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STABILITY OF G-FRAMES

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ABSTRACT. G-frames are natural generalizations of frames which cover many other recent generalizations of frames, e.g., bounded quasi-projectors, frames of subspaces, outer frames, oblique frames, pseudo-frames and a class of time-frequency localization operators. Moreover, it was shown that g-frames are equivalent to stable space splittings. In this paper, we study the stability of g-frames. We first present some properties for g-Bessel sequences. Then we prove that g-frames are stable under small perturbations. We also study the stability of dual g-frames.

1. INTRODUCTION

The frame has many nice properties which make it very useful in the characterization of function spaces, signal processing and many other fields. We refer to [5, 7, 9, 22] for an introduction to the frame theory and its applications.

In [19], a generalization of the frame concept was introduced. Let $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert spaces and $\{\mathcal{V}_j : j \in \mathbb{J}\}$ be a sequence of closed subspaces of $\mathcal{V}$, where $\mathbb{J}$ is a subset of $\mathbb{Z}$. Let $\mathcal{L}(\mathcal{U}, \mathcal{V}_j)$ be the collection of all bounded linear operators from $\mathcal{U}$ into $\mathcal{V}_j$.

Recall that a sequence $\{\Lambda_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j) : j \in \mathbb{J}\}$ is said to be a generalized frame, or simply a g-frame, for $\mathcal{U}$ with respect to $\{\mathcal{V}_j : j \in \mathbb{J}\}$ if there are two positive constants A and B such that

$$A\|f\|^2 \le \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \le B\|f\|^2, \qquad \forall f \in \mathcal{U}. \tag{1.1}$$

A and B are the lower and upper frame bounds, respectively. If the right-hand side of (1.1) holds, it is said to be a g-Bessel sequence.

G-frames are natural generalizations of frames which cover many other recent generalizations of frames, e.g., bounded quasi-projectors [13, 14], frames of subspaces [2, 3], outer frames [1], oblique frames [6, 10], pseudo-frames [17] and a class of time-frequency localization operators [8]. It was shown that g-frames are equivalent to stable space splittings studied in [18]. We refer to [19] for details. Note that there are also other generalizations of the frame concept, e.g., see [15, 16].

The stability of frames is important in practice and is therefore studied widely by many authors, e.g., see [4, 5, 11, 20, 21, 22]. The following is a fundamental result in the study of the stability of frames.

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Proposition 1.1. [4, Theorem 2] *Let* $\{f_i : i \ge 1\}$ *be a frame for some Hilbert space* $\mathcal{H}$ *with bounds* A, B. *Let* $\{g_i : i \ge 1\} \subset \mathcal{H}$ *and assume that there exist constants* $\lambda_1, \lambda_2, \mu \ge 0$ *such that* $\max\{\lambda_1 + \mu/\sqrt{A}, \lambda_2\} < 1$ *and*

$$\Big\|\sum_{i=1}^{n} c_i(f_i - g_i)\Big\| \le \lambda_1\Big\|\sum_{i=1}^{n} c_i f_i\Big\| + \lambda_2\Big\|\sum_{i=1}^{n} c_i g_i\Big\| + \mu\Big(\sum_{i=1}^{n}|c_i|^2\Big)^{1/2} \tag{1.2}$$

for all $c_1, \cdots, c_n (n \ge 1)$. *Then* $\{g_i : i \ge 1\}$ *is a frame for* $\mathcal{H}$ *with bounds*

$$A\Big(1 - \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{A}}{1+\lambda_2}\Big)^2, \qquad B\Big(1 + \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{A}}{1-\lambda_2}\Big)^2.$$

In this paper, we study the stability of g-frames. We first present some properties of g-Bessel sequences. Then we give analogs of Proposition 1.1 for the case of g-frames.

The stability of dual frames is also needed in practice. However, there are relatively few of results on this topic. In [12], the authors proved that if two frames are close to each other, so are their dual frames in the same sense. In this paper, we give a similar result for the case of g-frames.

