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# N–Black Hole Stationary and Axially Symmetric Solutions of the Einstein–Maxwell Equations

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# $N$-BLACK HOLE STATIONARY AND AXIALLY SYMMETRIC SOLUTIONS OF THE EINSTEIN-MAXWELL EQUATIONS

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ABSTRACT. It is well-known that the Einstein-Maxwell equations reduce in the stationary and axially symmetric case to an axially symmetric harmonic map with prescribed singularities $\varphi\colon \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$, where $\Sigma$ is a subset of the axis of symmetry, and $\mathbb{H}^2_{\mathbb{C}}$ is the complex hyperbolic plane. Motivated by this problem, we prove the existence and uniqueness of harmonic maps with prescribed singularities $\varphi\colon \mathbb{R}^n \setminus \Sigma \to \mathbb{H}$, where $\Sigma$ is a submanifold of $\mathbb{R}^n$ of co-dimension $\geqslant 2$, and $\mathbb{H}$ is a classical Riemannian globally symmetric space of noncompact type and rank one. This result, when applied to the black hole problem yields solutions of the reduced equations which can be interpreted as equilibrium configurations of multiple co-axially rotating charged black holes held apart by singular struts.

## 1. INTRODUCTION

Let $(M, g)$ be a four-dimensional Lorentzian manifold, and let $F$ be a two-form on $M$. Consider the Einstein-Maxwell field equations:

$$\operatorname{Ric}_g - \frac{1}{2} R_g\, g = 2\, T_F \tag{1.1}$$

$$F = dA \tag{1.2}$$

$$d{*}F = 0, \tag{1.3}$$

where $\operatorname{Ric}_g$ is the Ricci curvature tensor of the metric $g$, $R_g$ its scalar curvature, and $T_F$ is the energy-momentum-stress tensor of the electromagnetic

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field $F$:

$$\begin{aligned} T_F(X,Y) &= \frac{1}{2}(i_X F\cdot i_Y F + i_X{*}F\cdot i_Y{*}F) \\ &= i_X F\cdot i_Y F - \frac{1}{2}\left|F\right|^2 g(X,Y), \end{aligned} \tag{1.4}$$

Here, $i_X$ denotes inner multiplication by the vector $X$, $*$ is the Hodge star operator mapping $\wedge^k M$ to $\wedge^{4-k}M$, $\sigma\cdot\tau$ denotes the inner product, and $|\sigma|^2$ the norm of $k$-forms. These operators are given in local coordinates by:

$$*\,\sigma_{\mu_1\dots\mu_{4-k}} = \frac{1}{k!}\sigma^{\nu_1\dots\nu_k}\epsilon_{\nu_1\dots\nu_k\mu_1\dots\mu_{4-k}}, \tag{1.5}$$

$$\sigma\cdot\tau = -*(\sigma\wedge *\tau) = \frac{1}{k!}\sigma^{\mu_1\dots\mu_k}\tau_{\mu_1\dots\mu_k}, \tag{1.6}$$

$$|\sigma|^2 = \sigma\cdot\sigma = \frac{1}{k!}\sigma^{\mu_1\dots\mu_k}\sigma_{\mu_1\dots\mu_k}, \tag{1.7}$$

where $\epsilon$ is the volume form of $g$, and according to the Einstein summation convention, repeated indices are summed over. Note that $\operatorname{tr} T_F = 0$, hence taking the trace of Equations (1.1) yields $R_g = 0$. Thus, Equation (1.1) can be rewritten as

$$\operatorname{Ric}_g = 2\,T_F. \tag{1.8}$$

We are using rationalized units in which $4\pi G = 1$, where $G$ is the gravitational constant. We will seek asymptotically flat, stationary and axially symmetric solutions of Equations (1.1)–(1.4).

It is well-known that these equations reduce, using an idea originally due to Ernst [Er], to an axially symmetric harmonic map $\varphi\colon \mathbb{R}^3\backslash\Sigma \to \mathbb{H}^2_{\mathbb{C}}$, where $\Sigma = \mathbb{A}\setminus\bigcup_{j=1}^N I_j$ consists of the axis of symmetry $\mathbb{A}\subset\mathbb{R}^3$ with $N$ closed intervals $I_j$ removed, and $\mathbb{H}^2_{\mathbb{C}} = SU(1,2)/S(U(1)\times U(2))$ is the complex hyperbolic plane, see [Mz, C2]. Each interval $I_j$ corresponds in $(M,g)$ to a connected component of the event horizon, thus $N$ will throughout denote the number of black holes. The analytic interest of this problems lies in the fact that the boundary conditions for $\varphi$ on $\Sigma$ and as $r\to\infty$ are singular. For the case $N=1$, there is a family of solutions known in closed form, the Kerr-Newman black holes, see [C1]. We note that the problem has $4N-1$ parameters $\{m_j, d_j, L_j, q_j\}$: $N$ masses $m_j$, $N-1$ distances $d_j$, $N$ angular momenta $L_j$, and $N$ electric charges $q_j$.

Special cases of the Ernst reduction have been used to prove non-existence results for the Einstein equations. In early work along this line, Weyl investigated the vacuum static case and showed that the equations reduce to a single linear equation. The case $N=1$ was known to have the Schwarzschild solution. Superimposing two such solutions, Weyl obtained new solutions which could be interpreted as equilibrium configurations of a pair of black holes. However, with Bach [BW] in 1921, they showed that an obstruction arose as a conical singularity along the axis separating the two black holes. Having interpreted this singularity as the gravitational force, they computed its value and verified that the result was asymptotic to the Newtonian gravitational force in the appropriate limit. The reduction was also used by Robinson [R], following work of Carter [C1], to prove that within the $N = 1$ vacuum case, the Kerr solutions were unique. Mazur [Mz], and Bunting [C2] independently, generalized this uniqueness result to the charged $N = 1$ case.

In [We1, We2], we used the Ernst reduction to construct vacuum solutions which could be viewed as nonlinear generalizations of the Weyl solutions. Clearly, it is of great interest to find out whether the obstruction found by Weyl in the non-rotating case can be removed by some choice of the parameters. It has been conjectured for some time that this is impossible, see [Pe, Problem 14], but the evidence is limited. Indeed, in this case, the angle deficiencies most likely cannot be computed exactly. We mention [LTi2], where the small angular momentum case is treated, and [We3], where nonexistence is proved in the extreme regime. Also, in [LTi1], Li and Tian proved non-existence in the case where the solutions admits an involutive symmetry.

In this paper, we begin the generalization of this work to the Einstein-Maxwell equations. Our main motivation is eventually the study of the force. With the Maxwell field, it is expected to be different than with vacuum, since it is already known that in the static case equilibrium can be achieved with masses and charges equal, see [Mj, Pa]. However it is important to note that this configuration is extreme, see Section 2.3. Here, we prove the existence of a unique $(4N - 1)$-parameter family of solutions to the reduced problem. Our main tool is the study of harmonic maps with prescribed singularities into classical globally symmetric spaces of noncompact and type rank one, see [We4]. These are the real-, complex-, and quaternion-hyperbolic spaces, see [Mo]. The results obtained in [We4] however do not apply directly to the problem considered here. In that paper, we restricted our attention to bounded domains $\Omega \subset \mathbb{R}^n$, and to singular sets $\Sigma$ compactly contained in $\Omega$, while here $\Omega = \mathbb{R}^n$ and $\Sigma$ is unbounded. The necessary generalization is achieved in two steps. First, we allow the singular set $\Sigma$ to extend to $\partial\Omega$, and then we let $\Omega$ exhaust $\mathbb{R}^n$. In this second step, we require for the prescription of boundary data as $r \to \infty$ in $\mathbb{R}^n$, a given harmonic map which admits those singularities lying along the unbounded components of $\Sigma$. In the application to black holes, this map is given by the $N = 1$ case, i.e. the Kerr-Newman solutions.

In a forthcoming paper, we will study the regularity properties of these maps. This is necessary in order to complete the application of the results presented here to black holes.

## 2. The Ernst Reduction for the Einstein-Maxwell Equations

In this section, we recall the Ernst reduction for the Einstein-Maxwell equations in the stationary and axially symmetric case. The main result is Theorem 1 which allows us to state the Reduced Problem in terms of harmonic maps with prescribed singularities. The computations are carried out mostly with the exterior algebra formalism which is particularly well suited to treat the Maxwell Equations. We put particular emphasis on those points where the reduction differs significantly from the vacuum case, see [We1].

**2.1. Axial Symmetry.** We first describe the connection between the axially symmetric Einstein-Maxwell Equations and harmonic maps into $\mathbb{H}^2_{\mathbb{C}}$. It is important to note that the arguments could equally well apply to any non-null Killing field.

**Definition 1.** Let $(M, g)$ be an oriented four-dimensional Lorentzian manifold, and let $F$ be a two-form on $M$. We say that $(M, g, F)$ is *axially symmetric* if $SO(2)$ acts effectively on $(M, g)$ as a group of isometries leaving $F$ invariant.

Let $(M, g, F)$ be a simply connected axially symmetric solution of the Einstein-Maxwell Equations. Let $\xi$ be the Killing field generator, then $\mathcal{L}_\xi F = 0$. Note that if $(M, g)$ is causal, i.e. admits no closed causal curves, then $\xi$ either is spacelike, or vanishes. Define the one-forms $\alpha = i_\xi F$, and $\beta = i_\xi {*F}$. Then, we find $d\alpha = -i_\xi dF + \mathcal{L}_\xi F = 0$. Thus, there is a function $\chi$ such that $d\chi = \alpha$. Similarly $d\beta = 0$, hence there is a function $\psi$ such that $d\psi = \beta$. Note that $\chi$ and $\psi$ are determined only up to constants. Now, define the one-form $\gamma = \chi\, d\psi - \psi\, d\chi$, and observe that $d\gamma = 2\alpha \wedge \beta$.

It is easy to see from (1.5) that for any $k$-form $\sigma$ and one-form $\theta$, we have:

$$i_\theta {*\sigma} = *(\sigma \wedge \theta),$$

where we have used the metric $g$ to identify the tangent space $T_pM$ and its dual $T_p^*M$. Using $[\mathcal{L}_\xi, *] = 0$, and $** = (-1)^{k+1}$ on $k$-forms, it follows that

$$\delta(\sigma \wedge \xi) = (-1)^{k+1} \mathcal{L}_\xi \sigma + \delta\sigma \wedge \xi,$$

where $\delta = *d*$ is the divergence operator on forms. Thus, if we define the *twist* of $\xi$ by $\omega = *(d\xi \wedge \xi)$, we find

$$*d\omega = \delta(d\xi \wedge \xi) = -\mathcal{L}_\xi d\xi + \delta d\xi \wedge \xi = \delta d\xi \wedge \xi.$$

Now for any Killing field $\xi$, we have $\delta\xi = 0$, and

$$\delta d\xi = -2 \operatorname{tr} \nabla^2 \xi = 2\, i_\xi \operatorname{Ric}_g . \tag{2.1}$$

Hence in view of Equation (1.8), we obtain

$$*d\omega = 4\, i_\xi T_F \wedge \xi.$$

Furthermore, in view of (1.4), $i_\xi T_F = -i_\alpha F - (1/2)\,|F|^2\, \xi$. Thus, we have $*d\omega = 4\, \xi \wedge i_\alpha F$. On the other hand, $\beta = *(F \wedge \xi)$, hence

$$*(\alpha \wedge \beta) = -i_\alpha * \beta = \xi \wedge i_\alpha F.$$

It follows that

$$d\omega = 2\, d\gamma, \tag{2.2}$$

i.e. the one-form $\omega - 2\gamma$ is closed. We conclude that there is a function $v$, also determined up to a constant, such that $2\,dv = \omega - 2\gamma$.

Let $M' = \{x \in M;\ |\xi|^2 > 0\}$, and let $u = -\log|\xi|$ on $M'$. Then, we have $i_\xi d\xi = -d i_\xi \xi + \mathcal{L}_\xi \xi = 2e^{-2u} du$. Thus, since $\delta du = -\Delta u$, we have $\delta(e^{-2u} du) = 2e^{-2u}|du|^2 - e^{-2u}\Delta u$. On the other hand, $i_\xi d\xi = -*(*d\xi \wedge \xi)$, hence using Equations (2.1), (1.8), and (1.4), we obtain:

$$\delta(i_\xi d\xi) = -i_\xi \delta d\xi + |d\xi|^2 = -2(|\alpha|^2 + |\beta|^2) + |d\xi|^2 .$$

Furthermore, since $\omega = i_\xi * d\xi$, we find

$$|\omega|^2 = *(*d\xi \wedge i_\xi d\xi \wedge \xi) + *(*d\xi \wedge d\xi \wedge i_\xi \xi) = 4e^{-4u}|du|^2 - e^{-2u}|d\xi|^2$$

i.e. $|d\xi|^2 = 4\,e^{-2u}|du|^2 - e^{2u}|\omega|^2$. We conclude that

$$\Delta u - \frac{1}{2}e^{4u}|\omega|^2 - e^{2u}(|\alpha|^2 + |\beta|^2) = 0. \tag{2.3}$$

Now, $\delta\omega = *(d\xi \wedge d\xi)$, and

$$2\,du\cdot\omega = e^{2u} * (d\xi \wedge i_\xi d\xi \wedge \xi) + e^{2u} * (d\xi \wedge d\xi \wedge i_\xi \xi) = -2\,du\cdot\omega + *(d\xi \wedge d\xi),$$

i.e. $4\,du\cdot\omega = *(d\xi \wedge d\xi)$. Thus, we find that $\delta(e^{4u}\omega) = -4e^{4u}du\cdot\omega + e^{4u}\delta\omega = 0$, which we write as

$$\operatorname{div}(e^{4u}\omega) = 0, \tag{2.4}$$

where we have put $\operatorname{div}\sigma = -\delta\sigma$ for any one-form $\sigma$. In addition, since $\alpha = -*(*F \wedge \xi)$, we find $\delta\alpha = -*d(*F \wedge \xi) = d\xi \cdot F$, and

$$2\,du\cdot\alpha = -e^{2u} * (d\xi \wedge i_\xi * F \wedge \xi) - e^{2u} * (d\xi \wedge *F \wedge i_\xi \xi) = e^{2u}\beta\cdot\omega + d\xi\cdot F.$$

It follows that $\delta(e^{2u}\alpha) = -2e^{2u}du\cdot\alpha + e^{2u}\delta\alpha = -e^{4u}\beta\cdot\omega$, i.e.

$$\operatorname{div}(e^{2u}\alpha) - e^{4u}\beta\cdot\omega = 0. \tag{2.5}$$

Similarly, $\delta(e^{2u}\beta) = e^{4u}\alpha\cdot\omega$, i.e.

$$\operatorname{div}(e^{2u}\beta) + e^{4u}\alpha\cdot\omega = 0. \tag{2.6}$$

Substituting the definitions of $v$, $\chi$ and $\psi$ into Equations (2.3)–(2.6), it follows that $\varphi = (u, v, \chi, \psi)$ satisfies the following system of equations:

$$\Delta u - 2e^{4u}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 - e^{2u}(|\nabla\chi|^2 + |\nabla\psi|^2) = 0 \tag{2.7}$$

$$\operatorname{div}\left(e^{4u}(\nabla v + \chi\nabla\psi - \psi\nabla\chi)\right) = 0 \tag{2.8}$$

$$\operatorname{div}\left(e^{2u}\nabla\chi\right) - 2e^{4u}\nabla\psi\cdot(\nabla v + \chi\nabla\psi - \psi\nabla\chi) = 0 \tag{2.9}$$

$$\operatorname{div}\left(e^{2u}\nabla\psi\right) + 2e^{4u}\nabla\chi\cdot(\nabla v + \chi\nabla\psi - \psi\nabla\chi) = 0. \tag{2.10}$$

It is now clear that for every subset $\Omega \subset\subset M'$, $\varphi = (u, v, \chi, \psi)$ is a critical point of the functional

$$E_\Omega(\varphi) = \int_\Omega \left\{|\nabla u|^2 + e^{4u}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 + e^{2u}(|\nabla\chi|^2 + |\nabla\psi|^2)\right\} d\mu_g,$$

where $d\mu_g$ is the volume element of the metric $g$. Thus, if we choose an 'upper half-space model' for $\mathbb{H}^2_{\mathbb{C}}$, i.e. $\mathbb{R}^4$ with the metric given by the line element:

$$ds^2 = du^2 + e^{4u}(dv + \chi\, d\psi - \psi\, d\chi)^2 + e^{2u}(d\chi^2 + d\psi^2),$$

then the map $\varphi\colon (M', g) \to \mathbb{H}^2_{\mathbb{C}}$ is a harmonic map, see Section 3 and [We4].

