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Abstract. Density conditions for wavelet systems with arbitrary sampling points to be frames are studied. We show that for a wavelet system generated by admissible functions and irregular affine lattices to be a frame, the sampling points must have a positive lower Beurling density. The same is true for wavelet systems with arbitrary sampling points and nice generating functions.

Keywords Wavelet frames, affine Beurling density.
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1 Introduction and Main Results

Density is a useful tool in the study of frames. For example, in $L^2([-1/2,1/2])$, frames generated by exponentials of the form $\{e^{i2\pi\lambda_n\omega} : n\in\mathbb{Z}\}$ can be completely characterized by the density of the sequence $\{\lambda_n : n\in\mathbb{Z}\}$ (see [8]). In [3], Christensen, Deng and Heil studied the density of Gabor frames and proved that for a Gabor system $\{e^{i2\pi b_n x}g(x-a_n) : n\in\mathbb{Z}\}$ to be a frame for $L^2(\mathbb{R})$, the time-frequency parameters (a_n,b_n) must have a finite upper Beurling density and possess a lower Beurling density no less than 1.

For the case of wavelet systems, one can consider similar problems. In [6, 13], it was shown that for a wavelet system with arbitrary sampling points to be a frame for $L^2(\mathbb{R})$, the sampling points must be relatively uniformly discrete, or equivalently, they must have a finite upper affine Beurling density.

For the lower affine Beurling density, however, there is no general result. In [14], the authors studied density conditions for irregular multi-generated wavelet systems of the

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form

$$\{\tau(s_{\ell,j}, s_{\ell,j}t_{\ell,k})\psi_\ell : \ j \in \mathbb{J}_\ell, k \in \mathbb{K}_\ell, 1 \le \ell \le r\}$$

to be frames, where r is a fixed positive integer, $\psi_\ell \in L^2(\mathbb{R})$, $s_{\ell,j} > 0, t_{\ell,k} \in \mathbb{R}$, $\mathbb{J}_\ell, \mathbb{K}_\ell \subset \mathbb{Z}$, and

$$(\tau(s,t)\psi)(x) := s^{-1/2}\psi((x-t)/s).$$

For convenience, we call a sequence of sampling points of the form $\{(s_{\ell,j}, s_{\ell,j}t_{\ell,k}) : \ j \in \mathbb{J}_\ell, k \in \mathbb{K}_\ell, 1 \le \ell \le r\}$ an irregular affine lattice. Even for this special case, it is not clear whether the sampling points have a positive lower affine Beurling density for the wavelet system to be a frame.

On the other hand, Example 2.1 in [14] shows that the (one-dimensional) lower Beurling density of the translation parameters $\{t_{\ell,k} : \ k \in \mathbb{K}_\ell, 1 \le \ell \le r\}$ may be zero even if the corresponding wavelet system forms a frame. Specifically, let

$$\hat{\psi}(\omega) = \begin{cases} |\omega|^{1/4}(1-|\omega|), & |\omega| \le 1/2\pi, \\ 0, & otherwise, \end{cases}$$

and $\{t_k : \ k \in \mathbb{Z}\}$ be a rearrangement of $\{k \in \mathbb{Z} : \ k \le 2 \text{ or } 2^{2l} \le k \le 2^{2l+1} \text{ for some } l \ge 1\}$ such that $t_k \le t_{k+1}$, here $\hat{\psi}(\omega) = \int_{\mathbb{R}} \psi(x)e^{-i2\pi x\omega}dx$ is the Fourier transform of ψ. Then $\{\tau(2^j, 2^j t_k)\psi : \ j,k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. Obviously, $\sup_k(t_{k+1} - t_k) = +\infty$. This seems to suggest that the lower affine Beurling density of the sampling points might be zero. Fortunately, when the "affine density" is considered, it is also positive in this example. In fact, it is true for every wavelet frame generated by admissible functions with irregular affine lattices. Before stating our results, we introduce some notations.

The group action in $\mathcal{G} := \{(s,t) : \ s > 0, t \in \mathbb{R}\}$ is defined by

$$(a,b)(a',b') = (aa', b + ab').$$

For any $(x,y) \in \mathcal{G}$, its (a,b)-neighborhood is defined by $Q_{a,b}(x,y) = (x,y)V$ with $V = [a^{-1/2}, a^{1/2}] \times [-\frac{b}{2}, \frac{b}{2}]$, $a > 1, b > 0$. It is easy to check that

$$Q_{a,b}(x,y) = [a^{-1/2}x, a^{1/2}x] \times [y - \frac{bx}{2}, y + \frac{bx}{2}].$$

Let $\Gamma = \{(x_n, y_n) : \ n \in \Lambda\}$ be a sequence of elements of $\mathcal{G}$.

