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A Note on the Cohomology of the Complex of G–Invariant Forms on G–Manifolds

Mark V. Losik

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A NOTE ON THE COHOMOLOGY OF THE COMPLEX OF G-INVARIANT FORMS ON G-MANIFOLDS

Mark V.Losik

Saratov State University, Russia.

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Abstract. The cohomology of a certain class of G-manifolds, which are locally trivial fiber bundles over the orbit space, is considered. For a reductive Lie group G the conditions are indicated which allows to construct a convenient cochain complex with the cohomology isomorphic to the cohomology of the complex of G-invariant forms on M. In particular, these conditions are satisfied for a compact Lie group G.

0. Introduction

Let G be a reductive Lie group, H and N its closed subgroups such that H is a normal subgroup of N, and $K = N/H$. Consider a class of G-manifold M, for which M, under the projection $M \to M/G$, is a smooth locally trivial fiber bundle with fiber G/H. In particular, for a smooth principal K-bundle P, $M = P \times_K (G/H)$ is a G-manifold of this class. Any G-manifold of this class can be represented like this, if one take the normalizer of H in G for N.

Suppose that the Lie algebras $\mathfrak{h}$ and $\mathfrak{n}$ of the Lie groups H and N are reductive subalgebras of $\mathfrak{g}$. In [8] the convenient differential graded algebra (DG-algebra) $C(M)$ is constructed and, for a connected K, it is proved that the minimal models of the DG-algebra $\Omega(M)^G$ of G-invariant forms on M and $C(M)$ coincide. In particular, for a compact G a generalization of the Cartan theorem on the cohomology of homogeneous spaces [4] is proved for the G-manifold of the above class.

In this paper the extension of these results for a disconnected K is obtained in the case when there exists a K-invariant connection on P. Particularly, it is proved for a compact G that the cohomology $H(C(M))$ is isomorphic to the real cohomology of M.

Throughout the paper all manifolds, differential forms, maps of manifolds, and so on belong to the class C^∞; a Lie group means a finite-dimensional real Lie group.

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1. The definition of a class of G-manifolds

Let G be a Lie group and M a connected G-manifold such that all orbits of G have the type G/H, where H is a closed subgroup of G, M/G is a manifold, and, under the projection $M \to M/G$, M is a smooth locally trivial fiber bundle. Further we consider the class of G-manifolds satisfying these conditions. Note that for a compact G all above conditions follows from the above condition on orbits [9].

Denote by $N_G(H)$ the normalizer of H and by K_H the quotient group $N_G(H)/H$. It is clear that G/H is a principal K_H-bundle over $G/N_G(H)$. Let M^H be the set of points of M with the stabilizer H. There is the following theorem due to Borel [1].

1.1. Theorem. *M^H is a smooth subbundle of the bundle $M \to M/H$ and a smooth principal K_H-bundle under the natural action of N_H/H on M^H. $M^H \times_{K_H} (G/H)$ is a G-manifold canonically equivariant isomorphic to M and M with the projection $M = M^H \times_{K_H} (G/H) \to G/N_H(G)$ is a smooth locally trivial fiber bundle with fiber M^H.*

Suppose that the group K_H of the principal K_H-bundle M^H can be reduced to a subgroup $K = N/H$ of K_H, where N is a closed subgroup of N_H. Then $M^H = P \times_K (N_G(H)/H)$, where P is a smooth principal K-bundle, and $M = P \times_K (G/H)$. Thus, for any G-manifold M of the above class, one has the representation $M = P \times_K (G/H)$, where $K = N/H$ and N is a closed subgroup of G, for which H is a normal subgroup.

Let N_0 be the component of the neutral element of N, $K_0 = N_0/H$, and $\widetilde{M} = P \times_{K_0} (G/H)$. Put $\widetilde{B} = P/K_0$. It is evident that the discrete group $D = K/K_0$ acts freely on $\widetilde{M}$ and on $\widetilde{B}$. The following assertion is obvious.