**2.2. Stationary and Axially Symmetric Solutions.** We now turn to the case where $(M, g, F)$ admits an additional symmetry.

**Definition 2.** Let $(M, g)$ be an oriented and time-oriented four-dimensional Lorentzian manifold and let $F$ be a two-form on $M$. We say that $(M, g, F)$ is *stationary and axially symmetric* if $G = \mathbb{R} \times SO(2)$ acts effectively on $(M, g)$ as a group of isometries leaving $F$ invariant and such that each non-degenerate orbit is a timelike two-surface.

Here, we say that the orbit of a point $p \in M$ is *degenerate* if the isotropy subgroup at $p$ is non-trivial. The set of points whose orbit is degenerate will be called the *axis*. We let $\xi$ be the Killing field generator normalized so that its orbits are closed circles of length $2\pi\,|\xi|$, and we let $\tau$ be a linearly independent generator. Clearly, we have $[\xi, \tau] = 0$. As before, if $(M, g)$ is causal then $\xi$ is either spacelike or vanishes, i.e. $\xi$ is spacelike outside the axis.

We now prove that if $(M, g, F)$ is stationary and axially symmetric, satisfies the Einstein-Maxwell Equations, and has non-empty axis, then:

(i) $F$ and $*F$ vanish on the orbits, i.e. $F(\xi, \tau) = *F(\xi, \tau) = 0$;

(ii) the distribution of planes orthogonal to the orbits of $G$ is integrable.

To see (i), note that $[\mathcal{L}_\tau, i_\xi] = i_{[\tau,\xi]} = 0$, hence we have

$$di_\tau\alpha = -i_\tau d\alpha + \mathcal{L}_\tau i_\xi F = 0.$$

Since $i_\tau\alpha = 0$ on the axis it follows that $F(\xi, \tau) = i_\tau\alpha = 0$ everywhere. Similarly, $*F(\xi, \tau) = i_\tau\beta = 0$. To show (ii), it suffices by Frobenius Theorem to show that:

$$*\,(\xi \wedge \tau \wedge d\xi) = 0, \qquad *(\xi \wedge \tau \wedge d\tau) = 0. \tag{2.11}$$

However, in view of (2.2) and (i), we have

$$d * (\xi \wedge \tau \wedge d\xi) = di_\tau\omega = -i_\tau d\omega - \mathcal{L}_\tau\omega = 4i_\tau(\alpha \wedge \beta) = 0,$$

where $\omega$ is the twist of $\xi$. Thus, $*(\xi \wedge \tau \wedge d\xi)$ is constant, but since it vanishes on the axis, it is identically zero. Similarly, $*(\xi \wedge \tau \wedge d\tau) = 0$.

Let $(Q, h)$ be an integral surface of the distribution of planes orthogonal to the orbits with the induced metric $h$, then the quotient space $M/G$ with its quotient metric can be identified with $(Q, h)$. We choose the orientation on $Q$ so that $*(\xi \wedge \tau)$ is positively oriented. We now proceed to prove that the map $\varphi\colon M' \to \mathbb{H}^2_{\mathbb{C}}$ is invariant under $G$ hence reduces to a map, which we also denote by $\varphi$ on the quotient $Q' = Q \cap M'$. Let $\zeta$ be an arbitrary Killing field generator in the Lie algebra of $G$. We have $\zeta e^{-2u} = 2g([\zeta, \xi], \xi) = 0$, and hence $\zeta u = 0$. Also, $\zeta \chi = i_\zeta \alpha = F(\xi, \zeta) = 0$, and similarly $\zeta \psi = 0$. Finally, $\gamma(\zeta) = \chi \beta(\zeta) - \psi \alpha(\zeta) = 0$, hence $2\,\zeta v = i_\zeta(\omega - 2\gamma) = *(d\xi \wedge \xi \wedge \zeta) - 2\gamma(\zeta) = 0$.

Define $\sigma = \xi \wedge \tau$, then on $M'$ we have $|\sigma|^2 = |\xi|^2 |\tau|^2 - (\xi \cdot \tau)^2 < 0$. Let $\rho^2 = -|\sigma|^2$, then $\rho$ is invariant under $G$, hence reduces to a function on $Q'$. Therefore, it follows that for every subset $\Omega \subset\subset Q'$, the map $\varphi\colon (Q', h) \to \mathbb{H}^2_{\mathbb{C}}$ is a critical point of the functional:

$$E'_\Omega(\varphi) = \int_\Omega \Big\{ |\nabla u|_h^2 + e^{4u} \,|\nabla v + \chi \nabla \psi - \psi \nabla \chi|_h^2 \\ + e^{2u} \left( |\nabla \chi|_h^2 + |\nabla \psi|_h^2 \right) \Big\} \rho \, d\mu_h,$$

where $|\cdot|_h$ is the norm with respect to the metric $h$, and $d\mu_h$ is the volume element of the metric $h$. Indeed, if $\Omega \subset Q'$ is such a subset, and $f$ is a function on $M'$ invariant under $G$, then $2\pi \int_\Omega f \rho \, d\mu_h = \int_{G \cdot \Omega} f \, d\mu_g$, where $G \cdot \Omega$ is the orbit of $\Omega$ under $G$.

Next, we show that $\Delta_h \rho = 0$. Since $\Delta_g \rho = \rho^{-1} \operatorname{div}_h(\rho \nabla \rho)$, it suffices to prove that $\Delta_g \rho = \rho^{-1} |d\rho|^2$. To see this, note first that $\rho^2 = -\sigma(\xi, \tau) = i_\xi i_\tau \sigma$. Thus, we obtain:

$$2\rho \, d\rho = d i_\xi i_\tau \sigma = i_\xi i_\tau d\sigma = - * (*d\sigma \wedge \sigma). \tag{2.12}$$

Therefore, we see that

$$2\,\rho\, \Delta_g \rho + 2\,|d\rho|^2 = -\delta(2\,\rho\, d\rho) = *d(*d\sigma \wedge \sigma) = \sigma \cdot \delta d\sigma - |d\sigma|^2,$$

or equivalently:

$$\Delta_g \rho = \frac{1}{2\rho}(\sigma \cdot \delta d\sigma - |d\sigma|^2 - 2\,|d\rho|^2). \tag{2.13}$$

On the other hand, since $d\sigma = d\xi \wedge \tau - \xi \wedge d\tau$, we have, in view of Equation (2.11), that $i_\xi * d\sigma = *(d\sigma \wedge \xi) = 0$, and similarly $i_\tau * d\sigma = 0$. We deduce, using (2.12), that

$$4\rho^2\,|d\rho|^2 = *(i_\tau d\sigma \wedge *d\sigma \wedge i_\xi \sigma) = - * (d\sigma \wedge *d\sigma \wedge i_\tau i_\xi \sigma) = -\rho^2\,|d\sigma|^2,$$

i.e.

$$|d\sigma|^2 = -4\,|d\rho|^2 \tag{2.14}$$

Furthermore, we claim that $\sigma \cdot \delta d\sigma = 0$. Indeed, in view of Equation (2.1):

$$\delta d\sigma = *d(i_\tau * d\xi - i_\xi * d\tau) = \delta d\xi \wedge \tau + \xi \wedge \delta d\tau = 2\,(i_\xi \operatorname{Ric}_g \wedge \tau + \xi \wedge i_\tau \operatorname{Ric}_g),$$

and consequently:

$$\sigma \cdot \delta d\sigma = i_\tau i_\xi \delta d\sigma = 2\,|\tau|^2 \operatorname{Ric}_g(\xi, \xi) - 4\,(\xi \cdot \tau) \operatorname{Ric}_g(\xi, \tau) + 2\,|\xi|^2 \operatorname{Ric}_g(\tau, \tau).$$

Introducing $\tilde{\alpha} = i_\tau F$, and $\tilde{\beta} = i_\tau * F$, we can use Equations (1.8) and (1.4) to write

$$\operatorname{Ric}_g(\xi, \xi) = |\alpha|^2 + |\beta|^2$$

$$\operatorname{Ric}_g(\xi, \tau) = \alpha \cdot \tilde{\alpha} + \beta \cdot \tilde{\beta}$$

$$\operatorname{Ric}_g(\tau, \tau) = |\tilde{\alpha}|^2 + |\tilde{\beta}|^2,$$

from which it follows that:
(2.15)

$$\sigma\cdot\delta d\sigma = 2\,|\tau|^2\,(|\alpha|^2+|\beta|^2) - 4\,(\xi\cdot\tau)(\alpha\cdot\tilde\alpha+\beta\cdot\tilde\beta) + 2\,|\xi|^2\,(|\tilde\alpha|^2+\left|\tilde\beta\right|^2).$$

In addition, we have:

$$\begin{aligned}|\xi|^2\,F &= \xi\wedge\alpha + i_\xi *\beta\\ |\xi|^2 *F &= \xi\wedge\beta - i_\xi*\alpha,\end{aligned}$$

and hence:

$$|\xi|^2\,\tilde\alpha - (\xi\cdot\tau)\,\alpha = i_\tau i_\xi*\beta \tag{2.16}$$

$$|\xi|^2\,\tilde\beta - (\xi\cdot\tau)\,\beta = -i_\tau i_\xi*\alpha. \tag{2.17}$$

A computation similar to the one leading to (2.14) yields

$$|i_\tau i_\xi*\alpha|^2 = \rho^2\,|\alpha|^2\,,\qquad |i_\tau i_\xi*\beta|^2 = \rho^2\,|\beta|^2\,.$$

Thus, taking the norm squared of both sides of (2.16) and (2.17), and adding the results, we obtain:

$$|\xi|^4\,(|\tilde\alpha|^2+|\tilde\beta|^2) - 2\,|\xi|^2\,(\xi\cdot\tau)(\alpha\cdot\tilde\alpha+\beta\cdot\tilde\beta) + |\xi|^2\,|\tau|^2\,(|\alpha|^2+|\beta|^2) = 0,$$

which, in view of (2.15), implies $\sigma\cdot\delta d\sigma = 0$ as claimed. Substituting this result back into (2.13), and taking into account (2.14), we obtain $\Delta_g\rho = \rho^{-1}\,|d\rho|^2$ as required.

Therefore, if $d\rho\neq 0$ in $Q'$, the function $\rho$ can be used as an isothermal coordinate for the metric $h$ on $Q'$. Let $z$ be a conjugate isothermal coordinate, i.e. a function on $Q'$ such that $|dz|^2 = |d\rho|^2$, $dz\cdot d\rho = 0$, and $d\rho\wedge dz$ is positively oriented. Then the metric $h$ takes in the $(\rho,z)$-coordinate system the form:

$$h_{ab}dx^a dx^b = e^{2\lambda}(d\rho^2+dz^2),$$

where $\lambda = -\log|d\rho|$. Let $\tilde h = e^{-2\lambda}h$, then $\tilde h$ is a flat metric on $Q'$, and since the functional $E'_\Omega$ is conformally invariant, we have that $\varphi$ is a critical point of:

$$\begin{aligned}\tilde E_\Omega(\varphi) = \int_\Omega \Big\{&|\nabla u|^2_{\tilde h} + e^{4u}\,|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2_{\tilde h}\\ &+ e^{2u}\,(|\nabla\chi|^2_{\tilde h}+|\nabla\psi|^2_{\tilde h})\Big\}\rho\,d\mu_{\tilde h}.\end{aligned}$$

This gives a semilinear elliptic system of partial differential equations for the four unknowns $(u,v,\chi,\psi)$.

We have obtained that the metric $g$ must be of the form given by the line element
(2.18)

$$ds^2 = -\rho^2 e^{2u}\,dt^2 + e^{-2u}\,(d\phi - w\,dt)^2 + e^{2\lambda}(d\rho^2+dz^2),$$

where $w = -e^{2u}(\xi\cdot\tau)$. All the metric coefficients can be determined from the map $\varphi$. Indeed, $u$ is obtained directly from $\varphi$, and we will now obtain equations for the gradient of $w$ and $\lambda$. These quadratures are to be integrated after the harmonic map system has been solved.

Define $\eta = \tau + w\xi$, and observe that $\eta \cdot \xi = 0$, while $|\eta|^2 = \eta \cdot \tau = -e^{2u}\rho^2$. Furthermore, we have $dw = 2\,w\,du + e^{2u} i_\tau d\xi = e^{2u} i_\eta d\xi$. Since $d\xi = 2\,\xi \wedge du - e^{2u} * (\xi \wedge \omega)$, and $\xi \wedge \eta = \sigma$, we find $i_\eta d\xi = e^{2u} * (\omega \wedge \sigma)$, and hence

$$i_\tau i_\xi * dw = -\rho^2 e^{4u} \omega \tag{2.19}$$

The operator $\rho^{-1} i_\tau i_\xi *$ restricted to forms tangential to $Q$ is the Hodge star operator of the metric $h$. Since in two dimensions, the star operator restricted to one-forms is conformally invariant, $\rho^{-1} i_\tau i_\xi *$ is also the Hodge star operator $\star$ of the flat metric $\tilde{h}$ restricted to one-forms. Now, on one-forms in two dimensions $\star\star = -1$, hence we can rewrite Equation (2.19) as:

$$dw = e^{4u} \rho \star \omega. \tag{2.20}$$

To derive the equations for $d\lambda$, we must now use of the Einstein-Maxwell Equations in the $\rho z$-plane. Thanks to the fact that the metric in the $\rho z$-plane is conformally flat, these equations yield:

$$\begin{gathered}
\mathrm{Ric}_g(\partial_\rho, \partial_\rho) - \mathrm{Ric}_g(\partial_z, \partial_z) = 2\left(i_{\partial_\rho} F \cdot i_{\partial_\rho} F - i_{\partial_z} F \cdot i_{\partial_z} F\right) \\
\mathrm{Ric}_g(\partial_\rho, \partial_z) = 2\, i_{\partial_\rho} F \cdot i_{\partial_z} F.
\end{gathered}$$

A straightforward coordinate computation now leads to:

$$\begin{aligned}
\lambda_\rho - u_\rho &= \rho\left[u_\rho^2 - u_z^2 + \frac{1}{4} e^{4u}(\omega_\rho^2 - \omega_z^2) + e^{2u}(\chi_\rho^2 - \chi_z^2 + \psi_\rho^2 - \psi_z^2)\right] \\
\lambda_z - u_z &= 2\rho\left[u_\rho u_z + \frac{1}{4} e^{4u} \omega_\rho\, \omega_z + e^{2u}(\chi_\rho \chi_z + \psi_\rho \psi_z)\right].
\end{aligned} \tag{2.21}$$

Recall the following important definitions.

