

**On the Existence of Compact
Scalar-Flat Kähler Surfaces****Massimiliano Pontecorvo**

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ON THE EXISTENCE OF COMPACT SCALAR-FLAT KÄHLER SURFACES

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1. Preliminaries

We would like to report on some recent progress made by the author, Jongsu Kim and Claude LeBrun about the following:

Main Question. *Which compact complex surfaces admit Kähler metrics of zero scalar curvature?*

The object of our study is then a compact four-dimensional manifold (M, g, J) equipped with a riemannian metric g and a compatible almost complex structure J - i.e. an endomorphism of the tangent bundle $J : TM \rightarrow TM$ satisfying $J^2 = -id$. and $J^*g = g$ - such that J is parallel with respect to the Levi-Civita connection of g : $\nabla J = 0$. It is well known that this condition forces J to be a complex structure on M so that there are complex coordinates z_1, z_2 with respect to which J becomes multiplication by i on $\langle \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2} \rangle$; the fundamental $(1, 1)$ -form $\omega(X, Y) := g(X, JY)$ is also parallel and therefore closed. In this way (M, g, J) becomes a compact Kähler surface and from now on we will fix on M the orientation defined by the complex structure J .

Finally, we require that the scalar curvature of the metric g - i.e the trace of the Ricci-curvature - vanishes identically: $s = Scal(g) = g^{kl} R_{kjl}^j = 0$. Geometrically, this means that up to order 2, the volume of geodesics balls in M grows like the volume of geodesic balls in flat-space. A scalar-flat Kähler surface is then a complex surface which admits a metric with the above properties.

1.1 MOTIVATIONS

We will take the point of view that one of the goals of riemannian geometry is to look for metrics with some “nice” property and then try to equip a given manifold with such a “canonical” metric. For example it is often very useful to know that a

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Riemann surface admits constant-curvature metrics or that a K3 surface admits a Ricci-Flat Kähler metric.

Let us now point out some nice properties of scalar-flat Kähler metrics on compact 4-manifolds:

1.1.1. Scalar-flat Kähler metrics are critical metrics in the sense of Calabi [C] because of course they are absolute minima of the L^2 -norm of the scalar curvature. Furthermore any Ricci-flat Kähler metric is certainly scalar-flat so that we can also consider the classification problem of scalar-flat Kähler surfaces as a generalization of the following problem due to Calabi and which was completely solved by Yau: in each Kähler class of a compact Kähler manifold M there exists a unique Ricci-flat Kähler metric if and only if the first Chern class vanishes in real cohomology: $c_1^{\mathbb{R}}(M) = 0$; if and only if, in the case of surfaces, M is a complex torus, a K3 surface or one of their finite quotients: hyperelliptic surfaces and Enriques surfaces. In light of this result we will only consider compact scalar-flat Kähler surfaces with $c_1^{\mathbb{R}} = 0$ and use the following terminology, see also Remark 2.6.

Notation 1.1.2. *From now on a SFK surface M will denote a compact complex surface admitting Kähler metrics of zero-scalar curvature which are not Ricci-flat - i.e. with $c_1(M) \neq 0 \in H^2(M, \mathbb{R})$.*

1.1.3. Let M be a compact 4-manifold and consider the space of Riemannian metrics on M modulo homothety then: scalar-flat Kähler metrics are absolute minima of the following riemannian L^2 -functionals: the total energy functional

$$(1) \quad \int_M \|R\|^2 dvol$$

and the conformal energy functional

$$(2) \quad \int_M \|W\|^2 dvol$$

where R and W are respectively the Riemann and the Weyl curvature tensor of the metric. The above is a consequence of the Gauss-Bonnet formulas - given below - which relate the Euler characteristic χ and the topological signature τ of the four-manifold M with the decomposition - also given below - of the curvature tensor R into irreducible components under the action of $SO(4)$, see [L2] for all the details. This curvature decomposition stems from the fact that $\mathfrak{so}(4)$ is not a simple Lie algebra: on any oriented riemannian four-manifold $(M, g, or.)$ the bundle of 2-forms splits into two subbundles

$$(3) \quad \Lambda^2 M = \Lambda_+^2 \oplus \Lambda_-^2$$

where Λ_{\pm}^2 denotes the ± 1 -eigenspace of the Hodge-star operator $\star : \Lambda^2 M \rightarrow \Lambda^2 M$ since $\star^2 = id.$; sections of Λ_{\pm}^2 are called (anti-)self-dual 2-forms.

The famous gauge theory of Donaldson and the very recent one of Seiberg-Witten [W] show that the above decomposition has very strong consequences on

the differential geometry and topology of four-manifolds coming from the fact that the curvature of a connection is a bundle-valued 2-form.

In a similar way one can consider the curvature operator of a riemannian manifold as a symmetric endomorphism of 2-forms $\mathcal{R} : \Lambda^2 \rightarrow \Lambda^2$ and in the four-dimensional case use the decomposition $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ to split \mathcal{R} into four blocks of 3×3 matrices:

$$(4) \quad \mathcal{R} = \begin{pmatrix} W_+ + \frac{s}{12}I & R_0 \\ {}^t R_0 & W_- + \frac{s}{12}I \end{pmatrix}$$

here s is the scalar curvature, R_0 the trace-free part of the Ricci tensor and $W = W_+ + W_-$ is the Weyl tensor of the metric.