(i) Γ is called (p,q)-uniformly discrete if $|Q_{p,q}(x_n,y_n) \bigcap Q_{p,q}(x_m,y_m)| = 0, n \ne m$, where $|E|$ denotes the Lebesgue measure of measurable sets $E \subset \mathbb{R}^2$.

(ii) Γ is called relatively uniformly discrete if it is a finite union of uniformly discrete sequences.

It is easy to see that for any $a > 1, b > 0$, $\{(a^j, a^j bk) : \ j,k \in \mathbb{Z}\}$ is (a,b)-uniformly discrete. We denote $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : \ n \le x\}$ and $\lceil x \rceil = \min\{n \in \mathbb{Z} : \ n \ge x\}$, and, $\#E$ denotes the cardinality of a sequence or a set E.

$C_c^\infty(\mathbb{R})$ is the set of all functions which are compactly supported and infinite times differentiable.

We call a function $\psi \in L^2(\mathbb{R})$ admissible if

$$\int_{-\infty}^{+\infty} \frac{1}{|\xi|}|\hat{\psi}(\xi)|^2 d\xi < +\infty.$$

Let $d\mu = \frac{1}{s^2} dsdt$ be the left-invariant measure on $\mathcal{G}$ and ν be the weighted counting measure defined by

$$\nu(E) = \sum_{(s,t)\in E} s, \qquad \text{for any discrete set } E \subset \mathcal{G}.$$

For any sequence $\Gamma \subset \mathcal{G}$, its lower and upper affine Beurling density are defined by

$$D^-(\Gamma) = \varliminf_{a\to+\infty} \varliminf_{b\to+\infty} \inf_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \bigcap Q_{a,b}(x,y))}{\iint_{Q_{a,b}(x,y)} sd\mu} = \varliminf_{a\to+\infty} \varliminf_{b\to+\infty} \inf_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \bigcap Q_{a,b}(x,y))}{bx\ln a}$$

and

$$D^+(\Gamma) = \varlimsup_{a\to+\infty} \varlimsup_{b\to+\infty} \sup_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \bigcap Q_{a,b}(x,y))}{\iint_{Q_{a,b}(x,y)} sd\mu} = \varlimsup_{a\to+\infty} \varlimsup_{b\to+\infty} \sup_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \bigcap Q_{a,b}(x,y))}{bx\ln a},$$

respectively.

We are now ready to state the main results.

Theorem 1.1 *Let $\psi_\ell \in L^2(\mathbb{R})$ be admissible, S_ℓ and T_ℓ be real sequences, and S_ℓ consist of positive numbers, $1 \le \ell \le r$. If $\bigcup_{\ell=1}^r \{\tau(s,st)\psi_\ell : s \in S_\ell, t \in T_\ell\}$ is a frame for $L^2(\mathbb{R})$, then the lower affine Beurling density of $\bigcup_{\ell=1}^r \{(s,st) : s \in S_\ell, t \in T_\ell\}$ is positive.*

For wavelet frames with arbitrary sampling points, it was shown in [13] that the sampling points must have a positive lower affine Beurling density whenever $\psi_\ell, \psi'_\ell, X\psi'_\ell$, and $X\psi''_\ell$ are admissible. In this paper, we show that that ψ_ℓ and $X\psi'_\ell$ are admissible is sufficient.

Theorem 1.2 *Let $\Gamma_\ell \subset \mathcal{G}$ be sequences, $1 \le \ell \le r$, and $\{\tau(s,t)\psi_\ell : (s,t) \in \Gamma_\ell, 1 \le \ell \le r\}$ be a frame for $L^2(\mathbb{R})$. If $\psi_\ell(x)$ is local absolutely continuous and $\psi_\ell(x), x\psi'_\ell(x)$ are admissible, then*

$$D^-\Big(\bigcup_{1\le\ell\le r} \Gamma_\ell\Big) > 0.$$

2 Proofs of the main results

For fixed $\psi \in L^2(\mathbb{R})$, the continuous wavelet transform of a function $f \in L^2(\mathbb{R})$ is defined by

$$(W_\psi f)(s,t) = \int_{-\infty}^{+\infty} f(x)|s|^{-1/2}\overline{\psi(\frac{x-t}{s})}\, dx.$$