**Definition 3.** A spacetime is *globally hyperbolic* if it admits a Cauchy hypersurface. A spacetime is *asymptotically flat* if it is the Cauchy development of asymptotically flat data. A *domain of future outer communications* in an asymptotically flat spacetime $(M, g)$ is a maximal connected open set $O \subset M$ such that from each point $p \in O$ there are future directed timelike curves to asymptotically flat regions of $(M, g)$. A *future event horizon* is the boundary of a domain of future outer communications.

If $(\widetilde{M}, g)$ is asymptotically flat and globally hyperbolic, so is any domain of future outer communications $M$ in $(\widetilde{M}, g)$, since $M$ is then a past set. We will restrict our attention to such a domain. We may then assert that $(M, g)$ is causal. We assume that $M$ is simply connected. If in addition, $(M, g, F)$ is a solution of the Einstein-Maxwell Equations which is stationary and axially symmetric, then there is a unique Killing field generator $\tau$ such that $\tau$ is future directed and $|\tau|^2 \to -1$ at spacelike infinity. Defining now

$\xi$ and $\rho$ as before, it can be shown, using $\Delta_h \rho = 0$, that $\rho$ has no critical points, and hence can be used as an isothermal coordinate on $Q'$ as we have done above, see [We1, Section 6]. Furthermore, the future event horizon $H \subset \widetilde{M}$ can be characterized by the conditions $|\xi|^2 > 0$ and $\rho = 0$.

If $|d\sigma|^2$ vanishes identically on some component $H_0$ of $H$, then we say that $H_0$ is *degenerate*. If a component $H_0$ of $H$ is non-degenerate, then by (2.14), we have $|d\rho|^2 = |dz|^2 \not\equiv 0$ on $H_0$. We point out that $d\rho$ is null on $H$ since $H$ is a null surface, and therefore if a component $H_0$ of $H$ is non-degenerate, then $d\rho$ is actually singular on $H_0$. We will see in Section 2.3 that each component $H_0$ of $H$ is a *Killing horizon* in the sense that the null generators of $H_0$ are tangent to a Killing vector $\eta_0$ which is non-zero on $H_0$, see [C1, Ch]. Furthermore, $e^{2u}\,|d\rho|^2$ tends on $H_0$ to a constant $\kappa_0$, the *surface gravity* of $H_0$, defined by the equation $d(|\eta_0|^2) = -2\kappa_0\eta_0$ which holds on $H_0$. Thus, $H_0$ is degenerate if and only if $\kappa_0 = 0$ and the definition of a degenerate horizon adopted here coincides with the one in [C1]. If a component $H_0$ of $H$ is non-degenerate, then $H_0 \cap Q$ is an interval on the boundary of $Q$; otherwise, it is a point. Indeed, the intersection of $H_0$ with a spacelike hypersurface must have the topology of a 2-sphere, see [CW]. We will assume that the event horizon $H$ consists of $N$ non-degenerate components, which we denote by $H_j$, $1 \leqslant j \leqslant N$. The degenerate cases can be obtained as a limit.

Let $\mathbb{A}$ be the $z$-axis in $\mathbb{R}^3$, and consider $Q' \times SO(2) = \mathbb{R}^3 \setminus \mathbb{A}$ with the flat metric $\rho^2\, d\phi \otimes d\phi + \tilde{h}$. Let $\Sigma = \mathbb{A} \setminus \bigcup_{j=1}^N I_j$ where $I_j$ are the open intervals on the boundary of $Q'$ corresponding to the $N$ components $H_j$ of the event horizon $H$. Since $|\xi|^2 > 0$ on $I_j$, the map $\varphi$ can be extended across $I_j$, and it follows that $\varphi\colon \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ is an axially symmetric harmonic map. We have proved:

**Theorem 1.** *Let $(M, g, F)$ be a solution of the Einstein-Maxwell Equations which is stationary and axially symmetric. Assume that $M$ is simply connected, that $(M, g)$ is the domain of outer communications of an asymptotically flat globally hyperbolic spacetime, and that every component of the event horizon in $(M, g)$ is non-degenerate. Define $\varphi = (u, v, \chi, \psi)$, and $\Sigma$ as above. Then $\varphi\colon \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ is an axially symmetric harmonic map. In addition, the metric $g$ is of the form* (2.18)*, and the metric coefficients $w$ and $\lambda$ satisfy Equations* (2.20) *and* (2.21).

**2.3. Physical Parameters.** Let $S$ be an oriented spacelike two-sphere in $M$. We define the *mass*, *angular momentum*, and *charge* contained in $S$ by:

$$m(S) = -\frac{1}{8\pi}\int_S *d\tau,$$

$$L(S) = \frac{1}{16\pi}\int_S *d\xi,$$

$$q(S) = -\frac{1}{4\pi}\int_S *F.$$

If $\theta$ is a $k$-form and $\Lambda$ a $k$-dimensional submanifold in $M$ which is the orbit under the subgroup $SO(2) \subset G$ of a $(k-1)$-dimensional submanifold $\Gamma$ in $H$, then we have:

$$\int_\Lambda \theta = -2\pi \int_\Gamma i_\xi \theta, \tag{2.22}$$

Suppose $S$ and $S'$ are homologous, that is suppose there exists a three-dimensional submanifold $\Lambda$ such that $\partial\Lambda = S - S'$, then by Stokes Theorem and Equation (2.22), we find:

$$q(S) - q(S') = -\frac{1}{4\pi}\int_\Lambda d * F = 0. \tag{2.23}$$

In our situation, the second homology group is generated by spacelike cross-sections $S_j$ of the $N$ components $H_j$ of the event horizon $H$. Hence $q(S)$ is determined by the homology class of $S$ and the $N$ parameters $q_j = q(S_j)$. By (2.22), we find:

$$q_j = \frac{1}{2}\int_{I_j} d\psi,$$

and we may call $q_j$ the electric charge of the $j$-th black hole. In view of (2.23), $\psi$ is constant along each component of $\Sigma$. Note that according to Equation (1.2), we also have:

$$\frac{1}{2}\int_{I_j} d\chi = \frac{1}{4\pi}\int_{S_j} F = \frac{1}{4\pi}\int_{\partial S_j} A = 0.$$

This equation expresses the absence of magnetic charge. As a consequence, $\chi$ is constant throughout $\Sigma$, hence we may assume $\chi|_\Sigma = 0$.

Similarly, we define the angular momentum of the $j$-th black hole:

$$L_j = L(S_j) = \frac{1}{16\pi}\int_{S_j} *d\xi = \frac{1}{8}\int_{I_j} \omega = \frac{1}{4}\int_{I_j} (dv + \gamma). \tag{2.24}$$

However, $L(S)$ is not completely determined by these $N$ parameters. Indeed, let $\partial\Lambda = S - S'$ as before and suppose that $\Lambda$ is the orbit under $SO(2)$ of a domain $\Omega$ in $Q$, then:

$$L(S) - L(S') = \frac{1}{16\pi}\int_\Lambda d * d\xi = \frac{1}{2}\int_\Omega \alpha \wedge \beta.$$

This is due to the fact that the electromagnetic field carries angular momentum. Nevertheless, since $\gamma$ vanishes on $\Sigma$, it is easy to see from (2.24), that $v$ is constant on each component of $\Sigma$. We introduce the $N$ parameters $\lambda_j = \int_{I_j} dv$, and note that $L_j = (\lambda_j + l_j)/4$ where $l_j = \int_{I_j} \gamma$.

We define the mass of the $j$-th black hole:

$$m_j = m(S_j) = \frac{1}{4}\int_{I_j} i_\xi * d\tau.$$

Note that Equation (2.20) implies that $w$ is constant on each $I_j$. Denote $w_j = w|_{I_j}$, then $w_j$ may be interpreted as the *angular velocity* of the $j$-th black hole. Now, in view of (2.12), we have:

$$dz = \star d\rho = \frac{1}{2} * d\sigma,$$

and therefore,

$$i_\xi * d\tau = -2\, dz + *(d\xi \wedge \tau). \tag{2.25}$$

Since $d\xi = 2\,\xi \wedge du - e^{2u} * (\xi \wedge \omega)$, we have

$$* \, (d\xi \wedge \tau) = -2\,\rho \star du - w\omega. \tag{2.26}$$

Combining Equations (2.25) and (2.26), we obtain:

$$i_\xi * d\tau = -2\, dz - w\omega - 2\,\rho \star du.$$

Note that the last term vanishes on $I_j$. Consequently, we obtain:

$$m_j = \frac{1}{4} \int_{I_j} (2\, dz + w\omega) = \mu_j + 2 w_j L_j,$$

where $2\,\mu_j = \int_{I_j} dz$ is the length of $I_j$ in the metric $\tilde{h}$. Again, these do not determine $m(S)$ since the electromagnetic field carries energy. Let $\partial\Lambda = S - S'$ where $\Lambda$ is the orbit of $\Omega \subset Q$ under $SO(2)$ as before, then we find:

$$\begin{aligned} m(S) - m(S') &= \frac{1}{4} \int_\Omega d(w\omega + 2\rho \star du) \\ &= \frac{1}{2} \int_\Omega \left\{ e^{2u}(|\alpha|^2 + |\beta|^2)\rho\, d\rho\, dz + 2w\alpha \wedge \beta \right\}. \end{aligned}$$

For $0 \leqslant j \leqslant N$, let $\Sigma_j$ be the $N+1$ connected components of $\Sigma$, and let $r_j = \int_{\Sigma_j} dz$ be the length of $\Sigma_j$ in the flat metric $\tilde{h}$ for $1 \leqslant j \leqslant N-1$. The length of $\Sigma_j$ in the metric $h$ is $d_j = \int_{\Sigma_j} e^\lambda\, dz$. We will prescribe the $4N-1$ parameters $\{\mu_j > 0, r_j > 0, \lambda_j, q_j\}$; the $4N-1$ physical parameters $\{m_j, d_j, L_j, q_j\}$ are determined a posteriori. However, we note that the prescribed parameters $\{\mu_j, r_j, \lambda_j, q_j\}$ are nonetheless geometric invariants.

We conclude this section by verifying as mentioned earlier that $H_j$ is degenerate if and only if its surface gravity vanishes. If we let $\eta_j = \tau + w_j\xi$, then $\eta_j$ is a Killing field which is null but non-zero on $H_j$. Thus, $H_j$ is a Killing horizon. Let $\kappa_j$ be the surface gravity of $H_j$, then

$$\frac{\left|d(|\eta_j|^2)\right|^2}{4\,|\eta_j|^2} \to \kappa_j^2,$$

on $H_j$, see [C1, p. 150]. However, $|\eta_j|^2 = -\rho^2 e^{2u} + e^{-2u}(w - w_j)^2$, hence in view of (2.20),

$$\left|d(|\eta_j|^2)\right|^2 = 4\rho^2 e^{4u}\,|d\rho|^2 + O(\rho^4).$$

Therefore we conclude that $e^{2u}\,|d\rho|^2 \to \kappa_j$ on $H_j$ as claimed in Section 2.1, see also [C1, p. 192].

**2.4. Boundary Conditions.** To complete the reduction, we need to set-up boundary conditions for $\varphi$ on $\Sigma$ and as $r \to \infty$ in $\mathbb{R}^3$, where $r = \sqrt{\rho^2 + z^2}$. It is important to observe that, since $u$ behaves like $\log \rho$ near $\Sigma$ and as $r \to \infty$, these boundary conditions will be singular. They may be viewed as prescribed singularities for the map $\varphi$ on $\Sigma$ and at infinity. In particular, $\tilde{E}_{\mathbb{R}^3}(\varphi)$ the total energy of $\varphi$ will be infinite. In order to renormalize this energy, we first solve the linear problem obtained when neither rotation nor charge are present, i.e. when $\lambda_j = q_j = 0$. In this case, $v = \chi = \psi = 0$ solve Equations (2.8)–(2.10), hence by Equation (2.7), $u$ is harmonic. When $N = 1$ these solutions correspond to the Schwarzschild solutions. The solutions for $N \geqslant 2$ are derived from these by superposition, and are easily seen to be the potential of a uniform charge distribution on $\Sigma$ normalized so that $u - \log \rho \to 0$, as $r \to \infty$, i.e.

$$u(x) = -\int_\Sigma \Gamma(x-y)\, d\mathcal{H}^1(y) \tag{2.27}$$

where $\Gamma(x) = -(4\pi\,|x|)^{-1}$ is the fundamental solution of the Laplacian in $\mathbb{R}^3$, $d\mathcal{H}^1$ is the one-dimensional Hausdorff measure on $\mathbb{A}$, and the integral in (2.27) is renormalized. These are the *Weyl solutions*, see [BW, We2].

Given $2N-1$ positive numbers $\{\mu_j > 0, r_j > 0\}$, we define $N$ intervals $I_j$ of length $\mu_j$ distance $r_j$ apart on the $z$-axis $\mathbb{A}$ in $\mathbb{R}^3$, we let $\Sigma = \mathbb{A} \setminus \bigcup_{j=1}^N I_j$, and we denote by $\Sigma_j$ the $N+1$ connected components of $\Sigma$. We denote by $u_j$ the potential of a uniform charge distribution on $\Sigma_j$, normalized so that $u_j \to 0$ as $r \to \infty$ for $1 \leqslant j \leqslant N-1$, and $u_0 + u_N - \log \rho \to 0$ as $r \to \infty$. Let $u^{(0)} = \sum_{j=0}^N u_j$; $u^{(0)}$ is the corresponding Weyl solution. Given in addition $2N$ real numbers $\{\lambda_j, q_j\}$, we set $\lambda_0 = q_0 = 0$, and we write $\varphi_j = (u_j, \lambda_j, 0, q_j)\colon \mathbb{R}^3 \setminus \Sigma_j \to \mathbb{H}^2_{\mathbb{C}}$. We remark that each of these maps is harmonic and has its image contained in a geodesic of $\mathbb{H}^2_{\mathbb{C}}$. If $\varphi\colon \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ is a harmonic map obtained from a solution of the Einstein-Maxwell Equations in Theorem 1, then as $x \to \Sigma_j$ in $\mathbb{R}^3$, $\operatorname{dist}(\varphi(x), \varphi_j(x))$ in $\mathbb{H}^2_{\mathbb{C}}$ remains bounded, see the boundary conditions in [C1]. This motivated the following definition introduced in [We4]. Here, $\mathbb{H}$ is either real-, complex-, or quaternion-hyperbolic space, $\Omega \subset \mathbb{R}^n$ is any smooth bounded domain, $\Sigma$ any closed smooth submanifold of $\Omega$ of co-dimension $\geqslant 2$, and $L^\infty(\Omega \setminus \Sigma)$ is the space of essentially bounded measurable functions on $\Omega \setminus \Sigma$.

