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THE COMPLETENESS OF SOME HAMILTONIAN VECTOR FIELDS ON A POISSON MANIFOLD

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Abstract. We prove that under some restrictions the flow of a Hamiltonian vector field on a finite dimensional Poisson manifold exists for all time.

1. Introduction

Let (Q, g) be a Riemannian, n-dimensional, smooth $(= C^\infty)$ manifold, T^*Q its cotangent bundle and $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ canonical coordinates on T^*Q. Every smooth potential V on Q gives rise to a Hamiltonian H on T^*Q, namely

$$H = \sum_{i,j=1}^{n} g^{ij} p_i p_j + V(q^1, \ldots, q^n), \tag{1.1}$$

whose corresponding Hamiltonian vector field X_H is given locally by:

$$X_H = \sum_{i,j=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right). \tag{1.2}$$

Around 1969 W. Gordon [2] proved that X_H is complete if V is proper, i.e. V^{-1} (compact)=compact, and bounded below, say $V \geq 0$. The aim of our paper is to extend Gordon's result to arbitrary finite dimensional Poisson manifolds.

2. Generalities on vector fields and their completeness

Let M be a smooth n-dimensional manifold. We remind ourselves that (see [1], [3], [4] for details) that a vector field X on M is said to be complete if for every $x_0 \in M$, the maximal interval of existence (t_-, t_+) of every solution of:

$$\begin{cases} \dfrac{dx(t)}{dt} = X(x(t)) \\ x(0) = x_0 \end{cases} \tag{2.1}$$

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is given by $t_\pm = \pm\infty$. Hence if X is of class C^1, so that solutions of (2.1) are unique, to say that X is complete means that X generates a global flow on M. It is well known (see e.g. [1] Theorem 2.1.18 p. 70) that $t_\pm = \pm\infty$ if $x(t)$ remains in a compact set as t varies over any such neighbourhood. A sufficient condition which assures this property is provided by the following theorem:

Theorem 2.1. *(Gordon) Let X be a C^1-vector field on a manifold M of class C^1. Then X is complete if there exists a C^1-function E, a proper C^0-function f and the constants $\alpha, \beta \in \mathbb{R}$ such that for each $x \in M$ we have:*

$$|X(E(x))| \leq \alpha \cdot |E(x)| \tag{2.2}$$

$$|f(x)| \leq \beta \cdot |E(x)| \tag{2.3}$$

Proof. From the chain rule and the basic definitions we have:

$$\frac{dE(x(t))}{dt} = X(E)(x(t)) \tag{2.4}$$

Hence, if we define

$$h(t) = |E(x(t))| \tag{2.5}$$

then we have successively:

$$\begin{aligned} h(t) &\overset{(2.5)}{=} |E(x(t))| \\ &\overset{(2.5)}{=} |\int_0^t X(E)(x(s))ds + E(x(0))| \\ &\leq |E(x(0))| + |\int_0^t X(E)(x(s))ds| \\ &\leq |E(x(0))| + \int_0^t |X(E)(x(s))|ds \\ &\overset{(2.5)}{\leq} |E(x(0))| + \alpha \int_0^t h(s)ds. \end{aligned}$$

It follows that:

$$h(t) \leq |E(x(0))| + \alpha \int_0^t h(s)ds. \tag{2.6}$$

Now, using Gronwall's inequality (see the Appendix) we deduce that:

$$h(t) \leq |E(x(0))| \exp(\alpha|t|). \tag{2.7}$$

Then

$$\begin{aligned} |f(x(t))| &\overset{(2.3)}{\leq} \beta|E(x(t))| \\ &\overset{(2.7)}{\leq} \beta|E(x(0))| \cdot \exp(\alpha|t|), \end{aligned}$$

and therefore:

$$|f(x(t))| \leq \beta \cdot |E(x(0))| \cdot \exp(\alpha \cdot |t|). \tag{2.8}$$

Since f is proper, this means that $x(t)$ remains in a compact set as t varies over a bounded neighbourhood of zero (for which a solution is defined), so that X is complete. $\square$

3. The main result

Let $(P, \{\cdot,\cdot\})$ be a smooth n-dimensional Poisson manifold, i.e. P is a smooth n-dimensional manifold and $\{\cdot,\cdot\}$ is a bi-linear map

$$\{\cdot,\cdot\} : C^\infty(P,\mathbb{R}) \times C^\infty(P,\mathbb{R}) \to C^\infty(P,\mathbb{R})$$

such that the following conditions are satisfied

$$\begin{aligned}
&(P_1)\{f,g\} = -\{g,f\},\\
&(P_2)\{f,gh\} = g\{f,h\} + h\{f,g\},\\
&(P_3)\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0
\end{aligned}$$

for each $f, g, h \in C^\infty(P,\mathbb{R})$.

For each $H \in C^\infty(P,\mathbb{R})$ we denote by X_H the Hamiltonian vector field on P given by

$$X_H : f \in C^\infty(P,\mathbb{R}) \mapsto X_H(f) \overset{\text{def}}{=} \{f, H\} \in C^\infty(P,\mathbb{R}).$$

Then we can prove:

Theorem 3.1. *Let $(P, \{\cdot,\cdot\})$ be a smooth n-dimensional Poisson manifold. If $H \in C^\infty(M,\mathbb{R})$ is proper and bounded below, say $H \geq 0$, then X_H is complete.*

Proof. Let us take in Theorem 2.1

$$f = E = H.$$

Since

$$X_H(H) = \{H, H\} = 0$$

it follows that all conditions of the Theorem 2.1 are satisfied and then we can conclude that X_H is complete. $\square$

As a consequence we can obtain immediately the following theorem:

Theorem 3.2. *Let (M, ω) be a smooth, $2n$-dimensional symplectic manifold and $H \in C^\infty(M,\mathbb{R})$. If H is proper and bounded below, say $H \geq 0$, then X_H is complete.*

4. Appendix

In this last section we shall give, following [1], the proof of the well known Gronwall's inequality.

Theorem 4.1. *(Gronwall's inequality). Let $f, g : [a,b] \to \mathbb{R}$ be continuous and non-negative. Suppose that*

$$f(t) \leq A + \int_a^t f(s) g ds; \quad A \geq 0.$$

Then it follows that

$$f(t) \leq A \cdot \exp\left(\int_0^t g(s) ds\right),$$

for $t \in [a, b]$.

Proof. First, suppose that $A > 0$. Let

$$h(t) = A + \int_0^t f(s)g(s)ds. \tag{4.1}$$

Thus

$$h(t) > 0 \tag{4.2}$$

and

$$f(t) \leq h(t). \tag{4.3}$$

Then

$$\dot{h}(t) = f(t)g(t) \overset{(4.3)}{\leq} h(t) \cdot g(t)$$

or equivalently

$$\frac{\dot{h}(t)}{h(t)} \leq g(t).$$

Integration gives via (4.2):

$$ln \quad h(t)|_a^t \leq \int_a^t g(s)ds$$

or

$$ln\frac{h(t)}{A} \leq \int_a^t g(s)ds.$$

Hence:

$$h(t) \leq A \cdot \exp(\int_a^t g(s)ds)$$

and then via (4.3) we obtain the desired result

$$f(t) \leq A \cdot \exp(\int_a^t g(s)ds).$$

If $A = 0$, then we have the result replacing A by $\in > 0$ for every $\in > 0$, thus h and hence f is zero. $\square$

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