If ψ is admissible, then

$$\int_{-\infty}^{+\infty} \int_0^{+\infty} \frac{1}{s^2} |(W_\psi f)(s,t)|^2 dsdt < +\infty, \qquad \forall f \in L^2(\mathbb{R}).$$

Lemma 2.1 *Let $\{(s_n,t_n): n\in\Lambda\}\subset\mathcal{G}$ be an (a,b)-uniformly discrete sequence. Suppose that $h(s,t)\ge 0$ is a function defined on $\mathcal{G}$. Then for any $p>a^3$ and $q>b$,*

$$\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)}\iint_{E_n} h(s,t)dtds \le \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} 4\left(1+\frac{a^{3/2}}{s}\right)h(s,t)dtds,$$

where $E_n=[s_na^{-1/2},s_na^{1/2}]\times[t_n-\frac{b}{2},t_n+\frac{b}{2}]$.

Proof. Put $\Gamma=\{(s_n,t_n): n\in\Lambda\}$. For any $j\in\mathbb{Z}$, let $\Gamma_j=\{(s,t)\in\Gamma: a^{j-1/2}\le s< a^{j+1/2}\}$. We can write

$$\Gamma_j=\{(s_{j,k},t_{j,k}): k\in\Lambda_j\},$$

where $\Lambda_j\subset\mathbb{Z}$. Without loss of generality, we assume that $t_{j,k}\le t_{j,k+1}$. Let

$$E_{j,k}=[s_{j,k}a^{-1/2},s_{j,k}a^{1/2}]\times[t_{j,k}-\frac{b}{2},t_{j,k}+\frac{b}{2}].$$

Since Γ_j is (a,b)-uniformly discrete and $s_{j,k}a^{-1/2}<a^j\le s_{j,k}a^{1/2}$, we have

$$t_{j,k}-\frac{b}{2}s_{j,k}\ge t_{j,k'}+\frac{b}{2}s_{j,k'},\quad \forall k,k'\in\Lambda_j, k>k'.$$

Hence

$$t_{j,k}-t_{j,k'}\ge\frac{b}{2}(s_{j,k}+s_{j,k'})\ge a^{j-1/2}b.$$

Therefore, we can split Λ_j into $N_j:=\lceil a^{-j+1/2}\rceil$ subsets $\Lambda_{j,\ell}$ (may be empty) such that

$$t_{j,k}-t_{j,k'}\ge N_ja^{j-1/2}b\ge b,\qquad k,k'\in\Lambda_{j,\ell}, k\ne k'.$$

Note that $N_j\le a^{-j+1/2}+1$. We have

$$\begin{aligned}
\sum_{|j|\ge\frac{\ln p-\ln a}{2\ln a},k\in\Lambda_j}\iint_{E_{j,k}} h(s,t)dtds &= \sum_{|j|\ge\frac{\ln p-\ln a}{2\ln a}}\sum_{\ell=1}^{N_j}\sum_{k\in\Lambda_{j,\ell}}\int_{s_{j,k}a^{-1/2}}^{s_{j,k}a^{1/2}}\int_{t_{j,k}-b/2}^{t_{j,k}+b/2} h(s,t)dtds\\
&\le \sum_{|j|\ge\frac{\ln p-\ln a}{2\ln a}}\int_{a^{j-1}}^{a^{j+1}}\int_{-\infty}^{+\infty} N_jh(s,t)dtds\\
&\le \sum_{|j|\ge\frac{\ln p-\ln a}{2\ln a}}\int_{a^{j-1}}^{a^{j+1}}\int_{-\infty}^{+\infty}\left(1+\frac{a^{3/2}}{s}\right)h(s,t)dtds\\
&\le \iint_{\substack{s\notin[(p/a^3)^{-1/2},(p/a^3)^{1/2}]\\ t\in\mathbb{R}}} 2\left(1+\frac{a^{3/2}}{s}\right)h(s,t)dtds.
\end{aligned}$$

