ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

Classification of Bochner Flat Kähler Manifolds by Heisenberg, Spherical CR Geometry

Yoshinobu Kamishima

Vienna, Preprint ESI 1779 (2006)
February 20, 2006

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

CLASSIFICATION OF BOCHNER FLAT KÄHLER MANIFOLDS BY HEISENBERG, SPHERICAL CR GEOMETRY

YOSHINOBU KAMISHIMA

Abstract. A Bochner flat Kähler manifold is a Kähler manifold with vanishing Bochner curvature tensor. We shall give a uniformization of Bochner flat Kähler manifolds. One of the aims of this paper is to give a correction to the proof of our previous paper [9] concerning uniformization of Bochner flat Kähler manifolds. A Bochner flat locally conformal Kähler manifold is a locally conformal Kähler manifold with vanishing Bochner curvature tensor. We shall apply our result to Bochner flat locally conformal Kähler manifolds.

Contents

2000 *Mathematics Subject Classification.* Primary 57S25; Secondary 51C25, 53C55

Key words and phrases. Kähler structure, Chern-Moser- Webster curvature tensor, Pseudo Hermitian structure, Heisenberg Lie group, Bochner curvature tensor, Spherical CR-structure, Locally conformal Kähler structure.

Introduction

In 1949, S. Bochner has introduced the curvature tensor on Kähler manifolds, which is now called *Bochner curvature tensor.* This tensor can be also derived from the Weyl's principle on unitary group representation theory (cf. [20], [1], [17]). In 2000, R. Bryant [3] has exhibited miscellaneous results on Bochner flat Kähler metrics and determined complete Bochner flat Kähler metrics in which there are many complete Bochner flat Kähler manifolds but not locally homogeneous ones. Then it turned out that our classification [9] of Bochner flat Kähler manifolds was wrong. In this paper we provide with more explicit models of Bochner flat Kähler manifolds to correct our previous results and also reprove Bryant's classification of compelete case by our method. As an application, we shall extend our framework into the locally conformal Kähler geometry. Our proofs are purely topological by using the transformation groups of CR geometry $(\mathrm{Aut}_{CR}(S^{2n+1}), S^{2n+1})$ and complex hyperbolic geometry $(\mathrm{PU}(n+1,1), \mathbb{H}_{\mathbb{C}}^{n+1})$.

1. Geometric uniformization

1.1. **Review of Chern-Moser CR curvature tensor.** When we persist the Weyl's conformal geometry in the strictly pseudoconvex CR-manifolds, S. S. Chern and J. Moser [5] have found the fourth-order curvature tensor $S = (S_{\alpha\rho\bar{\beta}\bar{\sigma}})$ on a CR-manifold N^{2n+1} by making use of the structure equations modelled on the real hypersurface in $\mathbb{C}^{n+1}$. This CR-invariant tensor is a conformal invariant in the following sense: if two contact forms ω, ω' represnt the same CR structure (keeping the complex structure J on the CR-bundle fixed), then $\omega' = u \cdot \omega$ for some positive function u for which the Chern-Moser curvature tensor coincides $S(\omega, J) = S(\omega', J)$. The sphere S^{2n+1} is a CR-manifold viewed as a hyperquadric in $\mathbb{C}^{n+1}$, whose curvature tensor S vanishes identically. In fact, the complex analogue of conformal geometry states that if the Chern-Moser curvature tensor S of a CR-manifold N vanishes, then N is locally CR-equivalent to S^{2n+1} $(n > 1)$. In this case, N is said to be a *spherical CR-manifold.*

Note that the formula of S is given by

$$
\begin{aligned}
S_{\alpha\rho\bar{\beta}\bar{\sigma}} &= R_{\alpha\bar{\beta}\rho\bar{\sigma}} - \frac{1}{n+2}(R_{\alpha\bar{\beta}}g_{\rho\bar{\sigma}} + R_{\rho\bar{\beta}}g_{\alpha\bar{\sigma}} + g_{\alpha\bar{\beta}}R_{\rho\bar{\sigma}} + g_{\rho\bar{\beta}}R_{\alpha\bar{\sigma}}) \\
&\quad + \frac{R}{2(n+1)(n+2)}(g_{\alpha\bar{\beta}}g_{\rho\bar{\sigma}} + g_{\rho\bar{\beta}}g_{\alpha\bar{\sigma}}).
\end{aligned}
\tag{1.1}
$$

Here $R_{\alpha\bar{\beta}\rho\bar{\sigma}}$ is the Tanaka - Webster curvature tensor. Webster [18] has observed that S has the same formula as the Bochner curvature tensor B on a Kähler manifold. More precisely, when a CR manifold (N, ω, J) is the total space of the principal $\mathbb{R}$-bundle $p : N \to M$ over the Kähler manifold (M, g, J) with fundamental 2-form Ω for which the Levi form satisfies that $d\omega = p^*\Omega$, the Chern-Moser curvature tensor is obtained by the pull-back of the Bochner curvature tensor:

$$S(\omega, J) = p^* B(\Omega, J). \tag{1.2}$$

1.2. **Contactization.** We start with a simply connected Kähler manifold $(\tilde{M}, J, g)$ of real dimension $2n \geq 4$ where J is the complex structure and g is a Kähler metric. Let Ω be the fundamental two form of g. Suppose that $\tilde{M}$ is a Bochner flat Kähler manifold, i.e. the Bochner curvature tensor of g vanishes on $\tilde{M}$. We construct a uniformization for $\tilde{M}$ as in [9]. Choose a cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of $\tilde{M}$ such that each U_α is homeomorphic to a ball. For the local 2-form $\Omega_\alpha = \Omega | U_\alpha$, there is a 1-form θ_α with $d\theta_\alpha = \Omega_\alpha$ on each U_α. The product $M_\alpha = \mathbb{R} \times U_\alpha$ admits a contact form $\omega_\alpha = dt + p_\alpha^* \theta_\alpha$ where $p_\alpha : M_\alpha \to U_\alpha$ is the projection. If we define the complex structure $J_\alpha = p_{\alpha *}^{-1} \circ J \circ p_{\alpha *}$ on the contact subbundle $\mathrm{Null}\, \omega_\alpha$, then the pseudo-Hermitian structure $(\omega_\alpha, J_\alpha)$ gives a strictly pseudoconvex CR-structure on M_α for which the group $\mathbb{R}$ acts properly and freely as CR-transformations. As the Levi form satisfies that $d\omega_\alpha = p_\alpha^* \Omega_\alpha$, (1.2) shows that M_α is a spherical CR-manifold. There is the developing pair $(\rho_\alpha, \mathrm{dev}_\alpha) : (\mathbb{R}, M_\alpha) \longrightarrow (\mathrm{PU}(n+1, 1), S^{2n+1})$ up to composite by an element of the group of CR-transformations $\mathrm{Aut}_{CR}(S^{2n+1}) = \mathrm{PU}(n+1, 1)$. Fix α and put $\mathrm{dev}_\alpha = \mathrm{dev}, \rho_\alpha = \rho$. Let G be the closure of the holonomy group $\rho(\mathbb{R})$ in $\mathrm{PU}(n+1, 1)$. As G is a connected abelian Lie subgroup of the complex hyperbolic group $\mathrm{PU}(n+1, 1)$, it has the fixed point subset on the union $\mathbb{H}_{\mathbb{C}}^{n+1} \cup S^{2n+1}$ in which the boundary S^{2n+1} is viewed as the one point compactification of the Heisenberg nilpotent Lie group $\mathcal{N}$; $S^{2n+1} = \mathcal{N} \cup \{\infty\}$. (See [10].) If G is noncompact, then it has the unique fixed point $\{\infty\}$, or exactly two fixed points $\{0, \infty\}$ where $\{0\}$ is the origin of $\mathcal{N}$. If G is compact, it occurs either no fixed point on S^{2n+1} (it has the unique fixed point in $\mathbb{H}_{\mathbb{C}}^{n+1}$) or the fixed point set is the subsphere S^{2m-1} up to conjugacy, $m = 1, \cdots, n$. Denote by X one of the domains $\mathcal{N}$, $\mathcal{N} - \{0\}$, $S^{2n+1} - S^{2m-1}$ $(m = 0, 1, \cdots, n)$. Since $\mathbb{R}$ acts freely on M_α and dev is an immersion, $\rho(\mathbb{R})$ has no fixed point on the image $\mathrm{dev}(M_\alpha)$. It follows that $\mathrm{dev}(M_\alpha) \subset X$. We seek further the domain including the developing image. Let $\mathrm{Psh}(X)$ be the subgroup consisting of pseudo-Hermitian transformations of the group of all CR transformations $\mathrm{Aut}_{CR}(X)$. Choose the canonical contact form ω_0 on X which is invariant under $\mathrm{Psh}(X)$. (See § 2.) If J_0 is the complex structure to X from the resriction of the spherical CR-structure of S^{2n+1}, then (ω_0, J_0) represents the spherical CR-structure $(\mathrm{Null}\, \omega_0, J_0)$ on X. By the definition of CR-immersion, there exists a positive function u on M_α such that $\mathrm{dev}^* \omega_0 = u \cdot \omega_\alpha$ and $\mathrm{dev}_* \circ J_\alpha = J_0 \circ \mathrm{dev}_*$ on $\mathrm{Null}\, \omega_\alpha$. Let ξ be the vector field induced by the 1-parameter group $\rho(\mathbb{R})$ on X. Similarly d/dt denotes the vector field induced by $\mathbb{R}$ so that $\omega_\alpha(d/dt) = 1$. As the developing map is equivariant, $\xi = \mathrm{dev}_*(d/dt)$ and so $\omega_0(\xi) = u > 0$. This imposes a restriction on the developing image. If

$S = \{p \in X | \ \omega_0(\xi_p) = 0\}$ is the singular subset of X, then $\mathrm{dev}(M_\alpha) \subset (X - S)^0$ which is the connected component of the complement $X - S$.