We can now write the Gauss-Bonnet formula in four-dimension:

$$(5) \quad \tau(M) = \frac{1}{12\pi^2} \int_M \|W_+\|^2 - \|W_-\|^2$$

$$(6) \quad \chi(M) = \frac{1}{8\pi^2} \int_M \|W_+\|^2 + \|W_-\|^2 + \frac{s^2}{24} - \frac{1}{6} \|R_0\|^2.$$

1.1.4. Another motivation comes from Gravitational Physics because scalar-flat Kähler metrics give solutions of the Einstein-Maxwell equations, [L2] [LS1].

1.1.5. But for us the main technical motivation which was the key ingredient of the new constructions of SFK surfaces we wish to present here comes from the twistor theory of Penrose [P] which we briefly describe now:

Definition 1.1.6 A riemannian metric g on a four-manifold M is said to be anti-self-dual if $W_+ = 0$.

These metrics are of interest not only because they minimize the conformal energy functional (2) but also because they provide a strong link between four-dimensional riemannian geometry and three-dimensional complex geometry. This is achieved thanks to the twistor theory of Penrose [P] and the following basic result of Atiyah-Hitchin-Singer.

Theorem 1.1.7 [AHS]. *The twistor space (Z, \mathbb{J}) of the oriented riemannian 4-manifold $(M, g, or.)$ is a complex 3-manifold if and only if the metric g is anti-self-dual.*

At this point we should say that the twistor space (Z, \mathbb{J}) is a S^2 -bundle over $(M^4, g, or.)$ which can be defined as follows. As a smooth manifold Z is the bundle of almost complex structures on M which are compatible with the given metric and orientation

$$(7) \quad Z = Z(M, g, or.) := \{J \in SO(TM) \mid J^2 = -id.\} = SO(TM)/U(2)$$

and we will denote by $t : Z \rightarrow M$ the ‘twistor fibration’.

The ‘tautological’ almost complex structure \mathbb{J} of Z is constructed in the following way: the Levi-Civita connection of g gives a splitting of each tangent space of

$Z = SO(TM)/U(2)$ into horizontal and vertical part. If $z \in Z$ with $t(z) = p$ we then write $T_z Z = \mathcal{H} \oplus \mathcal{V}$; here \mathcal{V} is the tangent space to the fiber $t^{-1}(p) \cong \mathbb{C}\mathbb{P}_1$ and therefore comes equipped with a natural complex structure J_2 , while the horizontal space \mathcal{H} is identified to $T_p M$ via t_* and we let J_1 be the complex structure on $T_p M \cong \mathcal{H}$ defined by the point z itself. Finally we set $\mathbb{J} = J_1 \oplus J_2$ and call it the ‘tautological’ twistor complex structure on Z ; theorem 1.1.7 says that the Nijenhuis tensor of \mathbb{J} vanishes identically if and only if g is anti-self-dual.

Finally, the link between anti-self-dual - and therefore twistor theory - and SFK metrics is provided by the following result of Gauduchon.

Proposition 1.1.8 [G1]. *A Kähler surface (M, g, J) is anti-self-dual, with respect to the complex orientation, if and only if the scalar curvature of g is identically zero.*

As a consequence the twistor space of a SFK surface is a complex 3-fold and in fact our constructions are heavily based on Proposition 1.1.12 which characterizes SFK surfaces, compact or not, by means of a holomorphic property of their twistor spaces; first we need the following simple observation.

If we think of a compatible hermitian structure of $(M, g, or.)$ as a section of the twistor space Z it can be shown [BdB1] that the image $\Sigma := J(M)$ is a complex hypersurface of Z ; similarly let $\bar{\Sigma} := -J(M)$ be the image of the complex structure $-J$. Then we consider the effective divisor $X = \Sigma + \bar{\Sigma}$ of Z and let X also denote the associated holomorphic line bundle. Using the facts that X meets every fiber of the smooth fibration $t : Z \rightarrow M$ - notice that t is *never* holomorphic - in exactly 2 points and that Z is always a spin manifold, one can show [Po2] that $c_1(X) = -\frac{1}{2}c_1(K_Z)$ where as usual K_Z denotes the canonical bundle of holomorphic 3-forms on Z and has a square root because Z is spin. Finally we have:

Theorem [P2] 1.1.9. *X and $-\frac{1}{2}K_Z$ are isomorphic as holomorphic line bundles on Z if and only if the metric g is conformal to a SFK metric.*

The strength of the twistor construction is that Theorems 1.1.7 and 1.1.9 have a converse. More precisely let us pose the following definition, see [H]

Definition 1.1.10. *A complex 3-manifold Z is called a twistor space if the following conditions hold:*

- (i) *there exists a fixed-point free anti-holomorphic involution $\sigma : Z \rightarrow Z$.*
- (ii) *Z is foliated by σ -invariant $\mathbb{C}\mathbb{P}_1$'s with normal bundle $\nu = \mathcal{O}(1) \oplus \mathcal{O}(1)$.*

The reason is that starting from Z we can reconstruct the 4-manifold M , its orientation and conformal class $[g]$ as follows: the $\mathbb{C}\mathbb{P}_1$'s above form a smooth complex 4-dimensional family \mathcal{M} of submanifolds of Z by Kodaira's theory [K] and if we restrict our attention to those fibers which are invariant under σ we obtain a real 4-manifold M together with a smooth projection $t : Z \rightarrow M$. Finally, the conformal class $[g]$ of the anti-self-dual metric and the orientation of M come again from Kodaira's theory by identifying the tangent space of $T_p M$ with the real part of $H^0(\mathbb{C}\mathbb{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1)) \cong \mathbb{C}^4$. More precisely the anti-self-dual conformal metric $[g]$ is completely determined by the requirement that a *complex* tangent vector $v \in \mathbb{C} \otimes TM \cong T\mathcal{M}$ is null i.e. $g(v, v) = 0$ if and only if the corresponding section in $H^0(\mathbb{C}\mathbb{P}_1, \mathcal{O}(1) \oplus \mathcal{O}(1))$ has a zero.