2. G-BESSEL SEQUENCES

Let $\Lambda_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)$. Suppose that $\{e_{j,k} : k \in \mathbb{K}_j\}$ is an orthonormal basis for $\mathcal{V}_j$, where $\mathbb{K}_j$ is a subset of $\mathbb{Z}$, $j \in \mathbb{J}$. Let

$$u_{j,k} = \Lambda_j^* e_{j,k}, \qquad j \in \mathbb{J}, k \in \mathbb{K}_j. \tag{2.1}$$

It was shown in [19] that

$$\Lambda_j f = \sum_{k \in \mathbb{K}_j} \langle f, u_{j,k}\rangle e_{j,k}, \qquad \forall f \in \mathcal{U}. \tag{2.2}$$

and

$$\Lambda_j^* g = \sum_{k \in \mathbb{K}_j} \langle g, e_{j,k}\rangle u_{j,k}, \qquad \forall g \in \mathcal{V}_j. \tag{2.3}$$

We call $\{u_{j,k} : j \in \mathbb{J}, k \in \mathbb{K}_j\}$ the sequence induced by $\{\Lambda_j : j \in \mathbb{J}\}$ with respect to $\{e_{j,k} : j \in \mathbb{J}, k \in \mathbb{K}_j\}$.

With above representations of Λ_j and Λ_j^*, the following characterizations of generalized frames and Bessel sequences were proved in [19].

Proposition 2.1. *Let* $\Lambda_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)$ *and* $u_{j,k}$ *be defined as in (2.1). Then* $\{\Lambda_j : j \in \mathbb{J}\}$ *is a g-frame (g-Bessel sequence) for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_j : j \in \mathbb{J}\}$ *if and only if* $\{u_{j,k} : j \in \mathbb{J}, k \in \mathbb{K}_j\}$ *is a frame (Bessel sequence) for* $\mathcal{U}$ *with the same bounds.*

The following is an equivalent condition for a sequence of operators to be a g-Bessel sequence.

Lemma 2.2. $\{\Lambda_j : j \in \mathbb{J}\}$ *is a g-Bessel sequence with an upper bound* B *if and only if for any finite subset* $\mathbb{J}_1 \subset \mathbb{J}$,

$$\Big\|\sum_{j \in \mathbb{J}_1} \Lambda_j^* g_j\Big\|^2 \le B \sum_{j \in \mathbb{J}_1} \|g_j\|^2, \qquad g_j \in \mathcal{V}_j.$$

Proof. Since $\{e_{j,k} : k \in \mathbb{K}_j\}$ is an orthonormal basis for $\mathcal{V}_j$, every $g_j \in \mathcal{V}_j$ has an expansion of the form $g_j = \sum_{k\in\mathbb{K}_j} c_{j,k}e_{j,k}$, where $\{c_{j,k} : k \in \mathbb{K}_j\} \in \ell^2(\mathbb{K}_j)$. It follows that

$$\left\|\sum_{j\in\mathbb{J}_1} \Lambda_j^* g_j\right\|^2 \le B \sum_{j\in\mathbb{J}_1} \|g_j\|^2$$

is equivalent to

$$\left\|\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{K}_j} c_{j,k}u_{j,k}\right\|^2 \le B \sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{K}_j} |c_{j,k}|^2.$$

Now the conclusion follows from Proposition 2.1. □

Lemma 2.3.

(i). *Suppose that* $\{\Lambda_j : j \in \mathbb{J}\}$ *is a g-Bessel sequence with an upper bound* B. *Then for any finite sequence of complex numbers* $\{c_j : j \in \mathbb{J}\}$ *(all but finite* c_j *are zero),*

$$\left\|\sum_{j\in\mathbb{J}} c_j\Lambda_j\right\|^2 \le B\sum_{j\in\mathbb{J}} |c_j|^2. \tag{2.4}$$

(ii). *If (2.4) is satisfied,* $\mathcal{V}_j = \mathcal{V}, \forall j \in \mathbb{J}$ *and* $K := \dim \mathcal{V} < +\infty$, *then* $\{\Lambda_j : j \in \mathbb{J}\}$ *is a g-Bessel sequence with an upper bound* KB.