**Definition 4.** Let $\varphi, \varphi'\colon \Omega \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ be harmonic maps, and let $\Sigma' \subset \Sigma$. We say that $\varphi$ and $\varphi'$ are *asymptotic near* $\Sigma'$ if there is a neighborhood $\Omega'$ of $\Sigma'$ such that $\operatorname{dist}(\varphi, \varphi') \in L^\infty(\Omega' \setminus \Sigma')$.

Now let $\tilde{\Sigma} = \Sigma_0 \cup \Sigma_N$ be the union of the unbounded components of $\Sigma$. ¿From the Kerr-Newman solution, we obtain a harmonic map $\tilde{\varphi}\colon \mathbb{R}^3 \setminus \tilde{\Sigma} \to \mathbb{H}^2_{\mathbb{C}}$ which is asymptotic to $\varphi_0$ near $\Sigma_0$ and asymptotic to $\varphi_N$ near

$\Sigma_N$. In order to write down this solution explicitly, we introduce the global parameters $q$, $\mu$, $m$, and $a$ defined by:

$$q = \sum_{j=1}^{N} q_j, \quad 2\mu = \sum_{j=1}^{N} \mu_j + \sum_{j=1}^{N-1} r_j, \quad ma = \frac{1}{4}\sum_{j=1}^{N} \lambda_j, \quad m^2 - a^2 - q^2 = \mu^2,$$

and the Schwarzschild-type elliptical coordinates $(s, \theta)$ on $\mathbb{R}^3$ defined by:

$$\rho^2 = (s - m + \mu)(s - m - \mu)\sin^2\theta, \quad z = (s - m)\cos\theta,$$

where we have assumed without loss of generality that the point $z = 0$ on $\mathbb{A}$ is the center of the interval $\tilde{I} = \mathbb{A} \setminus \tilde{\Sigma}$. We define the function $\varsigma = \sqrt{s^2 + a^2\cos^2\theta}$. The harmonic map $\tilde{\varphi} = (\tilde{u}, \tilde{v}, \tilde{\chi}, \tilde{\psi})$ is then given by:

(2.28)

$$\begin{aligned}
\tilde{u} &= \log\sin\theta + \frac{1}{2}\log\left[s^2 + a^2 + a^2\varsigma^{-2}(2ms - q^2)\sin^2\theta\right] \\
\tilde{v} &= ma\cos\theta(3 - \cos^2\theta) - a\varsigma^{-2}(q^2 s - ma^2\sin^2\theta)\cos\theta\sin^2\theta + 2ma \\
\tilde{\chi} &= -qas\varsigma^{-2}\sin^2\theta \\
\tilde{\psi} &= q\varsigma^{-2}(s^2 + a^2)\cos\theta + q,
\end{aligned}$$

see [C1]. For the prescription of the singular behavior of $\varphi$ at infinity, we need the following definition.

**Definition 5.** Assume that $\Omega, \Sigma \subset \mathbb{R}^n$ are unbounded, and $\varphi, \varphi' \colon \Omega \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ are harmonic maps. We say that $\varphi$ and $\tilde{\varphi}$ are *asymptotic at infinity*, if $\operatorname{dist}(\varphi, \tilde{\varphi}) \to 0$ as $x \to \infty$ in $\Omega \setminus \Sigma$.

With this in hand, we can now formulate the reduced problem. Given a set $P = \{\mu_j > 0, r_j > 0, \lambda_j, q_j\}$ of $4N - 1$ parameters, we associate with $P$ the data $D_P = \{\varphi_j \colon \mathbb{R}^3 \setminus \Sigma_j \to \mathbb{H}^2_{\mathbb{C}},\ \tilde{\varphi} \colon \mathbb{R}^3 \setminus \tilde{\Sigma} \to \mathbb{H}^2_{\mathbb{C}}\}$ consisting of $N + 2$ harmonic maps constructed so far.

**Reduced Problem.** Let $P = \{\mu_j > 0, r_j > 0, \lambda_j, q_j\}$ be a given set of $4N - 1$ parameters, and let $D_P = \{\varphi_j \colon \mathbb{R}^3 \setminus \Sigma_j \to \mathbb{H}^2_{\mathbb{C}},\ \tilde{\varphi} \colon \mathbb{R}^3 \setminus \tilde{\Sigma} \to \mathbb{H}^2_{\mathbb{C}}\}$ be the associated data. Prove the existence of a unique axially symmetric harmonic map $\varphi \colon \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ such that $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$ for each $j = 0, \dots, N$, and $\varphi$ is asymptotic to $\tilde{\varphi}$ at infinity.

This problem will be studied in Section 3, see the Main Theorem and its Corollary. Clearly, of at least equal importance is the converse: given a solution of the Reduced Problem, is there a corresponding solution of the Einstein-Maxwell Equations which is stationary and axially symmetric, and which is a simply connected domain of outer communications of a globally hyperbolic and asymptotically flat spacetime with $N$ non-degenerate components to the event horizon? This question will be considered briefly in Section 4. We only mention here that this is almost true in the sense that the only possible obstruction is a conical singularity on the bounded components of $\Sigma$. This obstruction can be interpreted as the gravitational force, see [BW, We3]. It has been conjectured that this obstruction is always present, see [Pe]. Were this conjecture proved, the previously known black holes uniqueness theorems could be extended to the case $N \geqslant 2$.

## 3. Harmonic Maps with Prescribed Singularities

In this section, we prove the existence of a unique solution to the Reduced Problem. The result we state is somewhat more general than needed for the application to black holes, but its proof requires no additional effort. Let $\Omega \subset \mathbb{R}^n$, and let $\Sigma$ be a smooth closed submanifold of $\mathbb{R}^n$ of co-dimension $\geqslant 2$. Let $\mathbb{H}$ be one of the classical globally symmetric space of noncompact type and rank one, i.e. either real-, complex-, or quaternion-hyperbolic space of real dimension $m$. For our purpose, we take a *harmonic map* $\varphi = (\varphi^1, \dots, \varphi^m)\colon \Omega \setminus \Sigma \to \mathbb{H}$ to mean a map $\varphi \in C^\infty(\overline{\Omega} \setminus \Sigma; \mathbb{H})$ which for each $\Omega' \subset\subset \Omega \setminus \Sigma$ is a critical point of the energy functional:

$$E_{\Omega'} = \int_{\Omega'} |d\varphi|^2 ,$$

where $|d\varphi|^2 = \sum_{k=1}^n \langle \partial_k \varphi, \partial_k \varphi \rangle$ is the energy density, and $\langle \cdot, \cdot \rangle$ denotes the metric of $\mathbb{H}$. This is equivalent to requiring $\varphi$ to satisfy in $\Omega \setminus \Sigma$ the following elliptic system of nonlinear partial differential equations:

$$\Delta \varphi^a + \sum_{k=1}^n \Gamma^a_{bc}(\varphi)\, \partial_k \varphi^b\, \partial_k \varphi^c = 0,$$

where $\Gamma^a_{bc}$ are the Christoffel symbols of $\mathbb{H}$.

**Definition 6.** Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and let $\Sigma$ be a smooth closed submanifold of $\Omega$ of co-dimension $\geqslant 2$. Let $\gamma$ be a geodesic in $\mathbb{H}$. We say that a harmonic map $\varphi\colon \Omega \setminus \Sigma \to \mathbb{H}$ is a $\Sigma$-*singular map into* $\gamma$ if

(i) $\varphi(\Omega \setminus \Sigma) \subset \gamma(\mathbb{R})$
(ii) $\varphi(x) \to \gamma(+\infty)$ as $x \to \Sigma$
(iii) There is a constant $\delta > 0$ such that $|d\varphi(x)|^2 \geqslant \delta \operatorname{dist}(x, \Sigma)^{-2}$ in some neighborhood of $\Sigma$.

We assume that we are given $N + 1$ disjoint smooth closed connected submanifolds $\Sigma_j \subset \mathbb{R}^n$ of co-dimension $\geqslant 2$, and $N + 1$ harmonic maps $\varphi_j$ which are $\Sigma_j$-singular into $\gamma_j$. We set $\Sigma = \bigcup_j \Sigma_j$, let $J = \{0 \leqslant j \leqslant N\colon\ \Sigma_j \text{ bounded}\}$, and let $\widetilde{\Sigma} = \bigcup_{j \notin J} \Sigma_j$ be the union of the unbounded components. We need to add a technical condition on the data $\tilde{\varphi}$ at infinity. For $p \in \mathbb{H}$ and $\gamma$ a geodesic in $\mathbb{H}$ let $\operatorname{dist}(p, \gamma)$ denote the distance from $p$ to $\gamma(\mathbb{R})$, i.e. $\inf_t \operatorname{dist}(p, \gamma(t))$.

**Definition 7.** Let $\varphi_j\colon \mathbb{R}^n \setminus \Sigma_j \to \mathbb{H}$ be $\Sigma_j$-singular maps into $\gamma_j$, and let $\tilde{\varphi}\colon \mathbb{R}^n \setminus \widetilde{\Sigma} \to \mathbb{H}$ be a harmonic map. We say that $\tilde{\varphi}$ is *adapted to* $\{\varphi_j\}$ *at infinity* if for each $j \notin J$:

(i) $\tilde{\varphi}$ is asymptotic to $\varphi_j$ near $\Sigma_j$;

(ii) $\operatorname{dist}(\varphi(x), \gamma_j) \to 0$ as $x \to \Sigma_j$ outside some compact subset of $\mathbb{R}^n$.

The following result is our main existence theorem.

**Main Theorem.** *For each $j = 0, \ldots, N$, let $\gamma_j$ be a geodesic in $\mathbb{H}$, and let $\Sigma_j$ be disjoint closed smooth connected submanifolds of $\mathbb{R}^n$ of co-dimension $\geqslant 2$. Let $\Sigma = \bigcup_{j=0}^{N} \Sigma_j$, and let $\tilde{\Sigma} = \bigcup_{j \notin J} \Sigma_j$ be the union of the unbounded components. Let $\varphi_j \colon \mathbb{R}^n \setminus \Sigma_j \to \mathbb{H}$ be a $\Sigma_j$-singular map into $\gamma_j$, and let $\tilde{\varphi} \colon \mathbb{R}^n \setminus \tilde{\Sigma} \to \mathbb{H}$ be a harmonic map which is adapted to $\{\varphi_j\}$ at infinity. Then there exists a unique harmonic map $\varphi \colon \mathbb{R}^n \setminus \Sigma \to \mathbb{H}$ such that $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$ for each $j = 0, \ldots, N$, and $\varphi$ is asymptotic to $\tilde{\varphi}$ at infinity.*

In other words, given a harmonic map $\tilde{\varphi}$ with the correct asymptotic behavior at infinity, we can modify it to a harmonic map $\varphi$ with prescribed singular behavior near the compact components of $\Sigma$. It is easy to check that the maps $\varphi_j \colon \mathbb{R}^3 \setminus \Sigma_j \to \mathbb{H}^2_{\mathbb{C}}$ defined in Section 2.4 are $\Sigma_j$-singular into $\gamma_j$ where $\gamma_j$ are the geodesics in $\mathbb{H}^2_{\mathbb{C}}$ given by $\gamma_j(t) = (t, \lambda_j, 0, q_j)$, and the harmonic map $\tilde{\varphi} \colon \mathbb{R}^3 \setminus \tilde{\Sigma} \to \mathbb{H}^2_{\mathbb{C}}$ obtained from the Kerr-Newman solution is adapted to $\{\varphi_j\}$, see (2.28). Thus, as an immediate corollary to the Main Theorem, we obtain:

**Corollary.** *The Reduced Problem has a unique solution.*

We note that the axially symmetric property of the solution $\varphi$ follows immediately from the uniqueness statement in the Main Theorem.

In [We2] the case $\mathbb{H} = \mathbb{H}^2_{\mathbb{R}}$ corresponding to the vacuum equations was studied, and in [We4] we proved a version of this theorem for maps $\varphi \colon \Omega \setminus \Sigma \to \mathbb{H}$ with $\Omega$ is bounded and $\Sigma$ compactly contained in $\Omega$. The proof of our Main Theorem, which combines the ideas in both these papers, is divided into three steps. In step one we consider the problem on a ball $\Omega$, but allow $\Sigma$ to extend to $\partial\Omega$. Besides some minor technical points there are no new difficulties in this step. In step two, we use the known solution $\tilde{\varphi}$ to obtain a pointwise a priori bound on $d(\varphi, \varphi_j)$ and $d(\varphi, \tilde{\varphi})$ which is independent of the size of $\Omega$. This allows us to complete the proof in step three.

Clearly the same techniques can be used to obtain similar results. For instance, using the harmonic maps of P. Li and L.-F. Tam [LTa1, LTa2] as boundary data, one could construct harmonic maps between hyperbolic spaces with cusp-like singularities along prescribed compact submanifolds. Furthermore, just as in [We4], the results would apply to more general targets with pinched negative curvature were it not for Lemma 5.