Similarly we can prove that

$$\sum_{|t_{j,k}|\ge q/2}\iint_{E_{j,k}} h(s,t)dtds\le\iint_{\substack{s>0\\ |t|>(q-b)/2}} 2\left(1+\frac{a^{3/2}}{s}\right)h(s,t)dtds.$$

On the other hand, if $s_{j,k} \notin [p^{-1/2}, p^{1/2}]$, then $a^{j+1/2} > p^{1/2}$ or $a^{j-1/2} < p^{-1/2}$. Therefore, $|j| \ge \frac{\ln p - \ln a}{2\ln a}$. It follows that

$$\begin{aligned}\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} h(s,t)dtds &\le \sum_{|j|\ge\frac{\ln p-\ln a}{2\ln a},k\in\Lambda_j} \iint_{E_{j,k}} h(s,t)dtds + \sum_{|t_{j,k}|\ge q/2} \iint_{E_{j,k}} h(s,t)dtds \\ &\le \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} 4\left(1+\frac{a^{3/2}}{s}\right) h(s,t)dtds.\end{aligned}$$

□

The following lemma is a consequence of the Wirtinger's inequality[5].

Lemma 2.2 *If $f(x)$ is absolutely continuous on $[a,b]$, $f, f' \in L^2[a,b]$ and there is some $c \in [a,b]$ such that $f(c) = 0$, then $\int_a^b |f(x)|^2dx \le \frac{4(b-a)^2}{\pi^2}\int_a^b |f'(x)|^2dx$.*

Lemma 2.3 *Let $\{(s_n,t_n) : n \in \Lambda\} \subset \mathcal{G}$ be an (a,b)-uniformly discrete sequence. Suppose that $f \in C_c^\infty(\mathbb{R})$, $\psi \in L^2(\mathbb{R})$ is locally absolutely continuous and $\psi(x)$ and $x\psi'(x)$ are admissible. Then we have*

$$\begin{aligned}\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} |(W_\psi f)(s_n,t_n)|^2 \le C_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\Big(|(W_\psi f)(s,t)|^2 \\ + |(W_{\tilde\psi} f)(s,t)|^2 + |(W_\psi f')(s,t)|^2 + |(W_{\tilde\psi} f')(s,t)|^2\Big)dtds, \quad f \in L^2(\mathbb{R}),\end{aligned}$$

where $\tilde\psi(x) = \psi(x) + x\psi'(x)$ and $C_{a,b}$ is a constant.

Proof. Since $(W_\psi f)(s,t) = \langle f(\cdot + t), s^{-1/2}\psi(\cdot/s)\rangle$, it is easy to check that

$$\begin{aligned}\frac{\partial}{\partial t}(W_\psi f)(s,t) &= (W_\psi f')(s,t), \\ \frac{\partial}{\partial s}s^{-1/2}(W_\psi f)(s,t) &= -s^{-3/2}(W_{\tilde\psi} f)(s,t).\end{aligned}$$

Put $E_n = [s_n a^{-1/2}, s_n a^{1/2}] \times [t_n - \frac{b}{2}, t_n + \frac{b}{2}]$. We have

$$\begin{aligned}&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \frac{1}{s}\left|(W_\psi f)(s,t) - (W_\psi f)(s,t_n)\right|^2 dtds \\ =& \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \int_{s_n a^{-1/2}}^{s_n a^{1/2}} \frac{1}{s}\int_{t_n-b/2}^{t_n+b/2} \left|(W_\psi f)(s,t) - (W_\psi f)(s,t_n)\right|^2 dtds \\ \le& \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \int_{s_n a^{-1/2}}^{s_n a^{1/2}} \frac{1}{s}\cdot\frac{4b^2}{\pi^2}\int_{t_n-b/2}^{t_n+b/2} \left|(W_\psi f')(s,t)\right|^2 dtds \quad \text{(Lemma 2.2)} \\ =& \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{4b^2}{\pi^2}\iint_{E_n} \frac{1}{s}\left|(W_\psi f')(s,t)\right|^2 dtds \\ \le& \frac{4b^2}{\pi^2}\iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} 4\left(\frac{1}{s}+\frac{a^{3/2}}{s^2}\right)\left|(W_\psi f')(s,t)\right|^2 dtds, \qquad (2.1)\end{aligned}$$

where Lemma 2.1 is used in the last step. Using this lemma again, we get

$$\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \frac{1}{s}\,|(W_\psi f)(s,t)|^2\,dtds \le \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} 4\left(\frac{1}{s}+\frac{a^{3/2}}{s^2}\right)|(W_\psi f)(s,t)|^2\,dtds. \tag{2.2}$$