Proof. First, we prove (i). For any $\{c_j : j \in \mathbb{J}\}$, we have

$$\begin{aligned}
\left\|\sum_{j\in\mathbb{J}} c_j\Lambda_j\right\|^2 &= \sup_{\substack{f\in\mathcal{U}\\ \|f\|=1}} \left\|\sum_{j\in\mathbb{J}} c_j\Lambda_j f\right\|^2 \\
&= \sup_{\substack{f\in\mathcal{U}\\ \|f\|=1}} \sup_{\substack{g\in\mathcal{V}\\ \|g\|=1}} \left\|\sum_{j\in\mathbb{J}} c_j\langle\Lambda_j f, g\rangle\right\|^2 \\
&\le \sup_{\substack{f\in\mathcal{U}\\ \|f\|=1}} \sup_{\substack{g\in\mathcal{V}\\ \|g\|=1}} \sum_{j\in\mathbb{J}} |c_j|^2 \sum_{j\in\mathbb{J}} |\langle\Lambda_j f, g\rangle|^2 \\
&\le \sup_{\substack{f\in\mathcal{U}\\ \|f\|=1}} \sum_{j\in\mathbb{J}} |c_j|^2 \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \\
&\le B\sum_{j\in\mathbb{J}} |c_j|^2.
\end{aligned}$$

Next, we prove the second part. Suppose that $\{e_k : 1 \le k \le K\}$ is an orthonormal basis for $\mathcal{V}$. Let $u_{j,k} = \Lambda_j^* e_k$. Then we have

$$\Lambda_j f = \sum_{1\le k\le K} \langle f, u_{j,k}\rangle e_k.$$

It follows that

$$\begin{aligned}
&\sum_{1\le k\le K}\left|\sum_{j\in\mathbb{J}} c_j\langle f, u_{j,k}\rangle\right|^2 = \left\|\sum_{1\le k\le K}\sum_{j\in\mathbb{J}} c_j\langle f, u_{j,k}\rangle e_k\right\|^2 \\
= &\left\|\sum_{j\in\mathbb{J}} c_j\Lambda_j f\right\|^2 \le \left\|\sum_{j\in\mathbb{J}} c_j\Lambda_j\right\|^2 \|f\|^2 \le B\|f\|^2 \sum_{j\in\mathbb{J}} |c_j|^2
\end{aligned}$$

Hence

$$\left|\sum_{j\in\mathbb{J}} c_j\langle f, u_{j,k}\rangle\right|^2 \le B\|f\|^2\sum_{j\in\mathbb{J}}|c_j|^2, \qquad 1\le k\le K.$$

Since $\{c_j\}$ is arbitrary, we have

$$\sum_{j\in\mathbb{J}}|\langle f, u_{j,k}\rangle|^2 \le B\|f\|^2, \qquad 1\le k\le K.$$

Hence

$$\sum_{1\le k\le K}\sum_{j\in\mathbb{J}}|\langle f, u_{j,k}\rangle|^2 \le KB\|f\|^2.$$

By Proposition 2.1, $\{\Lambda_j : j\in\mathbb{J}\}$ is a g-Bessel sequence with an upper bound KB. $\square$

Remark 2.1. In general, (2.4) does not imply that $\{\Lambda_j : j\in\mathbb{J}\}$ is a g-Bessel sequence whenever $\dim V = +\infty$. The following is a counterexample.

Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\{e_j : j\in\mathbb{Z}\}$ be its orthonormal basis. Define $\Lambda_j : \mathcal{H}\to\mathcal{H}$ as follows:

$$\Lambda_j f = \langle f, e_1\rangle e_j.$$

Then we have

$$\left\|\sum_{j\in\mathbb{Z}} c_j\Lambda_j f\right\|^2 = \left\|\sum_{j\in\mathbb{Z}} c_j\langle f, e_1\rangle e_j\right\|^2 = |\langle f, e_1\rangle|^2\sum_{j\in\mathbb{Z}}|c_j|^2 \le \|f\|^2\cdot\sum_{j\in\mathbb{Z}}|c_j|^2.$$

Hence (2.4) is satisfied with $B=1$. But $\sum_{j\in\mathbb{Z}}\|\Lambda_j f\|^2 = \infty$ whenever $\langle f, e_1\rangle \ne 0$. Hence $\{\Lambda_j : j\in\mathbb{Z}\}$ is not a g-Bessel sequence.

Note that $\dim\Lambda_j\mathcal{H} = 1$. The above counterexample shows also that (2.4) does not imply that $\{\Lambda_j : j\in\mathbb{J}\}$ is a g-Bessel sequence even if $\dim V_j = 1$, $\forall j\in\mathbb{J}$.

Remark 2.2. In the second part of Lemma 2.3, the estimate for the upper bound of the g-Bessel sequence $\{\Lambda_j : j\in\mathbb{J}\}$ is best possible whenever $\mathbb{J}=\mathbb{Z}$.