Putting this discussion together with Proposition 1.1.8 and Theorem 1.1.9 we have that

Proposition 1.1.12. *Constructing a SFK surface (M, g, J) is equivalent to construct a twistor space Z , that is a complex 3-fold as in 1.1.10, together with a reducible hypersurface X which is linearly equivalent to $-\frac{1}{2}K_Z$.*

This proposition lies at the heart of our constructions which will be described in the last section.

Remark 1.1.13. We have insisted that SFK does not include the Ricci-flat Kähler case. However proposition 1.1.12 certainly holds for Kähler metrics with vanishing Ricci tensor as well. To distinguish between SFK and Ricci-flat metrics one has to look at the algebraic dimension $a(Z)$ of the twistor space. The result is that $a(Z) = 0$ for SFK surfaces while $a(Z) = 1$ in the Ricci-flat case [Po2] [P1].

1.2 SPIN STRUCTURES

It is also useful to describe the twistor space of $(M^4, g, or.)$ in terms of spin bundles. The first observation, see also 1.1.3, is that the double covering of $SO(4)$ splits as $Spin(4) = SU(2) \oplus SU(2)$ and the associated two irreducible representations of $SU(2)$ on $\mathbb{R}^4 = \mathbb{H}$ are given by left and right multiplication by unit quaternions - the unit sphere $SU(2) = S^3 \subset \mathbb{H}$ - and we indicate them by $SU_{\pm}(2)$.

Assume for a moment that M is spin - i.e. there is a principal $Spin(4)$ -bundle $Spin(M)$ which is a double covering of $SO(M)$ - this is a topological condition equivalent to the vanishing of the first two Stiefel-Whitney classes $w_1(M) = w_2(M) = 0$, then the spin bundle also splits as a fiber product over M :

$$Spin(M) = SU_2^+(M) \times SU_2^-(M)$$

where $SU_2^{\pm}(M)$ are the principal bundles associated to the above representations of $SU(2)$ on $\mathbb{R}^4 = \mathbb{C}^2 = \mathbb{H}$. Now for the twistor space of M we have

$$\begin{aligned} Z &= SO(M)/U(2) = Spin(M)/U(1) \times SU(2) \\ &= SU_2^+(M) \times SU_2^-(M)/U(1) \times SU(2) = SU_2^+(M)/U(1) = \mathbb{P}(\mathbb{S}_+) \end{aligned}$$

This shows that Z can also be thought of as the projectivization of the $+\frac{1}{2}$ -spin bundle \mathbb{S}_+ which is defined to be the rank-2 complex vector bundle on M associated to the principal bundle $SU_2^+(M)$. When M is oriented but not spin - i.e. $w_1(M) = 0$ and $w_2(M) \neq 0$ - the transition functions of \mathbb{S}_+ are defined only up to sign but its projectivization $\mathbb{P}(\mathbb{S}_+) = Z$ is always globally defined on M . A similar discussion holds for \mathbb{S}_- .

When M is not spin one would still like to have spin bundles in order to be able to define a Dirac operator. This can be done for every oriented 4-manifold by considering the more general notion of $Spin_c$ -structure. Such a structure on M^4 can be thought of as a complex rank-2 vector bundle \mathbb{V}_+ whose projectivization satisfies $\mathbb{P}(\mathbb{V}_+) = \mathbb{P}(\mathbb{S}_+)$ [W].

Since we will soon specialize to complex surfaces let us introduce the *canonical* Spin_c -structure of a hermitian manifold (M, g, J) . In general this is given by considering forms of type $(0, q)$. In the case of surfaces we have:

$$(8) \quad \mathbb{V}_+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \quad \text{while} \quad \mathbb{V}_- = \Lambda^{0,1}$$

This result follows by considering the interplay between spinors and the two decompositions of 2-forms given by the riemannian and the complex structures: $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ and also $\mathbb{C} \otimes \Lambda^2(M) = \Lambda^{0,2} \oplus \Lambda^{1,1} \oplus \Lambda^{0,2}$.

A useful observation is that this also gives an alternative description of the twistor space $Z = \mathbb{P}(\mathbb{V}_+) = \mathbb{P}(\Lambda^{0,0} \oplus \Lambda^{0,2})$ as the projectivization of a holomorphic bundle so that when (M, g, J) is a SFK surface Z also carries a complex structure $\mathbb{I} \neq \mathbb{J}$ which makes the twistor projection $t : (Z, \mathbb{I}) \rightarrow (M, J)$ into a *holomorphic* $\mathbb{C}\mathbb{P}_1$ -bundle. This can be used to show that there are many complex 3-manifolds Z which carry both a kählerian structure \mathbb{I} and a non-kählerian one \mathbb{J} and provide an answer to a problem posed by Catanese [Ct], see [P3].