**3.1. Preliminaries.** The primary tools for the proof of the Main Theorem are the *Busemann* functions on Cartan-Hadamard manifolds, see [Eb, HI]. We note that $\mathbb{H}$ is such a manifold with pinched sectional curvatures $-4 \leqslant \kappa \leqslant -1$. Let $\gamma$ be a geodesic in $\mathbb{H}$, then $f_\gamma(p)$ the Busemann function associated with $\gamma$ evaluated at $p \in \mathbb{H}$ is the *renormalized* distance between $p$ and the ideal point $\gamma(+\infty) \in \partial\mathbb{H}$. It is defined by:

$$f_\gamma(p) = \lim_{t\to\infty} \left(\operatorname{dist}(p, \gamma(t)) - t\right).$$

It is not difficult to see that the limit is obtained uniformly for $p$ in compact subsets of $\mathbb{H}$. It is well-known that $f_\gamma$ is analytic, convex, and has gradient of constant length one. The level sets, which we denote by $S_\gamma(t) = \{p \in \mathbb{H};\ f_\gamma(p) = t\}$, are called *horospheres*. They are diffeomorphic to $\mathbb{R}^{m-1}$. We will also use the *horoballs* defined by $\mathcal{B}_\gamma(t) = \{p \in \mathbb{H};\ f_\gamma(p) \leqslant t\}$, and the geodesic balls $B_R(p) = \{q \in \mathbb{H};\ \operatorname{dist}(p,q) \leqslant R\}$. Two geodesics $\gamma$ and $\gamma'$ are said to be *asymptotic* if $\operatorname{dist}(\gamma(t), \gamma'(t))$ is bounded for $t \geqslant 0$. This is clearly an equivalence relation. The boundary of $\mathbb{H}$ is defined to be the set of equivalence classes of geodesics in $\mathbb{H}$. We will denote the equivalence class of $\gamma$ in $\mathbb{H}$ by $\gamma(+\infty)$. Let $\gamma$ be a geodesic in $\mathbb{H}$. We denote the reverse geodesic $t \mapsto \gamma(-t)$ by $-\gamma$, and we also write $\gamma(-\infty) = -\gamma(+\infty)$. We will introduce an analytic coordinate system in $\mathbb{H}$ adapted to $\gamma$. Let $u = f_{-\gamma}$. Let $v_0\colon \mathcal{S}_{-\gamma}(0) \to \mathbb{R}^{m-1}$ be an analytic coordinate system on $\mathcal{S}_{-\gamma}(0)$ centered at $\gamma(0)$. The integral curves of $\nabla f_{-\gamma}$ are geodesics parameterized by arclength. We 'drag' the coordinate system $v_0$ along these integral curves. More precisely, let $\Phi_t$ be the analytic flow generated by this vector field, then $\Phi_{-t}$ maps $\mathcal{S}_{-\gamma}(t)$ to $\mathcal{S}_{-\gamma}(0)$, hence $v_t = v_0 \circ \Phi_{-t}$ is an analytic coordinate system on $\mathcal{S}_{-\gamma}(t)$. Define $v\colon \mathbb{H} \to \mathbb{R}^{m-1}$ by $v|_{\mathcal{S}_{-\gamma}(t)} = v_t$, and let $\Phi = (u,v)\colon \mathbb{H} \to \mathbb{R}^m$, then $\Phi$ is an analytic coordinate system on $\mathbb{H}$. In these coordinates, the metric of $\mathbb{H}$ reads:

$$ds^2 = du^2 + Q_p(dv),$$

where for each $p \in \mathbb{H}$, $Q_p$ is a positive quadratic form on $\mathbb{R}^{m-1}$. We note that the convexity of $f_{-\gamma}$ implies that for each non-zero $X \in \mathbb{R}^{m-1}$, and each fixed $v \in \mathbb{R}^{m-1}$, $Q_{\Phi^{-1}(u,v)}(X)$ is a positive increasing function of $u$.

The following lemmas were proved in [We4]. They are quoted here without proofs. Except for Lemma 5, they depend only on the bound $-4 \leqslant \kappa \leqslant -1$ for the sectional curvatures of $\mathbb{H}$.

**Lemma 1.** *For any $(u,v) \in \mathbb{R}^m$ and any $X \in \mathbb{R}^{m-1}$, there holds:*

$$2Q_{\Phi^{-1}(u,v)}(X) \leqslant \frac{\partial}{\partial u}\Big(Q_{\Phi^{-1}(u,v)}(X)\Big) \leqslant 4Q_{\Phi^{-1}(u,v)}(X). \tag{3.1}$$

**Lemma 2.** *Let $\gamma$ be a geodesic in $\mathbb{H}$. Then for any $t_0 \in \mathbb{R}$, and any $T \geqslant 0$, we have*

$$\mathcal{B}_\gamma(-t_0 + T) \cap \mathcal{B}_{-\gamma}(t_0 + T) \subset B_R(\gamma(t_0)),$$

*where $R = T + \log 2$.*

**Lemma 3.** *Let $\gamma$ and $\gamma'$ be geodesics in $\mathbb{H}$, such that $\gamma(-\infty) = \gamma'(-\infty)$ and $\gamma(+\infty) \neq \gamma'(+\infty)$. Then for some $d \in \mathbb{R}$, we have:*

$$\lim_{t\to\infty} (f_{-\gamma} - f_\gamma) \circ \gamma'(t) = d$$
$$\lim_{t\to-\infty} (f_{-\gamma} + f_\gamma) \circ \gamma'(t) = 0.$$

**Lemma 4.** *Let $\gamma$ be a geodesic in $\mathbb{H}$. Then, for any $t_0 \in \mathbb{R}$, and any $T \geqslant (1/2)\log 2$, we have*

$$\langle \nabla f_\gamma(p), \nabla f_{-\gamma}(p)\rangle > 0, \quad \forall p \in \mathcal{S}_\gamma(-t_0 + T) \setminus \mathcal{B}_{-\gamma}(t_0 + T)$$

*where $\langle \cdot, \cdot \rangle$ denotes the metric on $\mathbb{H}$.*

**Lemma 5.** *Let $\gamma$ be a geodesic in $\mathbb{H}$. Then there is an analytic coordinate system $\Phi = (u, v)\colon \mathbb{H} \to \mathbb{R}^m$ with $u = f_{-\gamma}$ such that in these coordinates, the metric of $\mathbb{H}$ is given by:*

$$ds^2 = du^2 + Q_p(dv),$$

*and satisfies the following conditions:*

(i) *Let $R > 0$, $t_0 \in \mathbb{R}$, and let $\gamma'$ be a geodesic in $\mathbb{H}$ with $\gamma'(-\infty) = \gamma(-\infty)$. Then, there exists a constant $c \geqslant 1$ such that for all $t \geqslant t_0$, and all $p \in B_R(\gamma'(t))$, there holds:*

$$\frac{1}{c}\, Q_{\gamma'(t)}(X) \leqslant Q_p(X) \leqslant c\, Q_{\gamma'(t)}(X), \qquad \forall X \in \mathbb{R}^{m-1}.$$

(ii) *For all $t, t' \in \mathbb{R}$, the set $\mathcal{S}_{-\gamma}(t) \cap \mathcal{B}_{\gamma'}(t')$ is star-shaped in these coordinates with respect to its 'center', the point where $\gamma'$ intersects $\mathcal{S}_{-\gamma}(t)$.*

Although in [We4], it was always assumed that $\Sigma$ is compactly contained in $\Omega$, the proof of the following lemma presented there is independent of this fact, see [We4, Lemma 8].

**Lemma 6.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, and let $\Sigma$ be a smooth closed submanifold of $\mathbb{R}^n$ of co-dimension $\geqslant 2$. Let $u \in C^\infty(\overline{\Omega} \setminus \Sigma)$ satisfy $\Delta u \geqslant 0$ and $0 \leqslant u \leqslant 1$ in $\Omega \setminus \Sigma$. If $u|_{\partial\Omega\setminus\Sigma} = 0$ then $u = 0$ in $\Omega \setminus \Sigma$.*

In fact, a slight variation of the same proof gives the next lemma.

**Lemma 7.** *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, and let $\Sigma$ be a smooth closed submanifold of $\mathbb{R}^n$ of co-dimension $\geqslant 2$. Let $u \in C^\infty(\bar{\Omega} \setminus \Sigma)$ satisfy $\Delta u \geqslant 0$ and $0 \leqslant u \leqslant 1$ in $\Omega\setminus\Sigma$. For $\delta > 0$, let $\Omega^\delta = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \delta\}$, and suppose $\partial\Omega^\delta$ is smooth. Then there holds:*

$$\int_{\Omega^\delta} |\nabla u|^2 < \infty, \tag{3.2}$$

$$\sup_{\Omega^\delta\setminus\Sigma} u \leqslant \sup_{\partial\Omega^\delta\setminus\Sigma} u. \tag{3.3}$$

*Proof.* If $\chi \in C_0^{0,1}(\Omega \setminus \Sigma)$ with $0 \leqslant \chi \leqslant 1$, then as in the proof of [We4, Lemma 8], we obtain from the inequality $\chi^2 u\, \Delta u \geqslant 0$ that there holds:

$$\int_\Omega \chi^2 \left|\nabla u\right|^2 \leqslant 4 \int_\Omega \left|\nabla \chi\right|^2 . \tag{3.4}$$

With the cut-off function

$$\chi_\epsilon = \begin{cases} 2 - \log r / \log \epsilon & \text{if } \epsilon^2 \leqslant r \leqslant \epsilon \\ 0 & \text{if } r \leqslant \epsilon^2 \\ 1 & \text{if } r \geqslant \epsilon, \end{cases}$$

where $r(x) = \operatorname{dist}(x, \Sigma)$, and with $\chi' \in C_0^\infty(\Omega)$ another cut-off satisfying $0 \leqslant \chi' \leqslant 1$ and $\chi' = 1$ on $\Omega^\delta$, take $\chi = \chi_\epsilon \chi'$ in (3.4), and let $\epsilon \to 0$; (3.2) follows immediately. Now, the same argument shows that $\Delta u \geqslant 0$ weakly throughout $\Omega$, hence (3.3) follows from a standard maximum principle. ☐

**3.2. Step One.** In this step, we restrict our attention to a ball $\Omega \subset \mathbb{R}^n$ sufficiently large that $\partial\Omega \cap (\Sigma \setminus \tilde{\Sigma}) = \emptyset$, and prove the existence of a solution $\varphi\colon \Omega \setminus \Sigma \to \mathbb{H}$ in a manner similar to the proof of [We4, Theorem 1]. We assume all the hypotheses of the Main Theorem. Some of the elements in the proof of Proposition 1 will be used also in the next steps.

**Proposition 1.** *Let $\Omega \subset \mathbb{R}^n$ be a ball large enough that $\partial\Omega \cap (\Sigma \setminus \tilde{\Sigma}) = \emptyset$. Then there is a unique harmonic map $\varphi\colon \Omega \setminus \Sigma \to \mathbb{H}$ such that $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$ for each $j = 0, \ldots, N$, and $\varphi = \tilde{\varphi}$ on $\partial\Omega \setminus \Sigma$.*

The proof follows closely that of Proposition 2 in [We4]. We begin with the uniqueness. Let $\varphi$ and $\varphi'$ be two such maps, then clearly $\operatorname{dist}(\varphi, \varphi') \in L^\infty(\Omega \setminus \Sigma)$, and $\varphi = \varphi'$ on $\partial\Omega \setminus \Sigma$. Consider the function $u = \operatorname{dist}(\varphi, \varphi')^2$. We have that $\Delta u \geqslant 0$ on $\Omega \setminus \Sigma$, see [SY], $u$ is bounded on $\Omega \setminus \Sigma$, and $u|_{\partial\Omega\setminus\Sigma} = 0$. Thus, by Lemma 6, it follows that $u = 0$ and hence $\varphi = \varphi'$ on $\Omega \setminus \Sigma$.

For the existence proof, we set-up a variational approach. Let $\{\Omega'_j\}_{j=0}^N$ be an open cover of $\mathbb{R}^n$, with $\partial\Omega'_j$ smooth, such that $\Sigma_j \subset \Omega'_j$, and $\Sigma_j \cap \Omega'_{j'} = \emptyset$ for $j \neq j'$. Let $\Omega_j = \Omega'_j \cap \Omega$. It can easily be arranged that if $j \in J$ then $\Omega_j \subset\subset \Omega$. Let $\{\chi_j\}_{j=0}^N$, be a partition of unity subordinate to the cover $\{\Omega_j\}_{j=0}^N$ of $\Omega$ such that $\chi_j = 1$ near $\Sigma_j$ for all $j = 0, \ldots, N$. We may assume without loss of generality that all the geodesics $\gamma_j$ have the same initial point on $\partial\mathbb{H}$. Let $\Phi = (u, v)$ be the coordinate system on $\mathbb{H}$ with $u = f_{-\gamma_0}$ given in Lemma 5. We will identify any map $\varphi\colon \Omega \setminus \Sigma \to \mathbb{H}$ with its parameterization $\Phi \circ \varphi = (u, v)$ whenever no confusion can arise. For $j = 0, \ldots, N$, let $\varphi_j = \gamma_j \circ u_j$ where $\Delta u_j = 0$ on $\Omega \setminus \Sigma_j$. Then, we have that $\varphi_j = (u_j, w_j)$ where $w_j \in \mathbb{R}^{m-1}$ are constants. Without loss of generality, we may assume that for $j \in J$, $u_j(x) = 0$ on $\partial\Omega$, and in particular $u_j > 0$ in $\Omega$.

Define the norms:

$$\|u\|_\Omega = \left(\int_\Omega \left\{u^2 + |\nabla u|^2\right\}\right)^{1/2}, \tag{3.5}$$

$$\|v\|_{\varphi_j;\Omega} = \left(\int_\Omega \left\{|v|^2 + Q_{\varphi_j}(\nabla v)\right\}\right)^{1/2}, \tag{3.6}$$

where $Q_{\varphi_j}(\nabla v)|_x = \sum_{k=1}^n Q_{\varphi_j(x)}(\nabla_k v(x))$ for $x \in \Omega \setminus \Sigma_j$. We will use the Hilbert spaces:

$$\begin{gathered} H_1(\Omega) = \{u \colon \Omega \to \mathbb{R};\ \|u\|_\Omega < \infty\} \\ H_1^{\varphi_j}(\Omega; \mathbb{R}^{m-1}) = \{v \colon \Omega \to \mathbb{R}^{m-1};\ \|v\|_{\varphi_j;\Omega} < \infty\}, \end{gathered}$$

and the subspaces $H_{1,0}(\Omega)$ and $H_{1,0}^{\varphi_j}(\Omega; \mathbb{R}^{m-1})$, defined to be the closure in $H_1(\Omega)$ of $C_0^\infty(\Omega)$ and the closure in $H_1^{\varphi_j}(\Omega; \mathbb{R}^{m-1})$ of $C_0^\infty(\Omega \setminus \Sigma_j; \mathbb{R}^{m-1})$ respectively.