1.3. **Closed holonomy case.** Suppose that $\rho(\mathbb{R})$ is closed in $\mathrm{PU}(n+1,1)$. Then, $\rho(\mathbb{R}) = \mathbb{R}$ or S^1. If $\rho(\mathbb{R}) = \mathbb{R}$, then either $\rho(\mathbb{R}) \subset \mathrm{Psh}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(n)$ or $\rho(\mathbb{R}) \subset \mathrm{Aut}_{CR}(\mathcal{N} - \{0\}) = \mathrm{U}(n) \times \mathbb{R}^+$ according to whether the fixed point set on S^{2n+1} is $\{\infty\}$ or $\{0, \infty\}$. In particular, $\rho(\mathbb{R})$ acts properly and freely on $X = \mathcal{N}$ or $X = \mathcal{N} - \{0\}$ respectively. When $\rho(\mathbb{R}) = S^1$, it does not necessarily act freely on $X = S^{2n+1} - S^{2m-1}$ $(m = 0, 1, \cdots, n)$ where $\mathrm{Aut}_{CR}(S^{2n+1} - S^{2m-1}) = P(\mathrm{U}(m,1) \times \mathrm{U}(n-m+1))$ such that $\rho(\mathbb{R}) = S^1 \subset \mathrm{U}(n-m+1)$ (see [9]). Let E be the set of all exceptional orbits of $\rho(\mathbb{R}) = S^1$ on X, see [2] for the definition. Consider the S^1-action on the image $\mathrm{dev}(M_\alpha) = S^1 \cdot \mathrm{dev}(U_\alpha)$. For $p \in \mathrm{dev}(U_\alpha)$, the slice theorem of compact Lie group actions [2] shows that there exists a slice $V \subset \mathrm{dev}(U_\alpha)$ such that the equivariant tubular neighborhood of the orbit at p has the form $S^1 \underset{S^1_p}{\times} V$. Choose $\tilde{U} \subset U_\alpha$ such that $\mathrm{dev} : \tilde{U} \to V$ is a diffeomorphism. Then $\mathbb{R} \times \tilde{U} \xrightarrow{\mathrm{dev}} S^1 \underset{S^1_p}{\times} V$ is a covering map. Since the stabilizer S^1_p is a finite cyclic group, $\rho^{-1}(S^1_p)$ is an infinite cyclic subgroup of $\mathbb{R}$. So dev induces an S^1-equivariant diffeomorphism from $\mathbb{R} \times \tilde{U}/\rho^{-1}(S^1_p) = S^1 \times \tilde{U}$ onto $S^1 \underset{S^1_p}{\times} V$. Hence $S^1_p = \{1\}$, i.e. S^1 acts freely on $\mathrm{dev}(M_\alpha)$. This shows that $\mathrm{dev}(M_\alpha) \subset (X - S - E)^0$.

1.4. **Non-closed holonomy case.** Suppose that $\rho(\mathbb{R})$ is not closed in $\mathrm{PU}(n+1,1)$. Then $\rho(\mathbb{R})$ does not act properly on X. Since dev is an immersion, there exists a maximal open interval $\triangle = (-\epsilon, \epsilon)$ of $\mathbb{R}$ such that $\mathrm{dev} : \triangle \times U_\alpha \to \mathrm{dev}(\triangle \times U_\alpha) = \rho(\triangle) \cdot \mathrm{dev}(U_\alpha)$ is equivariantly diffeomorphic. In particular, $\rho(\triangle)$ acts properly and freely on $\mathrm{dev}(\triangle \times U_\alpha)$. Choose a maximal domain W in X containing $\mathrm{dev}(\triangle \times U_\alpha)$ such that $\rho(\triangle)$ acts properly and freely.

1.5. **Bochner flat Kähler Model.** Put $\mathcal{W} = (X - S)^0$, $\mathcal{W} = (X - S - E)^0$ or $\mathcal{W} = W$ for the closed (respectively non-closed) case. Then $\mathrm{dev}(M_\alpha) \subset \mathcal{W}$ (respectively $\mathrm{dev}(\triangle \times U_\alpha) \subset \mathcal{W}$) from §1.3, 1.4. Since $\rho(\mathbb{R})$ (respectively $\rho(\triangle)$) acts properly and freely on $\mathcal{W}$, dev induces an immersion $\widehat{\mathrm{dev}}$ on the orbit space; $\widehat{\mathrm{dev}}(U_\alpha) \subset \mathcal{W}/\rho(\mathbb{R})$ (respectively $\widehat{\mathrm{dev}}(U_\alpha) \subset \mathcal{W}/\triangle$). Let $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ be the centralizer of $\rho(\mathbb{R})$ in $\mathrm{Aut}_{CR}(\mathcal{W})$ and denote its quotient group by $\mathcal{H} = \mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R})$. Similarly $\mathcal{H}$ is defined to be $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\triangle))/\rho(\triangle)$ for $\mathcal{W}/\rho(\triangle)$. As $\mathcal{H}$ acts on $\mathcal{W}/\rho(\mathbb{R})$, the projection $\nu : \mathcal{W} \to \mathcal{W}/\rho(\mathbb{R})$ (respectively $\mathcal{W}/\rho(\triangle)$) induces an equivariant principal bundle:

$$\rho(\mathbb{R}) \to (\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R})), \mathcal{W}) \xrightarrow{\nu} (\mathcal{H}, \mathcal{W}/\rho(\mathbb{R})) \tag{1.3}$$

(respectively local principal bundle for $(\mathcal{H}, \mathcal{W}/\rho(\triangle))$). We construct a Bochner flat Kähler metric on $\mathcal{W}/\rho(\mathbb{R})$ for which $\mathcal{H}$ acts as Kähler transformations. As $\omega_0(\xi) > 0$ on $\mathcal{W}$, define a 1-form η on $\mathcal{W}$ to be:

$$\eta(Z) = \frac{1}{\omega_0(\xi)} \cdot \omega_0(Z) \quad (\forall\ Z \in T\mathcal{W}). \tag{1.4}$$

By the definition, every element $h \in \mathrm{Aut}_{CR}(\mathcal{W})$ satisfies that $h^*\omega_0 = u\cdot\omega_0$ for some $u > 0$ on $\mathcal{W}$. If $h \in \mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$, then $h_*\xi = \xi$ and so $(h^*\eta)(Z_x) = \dfrac{1}{\omega_0(\xi_{hx})}\cdot (h^*\omega_0)(Z_x) = \dfrac{1}{u\cdot\omega_0(\xi_x)}\cdot(u\cdot\omega_0(Z_x)) = \eta(Z_x)$. The group $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ preserves η. In particular, $\mathcal{L}_\xi\eta = 0$. Obviously $\eta(\xi) = 1$ so that $d\iota_\xi\eta = 0$. By the formula $\mathcal{L}_\xi\eta = \iota_\xi d\eta + d\iota_\xi\eta$, it follows that

(1.5) $\qquad \iota_\xi d\eta = 0,\ \ $ that is ξ is the characteristic vector (Reeb) field for η.

As $\nu_* : \mathrm{Null}\,\eta \to T(\mathcal{W}/\rho(\mathbb{R}))$ is isomorphic at each point of $\mathcal{W}$, using J_0 on $\mathrm{Null}\,\eta = \mathrm{Null}\,\omega_0$, the complex structure $\hat J$ on $\mathcal{W}/\rho(\mathbb{R})$ is defined to be $\hat J\circ\nu_* = \nu_*\circ J_0$. Noting (1.5), we can define a two form $\hat\Omega$ on $\mathcal{W}/\rho(\mathbb{R})$ by

$$\nu^*\hat\Omega = d\eta \quad \text{on } \mathcal{W}. \tag{1.6}$$

Then $d\hat\Omega = 0$ and $\hat\Omega$ is $\hat J$-invariant because $d\eta$ is J_0-invariant. Since $d\eta$ is positive definite (i.e. strictly pseudo-convex), setting $\hat g(\hat X, \hat Y) = \hat\Omega(\hat J\hat X, \hat Y)$ $(\hat X, \hat Y \in T(\mathcal{W}/\rho(\mathbb{R}))$, we obtain a Kähler metric $\hat g$ on $(\mathcal{W}/\rho(\mathbb{R}), \hat J)$ with fundamental form $\hat\Omega$. As $(\mathrm{Null}\,\eta, J_0)$ is a *spherical* CR-structure with (1.6), the Bochner curvature tensor of $\hat g$ vanishes by (1.2). Since each element of $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ preserves η and commutes with J_0, $\mathcal{H}$ acts as holomorphic isometries of $\hat g$ on $\mathcal{W}/\rho(\mathbb{R})$. Therefore we obtain the model Bochner flat Kähler structure $(\mathcal{H}, \mathcal{W}/\rho(\mathbb{R}), \hat g, \hat J)$. Similary for $(\mathcal{H}, \mathcal{W}/\rho(\triangle), \hat g, \hat J)$.

Remark 1.1. *Our Bochner flat Kähler model is obtained from* $\mathcal{W} = (X - S)^0$, $(X - S - E)^0$ *or* $\mathcal{W} = W$ *according to whether* $\rho(\mathbb{R})$ *is closed or non-closed. Note that* X *and* S *are determined by the holonomy* $\rho(\mathbb{R})$. *In general the Kähler structure varies when* $\rho(\mathbb{R})$ *does. But the isomorphism classes of Bochner flat Kähler models are uniquely determined by the conjugacy classes of their holonomy groups. See Proposition 1.4.*

Remark 1.2. *If we note that* $\omega_0(\xi) > 0$ *on* $(X - S)^0$, *we can define the above contact form* η *on* $(X - S)^0$. *When* $\rho(\mathbb{R})$ *acts properly on* $(X - S)^0$, *the quotient space* $(X - S)^0/\rho(\mathbb{R})$ *is an orbifold in general. By the same argument as in* §1.5, *we obtain a Bochner flat singular Kähler metric* $\hat g^*$ *on the complex orbifold* $((X - S)^0/\rho(\mathbb{R}), \hat J^*)$.