By the triangle inequality, we have

$$\begin{aligned}
&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \frac{1}{s}\,|(W_\psi f)(s,t_n)|^2\,dtds \\
\le\ & M_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\Big(|(W_\psi f)(s,t)|^2+|(W_\psi f')(s,t)|^2\Big)dtds, \qquad (2.3)
\end{aligned}$$

where $M_{a,b}$ is a constants. Similarly we can prove that

$$\begin{aligned}
&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \left|s^{-1/2}(W_\psi f)(s,t_n)-s_n^{-1/2}(W_\psi f)(s_n,t_n)\right|^2 dtds \\
\le\ & \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{4s_n^2(a-1)^2}{\pi^2 a} \iint_{E_n} \frac{1}{s^3}|(W_{\tilde\psi} f)(s,t_n)|^2 dtds \\
\le\ & \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{4(a-1)^2}{\pi^2} \iint_{E_n} \frac{1}{s}|(W_{\tilde\psi} f)(s,t_n)|^2 dtds \\
\le\ & M'_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\Big(|(W_{\tilde\psi} f)(s,t)|^2+|(W_{\tilde\psi} f')(s,t)|^2\Big)dtds. \qquad (2.4)
\end{aligned}$$

where (2.3) is used in the last step. Putting (2.3) and (2.4) together, we get

$$\begin{aligned}
&\sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \frac{(a-1)b}{a^{1/2}}|(W_\psi f)(s_n,t_n)|^2 = \iint_{E_n} \left|s_n^{-1/2}(W_\psi f)(s_n,t_n)\right|^2 dtds \\
=\ & \sum_{(s_n,t_n)\notin Q_{p,q}(1,0)} \iint_{E_n} \left|s^{-1/2}(W_\psi f)(s,t_n)-\Big(s^{-1/2}(W_\psi f)(s,t_n)-s_n^{-1/2}(W_\psi f)(s_n,t_n)\Big)\right|^2 dtds \\
\le\ & M''_{a,b} \iint_{(s,t)\notin Q_{p/a^3,q-b}(1,0)} \left(\frac{1}{s}+\frac{1}{s^2}\right)\Big(|(W_{\tilde\psi} f)(s,t)|^2+|(W_{\tilde\psi} f')(s,t)|^2 \\
&\qquad + |(W_\psi f)(s,t)|^2+|(W_\psi f')(s,t)|^2\Big)dtds
\end{aligned}$$

□

Proof of Theorem 1.2. Put $\Gamma=\bigcup_{\ell=1}^r \Gamma_\ell$. Let A be the lower frame bound.

By [13, Theorem 3.2], Γ_ℓ is relatively uniformly discrete. Hence we can split Γ_ℓ into N_ℓ uniformly discrete sequences $\Gamma_{\ell,k}$. Therefore, we can find some $a_0>1, b_0>0$ such that $\Gamma_{\ell,k}$ is (a_0,b_0)-uniformly discrete, $1\le \ell\le r, 1\le k\le N_\ell$.

Take some $f \in L^2(\mathbb{R}) \bigcap C_c^\infty(\mathbb{R})$ such that $f \neq 0$ and f, f' are admissible. Since $(W_\psi f)(s,t) = \langle s^{1/2} f(s\cdot +t), \psi\rangle$, we have $(W_\psi f)(s,t) = \overline{(W_f\psi)}(\frac{1}{s}, -\frac{t}{s})$. It follows that

$$\begin{aligned}
\int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{s}|(W_\psi f)(s,t)|^2 dtds &= \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{s}\left|(W_f\psi)\left(\frac{1}{s}, -\frac{t}{s}\right)\right|^2 dtds \\
&= \int_0^{+\infty}\int_{-\infty}^{+\infty} \left|(W_f\psi)\left(\frac{1}{s}, t\right)\right|^2 dtds \\
&= \int_0^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{s^2}\left|(W_f\psi)(s,t)\right|^2 dtds \\
&< \infty.
\end{aligned}$$