To see this, let $\{u_j : j\in\mathbb{Z}\}$ and $\{e_k : 1\le k\le K\}$ be orthonormal bases for $\mathcal{U}$ and $\mathcal{V}$ respectively. Define

$$\Lambda_{jK+k} f = \langle f, u_j\rangle e_k, \qquad j\in\mathbb{Z}, 1\le k\le K.$$

We prove that

$$\sup_{\|c\|=1}\left\|\sum_{j\in\mathbb{Z}} c_j\Lambda_j\right\| = 1 \quad\text{and}\quad \sup_{\|f\|=1}\sum_{j\in\mathbb{Z}}\|\Lambda_j f\|^2 = K.$$

In fact, for any $c := \{c_j : j\in\mathbb{Z}\}\in\ell^2$ anf $f\in\mathcal{U}$, we have

$$\begin{aligned}
\left\|\sum_{j\in\mathbb{Z}} c_j\Lambda_j f\right\|^2 &= \left\|\sum_{j\in\mathbb{Z}}\sum_{1\le k\le K} c_{jK+k}\Lambda_{jK+k} f\right\|^2 \\
&= \left\|\sum_{j\in\mathbb{Z}}\sum_{1\le k\le K} c_{jK+k}\langle f, u_j\rangle e_k\right\|^2 \\
&= \sum_{1\le k\le K}\left|\sum_{j\in\mathbb{Z}} c_{jK+k}\langle f, u_j\rangle\right|^2 \\
&\le \sum_{1\le k\le K}\sum_{j\in\mathbb{Z}}|c_{jK+k}|^2\sum_{j\in\mathbb{Z}}|\langle f, u_j\rangle|^2
\end{aligned} \tag{2.5}$$

$$= \quad \|f\|^2 \cdot \|c\|^2.$$

Hence

$$\sup_{\|c\|=1} \Big\| \sum_{j\in\mathbb{Z}} c_j \Lambda_j \Big\| \le 1. \tag{2.6}$$

Fix some $f_0 \neq 0$. By setting $c_{jK+k} = \langle u_j, f_0\rangle$, $j \in \mathbb{Z}, 1 \le k \le K$, similar to (2.5) we can get that

$$\Big\| \sum_{j\in\mathbb{Z}} c_j \Lambda_j f_0 \Big\|^2 = \|f_0\|^2 \|c\|^2.$$

Hence

$$\sup_{\|c\|=1} \Big\| \sum_{j\in\mathbb{Z}} c_j \Lambda_j \Big\| \ge 1. \tag{2.7}$$

Putting (2.6) and (2.7) together, we get

$$\sup_{\|c\|=1} \Big\| \sum_{j\in\mathbb{Z}} c_j \Lambda_j \Big\| = 1.$$

On the other hand, it is easy to see that

$$\sum_{j\in\mathbb{Z}} \|\Lambda_j f\|^2 = \sum_{j\in\mathbb{Z}} \sum_{1\le k\le K} \|\Lambda_{jK+k} f\|^2 = \sum_{1\le k\le K} \sum_{j\in\mathbb{Z}} |\langle f, u_j\rangle|^2 = K\|f\|^2.$$

Hence $\{\Lambda_j : j \in \mathbb{Z}\}$ is a g-Bessel sequence with an upper bound K.

3. Stability of g-frames

In this section, we study the stability of g-frames. Similar to ordinary frames, g-frames are stable under small perturbations. Specifically, we have the following.

Theorem 3.1. *Let $\{\Lambda_j : j \in \mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j : j \in \mathbb{J}\}$. Let A, B be the frame bounds. Suppose that $\Gamma_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)$ and there exist constants $\lambda_1, \lambda_2, \mu \ge 0$ such that $\max\{\lambda_1 + \mu/\sqrt{A}, \lambda_2\} < 1$ and one of the following two conditions is satisfied,*

$$\begin{aligned}\Big(\sum_{j\in\mathbb{J}} \|(\Lambda_j - \Gamma_j)f\|^2\Big)^{1/2} \quad &\le \quad \lambda_1 \Big(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\Big)^{1/2} + \lambda_2 \Big(\sum_{j\in\mathbb{J}} \|\Gamma_j f\|^2\Big)^{1/2} \\ &\qquad + \mu\|f\|, \qquad \forall f \in \mathcal{H},\end{aligned} \tag{3.1}$$