1.6. **Construction of charts and Geometric uniformization.** Fix α and suppose that $U_\alpha\cap U_\beta \neq \emptyset$ for $\beta\neq\alpha$. As $d\theta_\beta = \Omega_\beta = \Omega|U_\beta$, $d(\theta_\beta - \theta_\alpha) = 0$ on $U_\alpha\cap U_\beta$. We may assume that $U_\alpha\cap U_\beta$ is contractible (shrinking small on each component if necessary). There is a smooth map $\chi : U_\alpha\cap U_\beta \to \mathbb{R}$ such that $\theta_\beta - \theta_\alpha = d\chi$. Then the map $f : \mathbb{R}\times U_\alpha\cap U_\beta \longrightarrow \mathbb{R}\times U_\alpha\cap U_\beta$ defined by $f(t,x) = (t + \chi(x), x)$ satisfies that $f^*\omega_\alpha = \omega_\beta$, $f_*\circ J_\beta = J_\alpha\circ f_*$ and $f_*(d/dt) = d/dt$. Hence f is a pseudo-Hermitian diffeomorphim between (ω_β, J_β) and $(\omega_\alpha, J_\alpha)$ which implies that the developing pairs $(\rho_\beta, \mathrm{dev}_\beta)$ and $(\rho_\alpha, \mathrm{dev}_\alpha\circ f)$ represent the same CR-structure $(\mathrm{Null}\,\omega_\beta, J_\beta)$. By definition of the equivalence, there exists $g_{\alpha\beta}\in \mathrm{PU}(n+1,1)$ such that $g_{\alpha\beta}\circ\mathrm{dev}_\beta = \mathrm{dev}_\alpha\circ f$ on $\mathbb{R}\times U_\alpha\cap U_\beta$. Since the developing map is equivariant, it follows that $\rho(t)(=\rho_\alpha(t)) = g_{\alpha\beta}\cdot\rho_\beta(t)\cdot {g_{\alpha\beta}}^{-1}$, $t\in\mathbb{R}$. If we replace dev_β by $\tilde{\mathrm{dev}}_\beta = g_{\alpha\beta}\circ\mathrm{dev}_\beta$, the holonomy map of $\tilde{\mathrm{dev}}_\beta$ changes into ρ. Perform this alternation to each developing pair $(\rho_\gamma, \mathrm{dev}_\gamma)$ $(\gamma \neq \alpha, \beta)$ one after another. Then we obtain a collection of charts $\{M_\gamma, \tilde{\mathrm{dev}}_\gamma\}_{\gamma\in\Lambda}$ satisfying that $\tilde{\mathrm{dev}}_\gamma(M_\gamma) \subset \mathcal{W}$, $\tilde{\mathrm{dev}}_\gamma^*\omega_0 = \tilde{u}_\gamma \cdot \omega_\gamma$ for some positive function $\tilde{u}_\gamma$, and that $\tilde{\mathrm{dev}}_{\gamma_*} \circ J_\gamma = J_0 \circ \tilde{\mathrm{dev}}_{\gamma_*}$ on $\mathrm{Null}\,\omega_\gamma$. As $\omega_\gamma(d/dt) = 1$, note that $\tilde{u}_\gamma(x) = \omega_0(\xi_{\tilde{\mathrm{dev}}_\gamma(x)})$. Then (1.4) implies that **(i)** $\tilde{\mathrm{dev}}_\gamma^*\eta = \omega_\gamma$. Moreover, since $\tilde{\mathrm{dev}}_\gamma(t,x) = \rho(t)\cdot\tilde{\mathrm{dev}}_\gamma(x)$, passing to the quotient for each γ, there is an immersion **(ii)** $\widehat{\mathrm{dev}}_\gamma : U_\gamma \to \mathcal{W}/\rho(\mathbb{R})$. When $U_\gamma \cap U_\delta \neq \emptyset$, it follows as above that $\tilde{g}_{\gamma\delta} \circ \tilde{\mathrm{dev}}_\delta = \tilde{\mathrm{dev}}_\gamma \circ f'$ for some f' and $\tilde{g}_{\gamma\delta}$, and so $\rho(t) = \tilde{g}_{\gamma\delta}\cdot\rho(t)\cdot\tilde{g}_{\gamma\delta}^{-1}$. Hence $\tilde{g}_{\gamma\delta} \in \mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ which induces an element $\hat{g}_{\alpha\beta} \in \mathcal{H}$. Noting that $f' : \mathbb{R}\times U_\gamma \cap U_\delta \to \mathbb{R}\times U_\gamma \cap U_\delta$ induces the identity map on the quotient, it follows that **(iii)** $\hat{g}_{\gamma\delta} \circ \widehat{\mathrm{dev}}_\delta = \widehat{\mathrm{dev}}_\gamma$ on $U_\gamma \cap U_\delta$. With these **(ii), (iii)**, the collection of charts $\{U_\gamma, \widehat{\mathrm{dev}}_\gamma\}_{\gamma\in\Lambda}$ gives rise to a uniformization of $\tilde{M}$ with respect to the geometry $(\mathcal{H}, \mathcal{W}/\rho(\mathbb{R}))$. The same is true for $(\mathcal{H}, \mathcal{W}/\rho(\triangle))$.

We now state a geometric uniformization. This is a correction of Theorem A [9]. Denote by $\mathrm{Iso}(\tilde{M}, g, J)$ the group of holomorphic isometries of a Kähler manifold $(\tilde{M}, g)$.

Theorem 1.3 (Geometric uniformization). *Let $\tilde{M}$ be a simply connected Bochner flat Kähler manifold of dimension $2n \geq 4$. Then $\tilde{M}$ is* geometrically *uniformized with respect to the Bochner flat Kähler geometry $(\mathcal{H}, \mathcal{W}/\rho(\mathbb{R}), \hat{g}, \hat{J})$ (respectively $(\mathcal{H}, \mathcal{W}/\rho(\triangle), \hat{g}, \hat{J})$ according as $\rho(\mathbb{R})$ is closed or non-closed. More precisely, there exist a holomorphically isometric immersion $D : \tilde{M} \longrightarrow \mathcal{W}/\rho(\mathbb{R})$ (or $\mathcal{W}/\rho(\triangle)$) and a holonomy homomorphism $\Psi : \mathrm{Iso}(\tilde{M}, g, J) \to \mathcal{H}$ such that $D^*\hat{g} = g$, $D_* \circ J = \hat{J} \circ D_*$ and $\Psi(\gamma) \circ D = D \circ \gamma$, $\forall\ \gamma \in \mathrm{Iso}(\tilde{M}, g, J)$.*

Proof. By the usual monodromy argument, a uniformization $\{U_\gamma, \widehat{\mathrm{dev}}_\gamma\}_{\gamma\in\Lambda}$ produces an immersion $D : \tilde{M} \longrightarrow \mathcal{W}/\rho(\mathbb{R})$ (respectively $\mathcal{W}/\rho(\triangle)$) such that $D|U_\gamma = \widehat{\mathrm{dev}}_\gamma$. Using **(i)** and (1.6), it follows that $d\omega_\gamma = (\nu \circ \tilde{\mathrm{dev}}_\gamma)^*\hat{\Omega} = p_\gamma^* \circ \widehat{\mathrm{dev}}_\gamma^*\hat{\Omega}$. As $p_\gamma^*\Omega_\gamma = d\omega_\gamma$, we have $\widehat{\mathrm{dev}}_\gamma^*\hat{\Omega} = \Omega_\gamma = \Omega|U_\gamma$ and so $D^*\hat{\Omega} = \Omega$ on $\tilde{M}$ (respectively $D^*\hat{g} = g$). Recall the commutativity that $\tilde{\mathrm{dev}}_{\gamma*} \circ J_\gamma = J_0 \circ \tilde{\mathrm{dev}}_{\gamma*}$ and $\nu \circ \tilde{\mathrm{dev}}_\alpha = \widehat{\mathrm{dev}}_\alpha \circ p_\alpha$, then the complex structures induce the equality $\widehat{\mathrm{dev}}_{\gamma*} \circ J = \hat{J} \circ \widehat{\mathrm{dev}}_{\gamma*}$ which shows also $D_* \circ J = \hat{J} \circ D_*$. When $h \in \mathrm{Iso}(\tilde{M}, g, J)$, we may suppose $h : U_\alpha \to U_\beta$ locally for some α, β. Then the equality $h^*\Omega = \Omega$ implies that $d(\theta_\alpha - h^*\theta_\beta) = 0$. Thers is a smooth map $\chi : U_\alpha \to \mathbb{R}$ such that $\theta_\alpha - h^*\theta_\beta = d\chi$ as before. The map $f' : \mathbb{R}\times U_\alpha \to \mathbb{R}\times U_\beta$ defined by $f'(t,x) = (t + \chi(x), h(x))$ is a pseudo-Hermitian diffeomorphim whose orbit map is h. Then the same argument as in §1.6 shows that there exists a unique global element $\tilde{\Psi}(h) \in \mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ (respectively $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\triangle))$) such that $\tilde{\Psi}(h) \circ \tilde{\mathrm{dev}}_\alpha = \tilde{\mathrm{dev}}_\beta \circ f'$. As $\tilde{\Psi}(h)$ induces an element $\Psi(h) \in \mathcal{H}$, passing to the quotient, the commutativity shows $\Psi(h) \circ D = D \circ h$. By uniqueness, Ψ is a homomorphism of $\mathrm{Iso}(\tilde{M}, g, J)$ into $\mathcal{H}$. □

Proposition 1.4. *Among those Bochner flat Kähler models $(\mathcal{H}, \mathcal{W}/\rho(\mathbb{R}), \hat{g}, \hat{J})$, they are isometric if and only if the corresponding holonomy groups are conjugate in* $\mathrm{PU}(n+1, 1)$. *(Similarly for $(\mathcal{H}, \mathcal{W}/\rho(\triangle), \hat{g}, \hat{J})$).*

Proof. Suppose that two Kähler manifolds $(\mathcal{W}_1/\rho_1(\mathbb{R}), \hat{g}_1)$, $(\mathcal{W}_2/\rho_2(\mathbb{R}), \hat{g}_2)$ are (locally) holomorphically isometric. By the same argument as in the proof of Theorem

1.3, a (local) holomorphic isometry h between $\mathcal{W}_1/\rho_1(\mathbb{R})$ and $\mathcal{W}_2/\rho_2(\mathbb{R})$ induces a (local) pseudo-Hermitian diffeomorphim $\bar{f}$ between $\mathrm{dev}_1(\mathbb{R}\times U_1)$ $(\subset \mathcal{W}_1)$ and $\mathrm{dev}_2(\mathbb{R}\times U_2)$ $(\subset \mathcal{W}_2)$. From the construction of f' of the above proof, note that f' is equivariant with respect to $\mathbb{R}$-action and also dev_i is $\rho_i(\mathbb{R})$-equivariant; $\mathrm{dev}_i(tx) = \rho_i(t)\,\mathrm{dev}_i(x)$, $i = 1, 2$. This implies that $\bar{f}$ is equivariant, $\bar{f}(\rho_1(t)p) = \rho_2(t)\bar{f}(p)$. On the other hand, as $\mathcal{W}_1$ and $\mathcal{W}_2$ are CR-isomorphic by $\bar{f}$, our construction implies that $\mathcal{W}_1 = \mathcal{W}_2$. Since $\mathcal{W}_1$ is a domain of S^{2n+1}, $\bar{f}$ extends to a global CR-transformation of S^{2n+1}. Hence $\bar{f}\in \mathrm{PU}(n+1,1)$ such that $\bar{f}\rho_1(t)\bar{f}^{-1} = \rho_2(t)$, $t\in\mathbb{R}$. Conversely, the equality $f\circ\rho_1\circ f^{-1} = \rho_2$ for some $f\in\mathrm{PU}(n+1,1)$ induces an isometry $\hat{f}: \mathcal{W}_1/\rho_1(\mathbb{R})\rightarrow\mathcal{W}_2/\rho_2(\mathbb{R})$. The same is true for $\mathcal{W}/\rho(\triangle)$. □