1.2. Proposition. *The projections $\widetilde{M} \to \widetilde{M}/D = M$ and $\widetilde{B} \to \widetilde{B}/D = B$ are smooth covering projections. Moreover, the projection $\widetilde{M} \to M$ is a G-equivariant map.*

2. The Weil complex

Recall some known facts on the Weil complex of a reductive Lie group G and its Lie algebra $\mathfrak{g}$ (see [3], [4] and [7]). Let $P_{\mathfrak{g}}$ be the space of primitive elements of the cohomology $H(\mathfrak{g};\mathbf{R})$. Then $P_{\mathfrak{g}}$ possesses a basis consisting of homogeneous elements of odd degrees and $H(\mathfrak{g};\mathbf{R})$ is the exterior algebra $\Lambda(P_{\mathfrak{g}})$. For reductive Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ the cohomology homomorphism $H(\mathfrak{g};\mathbf{R}) \to H(\mathfrak{h};\mathbf{R})$, which corresponds to a Lie algebra homomorphism $\mathfrak{h} \to \mathfrak{g}$, induces a map $P_{\mathfrak{g}} \to P_{\mathfrak{h}}$.

Consider the Weil DG-algebra $W(\mathfrak{g}) = \Lambda(\mathfrak{g}') \otimes S(\mathfrak{g}')$, where $\mathfrak{g}'$ is the dual space of $\mathfrak{g}$ and $S(\mathfrak{g}')$ is its symmetrical algebra, and its subalgebra of G-invariant elements $IW(\mathfrak{g})$. It is known that the space $I\Lambda(\mathfrak{g}')$ of G-invariant skew-symmetric forms on $\mathfrak{g}$ is equal to $\Lambda(P_{\mathfrak{g}})$.

Consider the graded algebra $IS(\mathfrak{g}')$ of G-invariant symmetric forms on $\mathfrak{g}$. For each $b \in IS(\mathfrak{g}')$ there is an element $c(b) \in IW(\mathfrak{g})$ such that the differential $d\,c(b)$ of $c(b)$ is equal to b. Let assign to b the component of $c(b)$ in $\Lambda(P_{\mathfrak{g}})$. It is known [3] that this component does not depend on a choice of $c(b)$. This assignment define a linear map of $IS(\mathfrak{g}')$ onto $P_{\mathfrak{g}}$. One can chose the collection (b_i) of homogeneous elements of $IS(\mathfrak{g}')$ such that their images under the above map form a basis of $P_{\mathfrak{g}}$ consisting of homogeneous elements of $P_{\mathfrak{g}}$. Then, the subspace $\overline{P_{\mathfrak{g}}}$ of $SI(\mathfrak{g}')$ spanned by (b_i) is isomorphic to $P_{\mathfrak{g}}$ and it is known [3] that $IS(\mathfrak{g}')$ is isomorphic to the symmetrical algebra of $\overline{P_{\mathfrak{g}}}$. Thus, one has a non-canonical isomorphism $IS(\mathfrak{g}') = S(\overline{P_{\mathfrak{g}}})$ and, for any element $x \in P_{\mathfrak{g}}$ there is an element $t(x) \in IW(\mathfrak{g})$ such that $dt(x) \in \overline{P_{\mathfrak{g}}}$. Denote $dt(x)$ by $\overline{x}$ and call $t(x)$ the *transgression cochain* of x.

Definition. Let U be a DG-algebra with a given left action of G by means of automorphisms of U. Denote by $\theta_x : U \to U$ $(x \in \mathfrak{g})$ the derivation of degree 0 of U such that $x \mapsto \theta_x$ is the representation of $\mathfrak{g}$ in U associated with the action of G. U is called a G-DG-algebra if, for each $x \in \mathfrak{g}$, an antiderivation i_x of degree -1 of U satisfying the following Cartan conditions:

$$i_{[x,y]} = \theta_x\, i_y - i_y\, \theta_x, \quad \theta_x = i_x\, d + d\, i_x,$$

is given [3].

An element of U annulated by all operators θ_x and i_x $(x \in \mathfrak{g})$ is called basic. The set $B_G(U)$ of basic elements is a DG-subalgebra of U.

Let U_1 and U_2 are two G-DG-algebras, then $U_1 \otimes U_2$ has the natural diagonal structure of a G-DG-algebra.