or

$$\Big\| \sum_{j\in\mathbb{J}_1} (\Lambda_j^* - \Gamma_j^*) g_j \Big\| \le \lambda_1 \Big\| \sum_{j\in\mathbb{J}_1} \Lambda_j^* g_j \Big\| + \lambda_2 \Big\| \sum_{j\in\mathbb{J}_1} \Gamma_j^* g_j \Big\| + \mu \Big(\sum_{j\in\mathbb{J}_1} \|g_j\|^2\Big)^{1/2} \tag{3.2}$$

for any finite subset $\mathbb{J}_1 \subset \mathbb{J}$ and $g_j \in \mathcal{V}_j$. Then $\{\Lambda_j : j \in \mathbb{J}\}$ is a g-frame for $\mathcal{U}$ with bounds

$$A\Big(1 - \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{A}}{1+\lambda_2}\Big)^2, \qquad B\Big(1 + \frac{\lambda_1 + \lambda_2 + \mu/\sqrt{A}}{1-\lambda_2}\Big)^2.$$

Proof. First, we assume that (3.1) is satisfied. Observe that

$$\|f\|^2 \le \frac{1}{A}\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2.$$

We see from (3.1) that

$$\begin{aligned}\Big(\sum_{j\in\mathbb{J}} \|(\Lambda_j-\Gamma_j)f\|^2\Big)^{1/2} \quad &\le \quad \Big(\lambda_1+\frac{\mu}{\sqrt{A}}\Big)\Big(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\Big)^{1/2} \\ &\qquad +\lambda_2\Big(\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2\Big)^{1/2}.\end{aligned}$$

By the triangle inequality, we have

$$\Big(\sum_{j\in\mathbb{J}} \|(\Lambda_j-\Gamma_j)f\|^2\Big)^{1/2} \ge \Big(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\Big)^{1/2} - \Big(\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2\Big)^{1/2}$$

Hence

$$\begin{aligned}(1+\lambda_2)\Big(\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2\Big)^{1/2} \quad &\ge \quad \Big(1-\lambda_1-\frac{\mu}{\sqrt{A}}\Big)\Big(\sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2\Big)^{1/2} \\ &\ge \quad \Big(1-\lambda_1-\frac{\mu}{\sqrt{A}}\Big)\sqrt{A}\|f\|.\end{aligned}$$

Therefore,

$$\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2 \ge A\Big(1-\frac{\lambda_1+\lambda_2+\mu/\sqrt{A}}{1+\lambda_2}\Big)^2\|f\|^2.$$

Similarly we can prove that

$$\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2 \le B\Big(1+\frac{\lambda_1+\lambda_2+\mu/\sqrt{A}}{1-\lambda_2}\Big)^2\|f\|^2.$$

Next, we assume that (3.2) is satisfied. For this case, the conclusion is a consequence of Propositions 1.1 and 2.1. □

Remark 3.1. In general, the inequality

$$\Big\|\sum_{j\in\mathbb{J}_1} c_j(\Lambda_j-\Gamma_j)\Big\| \le \lambda_1\Big\|\sum_{j\in\mathbb{J}_1} c_j\Lambda_j\Big\| + \lambda_2\Big\|\sum_{j\in\mathbb{J}_1} c_j\Gamma_j\Big\| + \mu\Big(\sum_{j\in\mathbb{J}_1}|c_j|^2\Big)^{1/2}$$

does not imply that $\{\Gamma_j : j\in\mathbb{J}\}$ is a g-frame. The following is a counterexample.

Assume that $\mathcal{U}=\mathcal{V}$ and $\{e_j : j\in\mathbb{J}\}$ is an orthonormal basis for $\mathcal{U}$. Fix some $\varepsilon>0$. Let

$$\Lambda_j f = \langle f, e_j\rangle e_1 \qquad \text{and} \qquad \Gamma_j f = \Lambda_j f + \varepsilon\langle f, e_1\rangle e_j.$$

For any $c := \{c_j : j\in\mathbb{Z}\}\in\ell^2$, we have

$$\Big\|\sum_{j\in\mathbb{Z}} c_j\Lambda_j\Big\| = \sup_{\|f\|=1}\Big\|\sum_{j\in\mathbb{Z}} c_j\Lambda_j f\Big\| = \sup_{\|f\|=1}\Big|\sum_{j\in\mathbb{Z}} c_j\langle f, e_j\rangle\Big| = \|c\|.$$