2. Review of standard contact structures ω_0 on X

2.1. **Heisenberg CR-geometry.** The Heisenberg nilpotent Lie group $\mathcal{N}$ is the product $\mathbb{R}\times\mathbb{C}^n$ with group law:

$$(a,z)\cdot(b,w) = (a+b-\mathrm{Im}\langle z,w\rangle,\ z+w), \tag{2.1}$$

where $\mathrm{Im}\langle z,w\rangle$ is the imaginary part of the Hermitian inner product on $\mathbb{C}^n$

$$\langle z,w\rangle = \bar{z}_1\cdot w_1 + \bar{z}_2\cdot w_2 + \cdots + \bar{z}_n\cdot w_n.$$

It is easy to see that $\mathcal{N}$ is 2-step nilpotent, i.e. the commutator $[\mathcal{N},\mathcal{N}] = \mathbb{R}$. Put $\mathcal{R} = \mathbb{R}$ which is the central subgroup of $\mathcal{N}$. Let $\mathrm{Aut}_{CR}(\mathcal{N})$ be the subgroup of CR transformations preserving $\mathcal{N}$. Then, $\mathrm{Aut}_{CR}(\mathcal{N}) = \mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+)$ whose action on $\mathcal{N}$ $(=\mathbb{R}\times\mathbb{C}^n)$ is defined (cf. [10], [7]):

$$((a,z),\lambda\cdot A)\cdot(b,w) = (a+\lambda^2\cdot b - \mathrm{Im}\langle z,\lambda\cdot A\cdot w\rangle, z+\lambda\cdot A\cdot w). \tag{2.2}$$

On the other hand, the contact form ω_0 on $\mathcal{N}$ is described as follows. Put $\omega_0 = \omega_{\mathcal{N}}$. If $(t,(z_1,\cdots,z_n))$ is the coordinate of $\mathcal{N} = \mathbb{R}\times\mathbb{C}^n$, then

$$\omega_{\mathcal{N}} = dt + \sum_{j=1}^{n}(x_j dy_j - y_j dx_j) = dt + \mathrm{Im}\langle z, dz\rangle. \tag{2.3}$$

The group of pseudo-Hermitian transformations $\mathrm{Psh}(\mathcal{N})$ is $\mathcal{N}\rtimes\mathrm{U}(n)$ which is the subgroup of $\mathrm{Aut}_{CR}(\mathcal{N})$. Then $\omega_{\mathcal{N}}$ is $\mathrm{Psh}(\mathcal{N})$-invariant. For this, if $\gamma = ((a,w),A)\in\mathcal{N}\rtimes\mathrm{U}(n)$, then $((a,w),A)\cdot(t,z) = (a+t-\mathrm{Im}\langle w,Az\rangle, w+Az)$, and so $\gamma^*\omega_{\mathcal{N}} = dt - d\mathrm{Im}\langle w,Az\rangle + \mathrm{Im}\langle w+Az, d(w+Az)\rangle$. Since $d\mathrm{Im}\langle w,Az\rangle = \mathrm{Im}\langle w,dAz\rangle$, it is easy to see that $\gamma^*\omega_{\mathcal{N}} = dt + \mathrm{Im}\langle z,dz\rangle = \omega_{\mathcal{N}}$. In general, if $h\in\mathrm{Aut}_{CR}(\mathcal{N})$, then there exists a positive function u on $\mathcal{N}$ such that $h^*\omega_{\mathcal{N}} = u\cdot\omega_{\mathcal{N}}$. Moreover, by definition, h is holomorphic (Cauchy-Riemann) on $\mathrm{Null}\,\omega_{\mathcal{N}}$. Hence, every element h of $\mathrm{Aut}_{CR}(\mathcal{N})$ preserves the CR-structure $(\mathrm{Null}\,\omega_{\mathcal{N}}, J_0)$ on $\mathcal{N}$ (cf. §1.2). Note that by the (trivial) fibration $\mathcal{R}\rightarrow\mathcal{N}\xrightarrow{P}\mathbb{C}^n$, P_* maps $\mathrm{Null}\,\omega_{\mathcal{N}}$ isomorphically onto $T\mathbb{C}^n$ at each point. Then the complex structure J_0 on $\mathrm{Null}\,\omega_{\mathcal{N}}$ is obtained from the standard complex structure $J_{\mathbb{C}}$ on $\mathbb{C}^n$ by the commutativity $J_{\mathbb{C}}\circ P_* = P_*\circ J$.

2.2. CR**-structure on** $S^{2n-1} - S^{2m-1}$**.** Let $S^{2n+1} - S^{2m-1}$ be the sphere complement for $m = 0, 1, \cdots, n$. ($S^{-1} = \emptyset$ for $m = 0$.) The contact form $\omega_0 = \omega_{\mathcal{S}}$ on $S^{2n+1} - S^{2m-1}$ is obtained as follows. First recall that V_{-1}^{2m+1} is the $(2m+1)$-dimensional Lorentz standard space form of constant sectional curvature -1 with transitive unitary Lorentz group $\mathrm{U}(m, 1)$ (cf. [13], [10]). The center $\mathcal{Z}\mathrm{U}(m, 1)$ of $\mathrm{U}(m, 1)$ is S^1. Then V_{-1}^{2m+1} is the total space of the principal S^1-bundle over the complex hyperbolic space:

$$\mathcal{Z}\mathrm{U}(m,1) \rightarrow V_{-1}^{2m+1} \longrightarrow \mathbb{H}_{\mathbb{C}}^m. \tag{2.4}$$

Denote by $\omega_{\mathbb{H}}$ the connection form of the above principal bundle. Then it is a contact form on V_{-1}^{2m+1}. If $\mathcal{Z}\mathrm{U}(n-m+1)$ is the center S^1 of the unitary compact group $\mathrm{U}(n-m+1)$, then there is the Hopf bundle:

$$\mathcal{Z}\mathrm{U}(n-m+1) \rightarrow S^{2(n-m)+1} \longrightarrow \mathbb{CP}^{n-m}.$$

As usual, the standard contact form on $S^{2(n-m)+1}$ is defined to be

$$\omega_{\mathbb{S}} = \sum_{j=1}^{n-m+1} (x_j dy_j \; - \; y_j dx_j) \;\; (z_k = x_k + i y_k). \tag{2.5}$$

Denote by $S^1 = \mathcal{Z}(\mathrm{U}(m,1) \times \mathrm{U}(n-m+1))$ the subgroup of the torus $T^2 = \mathcal{Z}\mathrm{U}(m,1) \times \mathcal{Z}\mathrm{U}(n-m+1)$. We put

$$T^1 = P(\mathcal{Z}\mathrm{U}(m,1) \times \mathcal{Z}\mathrm{U}(n-m+1)) = T^2/S^1.$$

There is the following principal circle bundle:

$$T^1 \rightarrow P(V_{-1}^{2m+1} \times S^{2(n-m)+1}) \stackrel{\nu}{\longrightarrow} \mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}. \tag{2.6}$$

Then the sphere complement is realized as

$$S^{2n+1} - S^{2m-1} = P(V_{-1}^{2m+1} \times S^{2(n-m)+1}).$$

(Compare [10].) Since the product $\omega_{\mathbb{H}} \times \omega_{\mathbb{S}}$ is invariant under the torus T^2, the subgroup S^1 induces a T^1-invariant contact form $\widehat{\omega_{\mathbb{H}} \times \omega_{\mathbb{S}}}$ on $S^{2n+1} - S^{2m+1}$ which satisfies $d(\widehat{\omega_{\mathbb{H}} \times \omega_{\mathbb{S}}}) = \nu^*(\Omega_{\mathbb{H}} \times \Omega_{\mathbb{CP}})$. By our definition, this is the desired contact form:

$$\omega_{\mathcal{S}} = \widehat{\omega_{\mathbb{H}} \times \omega_{\mathbb{S}}}. \tag{2.7}$$

3. Classification to the noncompact holonomy $\rho(\mathbb{R})$

We first determine the holonomy image $\rho(\mathbb{R})$ in $\mathrm{PU}(n+1, 1)$. Let G be the closure of the holonomy group $\rho(\mathbb{R})$ in $\mathrm{PU}(n+1, 1)$ as in §1.2.

3.1. **Non-compact holonomy with one fixed point.**

Proposition 3.1. *Suppose that G is a noncompact group which fixes the unique point $\{\infty\}$ in S^{4n+3}. Then G is contained in $\mathrm{Psh}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(n)$ up to conjugacy. Moreover, the holonomy $\rho(\mathbb{R})$ is a closed noncompact subgroup isomorphic to $\mathbb{R}$, i.e. $G = \rho(\mathbb{R}) = \mathbb{R}$ in which $\rho(\mathbb{R})$ has the form $\rho(t) = (t, t \cdot v, A_t)$ for some $v \in \mathbb{C}^n$, $A_t \in T^n$.*

Proof. Since G is a noncompact connected abelian Lie group, it is contained in $\mathrm{Aut}_{CR}(\mathcal{N}) = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ up to conjugay. Let $L : \mathrm{Aut}(\mathcal{N}) = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+) \longrightarrow \mathrm{Sim}_{\mathbb{C}}(\mathbb{C}^n) = \mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ be the projection with kernel $\mathcal{R}$. We show that $\rho(\mathbb{R})$ has no $\mathbb{R}^+$-summand. Suppose that $L \circ \rho(\mathbb{R})$ has a nontrivial $\mathbb{R}^+$-summand in $\mathrm{Sim}_{\mathbb{C}}(\mathbb{C}^n)$. We write such an element $L \circ \rho(t)$ as

$$L \circ \rho(t) = (x_t, \lambda_t \cdot A_t) \ \text{ for some } \lambda_t \neq 1 \ (t \neq 0).$$

It is able to conjugate the holonomy group by an element of $\mathrm{PU}(n+1, 1)$ because the CR-developing pair is unique up to the composite by any element of $\mathrm{PU}(n + 1, 1)$. As $\lambda_t \neq 1$ and $A_t \in \mathrm{U}(n)$, the matrix $\lambda_t \cdot A_t$ has no eigenvalue 1. Fix some t_0 so that the affine transformation $L \circ \rho(t_0)$ has a unique fixed point z_0 in $\mathbb{C}^n$. Then z_0 is also the unique fixed point for all $L \circ \rho(t)$ because $\{L \circ \rho(t)\}_{t \in \mathbb{R}}$ is abelian. Conjugate $L \circ \rho(t)$ by the translation $-z_0 \in \mathbb{C}^n$, we can assume that $L \circ \rho(t)$ has a fixed point at the origin of $\mathbb{C}^n$; $L \circ \rho(t) = (0, \lambda_t \cdot A_t)$. Hence, $L \circ \rho(\mathbb{R}) \subset \mathrm{U}(n) \times \mathbb{R}^+$, which implies that $\rho(\mathbb{R}) \subset \mathcal{R} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$. We may write $\rho(t) = ((a_t, 0), \lambda_t \cdot A_t)$. For t_0, let $b_0 = \dfrac{-a_{t_0}}{\lambda_{t_0}^2 - 1}$ and put $(b_0, 0) \in \mathbb{R} \times \mathbb{C}^n = \mathcal{N}$. Noting the action (2.2), it is easy to see that $\rho(t_0)(b_0, 0) = (b_0, 0)$. Since $(b_0, 0)$ is the unique fixed point of $\rho(t_0)$, so is $(b_0, 0)$ for all $\{\rho(t)\}_{t \in \mathbb{R}}$ as above. Again conjugate $\rho(t)$ by the translation element $(-b_0, 0) \in \mathcal{N}$, so $\rho(t)$ has the fixed point $(0, 0) \in \mathcal{N}$. This implies that $\rho(\mathbb{R}) \subset \mathrm{U}(n) \times \mathbb{R}^+$. However the subgroup $\mathrm{U}(n) \times \mathbb{R}^+$ of $\mathrm{Aut}_{CR}(\mathcal{N})$ fixes exactly two points $\{0, \infty\}$, so this is not our case. (Compare §3.10.) We conclude that $\rho(\mathbb{R})$ has no $\mathbb{R}^+$-summand.