Similarly, we can prove that $\frac{1}{s}|(W_\psi f')(s,t)|^2$, $\frac{1}{s}|(W_{\tilde\psi} f)(s,t)|^2$, and $\frac{1}{s}|(W_{\tilde\psi} f')(s,t)|^2$ are integrable on $\mathcal{G}$. By Lemma 2.3, there is some $p > 1$ and $q > 0$ such that for any (a_0, b_0)-uniformly discrete sequence $\Gamma' \subset \mathcal{G}$,

$$\sum_{(s,t)\in\Gamma'\setminus Q_{p,q}(1,0)} |(W_\psi f)(s,t)|^2 \le \frac{A}{2r(N_1+\cdots+N_r)}\|f\|_2^2.$$

On the other hand, it is easy to check that $\{(s/x, (t-y)/x) : (s,t) \in \Gamma_{\ell,k}\}$ is (a_0, b_0)-uniformly discrete for any $(x,y) \in \mathcal{G}$ since $\Gamma_{\ell,k}$ is, $1 \le \ell \le r, 1 \le k \le N_\ell$. Furthermore, $(s/x, (t-y)/x) \notin Q_{p,q}(1,0)$ whenever $(s,t) \notin Q_{p,q}(x,y)$. Hence

$$\begin{aligned}
A\|f\|_2^2 &= A\left\|x^{-\frac{1}{2}} f\left(\frac{\cdot - y}{x}\right)\right\|^2 \\
&\le \sum_{\ell=1}^{r}\sum_{(s,t)\in\Gamma_\ell}\left|\left\langle x^{-1/2} f\Big(\frac{\cdot-y}{x}\Big),\, s^{-1/2}\psi_\ell\Big(\frac{\cdot - t}{s}\Big)\right\rangle\right|^2 \\
&= \sum_{\ell=1}^{r}\sum_{(s,t)\in\Gamma_\ell}\left|\left\langle f,\, \Big(\frac{s}{x}\Big)^{-1/2}\psi_\ell\Big(\frac{\cdot - \frac{t-y}{x}}{\frac{s}{x}}\Big)\right\rangle\right|^2 \\
&= \sum_{\ell=1}^{r}\sum_{k=1}^{N_\ell}\sum_{(s,t)\in\Gamma_{\ell,k}\setminus Q_{p,q}(x,y)}\left|(W_{\psi_\ell} f)\left(\frac{s}{x}, \frac{t-y}{x}\right)\right|^2 \\
&\qquad + \sum_{\ell=1}^{r}\sum_{k=1}^{N_\ell}\sum_{(s,t)\in\Gamma_{\ell,k}\cap Q_{p,q}(x,y)}\left|(W_{\psi_\ell} f)\left(\frac{s}{x}, \frac{t-y}{x}\right)\right|^2 \\
&\le \frac{A}{2}\|f\|_2^2 + \sum_{\ell=1}^{r}\sum_{k=1}^{N_\ell}\sum_{(s,t)\in\Gamma_{\ell,k}\cap Q_{p,q}(x,y)}\left|(W_{\psi_\ell} f)\left(\frac{s}{x}, \frac{t-y}{x}\right)\right|^2.
\end{aligned}$$

Therefore,

$$\Gamma \bigcap Q_{p,q}(x,y) = \bigcup_{\ell=1}^{r}\bigcup_{k=1}^{N_\ell}\Big(\Gamma_{\ell,k}\bigcap Q_{p,q}(x,y)\Big) \neq \emptyset, \qquad \forall (x,y) \in \mathcal{G}.$$

For any $a > p, b > a^{-1/2}p^{1/2}q$ and $(x, y) \in \mathcal{G}$, we have

$$Q_{p,q}\left(a^{-1/2}xp^{j+1/2}, y - \frac{bx}{2} + (k + \frac{1}{2})qa^{-1/2}xp^{j+1/2}\right) \subset Q_{a,b}(x, y),$$
$$0 \le j \le \lfloor \frac{\ln a}{\ln p} \rfloor - 1, \quad 0 \le k \le \frac{b}{qa^{-1/2}p^{j+1/2}} - 1.$$

Hence

$$\begin{aligned} \nu\left(\Gamma \bigcap Q_{a,b}(x, y)\right) &\ge \sum_{j=0}^{\lfloor \ln a/\ln p\rfloor - 1} \left(\frac{b}{qa^{-1/2}p^{j+1/2}} - 1\right) \cdot a^{-1/2}xp^j \\ &\ge \frac{bx}{p^{1/2}q}\left(\frac{\ln a}{\ln p} - 1\right) - a^{-1/2}x \cdot \frac{a-1}{p-1}. \end{aligned}$$

Therefore,

$$D^-(\Gamma) = \underline{\lim}_{a\to+\infty}\, \underline{\lim}_{b\to+\infty} \inf_{(x,y)\in\mathcal{G}} \frac{\nu(\Gamma \bigcap Q_{a,b}(x, y))}{bx \ln a} \ge \frac{1}{p^{1/2}q \ln p}.$$

□

The following lemma can be proved similarly to Lemma 2.1.