Consider the Weil algebra $W(\mathfrak{g})$. The adjoint representation of $\mathfrak{g}$ induces the action of $\mathfrak{g}$ on $W(\mathfrak{g})$ and the corresponding operators θ_x $(x \in \mathfrak{g})$. Define the antiderivation i_x of $W(\mathfrak{g})$ of degree -1 putting it equal to the inner product of a vector $x \in \mathfrak{g}$ and a form on $\Lambda(\mathfrak{g}')$ and zero on $S(\mathfrak{g}')$. It is easily checked that $W(\mathfrak{g}) = \{W(\mathfrak{g}), d, \theta_x, i_x\}$ is a G-DG-algebra. It is clear that, for each closed subgroup H of G, every G-DG-algebra is an H-DG-algebra.

Let P be a smooth principal G-bundle with a base B and with a connection ω. Let (e_i) $(i = 1, \ldots, dim\, G)$ be a basis of $\mathfrak{g}$, (e^i) the dual basis, and $\omega = \sum_i \omega^i e_i$. Then there is a natural homomorphism h of DG-algebras $W(\mathfrak{g}) \to \Omega(P)$ assigning to every e^i the 1-form ω^i on P. The following properties of h are well known:

(1) Let F be a fiber of P and $i : F \to P$ the inclusion. Then, $i^* \circ h$ is an isomorphism between $\Lambda(P_{\mathfrak{g}})$ and the space of G-invariant forms on F;
(2) The image of $IS(\mathfrak{g}')$ under h is contained in $\Omega(B)$ and the corresponding homomorphism $IS(\mathfrak{g}') \to \Omega(B)$ induces the characteristic Weil homomorphism $H(IS(\mathfrak{g}')) \to H(B; \mathbf{R})$ for P.

For each homogeneous element $x \in P_{\mathfrak{g}}$, the form $h(t(x))$ is called the *transgression cochain* of x in $\Omega(P)$ and we denote by $b(x)$ the image of $\overline{x}$ under h.

3. The transgression in the cross product of two Weil algebras and its applications

Denote by $C(\mathfrak{g})$ the DG-algebra $B_G(W(\mathfrak{g}_1) \otimes W(\mathfrak{g}_2))$, where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are two copies of $\mathfrak{g}$. Let $e^i = \frac{1}{2}(e_1^i - e_2^i)$ and $\Lambda(\mathfrak{g})$ the subspace of $B_G(W(\mathfrak{g}_1) \otimes W(\mathfrak{g}_2)$

spanned by e^i $(i = 1, \dots, deg\, G)$. It is easily seen that

$$C(\mathfrak{g}) = I(\Lambda(\mathfrak{g}') \otimes S(\mathfrak{g}'_1) \otimes S(\mathfrak{g}'_2)),$$

where the action of G on $\Lambda(\mathfrak{g}')\otimes S(\mathfrak{g}'_1)\otimes S(\mathfrak{g}'_2)$ induced by the adjoint representation of G is considered. In [8] the following theorem is proved.

3.1. Theorem. *Let G be a reductive Lie group with the Lie algebra $\mathfrak{g}$ and a an element of $P_{\mathfrak{g}}$ of degree 2p-1. Then, there exists $c \in C(\mathfrak{g})$ such that its component in $I\Lambda(\mathfrak{g}')$ coincides with a and $dc = b_1 + b_2$, where $b_i \in IS(\mathfrak{g}'_i)$ $(i = 1, 2)$.*

By analogy with the Weil complex we call the cochain c from Theorem 3.1 the *transgression cochain* of the element $a \in P_{\mathfrak{g}}$ in the DG-algebra $C(\mathfrak{g})$.

Further we write K and $\mathfrak{k}$ instead of G and $\mathfrak{g}$, since G will denote another Lie group.

Let P_i $(i = 1, 2)$ be two smooth principal K-bundles, where K is a connected Lie group and ω_i a connection form on P_i. Consider the natural morphism of K-DG-algebras $h_i : W(\mathfrak{k}) \to \Omega(P_i)$ defined as above by means of ω_i, and the morphism of DG-algebras

$$h : C(\mathfrak{k}) = B_K(W(\mathfrak{k}) \otimes W(\mathfrak{k})) \to B_K(\Omega(P_1) \otimes \Omega(P_2))$$

induced by $h_1 \otimes h_2$.