Finally, the Dirac operator of the canonical Spin_c -structure (8) can be identified with

$$\bar{\partial} + \bar{\partial}^* : \Lambda^{0,0} \oplus \Lambda^{0,2} \rightarrow \Lambda^{0,1}$$

where $\bar{\partial}^*$ is the adjoint of $\bar{\partial}$ with respect to the hermitian metric g . The index of this operator is the Todd genus of M [LM, p.400].

2. Ruled Surfaces

We now go back to the Main Question and try to understand the complex structure of (M, J) . Since we are looking for compact SFK surfaces the following vanishing theorem of Yau applies:

Theorem 2.1 [Y1][Y2]. *Let M be a compact Kähler manifold of non-negative total scalar curvature*

$$\int_M s \, d\text{vol} \geq 0$$

then one and only one of the following properties applies:

- (i) $c_1(M) = 0 \in H^2(M, \mathbb{R})$ and M admits Ricci-flat Kähler metrics.
- (ii) $H^0(M, K^n) = 0$ for all $n \in \mathbb{N}$. That is all plurigenera of M vanish or in other words the Kodaira dimension $\text{Kod}(M) = -\infty$.

Since we are looking for scalar-flat Kähler surfaces M with $c_1^{\mathbb{R}}(M) \neq 0$ - SFK surfaces - we can exclude possibility (i) and conclude that our surfaces must have Kodaira dimension $-\infty$. At this point, the Enriques-Kodaira classification of surfaces [BPV, p.188] tell us that (M, J) is either $\mathbb{C}\mathbb{P}_2$ or it can be obtained by blowing up a geometrically ruled surface. However, the topological signature $\tau(\mathbb{C}\mathbb{P}_2) = 1$ and therefore $\mathbb{C}\mathbb{P}_2$ cannot admit any anti-self-dual metric with respect to the complex orientation by (5); notice that the Fubini-Study metric is self-dual indeed. We can then conclude that:

Proposition 2.2. *A SFK surface (M, J) is obtained by blowing up a geometrically ruled surface N of genus γ .*

Let us recall here that a geometrically ruled surface is nothing else than the projectivization of a holomorphic rank-2 vector bundle E over a Riemann surface S_γ of genus γ [BPV]. That is $N = \mathbb{P}(E) \rightarrow S_\gamma$ and then N is said to have genus γ . More generally, a ruled surface of genus γ is the blow up of a geometrically ruled surface of the same genus.

We can also give a lower bound on the number of points to be blown up in terms of the genus γ . In fact the Chern number c_1^2 of an almost complex 4-manifold is a topological invariant: $c_1^2 = 2\chi + 3\tau$ and from the Chern-Weil formulas (5) and (6) we deduce that

$$(9) \quad c_1^2 = \frac{1}{4\pi^2} \int 2\|W_+\|^2 + \frac{1}{24}s^2 - \frac{1}{6}\|R_0\|^2$$

and therefore:

Proposition 2.3 [Bo]. *If (M, J) is a SFK surface then $c_1^2(M) < 0$. As a consequence M is obtained by blowing up m points on a geometrically ruled surface N of genus γ where $m \geq 9$ if $\gamma = 0$ while $m \geq 1$ if $\gamma = 1$.*

Proof. Since M is SFK we have $s = 0$ but $R_0 \neq 0$, while $W_+ = 0$ by 1.1.8 and therefore $c_1^2(M) < 0$. On the other hand $c_1^2(N) = 2\chi(N) = 4(2 - 2\gamma)$ and blowing up a point is topologically equivalent to perform the connected sum with $\mathbb{C}\mathbb{P}_2$ - by which we mean $\mathbb{C}\mathbb{P}_2$ equipped with the opposite orientation - therefore $c_1^2(M) = 8(1 - \gamma) - m$. \square

Notice that there is no constrain on m if $\gamma \geq 2$. In fact a complete list of existence results which where known before [KP], [KLP] is:

Examples 2.4.

- (i) Let S_γ be a compact Riemann surface of genus $\gamma \geq 2$ and on $M = S_\gamma \times \mathbb{C}\mathbb{P}_1$ consider the product of the ∓ 1 -constant curvature metrics. It is not hard to show that the scalar curvature vanishes identically and therefore M is a SFK surface.
- (ii) The twisted version of the above example is the following: if $E \rightarrow S_\gamma$ is either a split or a stable rank-2 vector bundle of degree zero over a compact Riemann surface of genus $\gamma \geq 2$ then $M = \mathbb{P}(E)$ can be equipped with a metric which is locally a product as above and is therefore SFK, [NS], [BdB2].
- (iii) LeBrun [L1] constructed explicit examples of SFK metrics on the blow up of $\mathbb{C}\mathbb{P}_1 \times S_\gamma$, with $\gamma \geq 2$, at $m \geq 2$ points placed in special position.
- (iv) Lebrun and Singer [LS1] have shown that any versal family of deformations of the complex structure of a SFK surface of genus $\gamma \geq 2$ contains an open set of SFK surfaces.