Suppose that $\rho(\mathbb{R}) \subset \mathcal{N} \rtimes \mathrm{U}(n)$. For $t \in \mathbb{R}$, let $L \circ \rho(t) = (x_t, A_t)$. Then the map $t \rightarrow A_t$ is a homomorphism of $\mathbb{R}$ into an abelian subgroup of $\mathrm{U}(n)$. The closure of the image $\{A_t\}_{t \in \mathbb{R}}$ lies in the maximal torus T^n of $\mathrm{U}(n)$ up to conjugacy. It has the form:

$$A_t = \begin{pmatrix} e^{it \cdot a_1} & & \\ & \ddots & \\ & & e^{it \cdot a_n} \end{pmatrix}.$$

We may assume that $a_1 = \cdots = a_k = 0, a_{k+1} \leq \cdots \leq a_n$ $(0 \leq k)$ such that $A_t = (1, \cdots, 1, e^{it \cdot a_{k+1}}, \cdots, e^{it \cdot a_n})$. The matrix $A'_t = (e^{it \cdot a_{k+1}}, \cdots, e^{it \cdot a_n})$ has no eigenvalue 1. If the translation x_t is not zero, then we conjugate $L \circ \rho(t)$ by some translation element as above to make the corresponding translation part vanish. It follows that $x_t = (x_t^1, \cdots, x_t^k, 0, \cdots, 0) \in \mathbb{C}^n$ which is fixed by all $\{A_t\}$. Then the map $t \mapsto x_t$ is a homomorphism of $\mathbb{R}$ into $\mathbb{C}^n$ (this is obvious for $A_t = \mathrm{I}$). We choose a vector $v \in \mathbb{C}^n$ so that $x_t = t \cdot v$ and so $A_t(v) = v$. ($v = 0$ when $x_t = 0$.) As a consequence, we obtain that $\rho(t) = (\rho_1(t), t \cdot v, A_t)$ for some $\rho_1(t) \in \mathbb{R}$ $(t \in \mathbb{R})$. Using (2.1), calculate

$$\begin{aligned} \rho(t + s) &= \rho(t) \cdot \rho(s) \\ &= (\rho_1(t) + \rho_1(s) - \mathrm{Im}\langle tv, A_t(s \cdot v)\rangle, t \cdot v + A_t(s \cdot v), A_t A_s) \\ &= (\rho_1(t) + \rho_1(s), (t + s)v, A_{t+s}) \end{aligned}$$

where $\mathrm{Im}\langle tv, A_t(s \cdot v)\rangle = ts \cdot \mathrm{Im}\langle v, A_t(v)\rangle = ts \cdot \mathrm{Im}\langle v, v\rangle = 0$. Thus $\rho_1(t + s) = \rho_1(t) + \rho_1(s)$; $\rho_1 : \mathbb{R} \rightarrow \mathcal{R}$ is a nontrivial homomorphism. Normalize ρ_1 to write

$\rho(t) = (t, t\cdot v, A_t) \in \mathcal{N} \rtimes \mathrm{U}(n)$ for which $A_t v = v$, $\forall\, t \in \mathbb{R}$. In particular, $\rho(\mathbb{R})$ is a closed subgroup isomorphic to $\mathbb{R}$. □

Denote by $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$ the normalizer of $\rho(\mathbb{R})$ in $\mathrm{Aut}_{CR}(\mathcal{N})$ as well as the centralizer $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$.

Corollary 3.2. **(I)** *$\rho(t)$ can be reduced to the following form:*

(i) *$A_t = \mathrm{I}$, $v = 0$, $\rho(t) = ((t,0),\mathrm{I})$, i.e. $\rho(\mathbb{R}) = \mathcal{R}$.*
(ii) *$A_t \neq \mathrm{I}$, $v = 0$, $\rho(t) = ((t,0),A_t) \in \mathcal{R}\times T^n$, $A_t = (e^{it\cdot a_1},\cdots,e^{it\cdot a_n})$.*
(iii) *$A_t = \mathrm{I}$, $v \neq 0$, $\rho(t) = (0,(t,0,\cdots,0),\mathrm{I}) \in \mathbb{R}^1$.*
(iv) *$A_t \neq \mathrm{I}$, $v \neq 0$, $\rho(t) = (0,(t,0,\cdots,0),\begin{pmatrix} 1 & \\ & B_t\end{pmatrix}) \in \mathbb{R}^1\times T^{n-1}$ ($B_t = (e^{it\cdot b_2},\cdots,e^{it\cdot b_n})$, $b_i \in \mathbb{R}$).*

(II) **(1)** *If $A_t \neq \mathrm{I}$, then $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R})) = \mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R})) \subset \mathrm{Psh}(\mathcal{N})$.*

(2) *In case* **(i)**, *$\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R})) = \mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+)$ and the action of an element $h = ((a,z),\mu\cdot C) \in \mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+)$ on $\mathcal{R}$ is given by*

$$h\cdot\rho(t)\cdot h^{-1} = \rho(\mu^2\cdot t) = ((\mu^2\cdot t,0),\mathrm{I}).$$

(3) *In case* **(iii)**, *the action of $h = ((a,z),\mu\cdot B) \in \mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$ on $\mathbb{R}^1$ satisfies that*

$$h\cdot\rho(t)\cdot h^{-1} = \rho(\mu\cdot t) = (0,(\mu\cdot t,0,\cdots,0),\mathrm{I}).$$

Proof. The first two cases are obvious. Put $e = \begin{pmatrix} 1\\ 0\\ \vdots\\ 0\end{pmatrix}$ in $\mathbb{C}^n$. Suppose that $v \neq 0$. Choose $B \in \mathrm{U}(n)$ such that $Be = \dfrac{v}{||v||}$. We put $f = ((0,\dfrac{-\boldsymbol{i}}{2||v||^2}\cdot e), ||v||^{-1}B^{-1})$. Then f is an element of $\mathrm{Aut}_{CR}(\mathcal{N}) = \mathcal{N}\rtimes\mathrm{U}(n)\times\mathbb{R}^+$. As before conjugate $\rho(\mathbb{R})$ by f. Then a calculation shows that $f\cdot\rho(t)\cdot f^{-1} = ((0,t\cdot e),B^{-1}\cdot A_t\cdot B)$. Since $A_t v = v$, we have $B^{-1}\cdot A_t\cdot B\cdot te = te$. So we can write $B^{-1}\cdot A_t\cdot B = \begin{pmatrix} 1 & \\ & B_t\end{pmatrix}$ for some element $B_t \in T^{n-1}$. According as $A_t = \mathrm{I}$ or $A_t \neq \mathrm{I}$, we can reduce to case **(iii)** or case **(iv)**.

For **(II)**, if $h = ((a,z),\mu\cdot C)$ is an element of $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$, by the action (2.2), it follows that $h\cdot\rho(t)\cdot h^{-1} = \rho(\lambda_h\cdot t)$ for some $\lambda_h \in \mathbb{R}^+$. Using this equality, we have $A_{\lambda_h\cdot t} = A_t$. Since $t\mapsto A_t$ is a homomorphism, if $A_t \neq \mathrm{I}$, then $\lambda_h = 1$. This implies that $\mathbf{N}_{\mathrm{Aut}_{CR}(X)}(\rho(\mathbb{R})) = \mathbf{C}_{\mathrm{Aut}_{CR}(X)}(\rho(\mathbb{R}))$, proving **(1)**.

For **(2)**, if $\rho(t) = ((t,0),\mathrm{I}) \in \mathcal{R}$ then we know that the action of $\mathrm{U}(n)\times\mathbb{R}^+$ on $\mathcal{N}$ satsifies that $h\cdot(t,0,\mathrm{I})\cdot h^{-1} = (\mu^2\cdot t,0,\mathrm{I})$. Finally, suppose that $A_t = \mathrm{I}$ and $v \neq 0$. The above equality $h\cdot\rho(t)\cdot h^{-1} = \rho(\lambda_h\cdot t)$ shows that $\mu\cdot Ctv = \lambda_h\cdot tv$. As $\mu\cdot t|Cv| = \mu\cdot t|v| = \lambda_h\cdot t|v|$, we have $\mu = \lambda_h$. This shows **(3)**. □

3.2. **Heisenberg case $\mathcal{W} = \mathcal{N} - S$.** According to the cases (**i**) − (**iv**) of Corollary 3.2, we examine the Bochner flat Kähler structure $(\mathcal{H},\mathcal{W}/\rho(\mathbb{R}),\hat{g},\hat{J})$ for the Heisenberg Lie group $\mathcal{N}$. Let S be the singular subset in $\mathcal{N}$ consisting of the points p, $\omega_{\mathcal{N}}(\xi_p) = 0$. Recall that $\mathcal{W} = (\mathcal{N} - S)^0$. Put $\omega_{\mathcal{N}}(= \omega_0) = \omega$. Let $\rho(t) = ((t, 0), A_t)$ be the form of **(i)**,**(ii)** ($A_t = \mathrm{I}, A_t \neq \mathrm{I}$ respectively). It acts on $\mathcal{N}$ as $\rho(t)(s, z) = (t + s, A_t z)$. Then the vector field ξ induced by $\rho(\mathbb{R})$ is described:

$$\xi = \frac{d}{dt} + \sum_{j=1}^{k} a_j (x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j}) \quad \text{on} \ \ \mathcal{N}.$$