We write $\tilde\varphi = (\tilde u, \tilde v)$, and introduce the function $v_0$ obtained by truncating $\tilde v$ near the bounded components of $\Sigma$ and setting it to its prescribed values $w_j \in \mathbb{R}^{m-1}$:

$$v_0 = \left(\prod_{j \in J} (1 - \chi_j)\right) \tilde v + \sum_{j \in J} \chi_j w_j.$$

Note that $v_0 = w_j$ near $\Sigma_j$ for $j \in J$, hence clearly $v_0 - w_j \in H_1^{\varphi_j}(\Omega_j; \mathbb{R}^{m-1})$. We also separate $u^{(0)} = \sum_{j=0}^N u_j$ into $u^{(1)} = \sum_{j \in J} u_j$, the contribution from the bounded components of $\Sigma$, and $u^{(2)} = \sum_{j \notin J} u_j$, the contribution from the unbounded components. Clearly, $\Delta u^{(1)} = \Delta u^{(2)} = 0$ on $\Omega \setminus \Sigma$ and $u^{(1)} = 0$ on $\partial\Omega$. Since $\tilde\varphi$ is a harmonic map $\tilde u$ is subharmonic on $\mathbb{R}^n \setminus \widetilde\Sigma$, hence $\Delta(\tilde u - u^{(2)}) \geqslant 0$ on $\mathbb{R}^n \setminus \widetilde\Sigma$. Furthermore, $\left|\tilde u - u^{(2)}\right|$ is bounded on the larger ball $\Omega' = \{x \in \mathbb{R}^n;\ \operatorname{dist}(x, \Omega) < 1\}$. Indeed, on the one hand, for $j \notin J$ we have on $\Omega'_j \cap \Omega'$ that $\left|\tilde u - u^{(2)}\right| \leqslant |\tilde u - u_j| + \sum_{j' \neq j} |u_{j'}|$, and $|\tilde u - u_j| \leqslant \operatorname{dist}(\tilde\varphi, \varphi_j)$, while $|u_{j'}|$ is bounded, and on the other hand, for $j \in J$ we have that both $\tilde u$ and $u^{(2)}$ are bounded on $\Omega_j$. Hence it follows from Lemma 7 that $\tilde u - u^{(2)} \in H_1(\Omega)$. In fact $\tilde u$ satisfies the equation $\Delta \tilde u = (\partial/\partial u)(Q_{\tilde\varphi}(\nabla \tilde v))$, and Lemma 1 implies:

$$Q_{\tilde\varphi}(\nabla \tilde v) \leqslant \frac{1}{2}\Delta(\tilde u - u^{(2)}),$$

in $\Omega' \setminus \widetilde\Sigma$. Multiplying by a cut-off $\chi$ as in the proof of Lemma 7 and integrating over $\Omega$, we conclude, after taking the appropriate limit, that

$$\int_\Omega Q_{\tilde\varphi}(\nabla \tilde v) < \infty. \tag{3.7}$$

Define $\mathcal{H}$ to be the space of maps $\varphi = (u, v)\colon \Omega \setminus \Sigma \to \mathbb{H}$ satisfying:

$$\begin{cases} u - \tilde{u} - u^{(1)} \in H_{1,0}(\Omega); \\ v - v_0 \in \bigcap_{j=0}^{N} H_{1,0}^{\varphi_j}(\Omega; \mathbb{R}^{m-1}); \\ \operatorname{dist}(\varphi, \varphi_j) \in L^\infty(\Omega_j \setminus \Sigma_j), \quad \forall j = 0, \dots, N. \end{cases}$$

For maps $\varphi \in \mathcal{H}$ define:

$$F(\varphi) = \int_\Omega \left\{ \left| \nabla (u - u^{(0)}) \right|^2 + Q_\varphi(\nabla v) \right\}.$$

For $R > 0$ define the space $\mathcal{H}_R$ of maps $\varphi \in \mathcal{H}$ such that for $j = 0, \dots, N$, $\operatorname{dist}(\varphi, \varphi_j) \leqslant R$ for a.e. $x \in \Omega_j \setminus \Sigma_j$. We will need the following lemma which is a direct consequence of Lemma 5.

**Lemma 8.** *Let $R > 0$, then there is a $c \geqslant 1$ such that for all $\varphi \in \mathcal{H}_R$, there holds for each $j = 0, \dots, N$ and for all $X \in \mathbb{R}^{m-1}$:*

$$\frac{1}{c} Q_{\varphi_j(x)}(X) \leqslant Q_{\varphi(x)}(X) \leqslant c\, Q_{\varphi_j(x)}(X), \quad \textit{for a.e. } x \in \Omega_j \setminus \Sigma_j.$$

We already know that $\int_{\Omega_j} Q_{\varphi_j}(\nabla v_0) < \infty$ for $j \in J$, and it follows from Lemma 8 and (3.7) that the same holds also for $j \notin J$. Indeed, since for $j \notin J$, $\tilde{\varphi}$ is asymptotic to $\varphi_j$ near $\Sigma_j$, there is a constant $c \geqslant 1$ such that

$$\int_{\Omega_j} Q_{\varphi_j}(\nabla \tilde{v}) \leqslant c \int_{\Omega_j} Q_{\tilde{\varphi}}(\nabla \tilde{v}) < \infty.$$

Since $\chi_j$ is smooth and $\nabla \chi_j = 0$ near $\tilde{\Sigma}$, we obtain that $\int_{\Omega_j} Q_{\varphi_j}(\nabla v_0) < \infty$ as claimed. Now, let $\varphi = (u, v) \in \mathcal{H}$, then there is a constant $c \geqslant 1$ such that:

$$\int_\Omega Q_\varphi(\nabla v) \leqslant c \sum_{j=0}^{N} \int_{\Omega_j} Q_{\varphi_j}(\nabla v) < \infty. \tag{3.8}$$

Since also $u - u^{(0)} = (u - \tilde{u} - u^{(1)}) + (\tilde{u} - u^{(2)}) \in H_1(\Omega)$, we conclude that $F$ is finite on $\mathcal{H}$.

It is straightforward to check that a minimizer $\varphi \in \mathcal{H}$ of $F$ is a harmonic map on $\Omega \setminus \Sigma$, see [We4]. Hence, by the regularity theory for harmonic maps, see [SU1, SU2], $\varphi \in C^\infty(\overline{\Omega} \setminus \Sigma; \mathbb{H})$, $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$ for $j = 0, \dots, N$ by construction, and $\varphi = \tilde{\varphi}$ on $\partial\Omega \setminus \Sigma$. Thus, to prove Proposition 1, it suffices to show that $F$ admits a minimizer in $\mathcal{H}$.

It can be shown, exactly as in [We4, Proposition 1], that $F$ admits a minimizer in $\mathcal{H}_R$. The main steps are as follows. Consider a minimizing sequence $\hat{\varphi}_k = (\hat{u}_k, \hat{v}_k)$ in $\mathcal{H}_R$. Then for some subsequence $k'$, $\hat{u}_{k'} - u^{(0)}$ converges weakly in $H_1(\Omega)$ and pointwise a.e. in $\Omega$. Furthermore, using the bound $\operatorname{dist}(\hat{\varphi}_k, \varphi_j) \leqslant R$ in $\Omega_j$, it is shown that along perhaps a further subsequence $\hat{v}_{k'} - v_0$ also converges weakly in each $H_{1,0}^{\varphi_j}(\Omega, \mathbb{R}^{m-1})$, and pointwise a.e. in $\Omega$. Let $\varphi = (u, v) \in \mathcal{H}_R$ be the weak limit. Then clearly we have

$$\int_\Omega \left|\nabla(u - u^{(0)})\right|^2 \leqslant \liminf \int_\Omega \left|\nabla(\hat{u}_{k'} - u^{(0)})\right|^2. \tag{3.9}$$

Furthermore, $Q_\varphi$ is equivalent to $Q_{\varphi_j}$ on $\Omega_j$, hence we have

$$\begin{aligned}\int_\Omega Q_\varphi(\nabla v) &= \lim \int_\Omega Q_\varphi(\nabla v, \nabla \hat{v}_{k'}) \\ &\leqslant \liminf \left[\left(\int_\Omega \chi_{k'} Q_\varphi(\nabla v)\right)^{1/2} \left(\int_\Omega Q_{\hat{\varphi}_{k'}}(\nabla \hat{v}_{k'})\right)^{1/2}\right],\end{aligned} \tag{3.10}$$

where

$$\chi_k = \begin{cases} Q_\varphi(\nabla \hat{v}_k)/Q_{\hat{\varphi}_k}(\nabla \hat{v}_k) & \text{if } \hat{v}_k \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

However $\chi_{k'} \to 1$ pointwise a.e. in $\Omega$, and $\chi_{k'}$ is bounded. Thus, we conclude that

$$\lim \int_\Omega \chi_{k'} Q_\varphi(\nabla v) = \int_\Omega Q_\varphi(\nabla v),$$

and therefore (3.10) implies:

$$\int_\Omega Q_\varphi(\nabla v) \leqslant \left(\int_\Omega Q_\varphi(\nabla v)\right)^{1/2} \liminf \left(\int_\Omega Q_{\hat{\varphi}_{k'}}(\nabla \hat{v}_{k'})\right)^{1/2}.$$

Combining this with (3.9) we obtain

$$F(\varphi) \leqslant \liminf F(\hat{\varphi}_{k'}) = \inf_{\mathcal{H}_R} F,$$

and it follows that $\varphi$ is a minimizer in $\mathcal{H}_R$.

Thus, it remains to prove that for some $R > 0$ large enough $\inf_{\mathcal{H}} F = \inf_{\mathcal{H}_R} F$. This is the content of the next lemma.

**Lemma 9.** *There is a constant* $R > 0$ *such that for every* $\epsilon > 0$, *and every* $\varphi \in \mathcal{H}$, *there is* $\varphi' \in \mathcal{H}_R$ *such that* $F(\varphi') \leqslant F(\varphi) + \epsilon$.

*Proof of Lemma 9.* Let $\mathcal{H}^*$ be the space of maps $\varphi = (u, v) \in \mathcal{H}$ such that $v = w_j$ in a neighborhood of $\Sigma_j$ for $j \in J$, $\varphi = \tilde{\varphi}$ outside some compact set $K \subset \Omega$, and $v = w_j$ in a neighborhood of $\Sigma_j \cap K$ for $j \notin J$. Lemma 9 will follow immediately from: (i) for each $\varphi \in \mathcal{H}$ and each $\epsilon > 0$ there is $\varphi' \in \mathcal{H}^*$ such that $F(\varphi') \leqslant F(\varphi) + \epsilon$; and (ii) for each $\varphi \in \mathcal{H}^*$ there is $\varphi' \in \mathcal{H}_R$ such that $F(\varphi') \leqslant F(\varphi)$. The proof of (i), a standard approximation argument, is practically unchanged from [We4, Lemma 11]. We turn to the proof of (ii). The bound $\operatorname{dist}(\varphi, \varphi_j) \leqslant R$ will be achieved consecutively on each $\Omega_j$ beginning with all $j \notin J$. The case $j \in J$ is simpler, and in fact relies entirely on the results of [We4]. Assume without loss of generality that $0 \notin J$. Introduce a 'dual' coordinate system $\bar{\Phi} = (\bar{u}, \bar{v})$ with $\bar{u} = f_{\gamma_0}$ as given by Lemma 5. Write

$$ds^2 = d\bar{u}^2 + \bar{Q}_p(d\bar{v}).$$

for the metric of $\mathbb{H}$ in these coordinates. Set $\bar{u}^{(0)} = -u_0 + \sum_{j=1}^{N} u_j$. Let $\varphi \in \mathcal{H}^*$ and write $\bar{\Phi} \circ \varphi = (\bar{u}, \bar{v})$. Then, since $|\nabla u|^2 + Q_\varphi(\nabla v) = |\nabla \bar{u}|^2 + \bar{Q}_\varphi(\nabla \bar{v})$, we find that in $\Omega \setminus \Sigma$:

$$\begin{aligned} \left|\nabla(u - u^{(0)})\right|^2 + Q_\varphi(\nabla v) &= \left|\nabla(\bar{u} - \bar{u}^{(0)})\right|^2 + \bar{Q}_\varphi(\nabla \bar{v}) \\ &\quad - 2\,\nabla(u + \bar{u}) \cdot \nabla u_0 - 2\sum_{j=1}^{N} \nabla(u - \bar{u}) \cdot \nabla u_j + 4\sum_{j=1}^{N} \nabla u_j \cdot \nabla u_0. \end{aligned}$$

The idea is now to integrate this identity over $\Omega$, use the right hand side to truncate $\bar{u} - \bar{u}^{(0)}$, and the left hand side to truncate $u - u^{(0)}$. The convexity of the Busemann functions $u$ and $\bar{u}$ on $\mathbb{H}$, expressed in the monotonicity of $Q$ and $\bar{Q}$, ensures that these truncations do not increase $F$. This can be viewed as a weak form of a variational maximum principle.

Some care, however, needs to be taken because of the singularities on $\Sigma$. Integrating first over $\Omega^\epsilon = \{x \in \Omega;\ \mathrm{dist}(x, \Sigma) > \epsilon\}$, we obtain:

$$\int_{\Omega^\epsilon} \left\{ \left|\nabla(u - u^{(0)})\right|^2 + Q_\varphi(\nabla v) \right\} = \int_{\Omega^\epsilon} \left\{ \left|\nabla(\bar{u} - \bar{u}^{(0)})\right|^2 + \bar{Q}_\varphi(\nabla \bar{v}) \right\} + I_\epsilon(\varphi),$$

where:

$$\begin{aligned} I_\epsilon(\varphi) = \int_{\Omega^\epsilon} \Big\{ &-2\,\mathrm{div}\left((u + \bar{u})\,\nabla u_0\right) \\ &- 2\sum_{j=1}^{N} \mathrm{div}\left((u - \bar{u})\,\nabla u_j\right) + 4\sum_{j=1}^{N} \nabla u_j \cdot \nabla u_0 \Big\}. \end{aligned} \tag{3.11}$$

A tedious but straightforward verification shows that that when $\epsilon \to 0$, $I_\epsilon(\varphi) \to C$ where $C$ is a constant independent of $\varphi \in \mathcal{H}^*$, yielding an integral identity:

$$F(\varphi) = \int_\Omega \left\{ \left|\nabla(\bar{u} - \bar{u}^{(0)})\right|^2 + \bar{Q}_\varphi(\nabla \bar{v}) \right\} + C. \tag{3.12}$$

For example, note that if $\epsilon > 0$ is small enough, we can decompose $\partial\Omega^\epsilon$ into a disjoint union $\bigcup_{j=0}^{N} \partial_j^\epsilon \cup \partial^\epsilon\Omega$, where $\partial_j^\epsilon = \{x \in \Omega;\ \mathrm{dist}(x, \Sigma_j) = \epsilon\}$ and $\partial^\epsilon\Omega = \{x \in \partial\Omega;\ \mathrm{dist}(x, \Sigma) \geqslant \epsilon\}$. Then, the first term in (3.11) can be integrated by parts:

$$-2\int_{\Omega^\epsilon} \mathrm{div}\left((u + \bar{u})\,\nabla u_0\right) = -2\left( \int_{\partial^\epsilon\Omega} + \int_{\partial_0^\epsilon} + \sum_{j=1}^{N} \int_{\partial_j^\epsilon} \right) (u + \bar{u})\,\frac{\partial u_0}{\partial n}. \tag{3.13}$$

For the first integral on the right hand side of (3.13), we have as $\epsilon \to 0$:

$$\int_{\partial^\epsilon\Omega} (u + \bar{u})\,\frac{\partial u_0}{\partial n} \to \int_{\partial\Omega} (u + \bar{u})\,\frac{\partial u_0}{\partial n}, \tag{3.14}$$

which clearly is independent of $\varphi \in \mathcal{H}^*$. The integral on the right hand side of (3.14) is finite since $u + \bar{u}$ is bounded $\Omega_0 \setminus \Sigma_0$. Next, since $\varphi \in \mathcal{H}^*$, there is a compact set $K \subset \Omega$ such that $\varphi = \tilde{\varphi}$ outside $K$ and $\tilde{v} = w_0$ in a neighborhood of $\Sigma_0 \cap K$. We find:

$$\int_{\partial_0^\epsilon} (u + \bar{u}) \frac{\partial u_0}{\partial n} = \int_{\partial_0^\epsilon \setminus K} (u + \bar{u}) \frac{\partial u_0}{\partial n},$$

since for $\epsilon > 0$ small enough and $x \in \partial_0^\epsilon \cap K$, $\varphi(x)$ lies along the geodesic $\gamma_0$ where $u + \bar{u} = 0$. However for $x \in \partial_0^\epsilon \setminus K$, we have $\varphi(x) = \tilde{\varphi}(x)$ which implies $(u(x) + \bar{u}(x)) \leqslant \operatorname{dist}(\tilde{\varphi}(x), \gamma_0) \to 0$ as $\epsilon \to 0$. Thus, we obtain

$$\int_{\partial_0^\epsilon \setminus K} (u + \bar{u}) \frac{\partial u_0}{\partial n} \to 0.$$

Finally, in view of Lemma 3, we have for $1 \leqslant j \leqslant N$ that $\lim_{\epsilon \to 0} (u + \bar{u})|_{\partial_j^\epsilon} = d_j$, for some constants $d_j$. Hence the last sum in (3.13) also tends to zero since $u_0$ is regular in $\Omega_j$ for $j = 1, \ldots, N$. The other terms in (3.11) are handled similarly. See [We4, Proof of Lemma 13] for more details.