Remark 2.5. In relation with example (iii) above we should notice that the complex surface obtained by blowing up $\mathbb{C}\mathbb{P}_1 \times S_\gamma$, $\gamma \geq 2$ at only one point cannot admit Kähler metrics of constant-scalar curvature because its Lie algebra of holomorphic vector fields is not reductive, [Li], [M]. \triangle

Taking this remark into account and in light of the fact that the examples of (ii) are all *diffeomorphic* to $\mathbb{CP}_1 \times S_\gamma$ one is lead to wonder about:

Question 1. *Find sufficient conditions for the 1-point blow up of a SFK surface to admit SFK metrics.*

Question 2. *Does the smooth manifold $(\mathbb{CP}_1 \times S_\gamma) \# \overline{\mathbb{CP}}_2$ support SFK (or even just anti-self-dual) metrics?*

Perhaps, however a more compelling problem which is left open by the above list of examples is the following:

Question 3. *Can a ruled surface of genus $\gamma \leq 1$ admit SFK metrics?*

Remark 2.6. A consequence of Theorem 2.1 is that the blow up \tilde{M} of a Ricci-flat Kähler surface M cannot admit Kähler metrics of zero scalar curvature. The reason is that on the one hand if m is the number of blown up points then $c_1^2(\tilde{M}) = c_1^2(M) - m = -m$ - i.e. $c_1(M) \neq 0$; but on the other hand $Kod(\tilde{M}) = Kod(M) = 0$ because the Kodaira dimension is a bimeromorphic invariant. This shows that Question 1 is not interesting without the restriction of 1.1.2.

3. Main Results

In the joint work with Kim [KP] we give a satisfactory answer to Question 1, i.e. we find a weak sufficient condition, see Theorem 3.1 below. This condition is in fact necessary in the sense of Remark 3.2, and it also provides a positive answer to Question 2.

Question 3 receives a positive answer in [KLP], see Theorem 3.5 below. Furthermore the combination of Theorems 3.1 and 3.5 yields a powerful existence result à la Taubes which was conjectured to be true in [LS1], see Theorem 3.6 and 4.1.3.

Our answer to Question 1 is provided by the following result, let us explain the notation of the statement. With $H^0(\Theta_M)$ we denote the Lie algebra of holomorphic vector fields on M . By ‘any blow up’ we mean the blow up at any number $m \geq 0$ of points, placed in any arbitrary position and possibly with multiplicity. Finally we point out that it is not important the order in which the (possibly trivial) deformation and the blow up occur.

Theorem 3.1 [KP]. *Let M be a compact SFK surface with $c_1^{\mathbb{R}}(M) \neq 0$ then any blow up of M and any of its small deformation admits SFK metrics or else M satisfies one of the following equivalent conditions:*

- (i) *M is a minimal ruled surface of genus $\gamma \geq 2$ with non-trivial holomorphic vector fields: $H^0(\Theta_M) \neq 0$.*
- (ii) *M is the projectivization of a split rank-2 vector bundle of 0-degree over a Riemann surface of genus $\gamma \geq 2$ - i.e. $M = \mathbb{P}(L \oplus \mathcal{O}) \rightarrow S_\gamma$ with $\deg L = 0$ and $\gamma \geq 2$.*

Remarks 3.2. (i) The sufficient condition of theorem 3.1 is pretty weak because it holds, for examples, for any non-minimal SFK surface; in particular for any ruled surface of genus $\gamma \leq 1$ (at the moment however no such example was available).

(ii). The above condition is also necessary in the following weak sense: suppose $M = \mathbb{P}(L \oplus \mathcal{O}) \rightarrow S_\gamma$ is SFK and (therefore necessarily) satisfies condition (ii) of

Theorem 3.1; take any number of points on the zero-section of L and blow them up. The resulting surface \tilde{M} cannot admit SFK metrics because $H^0(\Theta_{\tilde{M}}) \cong \mathbb{C}$ is generated by the Euler vector field Ξ and the Futaki invariant $\mathcal{F}(\Xi, [\omega]) \neq 0$ for any admissible Kähler class $[\omega]$, [LS1, Corollary 3.4 (b)].

We are now ready to answer Question 2:

Corollary 3.3. *For any $n \geq 0$ there exist SFK metrics on the smooth 4-manifold $(\mathbb{C}\mathbb{P}_1 \times S_\gamma) \# \mathbb{C}\mathbb{P}_2$.*

Proof. Observe that the only open question was for $n = 1$. Take M as in Example 2.4 (ii) with E stable, then $H^0(\Theta_M) = 0$ so that the conclusion follows from Theorem 3.1. \square

The examples mentioned in the proof above lead to the following result which says that SFK surfaces are ‘generic’ among non-minimal ruled surfaces of genus $\gamma \geq 2$.

Corollary 3.4. *In any versal family of deformations of compact non-minimal ruled surfaces of genus $\gamma \geq 2$ there exists an open and dense subset of SFK surfaces.*

Proof. Any non-minimal ruled surface M can be obtained by bowing up a geometrically ruled surface $N = \mathbb{P}(E)$ with the property that $\deg E = 0$. (For example the blow up of the Hirzebruch surface Σ_1 is also the blow up of $\mathbb{C}\mathbb{P}_1 \times \mathbb{C}\mathbb{P}_1$ [BPV]. The result then follows from Example 2.4 (ii), Theorem 3.1 and the fact that stable bundles are open and dense among vector bundles of fixed degree over a given Riemann surface [NR]. \square

An ‘ad hoc’ geometric construction which first appeared in [LS2] gives a positive answer to Question 3. The result is the following:

Theorem 3.5 [KLP]. *There exist SFK metrics on the following surfaces:*

- (i) $\mathbb{C}\mathbb{P}_1 \times \mathbb{C}\mathbb{P}_1$ blown up at 13 suitably chosen points,
- (ii) $E \times \mathbb{C}\mathbb{P}_1$ blown up at 6 suitably chosen points, where E is any elliptic curve.