Using (2.3), it follows that

$$\omega(\xi) = 1 + (a_1|z_1|^2 + \cdots + a_n|z_k|^2). \tag{3.1}$$

Let $\rho(t) = ((0, (t, 0, \cdots, 0)), \begin{pmatrix} 1 & 0 \\ 0 & B_t \end{pmatrix})$ for **(iii)**,**(iv)** ($B_t = \mathrm{I}, B_t \neq \mathrm{I}$ respectively). Noting (2.2), we have $\rho(t)(s, z) = (s - ty_1, (t + x_1 + iy_1, B_t(z_2, \cdots, z_n)))$. A calculation shows

$$\begin{aligned} \xi &= -y_1 \frac{d}{dt} + \frac{d}{dx_1} + \sum_{j=2}^{n} b_j (x_j \frac{d}{dy_j} - y_j \frac{d}{dx_j}). \\ \omega(\xi) &= -2y_1 + (b_2|z_2|^2 + \cdots + b_n|z_n|^2). \end{aligned} \tag{3.2}$$

3.3. **Case (i).** If $\rho(\mathbb{R})$ is the center $\mathcal{R}$ of $\mathcal{N}$, then $\omega(\xi) = 1$ and the principal bundle (1.3) turns canonical one: $\mathcal{R} \to \mathcal{N} \xrightarrow{P} \mathbb{C}^n$ for which $d\omega = 2P^*(\sum_{j=1}^{2n} dx_j \wedge dy_j)$ is the lift of the standard Kähler form of $\mathbb{C}^n$. Therefore, the Bochner flat Kähler manifold $(\mathcal{N}/\rho(\mathbb{R}), \hat{g}, \hat{J})$ is the standard complex euclidean space $(\mathbb{C}^n, g_{\mathbb{C}}, J_{\mathbb{C}})$. Since $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R})) = \mathcal{N} \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$, it follows that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))/\rho(\mathbb{R}) = \mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$.

3.4. **Case (ii).** From (3.1), we see that $\omega(\xi) > 0$ on $\mathcal{N}$ if and only if all $a_j \geq 0$. In general, $S \neq \emptyset$ so that $\mathcal{W} = \{(s, z) \in \mathbb{R} \times \mathbb{C}^n \mid -1 < a_1|z_1|^2 + a_2|z_2|^2 + \cdots + a_k|z_k|^2\} \subset \mathcal{N}$. Since the complex structure $\hat{J}$ of $\mathcal{N}/\rho(\mathbb{R})$ is induced from J_0 of the contact subbundle $\mathrm{Null}\,\omega$ on $\mathcal{N}$ by the projection $\nu : \mathcal{N} \to \mathcal{N}/\rho(\mathbb{R})$, $(\mathcal{W}/\rho(\mathbb{R}), \hat{J})$ is a complex domain of $(\mathcal{N}/\rho(\mathbb{R}), \hat{J})$. We note that

Proposition 3.3. *The orbit space* $(\mathcal{N}/\rho(\mathbb{R}), \hat{J})$ *is biholomorphic to* $(\mathbb{C}^n, J_{\mathbb{C}})$.

Proof. Recall from §2.1 that if $P : \mathcal{N} \to \mathbb{C}^n$ is the projection, then the complex structure J_0 on $\mathrm{Null}\,\omega$ satisfies that $J_{\mathbb{C}} \circ P_* = P_* \circ J_0$. Let $f : \mathbb{C}^n \longrightarrow \mathcal{N}/\rho(\mathbb{R})$ be the map defined by

$$f((z_1, \cdots, z_n)) = \nu((\sum_{j=1}^{n} |z_j|^2/2, -\boldsymbol{i}(z_1, \cdots, z_n))). \tag{3.3}$$

Then f is holomorphic, i.e. $f_* \circ J_{\mathbb{C}} = \hat{J} \circ f_*$. For this, it is easy to check that the subbundle $\mathrm{Null}\,\omega$ is generated by the vector fields $\{v_j, w_j,\ j = 1, \cdots, n \mid v_j = y_j \frac{d}{dy_j} + x_j \frac{d}{dx_j}, w_j = (x_j^2 + y_j^2) \frac{d}{dt} + y_j \frac{d}{dx_j} - x_j \frac{d}{dy_j}\}$. Note that $J_{\mathbb{C}}(y_j \frac{d}{dy_j} + x_j \frac{d}{dx_j}) = y_j \frac{d}{dx_j} + x_j(-\frac{d}{dy_j})$. By the definition as above, J_0 satisfies that $J_0 v_j = w_j,\ J_0 w_j =$

$-v_j$. If $\tilde f : \mathbb{C}^n \to \mathcal{N}$ is defined by $\tilde f((z_1,\cdots,z_n)) = ((\sum_{j=1}^{n}|z_j|^2/2, -\boldsymbol{i}(z_1,\cdots,z_n))$, then it follows that $\tilde f_*(y_j\dfrac{d}{dy_j}+x_j\dfrac{d}{dx_j}) = w_j$, $\tilde f_*(y_j\dfrac{d}{dx_j}-x_j\dfrac{d}{dy_j}) = -v_j$. Hence, $\tilde f_*$ maps $T\mathbb{C}^n$ isomorphically onto $\mathrm{Null}\,\omega$ satisfying that $\tilde f_*\circ J_{\mathbb{C}} = J_0\circ \tilde f_*$. Since the complex structure $\hat J$ satisfies that $\nu_*\circ J_0 = \hat J\circ\nu_*$, the equality $\nu\circ\tilde f = f : \mathbb{C}^n\to\mathcal{N}/\rho(\mathbb{R})$ implies that $f_*\circ J_{\mathbb{C}} = \hat J\circ f_*$. □

From this, $\mathcal{W}/\rho(\mathbb{R})$ is an open subset of $\mathbb{C}^n$. Suppose that among a_j's of A_t, $a_1\leq\cdots\leq a_{\ell-1}<0<a_\ell\leq\cdots\leq a_k$ and the remaining $n-k$ are all zero. Put

$$\mathbf{b} = \sqrt{1+(a_\ell|z_1|^2+\cdots+a_k|z_k|^2)}.$$

We arrange the point $(s,(z_1,\cdots,z_n))\in\mathcal{W}$ as $(x,z_1,\cdots,z_{\ell-1},y)$. Here $x = (s,z_{k+1},\cdots,z_n)\in\mathbb{R}\times\mathbb{C}^{n-k}$, $y=(z_\ell,\cdots,z_k)\in\mathbb{C}^{k-\ell+1}$. Then $\mathcal{W}$ is diffeomorphic to $(\mathbb{R})\times\mathbb{C}^{n-k}\times\mathbb{B}^{\ell-1}_{\mathbb{C}}\times\mathbb{C}^{k-\ell+1}$ by the correspondence

$$(x,z_1,\cdots,z_{\ell-1},y)\mapsto(x,\sqrt{-a_1}\cdot z_1/\mathbf{b},\cdots,\sqrt{-a_{\ell-1}}\cdot z_{\ell-1}/\mathbf{b},y).$$

Hence, $\mathcal{W}/\rho(\mathbb{R})$ is biholomorphic to $\mathbb{C}^{n-k}\times\mathbb{B}^{\ell-1}_{\mathbb{C}}\times\mathbb{C}^{k-\ell+1}$ as a domain of $\mathbb{C}^n$.

We examine the normalizer $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$ of **(ii)** ; $\rho(t) = (t,0,A_t)$. Since $A_t\neq \mathrm{I}$, it follows from Corollary 3.2 that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R})) = \mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))\subset \mathrm{Psh}(\mathcal{N})$. Then any element $h\in\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$ satisfies that $h_*\xi=\xi$ and $h^*\omega=\omega$. And so $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$ leaves invariant $S\neq\emptyset$, which implies that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R})) = \mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$. Among a_j's of A_t, $j=1,\cdots,k$, we assume further that $\ell_1+\cdots+\ell_m = k$ and each ℓ_j is the number of the same diagonal component a_{ℓ_j} (cf. §3.6). Since $\{A_t\}$ belongs to the maximal torus T^n in $\mathrm{U}(n)$, the normalizer of $\{A_t\}$ in $\mathrm{U}(n)$ is $\mathrm{U}(\ell_1)\times\cdots\times\mathrm{U}(\ell_m)\times\mathrm{U}(n-k)$. Put

$$\mathrm{U}(\ell_1,\cdots,\ell_m) = \mathrm{U}(\ell_1)\times\cdots\times\mathrm{U}(\ell_m). \tag{3.4}$$

(For example, if $a_1,\cdots,a_k$ are mutually distinct, then $\mathrm{U}(1,\cdots,k) = T^k$.) By the formula of A_t, it follows that $\{z\in\mathbb{C}^n \mid A_t\cdot z = z\} = 0\times\mathbb{C}^{n-k}$. From these, the centralizer becomes

$$\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R})) = \left(\mathcal{R},\begin{pmatrix}0\\ \mathbb{C}^{n-k}\end{pmatrix}\right)\rtimes\begin{pmatrix}\mathrm{U}(\ell_1,\cdots,\ell_m) & 0\\ 0 & \mathrm{U}(n-k)\end{pmatrix}.$$

Here the product $(\mathcal{R},\mathbb{C}^{n-k})$ is the nilpotent Lie subgroup of $\mathcal{N}$. Therefore the quotient subgroup is isomorphic to the following Lie group:

$$\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))/\rho(\mathbb{R}) = \mathbb{C}^{n-k}\rtimes\mathrm{U}(n-k)\times\mathrm{U}(\ell_1,\cdots,\ell_m).$$

Putting $b=(a_1,\cdots,a_{\ell-1}<0<a_\ell,\cdots,a_k)$ and $\hat g = \hat g_b$, we obtain the Bochner flat Kähler metric $\hat g_b$ on $\mathcal{W}/\rho(\mathbb{R}) = \mathbb{C}^{n-k}\times\mathbb{B}^{\ell-1}_{\mathbb{C}}\times\mathbb{C}^{k-\ell+1}$.

Remark 3.4. *As is noted in the beginning,* $S=\emptyset$, *i.e.* $\mathcal{W}=\mathcal{N}$ *if and only if* $0\leq a_1\leq\cdots\leq a_n$. *It has been shown by Bryant* [3] *that* $\hat g_b$ *with all nonnegative numbers* a_j *are complete metrics on* $\mathbb{C}^n$. *We shall reprove this in* §3.6. *Note that our new Kähler metric* $\hat g_b$ *on* $\mathbb{C}^n$ *is obtained by deforming* $\omega=\omega_{\mathcal{N}}$ *to* $\eta = u\cdot\omega$ *where* $u=\dfrac{1}{\omega(\xi)}$. *In comparison with* η *under this change* $\eta = u\cdot\omega$,

(∗) *the effect on the original contact form* ω *is to produce a Hermitian metric* $\hat{g}_\omega$ *on* $\mathbb{C}^n$ *which is a globally conformal Kähler metric, i.e.* $\hat{g}_b = \hat{u} \cdot g_\omega$.