Hence,

$$\Big\|\sum_{j\in\mathbb{J}} c_j(\Lambda_j-\Gamma_j)\Big\| \quad = \quad \sup_{\|f\|=1}\Big\|\sum_{j\in\mathbb{J}} c_j\varepsilon\langle f, e_1\rangle e_j\Big\| \tag{3.3}$$

$$
\begin{aligned}
&= \sup_{\|f\|=1} \varepsilon\Big(\sum_{j\in\mathbb{J}} |c_j\langle f, e_1\rangle|^2\Big)^{1/2}\\
&= \varepsilon\Big(\sum_{j\in\mathbb{J}} |c_j|^2\Big)^{1/2}\\
&= \varepsilon\Big\|\sum_{j\in\mathbb{Z}} c_j\Lambda_j\Big\|.
\end{aligned}
$$

It follows that

$$\Big\|\sum_{j\in\mathbb{J}} c_j\Gamma_j\Big\| = \Big\|\sum_{j\in\mathbb{J}} c_j(\Gamma_j - \Lambda_j + \Lambda_j)\Big\| \ge (1-\varepsilon)\|c\|.$$

Therefore,

$$\Big\|\sum_{j\in\mathbb{J}} c_j(\Gamma_j - \Lambda_j)\Big\| = \varepsilon\|c\| \le \frac{\varepsilon}{1-\varepsilon}\Big\|\sum_{j\in\mathbb{J}} c_j\Gamma_j\Big\|. \tag{3.4}$$

On the other hand, it is easy to see that $\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 = \|f\|^2 < +\infty$ and

$$\sum_{j\in\mathbb{J}} \|\Gamma_j f\|^2 = \sum_{j\in\mathbb{J}} \|\Lambda_j f + \varepsilon\langle f, e_1\rangle e_j\|^2 \ge \sum_{j\in\mathbb{J}} \big(\|\Lambda_j f\| - \varepsilon|\langle f, e_1\rangle|\big)^2 = \infty$$

whenever $\langle f, e_1\rangle \ne 0$. Hence $\{\Lambda_j : j \in \mathbb{Z}\}$ is a g-frame for $\mathcal{U}$ while $\{\Gamma_j : j \in \mathbb{J}\}$ is not a g-frame for any $\varepsilon > 0$.

Now, we see from (3.3) and (3.4) that none of the following inequalities,

$$
\begin{aligned}
\Big\|\sum_{j\in\mathbb{J}_1} c_j(\Lambda_j - \Gamma_j)\Big\| &\le \lambda_1\Big\|\sum_{j\in\mathbb{J}_1} c_j\Lambda_j\Big\|,\\
\Big\|\sum_{j\in\mathbb{J}_1} c_j(\Lambda_j - \Gamma_j)\Big\| &\le \lambda_2\Big\|\sum_{j\in\mathbb{J}_1} c_j\Gamma_j\Big\|,\\
\Big\|\sum_{j\in\mathbb{J}_1} c_j(\Lambda_j - \Gamma_j)\Big\| &\le \mu\Big(\sum_{j\in\mathbb{J}_1} |c_j|^2\Big)^{1/2},
\end{aligned}
$$

implies that $\{\Gamma_j : j \in \mathbb{Z}\}$ is a g-frame.

For the case of $\mathcal{V}_j = \mathcal{V}$, $\forall j \in \mathbb{J}$ and $\dim\mathcal{V} < +\infty$, we have the following.

Theorem 3.2. *Let $\{\Lambda_j : j \in \mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\mathcal{V}$. Let A, B be the frame bounds. Suppose that $\mathcal{V}_j = \mathcal{V}$, $\forall j \in \mathbb{J}$, $K := \dim\mathcal{V} < \infty$ and $\Gamma_j \in \mathcal{L}(\mathcal{U}, \mathcal{V})$. If there exists some constant $\mu \ge 0$ such that $\mu < (A/K)^{1/2}$ and*

$$\Big\|\sum_{j\in\mathbb{J}_1} c_j(\Lambda_j - \Gamma_j)\Big\| \le \mu\Big(\sum_{j\in\mathbb{J}_1} |c_j|^2\Big)^{1/2} \tag{3.5}$$

for any finite subset $\mathbb{J}_1 \subset \mathbb{J}$ and complex numbers c_j, then $\{\Gamma_j : j \in \mathbb{J}\}$ is a g-frame for $\mathcal{U}$ with bounds

$$A\Big(1 - \mu\Big(\frac{K}{A}\Big)^{1/2}\Big)^2, \qquad B\Big(1 + \mu\Big(\frac{K}{A}\Big)^{1/2}\Big)^2.$$

Proof. This is a consequence of Lemma 2.3 and Theorem 3.1. □

Remark 3.2. The stability bound $(A/K)^{1/2}$ is the best possible whenever $\mathbb{J} = \mathbb{Z}$.