Now, truncate $\bar{u} - \bar{u}^{(0)}$ above at

$$\bar{T}_0 = \sup_{\partial\Omega} (\bar{u} - \bar{u}^{(0)}) + 1,$$

i.e. define a map $\varphi' = (u', v') \in \mathcal{H}$ by $\bar{\Phi} \circ \varphi' = (\bar{u}', \bar{v})$ where

$$\bar{u}' - \bar{u}^{(0)} = \min\{\bar{u} - \bar{u}^{(0)}, \bar{T}_0\}.$$

Clearly the map $\varphi' \in \mathcal{H}$ and satisfies:

$$\varphi'(x) \in \mathcal{B}_{\gamma_0}(\bar{u}^{(0)}(x) + \bar{T}_0), \qquad \forall x \in \Omega \setminus \Sigma, \tag{3.15}$$

and also, in view of (3.12), $F(\varphi') \leqslant F(\varphi)$. Let $c_0 = \sup_{\Omega_0} \sum_{j=1}^N u_j$, then we obtain from (3.15):

$$\varphi'(x) \in \mathcal{B}_{\gamma_0}(-u_0(x) + c_0 + \bar{T}_0), \qquad \forall x \in \Omega_0 \setminus \Sigma_0. \tag{3.16}$$

Next truncate $u' - u^{(0)}$ above at

$$T_0 = \max\left\{\sup_{\partial\Omega}(u - u^{(0)}) + 1,\, \bar{T}_0,\, (1/2)\log 2\right\},$$

to get a map $\varphi'' = (u'', v') \in \mathcal{H}$ satisfying:

$$\varphi''(x) \in \mathcal{B}_{-\gamma_0}(u^{(0)}(x) + T_0), \qquad \forall x \in \Omega \setminus \Sigma$$

and $F(\varphi'') \leqslant F(\varphi')$. As before it follows that:

$$\varphi''(x) \in \mathcal{B}_{-\gamma_0}(u_0(x) + c_0 + T_0), \qquad \forall x \in \Omega_0 \setminus \Sigma_0. \tag{3.17}$$

¿From Lemma 4 we obtain, as in [We4, Lemma 11], that the bound (3.16) still holds for $\varphi''$:

$$\varphi''(x) \in \mathcal{B}_{\gamma_0}(-u_0(x) + c_0 + T_0), \qquad \forall x \in \Omega_0 \setminus \Sigma_0. \tag{3.18}$$

Hence, combining (3.17) and (3.18) with Lemma 2, we conclude:

$$\varphi''(x) \in B_{R_0}(\varphi_0(x)), \qquad \forall x \in \Omega_0 \setminus \Sigma_0, \tag{3.19}$$

where $R_0 = c_0 + T_0 + \log 2$. Consider now the map $\varphi''|_{\bigcup_{j=1}^{N}\Omega_j}$, and notice that from (3.19) we can obtain a pointwise a priori estimate for $\varphi''$ on $\partial(\bigcup_{j=1}^{N}\Omega_j)$. Thus we can proceed by induction to obtain a map $\varphi''' \in \mathcal{H}$ satisfying $F(\varphi''') \leqslant F(\varphi)$, and for each $j = 0, \dots, N$:

$$\varphi'''(x) \in B_{R_j}(\varphi_j(x)), \qquad \forall x \in \Omega_j \setminus \Sigma_j,$$

for some constants $R_j$ depending only on the data, $\tilde{\varphi}$ and $\varphi_j$. Setting $R = \max_j R_j$, we have obtained $\varphi''' \in \mathcal{H}_R$ with $F(\varphi''') \leqslant F(\varphi)$. This completes the proof of Lemma 9 and of Proposition 1. □

*Remark.* It is important to note that the a priori bounds $\operatorname{dist}(\varphi, \varphi_j) \leqslant R$ in $\Omega_j \setminus \Sigma_j$ given by Lemma 9 depend on the size of $\Omega$, hence cannot be used to obtain a solution on $\mathbb{R}^n \setminus \Sigma$. This is remedied in the next step where we obtain a priori bounds independent of the size of $\Omega$.

### 3.3. Step Two.

In this step, we furnish the main ingredient in the proof of the Main Theorem. We establish pointwise a priori bounds for $\operatorname{dist}(\varphi, \tilde{\varphi})$ near $\infty$, and for $\operatorname{dist}(\varphi, \varphi_j)$ near $\Sigma_j$, where $\varphi\colon \Omega \setminus \Sigma \to \mathbb{H}$ is the solution given by Proposition 1. As in Proposition 1, $\Omega \subset \mathbb{R}^n$ is a sufficiently large ball, but we now use a slightly different open cover of $\mathbb{R}^n$. Since for $j \notin J$, $\tilde{\varphi}$ is asymptotic to $\varphi_j$ near $\Sigma_j$, we can choose $\Omega'_j$ to be a neighborhood of $\Sigma_j$ such that $\operatorname{dist}(\tilde{\varphi}, \varphi_j)$ is bounded on $\Omega'_j \setminus \Sigma_j$. However, to get an open cover of $\mathbb{R}^n$, we may need to add another open set $\widetilde{\Omega}'$ which we can choose so that $\Sigma_j \cap \widetilde{\Omega}' = \emptyset$ for all $0 \leqslant j \leqslant N$. We set $\Omega_j = \Omega'_j \cap \Omega$, and $\widetilde{\Omega} = \widetilde{\Omega}' \cap \Omega$. We also change the normalization of the harmonic functions $u_j$ for $j \in J$, so that $u_j(x) \to 0$ as $r = |x| \to \infty$ in $\mathbb{R}^n$.

**Proposition 2.** *There is a constant $R > 0$ independent of $\Omega$ such that if $\varphi$ is the solution given by Proposition 1, then*

$$\operatorname{dist}(\varphi, \tilde{\varphi}) \leqslant R, \quad \text{in } \widetilde{\Omega}, \tag{3.20}$$

$$\operatorname{dist}(\varphi, \varphi_j) \leqslant R, \quad \text{in } \Omega_j \setminus \Sigma_j, \text{ for } j = 0, \dots, N. \tag{3.21}$$

*Proof.* Define on $\Omega \setminus \Sigma$ the function:

$$\nu = \sqrt{1 + \operatorname{dist}(\varphi, \tilde{\varphi})^2} - u^{(1)}.$$

and note that since the function $\operatorname{dist}(\cdot, \cdot)^2$ is convex on $\mathbb{H} \times \mathbb{H}$, we have $\Delta\nu \geqslant 0$ on $\Omega \setminus \Sigma$, see [SY]. We claim that $\nu$ is bounded and therefore Lemma 7 applies. To see this, note first that for $j \notin J$, $\varphi$ is asymptotic to $\varphi_j$, which is asymptotic to $\tilde{\varphi}$ near $\Sigma_j$, hence $\operatorname{dist}(\varphi, \tilde{\varphi})$ is bounded in $\Omega_j$, and $u^{(1)}$ is also bounded there. Therefore $\nu$ is bounded on $\Omega_j$ for $j \notin J$. Now, let $j \in J$, and observe that $\operatorname{dist}(\gamma_j(0), \tilde{\varphi})$ is bounded on $\Omega_j$. Thus, since $\operatorname{dist}(\varphi_j, \gamma_j(0)) = u_j$, we see that

$$\begin{aligned}\nu &\leqslant 1 + \operatorname{dist}(\varphi, \varphi_j) + \operatorname{dist}(\varphi_j, \gamma_j(0)) + \operatorname{dist}(\gamma_j(0), \tilde{\varphi}) - u^{(1)} \\ &= 1 + \operatorname{dist}(\varphi, \varphi_j) + \operatorname{dist}(\gamma_j(0), \tilde{\varphi}) - \sum\nolimits_{\substack{j' \in J \\ j' \neq j}} u_{j'}\end{aligned}$$

Since $\varphi$ is asymptotic to $\varphi_j$ near $\Sigma_j$, and $u_{j'}$ is bounded in $\Omega_j \setminus \Sigma_j$ for $j' \neq j$, we obtain $\nu \leqslant C_j$ in $\Omega_j \setminus \Sigma_j$, hence $\nu \leqslant C$ in $\Omega \setminus \Sigma$. Similarly, we obtain

$$\nu \geqslant -\operatorname{dist}(\gamma_j(0), \tilde{\varphi}) - \operatorname{dist}(\varphi, \varphi_j) - \sum\nolimits_{\substack{j' \in J \\ j' \neq j}} u_{j'} \geqslant -C.$$

Applying Lemma 7, we deduce that:

$$\nu \leqslant \sup_{\partial\Omega^\delta \setminus \Sigma} \nu \tag{3.22}$$

Since $\varphi = \tilde{\varphi}$ on $\partial\Omega \setminus \Sigma$, $\nu \leqslant 1$ on $\partial\Omega \setminus \Sigma$, hence letting $\delta \to 0$, we conclude that:

$$\operatorname{dist}(\varphi, \tilde{\varphi}) \leqslant \sqrt{\left(1 + u^{(1)}\right)^2 - 1}. \tag{3.23}$$

Now, there is a $T$ such that $u^{(1)} \leqslant T$ on $\bigcup_{j \notin J} \Omega'_j \cup \widetilde{\Omega}'$. Thus, we obtain immediately $\operatorname{dist}(\varphi, \tilde{\varphi}) \leqslant \tilde{R}$ on $\widetilde{\Omega}$ with $\tilde{R} = 1 + T$. Furthermore, for $j \notin J$ there are constants $T_j > 0$ such that $\operatorname{dist}(\tilde{\varphi}, \varphi_j) \leqslant T_j$ on $\Omega'_j \setminus \Sigma_j$. Thus, from (3.23), we obtain for $j \notin J$ the required pointwise a priori estimate:

$$\operatorname{dist}(\varphi, \varphi_j) \leqslant R_j,$$

in $\Omega_j \setminus \Sigma_j$, where $R_j = 1 + T + T_j$ is clearly independent of $\Omega$. These estimates also provide us for $j \in J$ with pointwise a priori bounds for $\operatorname{dist}(\varphi, \varphi_j)$ on $\partial\Omega_j$. Hence we can now use the bounded subharmonic functions $\nu_j = \sqrt{1 + \operatorname{dist}(\varphi, \varphi_j)^2}$ in $\Omega_j \setminus \Sigma_j$ to obtain for $j \in J$ and for some $R_j$ independent of $\Omega$:

$$\operatorname{dist}(\varphi, \varphi_j) \leqslant R_j,$$

in $\Omega_j \setminus \Sigma_j$. Setting $R = \max_j R_j$, Proposition 2 follows. □

**3.4. Proof of the Main Theorem.** The proof of the uniqueness statement is almost the same as in Proposition 1. If $\varphi$ and $\varphi'$ are two solutions, then they are asymptotic near each $\Sigma_j$, and asymptotic at infinity, hence $\operatorname{dist}(\varphi, \varphi')^2$ is a bounded subharmonic function on $\mathbb{R}^n \setminus \Sigma$. From Lemma 7, it follows that for any ball $\Omega \subset \mathbb{R}^n$ of radius $R$ centered at the origin:

$$\sup_{\Omega} \operatorname{dist}(\varphi, \varphi') \leqslant \sup_{\partial\Omega} \operatorname{dist}(\varphi, \varphi'),$$

Since the right hand side tends to zero as the radius $R$ of $\Omega$ tends to $\infty$, it follows that $\operatorname{dist}(\varphi, \varphi') = 0$, and hence $\varphi = \varphi'$.

For the existence proof, we choose a sequence of radii $R_k \to \infty$, and denote by $\Lambda_k$ the sequence of balls of radius $R_k$ centered at the origin, and by $\hat{\varphi}_k = (\hat{u}_k, \hat{v}_k)$ the solutions given by Proposition 1 on $\Lambda_k$. We extend $\hat{\varphi}_k$ outside $\Lambda_k$ so that it coincides with $\tilde{\varphi}$. We will use the uniform pointwise a priori bounds given in Proposition 2 to prove the convergence of $\hat{\varphi}_k$ to a solution on $\mathbb{R}^n \setminus \Sigma$. We adopt the same notation as in Proposition 1, but use the open cover $\{\Omega'_j, \Omega'\}$ of $\mathbb{R}^n$ introduced in Section 3.3.

For any ball $\Omega$, define the space $\mathcal{H}(\Omega)$ as the space of maps $\varphi\colon \mathbb{R}^n \setminus \Sigma \to \mathbb{H}$ satisfying

$$\begin{cases} u - u^{(0)} \in H_1(\Omega); \\ v - v_0 \in \bigcap_{j=0}^{N} H_1^{\varphi_j}(\Omega; \mathbb{R}^{m-1}); \\ \operatorname{dist}(\varphi, \varphi_j) \in L^\infty(\Omega'_j \setminus \Sigma_j), \quad \forall j = 0, \dots, N, \end{cases}$$

and for $R > 0$ denote by $\mathcal{H}_R(\Omega)$ the space of maps $\varphi \in \mathcal{H}(\Omega)$ satisfying:

$$\begin{cases} \operatorname{dist}(\varphi, \tilde{\varphi}) \leqslant R, & \text{in } \widetilde{\Omega}, \\ \operatorname{dist}(\varphi, \varphi_j) \leqslant R, & \text{in } \Omega_j \setminus \Sigma_j, \text{ for } j = 0, \dots, N. \end{cases}$$

Now, fix $\Omega = \Lambda_{k_0}$, let $\Omega' = \{x \in \Omega;\ \operatorname{dist}(x, \partial\Omega) > 1\}$, and consider the sequence of maps $\hat{\varphi}_k|_\Omega$ for $k \geqslant k_0$. By Proposition 2, there is $R > 0$ such that $\hat{\varphi}_k \in \mathcal{H}_R(\Omega)$. It follows that on $\Omega_j \setminus \Sigma_j$, we have $|\hat{u}_k - u_j| \leqslant R$, and hence we obtain $\left|\hat{u}_k - u^{(0)}\right| \leqslant R'$, where $R' = R + \sup_{\Omega_j} \sum_{j' \neq j} |u_{j'}|$. Similarly, on $\widetilde{\Omega}$ we have $\left|\hat{u}_k - u^{(0)}\right| \leqslant |\hat{u}_k - \tilde{u}| + \left|\tilde{u} - u^{(0)}\right| \leqslant R''$ where $R'' = R + \sup_{\widetilde{\Omega}} \left|\tilde{u} - u^{(0)}\right|$. Thus, we conclude that $\left|\hat{u}_k - u^{(0)}\right| \leqslant C$ on $\Omega \setminus \Sigma$, where $C = \max\{R', R''\}$. Furthermore, we have $\Delta(\hat{u}_k - u^{(0)}) \geqslant 0$ on $\Omega \setminus \Sigma$. Now the argument in the proof of Lemma 7 shows that there is a constant $C'$ independent of $k$ such that

$$\int_{\Omega'} \left|\nabla(\hat{u}_k - u^{(0)})\right|^2 \leqslant C',$$

and similarly, using $Q_{\hat{\varphi}_k}(\nabla \hat{v}_k) \leqslant (1/2)\Delta(\hat{u}_k - u^{(0)})$, we deduce, as in the argument leading to (3.7), that

$$\int_{\Omega'} Q_{\hat{\varphi}_k}(\nabla \hat{v}_k) \leqslant C''.$$

Thus, we have a uniform bound $F(\hat{\varphi}_k) \leqslant C'''$. It is straightforward now to show that there is a subsequence, without loss of generality $\hat{\varphi}_k$, which converges weakly in $\mathcal{H}(\Omega')$ and pointwise a.e. in $\Omega'$, see [We4, Proof of Proposition 1].