Define now the DG-algebra $C(P_1, P_2; \mathfrak{k})$ in the following way. As a graded algebra

$$C(P_1, P_2; \mathfrak{k}) = \Omega(B_1) \otimes \Lambda(P_{\mathfrak{k}}) \otimes \Omega(B_2),$$

where B_i is the base of P_i, and the differential d is equal to the exterior derivative on $\Omega(B_i)$ and, for each basic element $a \in P_{\mathfrak{k}}$ of degree $2p-1$, $da = h(dc) = h(b_1)+h(b_2)$, where c is the transgression cochain of a chosen by means of Theorem 3.1. Since by Theorem 3.1 $h(b_i) \in \Omega(B_i)$, $da \in \Omega(B_1) \otimes \Omega(B_2)$.

Consider the manifold $M = P_1 \times_K P_2$ and the DG-algebra $\Omega(M)$. The composition of the natural inclusions

$$C(P_1, P_2; \mathfrak{k}) \to B(\Omega(P_1) \otimes \Omega(P_2)) \to \Omega(M)$$

is the inclusion of $C(P_1, P_2; \mathfrak{k})$ into $\Omega(M)$.

Consider now $M = P_1 \times_K P_2$ as a locally trivial fiber bundle over $B_1 \times B_2$. This fiber bundle is not principal but its standard fiber F is diffeomorphic to $K \times_K K = K$.

3.2. Theorem. *Let K be a connected compact Lie group, P_i $(i = 1, 2)$ two smooth connected principal K-bundles, $M = P_1 \times_K P_2$, and ω_i a connection form on P_i. Then*

(1) *Each homogeneous element of $P_{\mathfrak{k}}$ is a transgressive element of the fiber bundle $M \to B_1 \times B_2$;*
(2) *The inclusion of $C(P_1, P_2; \mathfrak{k})$ into $\Omega(M)$ is a quasi-isomorphism, i.e. induces a cohomology isomorphism.*

Further we consider the G-manifold $M = P \times_K (G/H)$, where G is a reductive Lie group with the Lie algebra $\mathfrak{g}$, H and N are its closed subgroups with the Lie algebras $\mathfrak{h}$ and $\mathfrak{n}$, respectively, such that H is a normal subgroup of N. Moreover, suppose that $\mathfrak{h}$ and $\mathfrak{n}$ are reductive subalgebras of $\mathfrak{g}$, i.e. their adjoint representations in $\mathfrak{g}$ are completely reducible. Denote by $\mathfrak{k}$ the Lie algebra of the quotient group $K = N/H$. Note that, for a compact Lie group G, all reductivity conditions follow from the above relations between G and its closed subgroups H and N.

Consider a smooth connected principal K-bundle P with a base B and the G-manifold $M = P \times_K (G/H)$, where G/H is considered as a principal K-bundle. Let us chose a connection on P and a G-invariant connection on G/H, which exists under our assumptions [6]. In [8] it is proved that

$$\Omega(M)^G = C(P, G/H; \mathfrak{k})^G = \Omega(B) \otimes \Lambda(P_{\mathfrak{k}}) \otimes C(\mathfrak{g}, \mathfrak{n}),$$

where $C(\mathfrak{g}, \mathfrak{n})$ is the complex of cochains of $\mathfrak{g}$ relative to subalgebra $\mathfrak{n}$ [7].

Consider now the following graded algebra

$$C(M) = \Omega(B) \otimes \Lambda(P_{\mathfrak{g}}) \otimes \Lambda(P_{\mathfrak{k}}) \otimes S(\overline{P_{\mathfrak{n}}}),$$

where $\Omega(B)$ denote the DG-algebra of differential forms on the base B of P. Define a differential d on $C(M)$ as an antiderivation of degree 1 satisfying the following conditions:

(1) d is equal to exterior derivative on $\Omega(B)$;
(2) d vanishes on $S(P_{\mathfrak{n}})$;
(3) for $x \in P_{\mathfrak{g}}$ dx is equal to the image of $\overline{x}$ in $\overline{P_{\mathfrak{n}}}$ under the map $P_{\mathfrak{g}} \to P_{\mathfrak{n}}$ induced by the inclusion $\mathfrak{n} \subset \mathfrak{g}$; for $y \in P_{\mathfrak{k}}$, dy is the sum of two components: the first one is equal to $b(y)$ and the second one is equal to the image of $\overline{y}$ in $\overline{P_{\mathfrak{n}}}$ under the map induced by the natural projection $\mathfrak{n} \to \mathfrak{k}$.