Although the above result only concerns two examples its proof involves several general theorems and together with Theorem 3.1 produces a simple proof of the following remarkable result which was conjectured by LeBrun-Singer [LS1]. It is the exact Kähler analogue of the powerful existence theorem 4.1.3 of Taubes on anti-self-dual metrics on 4-manifolds [T].

Theorem 3.6 [KLP]. *Let M be any ruled surface - i.e. a compact surface with a holomorphic map onto a compact Riemann surface with generic fiber $\mathbb{C}\mathbb{P}_1$ - then there exists a blow up \tilde{M} of M at sufficiently many points with the property that any blow up of \tilde{M} admits SFK metrics.*

Proof. Any two ruled surfaces of the same genus M and N are bimeromorphic [BPV] i.e. there exists a ruled surface \tilde{M} which is a blow up of both M and N . If the genus is $\gamma \geq 2$ let N be the projectivization of a stable rank-2 vector bundle. Otherwise $\gamma \leq 1$ and we let N be the corresponding example in Theorem 3.5. Now, in all these cases Theorem 3.1 applies to N and we conclude that any blow up of N and in particular of \tilde{M} admits SFK metrics. \square

An argument of LeBrun-Simanca [LSa] based on an inverse function theorem also gives:

Corollary 3.7 [KLP]. *Let M be a compact complex surface which admits a Kähler metric of positive total scalar curvature. Then M has a blow up \tilde{M} such that any blow up of \tilde{M} admits Kähler metrics of constant positive scalar curvature.*

Proof. By Theorem 2.1 and the classification of surfaces M is a ruled surface and therefore by Theorem 3.6 there is a blow up \tilde{M} with the property that any blow up of \tilde{M} is SFK; the result of LeBrun-Simanca [LSa] implies that there is also a Kähler metric of constant positive scalar curvature.

4. Idea of the Proofs

In this section we would like to sketch the proof of theorems 3.1 and 3.5. In both cases we have to generalize some techniques introduced by Donaldson-Friedman [DF] who were investigating the existence of anti-self-dual metrics on the connected sum $M_1 \# M_2$ of two anti-self-dual 4-manifolds.

4.1. THE DONALDSON-FRIEDMAN CONSTRUCTION

Because the Penrose construction is invertible it is enough to construct the twistor space - in the sense of 1.1.10 - of the desired metric on $M_1 \# M_2$. This is done in two steps which will be called ‘geometric construction’ and ‘deformation theory’. Let $t_i : Z_i \rightarrow M_1$ denote the twistor fibration for $i = 1, 2$.

The geometric construction. We build a singular complex 3-fold Z_0 which will be called the ‘singular twistor space’ of $M_1 \# M_2$ in the following way. First blow up Z_i , $i = 1, 2$ along a fiber of the twistor fibration and then identify the resulting two exceptional divisors, a quadric $Q_i = \mathbb{CP}_1 \times \mathbb{CP}_1$ in each Z_i with normal bundle $\nu = \mathcal{O}(1, -1)$, by switching the factors in such a way that the resulting complex space

$$Z_0 = \tilde{Z}_1 \cup_Q \tilde{Z}_2$$

is a singular 3-fold with only normal crossing singularities along Q satisfying the d -semistable condition [DF].

It is known that this condition is necessary in order for Z_0 to admit smooth deformations and the key result [DF] is that if Z_0 admits smooth deformations then it is possible to find small deformations Z_t of Z_0 which are smooth 3-folds satisfying Definition 1.1.10 - i.e. they are twistor spaces - and it is not too difficult to see that the underlying 4-manifold is diffeomorphic to $M_1 \# M_2$ which therefore admits anti-self-dual metrics, in this case.

Deformation theory. Because of what we just said it is enough to understand when the singular twistor space Z_0 admits smoothings. Using the deformation theory of complex spaces with normal crossings satisfying the d -semistable condition Donaldson-Friedman find that the obstruction to smooth out Z_0 lies in $H^2(\tau_{Z_0}^0)$ the second cohomology group of the sheaf of derivations of Z_0 . Then there is a natural ‘normalization’ exact sequence relating the cohomology of the tangent bundles Θ_{Z_i} of the twistor spaces Z_i , $i = 1, 2$ to the obstruction space $H^2(\tau_{Z_0}^0)$; this sequence yields the following sufficient condition:

Theorem 4.1.1 [DF]. *If $H^2(\Theta_{Z_i}) = 0$ for $i = 1, 2$ then Z_0 admits smoothings and the connected sum $M_1 \# M_2$ admits anti-self-dual metrics.*

Since the twistor space of the Fubini-Study metric on \mathbb{CP}_2 is the flag manifold $F_{12}(\mathbb{C}^3)$ one obtains for example:

Corollary 4.1.2 [DF]. *For every $n \in \mathbb{N}$ the connected sum $\mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2$ admits anti-self-dual metrics.*

The same result was proved around the same time by Floer [F] using PDE's techniques. Floer's method was then generalized by Taubes who proved that

Theorem 4.1.3 [T]. *Let M be any compact oriented 4-manifold. Then there exists $n_0 \geq 0$ such that for every natural number $n \geq n_0$ the connected sum $M \# \mathbb{CP}_2 \# \cdots \# \mathbb{CP}_2$ admits anti-self-dual metrics.*