Repeating this argument for each $k_0$, and using a diagonal sequence, it is clear that we can choose a subsequence, without loss of generality $\hat{\varphi}_k$ again, which converges pointwise a.e. in $\mathbb{R}^n \setminus \Sigma$, and which for each ball $\Omega \subset \mathbb{R}^n$, converges weakly in $\mathcal{H}(\Omega)$ to a map $\varphi \in \mathcal{H}_R(\Omega)$. For each open set $U \subset\subset \mathbb{R}^n \setminus \Sigma$, $\hat{\varphi}_k$ is a family of smooth harmonic maps with uniformly bounded energy into $\mathbb{H}$, and $\hat{\varphi}_k(U)$ is contained in a fixed compact subset of $\mathbb{H}$. Hence, using standard harmonic map theory, one can obtain uniform a priori bounds in $C^{2,\alpha}(U)$ for $\hat{\varphi}_k$, and hence we deduce that a subsequence converges uniformly in $U$ together with two of its derivatives. We conclude that $\varphi$ is a harmonic map. It remains to see that $\varphi$ is asymptotic to $\tilde{\varphi}$ at infinity. However, from the estimate (3.23) on $\nu$ which holds for each $\hat{\varphi}_h$, we deduce that the same holds of $\varphi$:

$$\operatorname{dist}(\varphi, \tilde{\varphi}) \leqslant \sqrt{u^{(1)}(2 + u^{(1)})}$$

Since $u^{(1)}(x) \to 0$ as $x \to \infty$, this shows that $\operatorname{dist}(\varphi, \tilde{\varphi}) \to 0$ as $x \to \infty$ in $\Omega \setminus \Sigma$. This completes the proof of the Main Theorem.

## 4. $N$- Black Hole Solutions of the Einstein-Maxwell Equations

In this section, starting from a solution $\varphi \colon \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ of the Reduced Problem given by the corollary of the Main Theorem we construct a solution $(M, g, F)$ of the Einstein-Maxwell Equations, which can be interpreted as representing $N$ co-axially rotating charged black holes in equilibrium held apart by singular struts.

**4.1. The Spacetime Metric $g$.** Let $\varphi \colon \mathbb{R}^3 \setminus \Sigma \to \mathbb{H}^2_{\mathbb{C}}$ be a solution of the Reduced Problem. Then $\varphi = (u, v, \chi, \psi)$ satisfies on $\mathbb{R}^3 \setminus \Sigma$ the following nonlinear elliptic system of partial differential equations:

(4.1)
$$\Delta u - 2e^{4u} \, |\nabla v + \chi \nabla \psi - \psi \nabla \chi|^2 - e^{2u} \left( |\nabla \chi|^2 + |\nabla \psi|^2 \right) = 0$$

$$\operatorname{div} \left( e^{4u} (\nabla v + \chi \nabla \psi - \psi \nabla \chi) \right) = 0 \tag{4.2}$$

$$\operatorname{div} \left( e^{2u} \nabla \chi \right) - 2e^{4u} \nabla \psi \cdot (\nabla v + \chi \nabla \psi - \psi \nabla \chi) = 0 \tag{4.3}$$

$$\operatorname{div} \left( e^{2u} \nabla \psi \right) + 2e^{4u} \nabla \chi \cdot (\nabla v + \chi \nabla \psi - \psi \nabla \chi) = 0. \tag{4.4}$$

Introduce cylindrical coordinates $(\rho, \phi, z)$ in $\mathbb{R}^3$, and denote by $\xi$ and $\zeta$ the vector fields $\partial/\partial\phi$ and $\partial/\partial z$ respectively. Let $*$ be the Hodge star operator of the Euclidean metric in $\mathbb{R}^3$ and $\star = \rho^{-1} i_\xi *$ the Hodge star operator of the flat metric $ds^2 = d\rho^2 + dz^2$ in the $\rho z$-plane. Setting $\omega = 2 \, (dv + \chi \, d\psi - \psi \, d\chi)$ it follows from (4.2) that the two-form $e^{4u} * \omega$ is closed on $\mathbb{R}^3 \setminus \Sigma$, and hence, since $\omega$ is invariant under $\xi$, the one-form $e^{4u} i_\xi * \omega$ is closed on $\mathbb{R}^3 \setminus \Sigma$. Since $\mathbb{R}^3 \setminus \Sigma$ is simply connected, there is a function $w$, defined up to an additive constant such that

$$dw = e^{4u} i_\xi * \omega = \rho \, e^{4u} \star \omega, \tag{4.5}$$

see (2.20). Now let $T_\varphi$ be the stress tensor of the map $\varphi$:

$$T_\varphi(X, Y) = \langle d\varphi(X), d\varphi(Y) \rangle - \frac{1}{2} \, |d\varphi|^2 \, X \cdot Y,$$

for $X, Y \in \mathbb{R}^3$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{H}^2_{\mathbb{C}}$. Clearly $T_\varphi$ is symmetric, and since $\varphi$ is a harmonic map $T_\varphi$ is also divergence free. Thus, since $\zeta$ is a Killing field in $\mathbb{R}^3$, $i_\zeta T_\varphi$ is a divergence free vector field on $\mathbb{R}^3 \setminus \Sigma$. As before, it follows that $i_\xi * i_\zeta T_\varphi$ is a closed one-form on $\mathbb{R}^3 \setminus \Sigma$, hence there is a function $\lambda$, defined up to an additive constant such that

$$d(\lambda - u) = -2 \, i_\xi * i_\zeta T_\varphi.$$

Note that

$$-2i_\xi{*}i_\zeta T_\varphi = \rho\left[\left(u_\rho^2 - u_z^2 + \frac{1}{4}e^{4u}(\omega_\rho^2 - \omega_z^2) + e^{2u}(\chi_\rho^2 - \chi_z^2 + \psi_\rho^2 - \psi_z^2)\right)d\rho \right.$$
$$\left. + 2\left(u_\rho u_z + \frac{1}{4}e^{4u}\omega_\rho\,\omega_z + e^{2u}(\chi_\rho\chi_z + \psi_\rho\psi_z)\right)dz\right],$$

and compare with Equations (2.21). It is clear that $w$ and $\lambda$ so defined are axially symmetric. Furthermore, we deduce from (4.5) that $e^{4u}{*}\omega = d\phi\wedge dw$. Thus, Equation (4.4) implies that the two-form $e^{2u} * d\psi + w\, d\chi \wedge d\phi$ is closed. It follows that the one-form $e^{2u}i_\xi * d\psi - w\, d\chi$ is closed, and hence there is a function $\tilde{\chi}$ such that

$$d\tilde{\chi} = e^{2u}i_\xi * d\psi - w\, d\chi.$$

Let $\mathbb{A} \subset \mathbb{R}^3$ be the $z$-axis and let $g$ be the metric on $M' = \mathbb{R}\times(\mathbb{R}^3 \setminus \mathbb{A})$ given by the line element:

$$ds^2 = -\rho^2 e^{2u}\, dt^2 + e^{-2u}(d\phi - w\, dt)^2 + e^{2\lambda}(d\rho^2 + dz^2). \tag{4.6}$$

Define the one-form $A$ on $M'$ by:

$$A = -(\chi\, d\phi + \tilde{\chi}\, dt), \tag{4.7}$$

and let

$$F = dA = dt \wedge d\tilde{\chi} + d\phi \wedge d\chi.$$

We have that $i_\xi F = d\chi$, and $i_\tau F = d\tilde{\chi}$, where $\tau = \partial/\partial t$. Clearly $\xi$ and $\tau$ are commuting Killing fields of $(M', g)$ which generate an abelian two-parameter group $G = \mathbb{R}\times SO(2)$ of isometries leaving $F$ invariant with timelike two-dimensional orbits, and hence $(M', g, F)$ is stationary and axially symmetric. Furthermore, it is not difficult to check that $\omega$ is the twist of $\xi$, and that $i_\xi * F = d\psi$ where here $*$ denotes the Hodge star operator of the metric $g$. It is then quite straightforward to verify that $(M', g, F)$ satisfies the Einstein-Maxwell equations. Finally, since the parameters $\{\mu_j, r_j, \lambda_j, q_j\}$ are geometric invariants, these solutions form a $(4N-1)$-parameter family of stationary and axially symmetric solutions of the Einstein-Maxwell equations.

**4.2. Conclusion.** We have shown the existence a $(4N-1)$-parameter family of solutions of the Einstein-Maxwell equations which are stationary and axially symmetric, and which should be interpreted as $N$ co-axially rotating charged black holes in equilibrium held apart by singular struts. In order to complete this interpretation, it is necessary to show that the metric $g$ and the field $F$ can be extended smoothly across the axis of symmetry $\Sigma$. This requires first the smoothness of the metric and field components across $\Sigma$, which would follow from the smoothness of $e^u$, $v$, $\chi$ and $\psi$ on $\Sigma \setminus \partial\Sigma$, where $\partial\Sigma$ denotes the set of $2N$ endpoints of $\Sigma$; less regularity is expected on $\partial\Sigma$, see [We2]. In fact, one expects to have some regularity properties for any maps $\varphi\colon \mathbb{R}^n \setminus \Sigma \to \mathbb{H}$ obtained from the Main Theorem. For the target $\mathbb{H} = \mathbb{H}^2_{\mathbb{R}}$ corresponding to the Vacuum Einstein Equations this was established in the axially symmetric case and $n = 3$ in [We1], and in general in [LTi2].

Even after smoothness is established, there is still the possibility of a conical singularity on the bounded components of $\Sigma$. As in the vacuum case, this is to be interpreted as a singular strut holding the black holes apart, and the angle deficiency can be related to the force between these. Finally, to prove that $g$ is asymptotically flat, the decay estimate obtained at infinity must be somewhat improved, see [We2].

These questions will be addressed in a forthcoming paper.

## References

[BW] R. Bach and H. Weyl, *Neue Lösungen der Einsteinschen Gravitationsgleichungen*, Mathematische Zeitschrift **13** (1921), 132–145.

[C1] B. Carter, *Black Hole Equilibrium States*, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt, Gordon and Breach Science Publishers, New York, 1973.

[C2] B. Carter, *Bunting Identity and Mazur Identity for Nonlinear Systems Including the Black Hole Equilibrium System*, Comm. Math. Phys. **99** (1985), 563–591.

[Ch] P. T. Chruściel, *"No Hair" Theorems — Folklore, Conjectures, Results*, to appear in Proc. of AMS/CMS Conf. on Diff. Geom. and Math. Phys., Vancouver, August 1993, Eds. J. Beem and K. Duggal, Contemporary Mathematics, AMS, 1994.

[CW] P. T. Chruściel and R. M. Wald, *On the Topology of Black Holes*, ESI preprint, Vienna, October 1994.

[Eb] P. Eberlein and B. O'Neill, *Visibility Manifolds*, Pacific J. Math. **46** (1973), No. 1, 45–109.

[Er] F. J. Ernst, *New Formulation of the Gravitational Field Problem*, Phys. Rev. Letters **167** (1968), 1175–1178.

[HI] E. Heintze and H. C. Im Hof, *Geometry of Horospheres*, J. Diff. Geom. **12** (1977), 481–491.

[LTa1] P. Li and L.-F. Tam, *The Heat Equation and Harmonic Maps of Complete Manifolds*, Invent. Math. **105** (1991), 159–192.

[LTa2] ———, *Uniqueness and Regularity of Proper Harmonic Maps*, Ann. Math. **137** (1993), 167–201.

[LTi1] Y. Li and G. Tian, *Nonexistence of Axially Symmetric, Stationary Solution of Einstein Vacuum Equation with Disconnected Symmetric Event Horizon*, Manuscripta Math. **73** (1991), 83–89.

[LTi2] ———, *Regularity of Harmonic Maps with Prescribed Asymptotic Behavior and Applications*, Comm. Math. Phys. **149** (1992), No. 1, 1–30.

[Mj] S. D. Majumdar, *A Class of Exact Solutions of Einstein's Field Equations*, Phy. Rev. **72** (1947), 390–398.

[Mz] P. O. Mazur, *Proof of Uniqueness of the Kerr-Newman Black Holes*, J. Phys. A **15** (1982), 3173–3180.

[Mo] G. D. Mostow, *Strong Rigidity of Locally Symmetric Spaces*, Annals of Mathematics Studies, No. 78, Princeton University Press, Princeton, 1973.

[Pa] A. Papapetrou, *A Static Solution of the Equations of the Gravitational Field for an Arbitrary Charge-Distribution*, Proc. Roy. Irish Acad. **A51** (1947), 191–204.

[Pe] R. Penrose, *Some Unsolved Problems in Classical General Relativity*, in *Seminar on Differential Geometry*, edited by S. T. Yau, Annals of Mathematics Studies no. 102, Princeton University Press, Princeton 1982.

[R] D. C. Robinson, *Uniqueness of the Kerr Black Hole*, Phys. Rev. Letters **34** (1975), 905–906.

[SU1] R. Schoen and K. Uhlenbeck, *A Regularity Theory for Harmonic Maps*, J. Diff. Geom. **17** (1982), 307–335.

[SU2] ———, *Boundary Regularity and the Dirichlet Problem for Harmonic Maps*, J. Diff. Geom. **18** (1983), 253–268.

[SY] R. Schoen and S. T. Yau, *Compact Group Actions and the Topology of Manifolds with Non-Positive Curvature*, Top. **18** (1979), 361–380.

[We1] G. Weinstein, *On Rotating Black Holes in Equilibrium in General Relativity*, Comm. Pure Appl. Math. **43** (1990), 903–948.

[We2] ———, *The Stationary Axisymmetric Two-Body Problem in General Relativity*, Comm. Pure Appl. Math. **45** (1992), 1183–1203.

[We3] ———, *On the Force between Rotating Co-Axial Black Holes*, Trans. Amer. Math. Soc. **343** (1994), No. 2, 899–906.

[We4] ———, *On the Dirichlet Problem for Harmonic Maps with Prescribed Singularities*, Duke Math. J. to appear.

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