To see this, let Λ_j be defined as in Remark 2.2. Then $\{\Lambda_j : j \in \mathbb{Z}\}$ is a tight g-frame with the bound K. Let $\Gamma_j = 0$ for $1 \le j \le K$, and Λ_j otherwise. It is easy to see that

$$\sup_{\|c\|=1} \left\| \sum_{j\in\mathbb{Z}} c_j(\Gamma_j - \Lambda_j) \right\| = 1.$$

Hence

$$\left\| \sum_{j\in\mathbb{Z}} c_j(\Gamma_j - \Lambda_j) \right\| \le \left(\sum_{j\in\mathbb{Z}} |c_j|^2 \right)^{1/2},$$

i.e., (3.5) is satisfied with $\mu = 1 = (A/K)^{1/2}$. However, since $\Gamma_j u_0 = 0$, $\forall j \in \mathbb{Z}$, $\{\Gamma_j : j \in \mathbb{Z}\}$ is not a g-frame.

4. STABILITY OF DUAL G-FRAMES

In this section, we study the stability of dual g-frames. The main result is the following.

Theorem 4.1. *Let* $\{\Lambda_j : j \in \mathbb{J}\}$ *and* $\{\tilde{\Lambda}_j : j \in \mathbb{J}\}$, $\{\Gamma_j : j \in \mathbb{J}\}$ *and* $\{\tilde{\Gamma}_j : j \in \mathbb{J}\}$ *be two pairs of canonical dual g-frames for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_j : j \in \mathbb{J}\}$. *Denote the g-frame bounds of* $\{\Lambda_j : j \in \mathbb{J}\}$ *and* $\{\Gamma_j : j \in \mathbb{J}\}$ *by* (A_1, B_1) *and* (A_2, B_2), *respectively.*

(i) *If* $\{\Lambda_j - \Gamma_j : j \in \mathbb{J}\}$ *is a g-Bessel sequence with an upper bound* δ, *so is* $\{\tilde{\Lambda}_j - \tilde{\Gamma}_j : j \in \mathbb{J}\}$ *with an upper bound* $\delta\left((A_1 + B_1 + B_1^{1/2}B_2^{1/2})/(A_1A_2)\right)^2$.

(ii) *If*

$$\left| \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 - \sum_{j\in\mathbb{J}} \|\Gamma_j f\|^2 \right| \le \delta\|f\|^2, \qquad \forall f \in \mathcal{U},$$

then

$$\left| \sum_{j\in\mathbb{J}} \|\tilde{\Lambda}_j f\|^2 - \sum_{j\in\mathbb{J}} \|\tilde{\Gamma}_j f\|^2 \right| \le \frac{\delta}{A_1A_2}\|f\|^2, \qquad \forall f \in \mathcal{U}.$$

Proof. Put

$$Sf = \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j f \quad \text{and} \quad Tf = \sum_{j\in\mathbb{J}} \Gamma_j^*\Gamma_j f.$$

Then S and T are self-adjoint, $\tilde{\Lambda}_j = \Lambda_j S^{-1}$, $\tilde{\Gamma}_j = \Gamma_j T^{-1}$, $A_1 I \le S \le B_1 I$ and $A_2 I \le T \le B_2 I$. For any $f \in \mathcal{H}$, we have

$$\begin{aligned} \|Sf - Tf\| &= \left\| \sum_{j\in\mathbb{J}} \left(\Lambda_j^*\Lambda_j f - \Gamma_j^*\Gamma_j f \right) \right\| \\ &\le \left\| \sum_{j\in\mathbb{J}} \Lambda_j^*(\Lambda_j - \Gamma_j) f \right\| + \left\| \sum_{j\in\mathbb{J}} (\Lambda_j^* - \Gamma_j^*)\Gamma_j f \right\| \end{aligned}$$

$$\begin{aligned} &\leq B_1^{1/2}\left(\sum_{j\in\mathbb{J}}\|(\Lambda_j-\Gamma_j)f\|^2\right)^{1/2}+\delta^{1/2}\left(\sum_{j\in\mathbb{J}}\|\Gamma_j f\|^2\right)^{1/2} \\ &\leq \delta^{1/2}(B_1^{1/2}+B_2^{1/2})\|f\|. \end{aligned}$$