4.2. PROOF OF THEOREM 3.1

The geometric construction. Given a SFK surface M we want to investigate the existence of SFK metrics on the 1-point blow up \tilde{M} of M . Since \tilde{M} is diffeomorphic to the connected sum $M \# \mathbb{CP}_2$ the Donaldson-Friedman construction applies. Let Z_1 be the twistor space of M and $Z_2 = F_{12}(\mathbb{C}^3)$ the twistor space of \mathbb{CP}_2 ; we are now going to construct a singular twistor space Z_0 as before. However, since we are looking for SFK metrics, rather than just anti-self-dual, it will not suffice to have a smoothing Z_t of Z_0 ; by Proposition 1.1.12 we also need to obtain a certain complex hypersurface X_t sitting inside Z_t . Because of this when we construct the singular twistor space Z_0 we have to use extra care and make sure to have an appropriate divisor X_0 in Z_0 . In fact it is always possible to achieve this because M is assumed to be SFK so that Z_1 comes with a hypersurface X_1 . Let \tilde{Z}_1 be the blow up of Z_1 as before and take \tilde{X}_1 to be the proper transform of X_1 . Similarly in $Z_2 = F_{12}(\mathbb{C}^3)$ we can easily find a divisor $X_2 = D_2 \cup \bar{D}_2$ such that $D_2 \cap \bar{D}_2 =$ the twistor line l_2 ; if we let \tilde{Z}_2 be the blow up of Z_2 along l_2 we will have a smooth hypersurface $\tilde{X}_2 \subset \tilde{Z}_2$.

Finally, $Z_0 = \tilde{Z}_1 \cup_Q \tilde{Z}_2$ contains a divisor $X_0 = \tilde{X}_1 \cup_{\bar{l}} \tilde{X}_2$ with only normal crossing singularities satisfying the d -semistable condition. Here l and \bar{l} are two disjoint lines in Q and it is not hard to see that X_t always admits smooth deformations of the right topological type. We also already understand smoothings of Z_0 from the previous section and our task now is to show that if Z_0 admits smoothings which contain smoothings of X_0 then we can choose small deformations Z_t containing X_t which are twistor spaces of a SFK metric on $M \# \mathbb{CP}_2$ because of proposition 1.1.12.

Deformation Theory. Given the above situation we now need to understand the deformations of the pair (Z_0, X_0) with $X_0 \subset Z_0$ and find a sufficient condition for the existence of simultaneous smoothings Z_t of Z_0 and X_t of X_0 with the property that $X_t \subset Z_t$. Such a theory of relative deformations of singular complex spaces does not appear to be a simple extension of the 'absolute' deformation theory of Donaldson-Friedman. Instead we have to use the theory of deformations of maps due to Ziv Ran [R1] [R2]. This involves the computation of various local $\mathcal{E}xt$ sheaves and global $\mathcal{E}xt$ groups and it yields the following:

Theorem 4.2.1 [KP, 4.6]. *Let M be a SFK surface with twistor space Z and hypersurface $X \subset Z$ as in Proposition 1.1.12. Let \mathcal{I}_X be the ideal sheaf of X in Z and Θ_Z the holomorphic tangent bundle. If $H^2(Z, \Theta_Z \otimes \mathcal{I}_X) = 0$ then the singular pair (Z_0, X_0) admits simultaneous smoothings. As a consequence the blow up of M at any point and any of its small deformations admit SFK metrics*

Theorem 3.1 is a direct consequence of the above result plus the following powerful vanishing theorem of LeBrun-Singer [LS1], see also the Appendix in [KLP].

Theorem 4.2.2 [LS1]. *Let M be a compact SFK surface with $c_1 \neq 0$ and hypersurface $X \subset Z$. Then $H^2(Z, \Theta_Z \otimes \mathcal{I}_X) = 0$ or else one of the following equivalent conditions apply:*

- (i) M is a minimal ruled surface of genus $\gamma \geq 2$ and $H^0(M, \Theta_M) \neq 0$.
- (ii) $M = \mathbb{P}(L + \mathcal{O}) \rightarrow S_\gamma$ is the projectivization of a split rank-2 vector bundle over a Riemann surface of genus $\gamma \geq 2$.

4.3. PROOF OF THEOREM 3.5

The basic idea of the proof is the same as before and it involves again a geometric construction followed by a relative-deformation argument. However in this case both steps are considerably more difficult to carry out.

The Geometric Construction. For the proof of Theorem 3.5 we not only need to generalize the geometric construction of Donaldson-Friedman to a relative situation as in the proof of 3.1; but we also need a generalized connected sum of \mathbb{Z}_2 -orbifolds. This new geometric idea is due to LeBrun-Singer [LS2] who used it to construct new examples of anti-self-dual 4-manifolds. Again we want to produce a singular twistor space Z_0 with a singular hypersurface X_0 . As before $Z_0 = Z_1 \cup_{Q_j} Z_2$ will be constructed by identifying a finite number of quadrics Q_j $j = 1, \dots, k$ contained in two smooth 3-folds Z_1 and Z_2 ; at the same time we also get a hypersurface $X_0 \subset Z_0$ with $X_0 = X_1 \cup_{l_j \bar{l}_j} X_2$ where l_j and \bar{l}_j are disjoint lines in Q_j .