3.5. **Carnot-Carathéodory metric and completeness.** As usual $\omega = \omega_X$ is a contact form on a CR-manifold X which represents a CR-structure $\mathrm{Null}\,\omega$. Let σ be a sectionally smooth curve between p and q in X satisfying that $\dot\sigma(t) \in (\mathrm{Null}\,\omega)_{\sigma(t)}$ *a.e.*. Then the length is defined as $L(\sigma) = \displaystyle\int_0^1 d\omega(J\dot\sigma(t), \dot\sigma(t))^{\frac{1}{2}} dt$. Let C_ω be the set of all such curves σ joining p and q. Set

$$d_\omega(p, q) = \inf_{\sigma \in C_\omega} L(\sigma).$$

Since $\mathrm{Null}\,\omega$ satisfies that $[\mathrm{Null}\,\omega, \mathrm{Null}\omega] \oplus \mathrm{Null}\omega = TX$, $d_\omega(p, q)$ is defined and finite on X, i.e. any point q sufficiently close to a point p can be joined by a curve σ with $\dot\sigma(t) \in \mathrm{Null}\,\omega$. (Compare [14].) The metric d_ω is called the Carnot-Carathéodory distance on X associated to ω. Letting $f(x) = \dfrac{1}{\omega(\xi_x)}$, recall from (1.4) that $\eta(Z) = f \cdot \omega(Z)$ $(Z \in T(\mathcal{W}))$. For $\tilde{X}, \tilde{Y} \in \mathrm{Null}\,\eta (= \mathrm{Null}\omega)$ at $x \in \mathcal{W}$, we have

$$d\eta(\tilde{X}, \tilde{Y}) = f(x) \cdot d\omega(\tilde{X}, \tilde{Y}) \;\text{ on }\; \mathcal{W}. \tag{3.5}$$

3.6. **Complete Kähler metrics on $\mathbb{C}^n$ after Bryant.** As in **(ii)**, let $\rho(t) = ((t, 0), A_t)$ with $A_t = (e^{it\cdot a_1}, \cdots, e^{it\cdot a_n})$. For real numbers $a_1, \cdots, a_n$, suppose that a_i's consist of m-distinct numbers $a_{i_1}, \cdots, a_{i_m}$ such that

$$(a_1, \cdots, a_k) = (\overbrace{a_{i_1} \cdots a_{i_1}}^{\ell_1-\text{times}}; \cdots; \cdots; \overbrace{a_{i_m} \cdots a_{i_m}}^{\ell_m-\text{times}}), \quad (\ell_1 + \cdots + \ell_m = k). \tag{3.6}$$

The rest of a_i's are all zero.

Proposition 3.5. *Suppose that* $0 < a_1 \leq a_2 \cdots \leq a_k$ $(1 \leq m \leq k \leq n)$ *and* m *is a number satisfying* (3.6). *Put* $a = (a_1, \cdots, a_k, 0, \cdots, 0)$. *Then* $\hat{g}_a$ *is complete on* $\mathbb{C}^n = \mathcal{N}/\rho(\mathbb{R})$. *Moreover, the cohomogeneity of* $\hat{g}_a$ *on* $\mathbb{C}^n$ *is* m. *The group of isometris* $\mathrm{Iso}(\mathbb{C}^n, \hat{g}_a) = (\mathbb{C}^{n-k} \rtimes \mathrm{U}(n-k)) \times \mathrm{U}(\ell_1) \times \cdots \times \mathrm{U}(\ell_m)$ *has the principal orbit of dimension* $2n - m$ *at least.*

Proof. Denote by $o = (0, 0, \cdots, 0)$ the origin of $\mathcal{N}$. Let $B^\eta(o, r)$ be the closed metric ball $\{x \in \mathcal{N} \mid d_\eta(o, x) \leq r\}$ of $\mathcal{N}$. If $\overline{B^\eta(o, r)}$ is the closure of $B^\eta(o, r)$ in $\mathcal{N} \cup \{\infty\} = S^{2n+1}$, then $\overline{B^\eta(o, r)}$ is compact. We prove that $B^\eta(o, r) = \overline{B^\eta(o, r)}$. It is valid when $\overline{B^\eta(o, r)}$ misses the point at infinity $\{\infty\}$. If $\{\infty\} \in \overline{B^\eta(o, r)}$ for some r, then there exists a sequence of points $\{p_m\} \in \mathcal{N}$ converging to $\{\infty\}$ such that

$$d_\eta(o, p_m) \leq r. \tag{3.7}$$

On the other hand, by the the principal Riemannian submersion: $\mathcal{R} \to (\mathcal{N}, g^0_{\mathcal{N}}) \xrightarrow{P} (\mathbb{C}^n, g_{\mathbb{C}})$ in which $g^0_{\mathcal{N}} = \omega_{\mathcal{N}} \cdot \omega_{\mathcal{N}} + d\omega_{\mathcal{N}}(J\;,\;)$, we have $d\omega_{\mathcal{N}}(JX, Y) = g_{\mathbb{C}}(P_*X, P_*Y)$ for $X, Y \in \mathrm{Null}\,\omega_{\mathcal{N}}$. Put $\omega_{\mathcal{N}} = \omega$ as usual. Choose a point $x = (x_0, x_1, \cdots, x_n) \in \mathcal{N}$. Let $\sigma : [0, 1] \to \mathcal{N}$ be a curve between $\sigma(0) = o$ and $\sigma(1) = x$ such that $\dot\sigma(t) \in$

Null η (= Null ω), i.e. $\sigma \in C_\eta$. Using (3.5) of § 3.5, calculate

$$
\begin{aligned}
d_\eta(o,x) &= \inf_{\sigma \in C_\eta} L(\sigma) \\
&= \inf_{\sigma \in C_\eta} \int_0^1 \frac{1}{\sqrt{\omega(\xi_{\sigma(t)})}} \cdot d\omega(J\dot{\sigma}(t), \dot{\sigma}(t))^{\frac{1}{2}} dt \\
&= \inf_{\sigma \in C_\eta} \int_0^1 \frac{1}{\sqrt{\omega(\xi_{\sigma(t)})}} \cdot g_{\mathbb{C}}(\dot{P\sigma}(t), \dot{P\sigma}(t))^{\frac{1}{2}} dt.
\end{aligned} \tag{3.8}
$$

If we write $\sigma(t) = (\sigma_0(t), \sigma_1(t), \cdots, \sigma_n(t)) \in \mathcal{N}$, then (3.1) implies that $\omega(\xi_{\sigma(t)}) = 1+(a_1|\sigma_1(t)|^2+a_2|\sigma_2(t)|^2+\cdots+a_k|\sigma_k(t)|^2)$. We may put $A = \max\{1, \sqrt{a_1}, \cdots, \sqrt{a_k}\}$. Then the following inequality holds.

$$
\begin{aligned}
\omega(\xi_{\sigma(t)}) &\le A^2(1 + |\sigma_1(t)|^2 + |\sigma_2(t)|^2 + \cdots + |\sigma_n(t)|^2) \\
&\le A^2(1 + |\sigma_1(t)| + |\sigma_2(t)| + \cdots + |\sigma_n(t)|)^2.
\end{aligned} \tag{3.9}
$$

As $P\sigma(t) = (\sigma_1(t), \cdots, \sigma_n(t)) \in \mathbb{C}^n$, it follows $(\dot{P\sigma})(t) = (\dot{\sigma_1}(t), \cdots, \dot{\sigma_n}(t)) \in \mathbb{C}^n$ for which $g_{\mathbb{C}}(\dot{P\sigma}(t), \dot{P\sigma}(t)) = \dot{\sigma_1}(t)^2 + \cdots + \dot{\sigma_n}(t)^2$. We obtain that

$$
g_{\mathbb{C}}(\dot{P\sigma}(t), \dot{P\sigma}(t))^{\frac{1}{2}} \ge \frac{1}{n}(|\dot{\sigma_1}(t)| + \cdots + |\dot{\sigma_n}(t)|). \tag{3.10}
$$

Using (3.9), (3.10), calculate that

$$
\begin{aligned}
&\int_0^1 \frac{1}{\sqrt{\omega(\xi_{\sigma(t)})}} \cdot g_{\mathbb{C}}(\dot{P\sigma}(t), \dot{P\sigma}(t))^{\frac{1}{2}} dt \\
&\quad \ge \frac{1}{nA} \int_0^1 \frac{|\dot{\sigma_1}(t)| + \cdots + |\dot{\sigma_n}(t)|}{1 + |\sigma_1(t)| + |\sigma_2(t)| + \cdots + |\sigma_n(t)|} dt \\
&\quad = \frac{1}{nA} \cdot \Big[\log(1 + |\sigma_1(t)| + |\sigma_2(t)| + \cdots + |\sigma_n(t)|)\Big]_0^1 \\
&\quad = \frac{1}{nA} \cdot \log(1 + |x_1| + |x_2| + \cdots + |x_n|).
\end{aligned} \tag{3.11}
$$

Since this does not depend on the choice of curves, we obtain that

$$
d_\eta(0,x) \ge \frac{1}{nA} \cdot \log(1 + |x_1| + |x_2| + \cdots + |x_n|). \tag{3.12}
$$

For the sequence of points $\{p_m\} \in \mathcal{N}$ in (3.7), put $P(p_m) = (x_1^m, \cdots, x_n^m) \in \mathbb{C}^n$. Since $\{p_m\} \in \mathcal{N}$ converges to $\{\infty\}$, the sequence $\{P(p_m)\} \in \mathbb{C}^n$ also converges to $\{\infty\}$. (Here the union $\mathbb{C}^n \cup \{\infty\}$ constitutes the sphere S^{2n}.) In particular, $\sum_{i=1}^n |x_i^m| \to \infty$ as $m \to \infty$. Hence $d_\eta(0, p_m) \longrightarrow \infty$ which contradicts (3.7). Therefore $\overline{B^\eta(o,r)} \subset \mathcal{N}$ for all $r \ge 0$. It follows that $B^\eta(o,r) = \overline{B^\eta(o,r)}$, which is compact $(r \ge 0)$. For the Kähler metric $\hat{g}_a$, there is the Riemannian submersion:

$$
\rho(\mathbb{R}) \to (\mathcal{N}, g^1_{\mathcal{N}}) \xrightarrow{\ \nu\ } (\mathcal{N}/\rho(\mathbb{R}), \hat{g}) = (\mathbb{C}^n, \hat{g}_a). \tag{3.13}
$$

where $g^1_{\mathcal{N}} = \eta \cdot \eta + d\eta(J\ ,\ )$. Noting (3.3), the map $\nu : \mathcal{N} \to \mathbb{C}^n$ is defined by $\nu(s, (z_1, \cdots, z_n)) = (e^{-\mathrm{i}sa_1} \cdot z_1, \cdots, e^{-\mathrm{i}sa_k} \cdot z_k, z_{k+1}, \cdots, z_n)$. Let v be an arbitrary vector of $T_o\mathbb{C}^n$. Suppose that the geodesic $\gamma(t) = \exp_o t \cdot v$ is defined for $0 \le t < T_0$

with respect to $\hat{g}_a$ where $a = (a_1, \cdots, a_k; 0, \cdots, 0)$. Let $t_m = T_0 - \dfrac{1}{m}$ and consider the straight line segments $\{\mu_m\}$ from o to each $(0, \gamma(t_m)) \in (0, \mathbb{C}^n) \subset \mathcal{N}$:

$$\mu_m(s) = (0, \gamma(t_m) \cdot s) = (0, (\gamma(t_m)_1 \cdot s, \cdots, \gamma(t_m)_n \cdot s)).$$