In [8], for a connected N, the following theorem was proved.

3.3. Theorem. *The DG-algebras $\Omega(M)^G$ of G-invariant forms on a G-manifold M and $C(M)$ have the same minimal model, in particular, their cohomologies are isomorphic.*

For a compact group Lie G there is a natural geometrical interpretation of the complex $C(M)$. Denote by $EG \to BG$ the universal principal G-bundle. It is known [3] that one can consider $IS(\mathfrak{g}')$ as the cohomology $H(BG; \mathbf{R})$ and the transgression in $IW(\mathfrak{g})$ as the transgression in the Leray-Serre spectral sequence of the fiber bundle $EG \to BG$. It is obvious that the action of the subgroup H of G on EG defines on EG the structure of an universal H-bundle. Therefore, the space $G \times_H EG$ is homotopy equivalent to G/H and the space $P \times_K (G \times_H EG)$ is homotopy equivalent to $M = P \times_K (G/H.)$ The commutative diagram

$$\begin{array}{ccc} G \times_H EG & \longrightarrow & G/H \\ \downarrow & & \downarrow \\ G \times_N EG & \longrightarrow & G/N, \end{array}$$

where all arrows are the natural projections, defines a homotopy equivalence between two principal K-bundles $G\times_H EG \to G\times_N EG$ and $G/H\to G/N$.

Consider $\Lambda(P_{\mathfrak{g}})\otimes\Lambda(P_{\mathfrak{k}})\otimes S(\overline{P_{\mathfrak{n}}})$ with the structure of a DG-algebra such that the projection

$$C(M)\to\Lambda(P_{\mathfrak{g}})\otimes\Lambda(P_{\mathfrak{k}})\otimes S(\overline{P_{\mathfrak{n}}})$$

is a homomorphism of DG-algebras. It is clear that this structure is unique and the cohomology of this DG-algebra is isomorphic to the real cohomology of $G\times_H EG$. Then, from Theorem 3.3 it follows that the minimal models of $C(M)$ and the de Rham complex $\Omega(M)$ coincide.

4. The proof of the main theorem

Let now M, G, H, N, and K are as in Section 2 but N is not connected. By Proposition 1.2, for the component N_0 of the neutral element of N and $K_0=N_0/H$, one has the covering G-manifold $\widetilde{M}=P\times_{K_0}(G/H)$ over M. Then, by Theorem 2.2 the minimal models of $C(\widetilde{M})$ and $\Omega(\widetilde{M})^G$ coincide.

Recall now the following standard construction of homological algebra for the definition of an equivariant cohomology (see, for example, [2]). Let $K=\{K^q,d^q\}$ be a D-cochain complex, i.e a cochain complex with a left action of D on K by means of automorphisms of K. Denote by $C^{p,q}$ the space of maps from D^{p+1} to K^q $(p,q\geq 0)$ and define the maps $d':C^{p,q}\to C^{p+1,q}$, and $d'':C^{p,q}\to C^{p,q+1}$ as follows:

$$(d'c)(g_0,\dots,g_{p+1})=\sum_{i=0}^{p+1}(-1)^i c(g_0,\dots,\widehat{g}_i,\dots,g_{p+1}),$$

$$(d''c)(g_0,\dots,g_p)=(^{-1})^p d^q\, c(g_0,\dots,g_p),$$

where $c\in C^{p,q}$ and $g_0,\dots,g_{p+1}\in D$.

It is easily seen that d' and d'' are D-equivariant under the diagonal action of D on D^{p+1} and the given action of D on K. It is easily checked that $(d')^2=0$, $(d'')^2=0$, and $d'd''+d''d'=0$. Denote by $C(D,K)$ the bigraded space of G-invariant elements of $\bigoplus_{p,q}C^{p,q}$. It is evident that $C(D,K)$ is a double complex under the differentials d' and d''. It is known that $C(D,K)$ under the first differential d' is the complex of standard homogeneous cochains of the group D with coefficients in D-module K.