To construct Z_1 we start with the twistor space $t : Z_N \rightarrow N$ of a SFK surface N and *assume* that N admits a holomorphic isometry Φ with only isolated fixed points p_1, \dots, p_k and satisfying $\Phi^2 = id$. Let $L_1, \dots, L_k \subset Z_N$ be the twistor lines corresponding to the fixed points i.e. $L_j = t^{-1}(p_j)$, $j = 1, \dots, k$; and let \tilde{Z}_N be the blow up of Z_N along the lines L_j 's. The exceptional divisors Q_1^1, \dots, Q_k^1 in \tilde{Z}_N are quadrics with normal bundle $\nu = \mathcal{O}(1, -1)$ just as in the previous proof; if we now consider the induced holomorphic involution $\tilde{\Phi} : \tilde{Z}_N \rightarrow \tilde{Z}_N$ we realize that $\tilde{\Phi}$ fixes each quadric Q_j and acts on its normal bundle by -1 ; the crucial point is that because $\Phi^2 = id$. the quotient space $Z_1 = \tilde{Z}_N / \tilde{\Phi}$ is a *smooth* complex 3-fold, the images $Q_j^1 \subset Z_1$ of the Q_j 's are again smooth quadrics but with normal bundle $\mathcal{O}(2, -2)$. We can also trace what happens to the reducible hypersurface $X_N \subset Z_N$. Recall that X_N is the disjoint union of D_N and $\bar{D}_N = \sigma(D_N)$ where D_N and \bar{D}_N are respectively biholomorphic to $(N, \pm J_N)$. If we fix our attention on D_N we can see that Z_1 has a smoothly imbedded hypersurface D_1 which is obtained by first taking the proper transform of D_N in \tilde{Z}_N and then projecting this hypersurface to Z_1 ; this exactly amounts to say that D_1 is obtained from N/Φ by replacing each singular point with a $\mathbb{C}\mathbb{P}_1$ of self-intersection -2 ; finally we let $l_j^1 = Q_j \cap D_1$.

Similarly for \overline{D}_N and we obtain the desired smooth hypersurface $X_1 = D_1 \amalg \overline{D}_1$ in Z_1 containing $2k$ rational curves l_j^1 and \overline{l}_j^1 of self-intersection -2 .

The next step is to construct Z_2 and we will just say here that Z_2 consists of k distinct copies of the smooth 3-fold \tilde{Z}_{EH} obtained from the orbifold twistor space of the conformally compactified Eguchi-Hanson metric by blowing up the twistor line at infinity. We refer to [LS2] and [KLP] for an explicit description of Z_2 and we just mention here that Z_2 will contain Q_1^2, \dots, Q_k^2 quadrics with normal bundle $\mathcal{O}(2, -2)$ and a smooth hypersurface X_2 intersecting each quadric in two disjoint lines $l_j^2 \amalg \overline{l}_j^2 = X_2 \cap Q_j$ of self-intersection -2 in X_2 .

Finally, our singular twistor space $Z_0 = Z_1 \cup_{Q_j} Z_2$ is obtained by identifying each quadric $Q_j^1 \cong \mathbb{CP}_1 \times \mathbb{CP}_1$ of Z_1 with the corresponding $Q_j^2 \in Z_2$ in such a way that the two factors are switched; in this way we also obtain a singular divisor $X_0 \subset Z_0$. As before the pair (Z_0, X_0) is a pair of singular complex spaces with only normal crossing singularities satisfying the d -semistable condition.

One important remark we wish to make at this point is that in order for Z_0 and X_0 to have only normal crossing singularities we need that all of the spaces Z_1, Z_2 and X_1, X_2 are smooth manifolds and the fact that this can be achieved by a simple blowing up process, described above, is only because we are considering Z_2 -orbifolds rather than more general orbifold singularities; this is analogous to the familiar Kummer construction of K3 surfaces.

Deformation Theory. The same deformation arguments used in the proof of 3.1 show that in the non-obstructed case - i.e. $H^2(Z_N, \Theta_{Z_N} \otimes \mathcal{I}_{X_N}) = 0$ - the singular pair (Z_0, X_0) admits smoothings (Z_t, X_t) which are twistor spaces of SFK surfaces; however the technical details of the computations are more complicated than before.

Finally, the above discussion proves the following quotient result:

Theorem 4.3.1 [KLP]. *Let N be a non-minimal compact SFK surface. Assume there is a holomorphic isometry $\Phi : N \rightarrow N$ with only isolated singularities and $\Phi^2 = id$. Then there are SFK metrics on the complex surface M which is obtained from N/Φ by replacing each singular point with a \mathbb{CP}_1 of self-intersection -2 .*

To conclude the proof of Theorem 3.5 we ‘only’ need to find a SFK surface N with a holomorphic isometry satisfying the conditions of the quotient Theorem. To find such an isometry we need to know the SFK metric explicitly and this can be achieved using the hyperbolic ansatz of LeBrun [L1][L2].

Finally, to prove Theorem 3.5 (i) we can take N to be the blow up pf $\mathbb{CP}_1 \times S_2$ at two points lying on the same \mathbb{CP}_1 -fiber, where S_2 is a hyperelliptic Riemann surface of genus 2. For the proof of part (ii) of the same theorem we construct a genus-2 Riemann surface S_2 as a 2-fold branched cover of E , $p : S_2 \rightarrow E$, consider a certain line bundle L over E and then N is obtained from $\mathbb{P}(p^*L \oplus \mathcal{O}) \rightarrow S_2$ by blowing up four points on the zero-section of $p^*L \subset \mathbb{P}(p^*L \oplus \mathcal{O})$.

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