
$\mathfrak{g}$ **is of type** A_ℓ:

- i) $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{R})$ *and $F = F_{p,q}(\mathbb{R}) = \mathrm{SL}_{\ell+1}(\mathbb{R})/P$ is the manifold of (p,q)-flags in the space $V = \mathbb{R}^{\ell+1}$, where P is the stabilizer of the standard (p,q)-flags $f_0 = (\mathbb{R}^p \subset \mathbb{R}^q)$, $1 \le p < q \le \ell$;*
- ii) $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{C})$ *and $F = F_{p,q}(\mathbb{C}) = \mathrm{SL}_{\ell+1}(\mathbb{C})/P$ is the manifold of (p,q)-flags in the space $V = \mathbb{C}^{\ell+1}$, where P is the stabilizer of the standard (p,q)-flag $f_0 = (\mathbb{C}^p \subset \mathbb{C}^q)$, $1 \le p < q \le \ell$;*
- iii) $\mathfrak{g} = \mathfrak{sl}_{\ell+1}(\mathbb{H})$ *and $F = F_{p,q}(\mathbb{H}) = \mathrm{SL}_{\ell+1}(\mathbb{H})/P$ is the manifold of (p,q)-flags in the space $V = \mathbb{H}^{\ell+1}$, where P is the stabilizer of the standard (p,q)-flags $f_0 = (\mathbb{H}^p \subset \mathbb{H}^q)$, $1 \le p < q \le \ell$.*

$\mathfrak{g}$ **is of type** D_ℓ:

i) $\mathfrak{g} = \mathfrak{so}_{2\ell}(\mathbb{C})$ *and* $F = \mathrm{SO}^+_{2\ell}(\mathbb{C})/P$ *is the manifold of all isotropic* $(1,\ell)$*-flags in the complex Euclidean space* $V = (\mathbb{C}^{2\ell}, <,>)$*, where* P *is the standard* $(1,\ell)$*-flag* $f_0 = \mathbb{C} \subset \mathbb{C}^\ell$*;*

ii) $\mathfrak{g} = \mathfrak{so}_{2\ell}(\mathbb{C})$ *and* $F = \mathrm{SO}^+_{2\ell}(\mathbb{C})/P$ *is the manifold of all isotropic* $(\ell-1,\ell)$*-flags in the complex Euclidean space* $V = (\mathbb{C}^{2\ell}, <,>)$*, where* P *is the standard* $(\ell-1,\ell)$*-flag* $f_0 = \mathbb{C}^{\ell-1} \subset \mathbb{C}^\ell$*;*

iii) $\mathfrak{g} = \mathfrak{so}_{\ell,\ell}$ *(the normal form of* D_ℓ*) and* $F = F_{1,\ell} = \mathrm{SO}_{\ell,\ell}/P$ *is the manifold of isotropic* $(1,\ell)$*-flags in the pseudo-Euclidean space* $\mathbb{R}^{\ell,\ell}$*;*

iv) $\mathfrak{g} = \mathfrak{so}_{\ell,\ell}$ *(the normal form of* D_ℓ*) and* $F = F_{\ell-1,\ell} = \mathrm{SO}_{\ell,\ell}/P$ *is the manifold of isotropic* $(\ell-1,\ell)$*-flags in the pseudo-Euclidean space* $\mathbb{R}^{\ell,\ell}$*.*

$\mathfrak{g}$ **is of type** E_6*:*

i) $\mathfrak{g} = \mathfrak{e}_6$ *(see subsection* 6.2 *for the description of the manifold* F*);*

ii) $\mathfrak{g} = \mathfrak{e}_6^{\mathrm{norm}} = E\,I$ *(the normal form of* E_6*) with the maximal compact subalgebra* $\mathfrak{sp}_4$ *and* $F = \mathrm{E}_6^{\mathrm{norm}}/P$ *is the flag manifold described like in the complex case;*

iii) $\mathfrak{g} = \mathfrak{e}_6(\mathfrak{f}_4) = E\,IV$ *the real form of* $\mathfrak{e}_6$ *with maximal compact subalgebra* $\mathfrak{f}_4$ *and* $F = \mathrm{E}_6(\mathfrak{f}_4)/P$ *is the flag manifold described like in the complex case .*

Moreover, we have

		$\dim H_-F$	$\dim H_+F$	$\dim F - \dim HF$
A_ℓ	*i), ii), iii)*	$p(q-p)$	$(q-p)(\ell+1-q)$	$p(\ell+1-q)$
D_ℓ	*i), iii)*	$\frac{(\ell-1)(\ell-2)}{2}$	$\ell-1$	$\ell-1$
	ii), iv)	$\ell-1$	$\ell-1$	$\frac{(\ell-1)(\ell-2)}{2}$
E_6	*i), ii), iii)*	8	8	8

(where the dimensions have to be intended over $\mathbb{C}$ *whenever* $\mathfrak{g}$ *has a complex structure).*

In particular, the weak para-CR structure is a para-CR structure in cases A_ℓ *for* $p+q = \ell+1$*,* D_ℓ *ii) and iv) and* E_6*.*

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