Lemma 2.4 *Let* $\{(s_j, t_{j,k}) : j \in \Lambda, k \in \Lambda_j\}$ *and* $\{(s_j, 0) : j \in \Lambda\}$ *be* (a, b)*-uniformly discrete sequence. Suppose that* $h(s, t) \ge 0$ *is a function defined on* $\mathcal{G}$*. Then for any* $p > a$ *and* $q > b$*,*

$$\sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)} \iint_{F_{j,k}} h(s, t)dtds \le \iint_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)} 2\left(1 + \frac{a^{1/2}}{s}\right) h(s, t)dtds,$$

where $F_{j,k} = [s_ja^{-1/2}, s_ja^{1/2}] \times [\frac{t_{j,k}}{s_j} - \frac{b}{2s_j}, \frac{t_{j,k}}{s_j} + \frac{b}{2s_j}]$.

Lemma 2.5 *Let* $\{(s_j, t_{j,k}) : j \in \Lambda, k \in \Lambda_j\}$ *and* $\{(s_j, 0) : j \in \Lambda\}$ *be* (a, b)*-uniformly discrete sequence. Suppose that* $\psi \in L^2(\mathbb{R})$*,* $f \in C_c^\infty(\mathbb{R})$ *and* $f, f', \tilde{f}$*, and* $\tilde{f}'$ *are admissible, where* $\tilde{f}(x) = \frac{1}{2}f(x) + xf'(x)$*. Then we have*

$$\begin{aligned} \sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)} |(W_\psi f)(s_j, t_{j,k})|^2 \le C'_{a,b} \iint_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)} \left(1 + \frac{1}{s}\right) \Big(|(W_\psi f)(s, st)|^2 \\ + |(W_\psi f')(s, st)|^2 + |(W_\psi \tilde{f})(s, st)|^2 |(W_\psi \tilde{f}')(s, st)|^2\Big) dtds, \end{aligned}$$

where $C'_{a,b}$ *is a constant.*

Proof. Since $(W_\psi f)(s, st) = \langle s^{1/2}f(s(\cdot + t)), \psi\rangle$, we have

$$\begin{aligned} \frac{\partial}{\partial t}(W_\psi f)(s, st) &= s(W_\psi f')(s, st), \\ \frac{\partial}{\partial s}(W_\psi f)(s, st) &= \frac{1}{s}(W_\psi \tilde{f})(s, st). \end{aligned}$$

Similar to the proof of Lemma 2.3, we can prove that

$$\sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)} \iint_{F_{j,k}} |(W_\psi f)(s,st)-(W_\psi f)(s,st_{j,k}/s_j)|^2 dtds$$
$$\leq \frac{4ab^2}{\pi^2} \iint_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)} 2\left(1+\frac{a^{1/2}}{s}\right)|(W_\psi f')(s,st)|^2 dtds$$

and

$$\sum_{(s_j,t_{j,k})\notin Q_{p,q}(1,0)} \iint_{F_{j,k}} |(W_\psi f)(s,st_{j,k}/s_j)-(W_\psi f)(s_j,t_{j,k})|^2 dtds$$
$$\leq M'''_{a,b} \iint_{(s,t)\notin Q_{p/a,p^{-1/2}(q-b)}(1,0)} 2\left(1+\frac{a^{1/2}}{s}\right)\Big(|(W_\psi \tilde{f})(s,st)|^2+|(W_\psi \tilde{f}')(s,st)|^2\Big) dtds.$$

Now the conclusion follows by the triangle inequality and Lemma 2.4. □

Proof of Theorem 1.1. By [14, Theorem 2.1], $S_\ell \times \{0\}$ is relatively uniformly discrete, $1 \leq \ell \leq r$. Using Lemma 2.5 instead of Lemma 2.3, the conclusion can be proved similarly to Theorem 1.2, which we leave to interested readers. □

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