Further we shall consider $C(D,K)$ as a cochain complex under the total differential $d'+d''$. Its cohomology is denoted by $H_D(K)$ and is called the *equivariant cohomology* of K. If K is a graded associative algebra and its differential is an antiderivation of degree 1, the Alexander-Whitney map for the standard resolution of D, i.e. the map

$$(g_0,\dots,g_n)\mapsto\sum_{p=0}^{n}(g_0,\dots,g_p)\otimes(g_p,\dots,g_n),$$

induces the structure of a graded associative algebra on $C(D,K)$ and the total differential of $C(D,K)$ is an antiderivation of this algebra of degree 1. Hence, there is the structure of an associative algebra on $H_D(K)$ and a multiplicative structure on $H_D(K)$ in this case.

4.1. Proposition. *Let D act freely on K. Assign to each D-invariant element $k \in K$ the constant map $c_k : D \to K$ with the value k. The inclusion $k \to c_k$ of the subcomplex K^D of D-invariant elements of K into $C(D, K)$ is a homomorphism of complexes inducing a cohomology isomorphism.*

Proof. It is evident that the above inclusion is a homomorphism of complexes. Consider the double complex $C(D, K)$. Since D acts freely on K one can consider a cochain $c \in C^{p,q}$ of this complex as a map from D^{p+1} to K^G. Then, the cohomology of the complex $C(D, K))$ under d' is trivial and, for the second spectral sequence of $C(D, K)$, $E_2^{p,q} = 0$, for $p \neq 0$, and $E_2^{0,q} = E_\infty^{0,q} = H^q(K^D)$. Thus, the above inclusion of K^D into $C(D, K)$ induces a cohomology isomorphism. $\square$

4.2. Theorem. *Assume that there exists a K-invariant connection on the principal K_0-bundle $\widetilde{P}$. Then, there is a natural isomorphism of cohomology algebras $H_D(C(\widetilde{M})) = H(\Omega(M)^G)$.*

Proof. Consider the double complex $C(D, \Omega(\widetilde{M})^G)$. Since D acts freely on $\widetilde{M}$ by Proposition 4.1 the inclusion $\Omega(M)^G \subset C(D, \Omega(\widetilde{M^G}))$ induces a cohomology isomorphism.

Let us use for the above construction of the transgression cochains of homogeneous elements of $P_{\mathfrak{k}}$ in the complex $\Omega(\widetilde{P})$ the K-invariant connection on $\widetilde{P}$. Then, the exterior derivatives of these transgression cochains are D-invariant forms on $\widetilde{B}$. Then, one can consider $C(M)$ as the subalgebra of D-invariant elements of $C(\widetilde{M})$ and, by Proposition 4.1, the inclusion $C(M) \subset C(D, C(\widetilde{M}))$ induces a cohomology isomorphism.

Remark that $C(D, K)$ and the first spectral sequence of the double complex $C(D, K)$ are covariant functors under K. Hence, for the first spectral sequence of $C(D, K)$, $E_2 = H^p(D, H^q(K))$, if K_1, K_2 are two D-cochain complexes and $f : K_1 \to K_2$ is an equivariant homomorphism of complexes inducing a cohomology isomorphism, the corresponding homomorphism $C(D, K_1) \to C(D, K_2)$ induces a cohomology isomorphism $H_D(K_1) \to H_D(K_2)$.

Consider $\Lambda(\mathfrak{g}')$ as a complex of cochains of the Lie algebra $\mathfrak{g}$ with coefficients in the trivial $\mathfrak{g}$-module $\mathbf{R}$ [6]. It is evident that the adjoint representation of $\mathfrak{g}$ and the inner product of a vector $x \in \mathfrak{g}$ and a form on $\mathfrak{g}$ define on $\Lambda(\mathfrak{g}')$ the structure of G-DG-algebra.