Hence

$$\|S-T\|\leq\delta^{1/2}(B_1^{1/2}+B_2^{1/2}).$$

Therefore,

$$\begin{aligned} \|S^{-1}-T^{-1}\| &= \|T^{-1}(T-S)S^{-1}\|\leq\|T^{-1}\|\cdot\|T-S\|\cdot\|S^{-1}\| \\ &\leq \frac{1}{A_1A_2}\delta^{1/2}(B_1^{1/2}+B_2^{1/2}). \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{j\in\mathbb{J}}\|\Lambda_j(S^{-1}-T^{-1})f\|^2 &\leq B_1\|(S^{-1}-T^{-1})f\|^2 \\ &\leq \frac{B_1}{A_1^2A_2^2}\delta(B_1^{1/2}+B_2^{1/2})^2\|f\|^2. \end{aligned}$$

On the other hand,

$$\sum_{j\in\mathbb{J}}\|(\Lambda_j-\Gamma_j)T^{-1}f\|^2\leq\delta\|T^{-1}f\|^2\leq\frac{\delta}{A_2^2}\|f\|^2.$$

Hence,

$$\begin{aligned} \sum_{j\in\mathbb{J}}\|(\tilde{\Lambda}_j-\tilde{\Gamma}_j)f\|^2 &= \sum_{j\in\mathbb{J}}\|(\Lambda_jS^{-1}-\Gamma_jT^{-1})f\|^2 \\ &= \sum_{j\in\mathbb{J}}\left\|\Lambda_j(S^{-1}-T^{-1})f+(\Lambda_j-\Gamma_j)T^{-1}f\right\|^2 \\ &\leq \delta\left(\frac{1}{A_2}+\frac{B_1^{1/2}}{A_1A_2}(B_1^{1/2}+B_2^{1/2})\right)^2\|f\|^2 \\ &= \delta\left(\frac{A_1+B_1+B_1^{1/2}B_2^{1/2}}{A_1A_2}\right)^2\|f\|^2. \end{aligned}$$

Next we prove (ii). Since both S and T are self-adjoint, we have

$$\begin{aligned} \|S-T\| &= \sup_{\|f\|=1}|\langle(S-T)f,f\rangle|=\sup_{\|f\|=1}|\langle Sf,f\rangle-\langle Tf,f\rangle| \\ &= \sup_{\|f\|=1}\left|\sum_{j\in\mathbb{J}}\|\Lambda_jf\|^2-\sum_{j\in\mathbb{J}}\|\Gamma_jf\|^2\right| \\ &\leq \delta. \end{aligned}$$

Therefore,

$$\|S^{-1}-T^{-1}\|\leq\|T^{-1}\|\cdot\|T-S\|\cdot\|S^{-1}\|\leq\frac{1}{A_1A_2}\delta.$$

Since $\tilde{\Lambda}_j=\Lambda_jS^{-1}$, we have

$$\sum_{j\in\mathbb{J}}\|\tilde{\Lambda}_jf\|^2 = \sum_{j\in\mathbb{J}}\langle\Lambda_jS^{-1}f,\Lambda_jS^{-1}f\rangle=\sum_{j\in\mathbb{J}}\langle\Lambda_j^*\Lambda_jS^{-1}f,S^{-1}f\rangle$$

$$= \quad \langle SS^{-1}f, S^{-1}f\rangle = \langle f, S^{-1}f\rangle.$$

Similarly,

$$\sum_{j\in\mathbb{J}} \|\tilde{\Gamma}_j f\|^2 = \langle f, T^{-1}f\rangle.$$

It follows that

$$\begin{aligned}\left|\sum_{j\in\mathbb{J}} \|\tilde{\Lambda}_j f\|^2 - \sum_{j\in\mathbb{J}} \|\tilde{\Gamma}_j f\|^2\right| &= |\langle f, (S^{-1} - T^{-1})f\rangle| \\ &\le \|S^{-1} - T^{-1}\| \cdot \|f\|^2 \\ &\le \frac{\delta}{A_1 A_2}\|f\|^2.\end{aligned}$$

□

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