Let $B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$ be the DG-subalgebra of basic elements of N-DG-algebra $B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$. There is a unique structure of DG-algebra on

$$C_1(\widetilde{M}) = \Omega(\widetilde{B}) \otimes \Lambda(P_{\mathfrak{k}}) \otimes B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}')).$$

such that its restriction to $B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$ coincides with the given one and the morphism of graded algebras

$$f_1 : C(\widetilde{P}, G, \mathfrak{k})^G \to C_1(\widetilde{M})$$

induces a morphism of DG-algebras.

Consider now the Cartan complex [4] $C(G/H) = \Lambda(P_{\mathfrak{g}}) \otimes S(\overline{P_{\mathfrak{h}}})$ be the DG-algebra with the differential δ defined as follows:

$$\delta y = \overline{j \circ l(y)}, \quad \delta z = 0,$$

where y is a homogeneous element of $P_{\mathfrak{g}}$ and $z \in S(\overline{P_{\mathfrak{h}}})$. There is a natural morphism of DG-algebras $C(G/N) \to B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}'))$, which induces a cohomology isomorphism [4]. Let $f_2 : C(\widetilde{M}) \to C_1(\widetilde{M})$ be the morphism of graded algebras induced by the morphism

$$C(G/N) = \Lambda(P_{\mathfrak{g}}) \otimes S(\overline{P_{\mathfrak{n}}}) \to B_N(W(\mathfrak{n}) \otimes \Lambda(\mathfrak{g}')).$$

Clearly, f_2 is a morphism of DG-algebras.

Since for the definition of $C(\widetilde{P}, G/H, \mathfrak{k})$ we used the D-invariant connection, $C(\widetilde{P}, G/H, \mathfrak{k})$ is a D-module. Besides, the action of D on $\Omega(B)$ induces the structures of D-modules on $C(\widetilde{M})$ and $C_1(\widetilde{M})$ such that the morphisms f_1 and f_2 are D-equivariant. Moreover, in [8] it is proved that f_1 and f_2 are quasi-isomorphisms.

Thus, from the above remark it follows that $H_D(\Omega(\widetilde{M})^G) = H_D(C(\widetilde{M}))$. Therefore, the inclusion $\Omega(M)^G \subset C(D, \Omega(\widetilde{M}^G))$ and the above isomorphism produce an isomorphism $H(\Omega(M^G)) = H_D(C(\widetilde{M}))$. □

The following proposition is well known (see, for example, [5]).

4.3. Proposition. *If D is a finite group and V is a vector D-module over a field of characteristic 0, $H^q(D; V) = 0$ for $q \geqq 0$.*

4.4. Theorem. *If D is a finite group, the cohomology algebra $H(\Omega(M)^G)$ is isomorphic to $H(C(M))$. In particular, for a compact Lie group G, $H(\Omega(M)^G) = H(C(M))$.*

Proof. At first, note that, for a finite D, there exists a D-invariant connection on $\widetilde{P}$. By Proposition 4.3, for the first spectral sequence of the double complex $C(D, C(\widetilde{M}))$, $E_2^{p,q} = 0$ for $p > 0$ and, as usual, $E_2^{0,q}$ is the space of D-invariant elements of $C(\widetilde{M})$ of degree q. From this the statement of the theorem follows at once. □

Remark. The geometric interpretation of the standard resolution of D is the contractible simplicial complex $S(D)$, for which D is the set of vertices, with the natural action of D. Under d', $C(D, K)$ is the complex of simplicial cochains of $S(D)$ with coefficients in the D-module K. It is clear that $C(D, K) = C(D, \mathbf{R}) \otimes K$, where $\mathbf{R}$ is a trivial D-module. The multiplication in $C(D, \mathbf{R})$ is defined by means of the Alexander-Whitney map and is not commutative in the graded sense. Hence, $C(D, K)$ is not DG-algebra even if K has such a structure. One can obtain the convenient DG-algebra instead of $C(D, K)$ if one replace $C(D, \mathbf{R})$ by the de Rham-Sullivan complex of forms on $S(D)$ [10]. This DG-algebra is rather complicated but from its existence it follows that, under the condition of Theorem 4.4, the minimal models of the DG-algebras $\Omega(M)^G$ and $C(M)$ coincide.

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Saratov State University, Astrakhanskaya 83, 410056 Saratov, Russia
E-mail address: losik@scnit.saratov.su