If we recall the contact form $\omega = dt + \mathrm{Im}\langle z, dz\rangle$ from (2.3), then it is easy to check that $\omega(\dot{\mu_m}(s)) = 0$, and thus

$$\dot{\mu_m}(s) \in \mathrm{Null}\,\eta, \;\; \mu_m \in C_\eta. \tag{3.14}$$

Since $\nu(\mu_m(s)) = \gamma(t_m) \cdot s$, we have $\nu(\dot{\mu_m}(s)) = \gamma(t_m)$. As before (cf. (3.5)), we note that

$$\begin{aligned} d\eta(J\dot{\mu_m}(s), \dot{\mu_m}(s)) &= \frac{1}{\omega(\xi_{\mu_m(s)})} \cdot d\omega(J\dot{\mu_m}(s), \dot{\mu_m}(s)) \\ &= \frac{g_{\mathbb{C}}(\nu(\dot{\mu_m})(s), \nu(\dot{\mu_m})(s))}{1 + (a_1|\gamma(t_m)_1|^2 + \cdots + a_k|\gamma(t_m)_k|^2) \cdot s^2} \\ &= \frac{|\gamma(t_m)|^2}{1 + (a_1|\gamma(t_m)_1|^2 + \cdots + a_k|\gamma(t_m)_k|^2) \cdot s^2}. \end{aligned}$$

Letting $A^2 = \min\{1, a_1, \cdots, a_k\}$, we have the inequality

$$d\eta(J\dot{\mu_m}(s), \dot{\mu_m}(s))^{\frac{1}{2}} \leq \frac{|\gamma(t_m)|}{A\sqrt{1 + |\gamma(t_m)|^2 \cdot s^2}}.$$

Calculate

$$\begin{aligned} d_\eta(o, \mu_m(1)) &= \inf_{\sigma \in C_\eta} L(\sigma) \leq L(\mu_m) \;\; \text{(by (3.14))}, \\ L(\mu_m) &= \int_0^1 d\eta(J\dot{\mu_m}(s), \dot{\mu_m}(s))^{\frac{1}{2}} ds \\ &\leq \frac{|\gamma(t_m)|}{A} \int_0^1 \frac{1}{\sqrt{1 + |\gamma(t_m)|^2 \cdot s^2}} ds \\ &= \frac{1}{A} \log(|\gamma(t_m)| + \sqrt{1 + |\gamma(t_m)|^2}) \leq \frac{1}{A} \log(2|\gamma(t_m)| + 1). \end{aligned} \tag{3.15}$$

If $d_\eta(o, \mu_m(1)) \to \infty$ (as $m \to \infty$), then $|\gamma(t_m)| \to \infty$ (as $t_m \to T_0$). Hence the geodesic $\gamma(t) = \exp_o t \cdot v$ were defined entirely on $T_o\mathbb{C}^n$. Otherwise, $\{d_\eta(o, \mu_m(1))\}$ is bounded for some $r > 0$ so that $d_\eta(o, \mu_m(1)) \leq r$, $\forall\, m$. As $\mu_m(1) \in B^\eta(o, r)$ which is compact, $\{\mu_m(1)\}$ has a limit point $(a, z) \in \mathcal{N}$. As $\mu_m(1) = (0, \gamma(t_m))$, $\lim_{m\to\infty} \gamma(t_m) = z$, hence the geodesic segment $\gamma(t) = \exp_o t \cdot v$ for $0 \leq t < T_0$ can be extended to $\gamma(T_0) = z$. Therefore $(\mathbb{C}^n, \hat{g}_a)$ is complete. □

Remark 3.6. (*Bryant* [3].) *The moduli of complete Bochner flat Kähler metrics on the complex euclidean space* $\mathbb{C}^n$ *is the parameter space*

$$\{(a_1, \cdots, a_n) \in \mathbb{R}^n \mid 0 \leq a_1 \leq a_2 \leq \cdots \leq a_n\}.$$

3.7. **Case (iii).** If $\rho(t) = ((0, (t, 0, \cdots, 0)), \mathrm{I})$, then the action on $\mathcal{N}$ is obtained as $\rho(t)(s, (z_1, \cdots, z_n)) = (s - ty_1, (t + x_1 + iy_1, z_2, \cdots, z_n)$. Then, $\xi = -y_1 \dfrac{d}{dt} + \dfrac{d}{dx_1}$ so that $\omega(\xi) = -2y_1$. By the definition, we have $\mathcal{W} = (\mathcal{N} - S)^0 = \{(s, (z_1, \cdots, z_n)) \in \mathbb{R} \times \mathbb{C}^n \mid \mathrm{Im} z_1 = y_1 < 0\}$. Put $\gamma_\theta = ((0, 0), e^\theta \cdot \mathrm{I}) \in \mathbb{R}^+ \subset \mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$. Recall from **(3)** of Corollary 3.2 that $\gamma_\theta\cdot\rho(t)\cdot\gamma_{-\theta}=((0,(e^\theta\cdot t,0,\cdots,0)),\mathrm{I})$. We obtain that

$$\begin{aligned}\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))&=\{(s,(x,z_2,\cdots,z_n)),\begin{pmatrix}1&\\&B\end{pmatrix}\}\rtimes\{\gamma_\theta\}\\&=\big(\mathbb{R},(\mathbb{R},\mathbb{C}^{n-1})\big)\rtimes(\mathrm{U}(n-1)\times\mathbb{R}^+).\end{aligned}$$

Let $\mathcal{M}$ be the $(2n-1)$-dimensional Heisenberg nilpotent Lie subgroup of $\mathcal{N}$ which is defined to be the product $\mathbb{R}\times\mathbb{C}^{n-1}$ with same group law. It follows that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))/\rho(\mathbb{R})=\mathcal{M}\rtimes(\mathrm{U}(n-1)\times\mathbb{R}^+)$. As the action of $\langle\gamma_\theta\rangle=\mathbb{R}^+$ on $\mathcal{N}$ satisfies that

$$\gamma_\theta(t,z)=(e^{2\theta}\cdot t,e^\theta\cdot z). \tag{3.16}$$

A calculation shows that ${\gamma_\theta}^*\omega=d(e^{2\theta}\cdot t)+\mathrm{Im}\langle e^\theta\cdot z,d(e^\theta\cdot z)\rangle=e^{2\theta}\cdot\omega$, and so γ_θ preserves S. We have that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))=\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$. As above note that $\gamma_{\theta*}\xi=e^\theta\cdot\xi$ on $\mathcal{W}$. Since $\eta=\dfrac{1}{\omega(\xi)}\cdot\omega$ on $\mathcal{W}$ from (1.4), it is easy to check that

$${\gamma_\theta}^*\eta=e^\theta\cdot\eta\ \text{ on }\mathcal{W}. \tag{3.17}$$

The map F defined by $F([s,(x_1+iy_1,z)])=(s-x_1y_1,(y_1,z)),\ z=(z_2,\cdots,z_n)$ is a diffeomorphism of $\mathcal{W}/\rho(\mathbb{R})$ onto $\mathbb{R}\times\mathbb{C}^{n-1}\times\mathbb{R}^-(=\mathcal{M}\times\mathbb{R}^-)$. Then F induces an action of $\mathcal{M}\rtimes(\mathrm{U}(n-1)\times\mathbb{R}^+)$ on $\mathcal{M}\times\mathbb{R}^-$, which can be described as: $(w,z'\in\mathbb{C}^{n-1})$

$$\begin{aligned}&((a,w),e^\theta\cdot B)\cdot(s,(y,z'))\\&\quad=(a+e^{2\theta}\cdot s-\mathrm{Im}\langle w,e^\theta\cdot Bz'\rangle,(e^\theta\cdot y,w+e^\theta\cdot Bz')).\end{aligned} \tag{3.18}$$

Therefore, we can identify $\mathcal{W}/\rho(\mathbb{R})=\mathcal{M}\times\mathbb{R}^-$ equipped with this action. If we note (3.17), then $\mathbb{R}^+$ acts as homotheties of the Kähler metric $\hat g$, i.e. ${\gamma_\theta}^*\hat g=e^\theta\cdot\hat g$, while the subgroup $\mathcal{H}=\mathcal{M}\rtimes\mathrm{U}(n-1)$ acts as isometries.

Remark 3.7. *We donot know whether the complex structure $\hat J$ is a restriction of $J_{\mathbb{C}}$ to $\mathcal{M}\times\mathbb{R}^-$.*

3.8. **Similarity geometry $(\mathcal{M}\rtimes(\mathrm{U}(n-1)\times\mathbb{R}^+),\mathcal{M}\times\mathbb{R}^-)$.** We have a Kähler metric $\hat g$ on $\mathcal{M}\times\mathbb{R}^-$, however $\mathbb{R}^+$ acts as nontrivial homothetic transformations with respect to $\hat g$. In this section, we seek a Riemannian metric (but not Kähler) on $\mathcal{M}\times\mathbb{R}^-$ invariant under the group $\mathcal{M}\rtimes(\mathrm{U}(n-1)\times\mathbb{R}^+)$. First note that $(\mathrm{Aut}_{CR}(\mathcal{N}),\mathcal{N})$ contains it as a subgeometry:

$$\begin{aligned}&\gamma=((t,(0,z)),\lambda\cdot A)\in\mathcal{M}\rtimes(\mathrm{U}(n-1)\times\mathbb{R}^+)\ \ (z\in\mathbb{C}^{n-1}),\\&p=(b,(x,w))\in\mathcal{M}\times\mathbb{R}^-\ \ (x<0,\ z\in\mathbb{C}^{n-1}),\end{aligned} \tag{3.19}$$

in which the action becomes:

$$\gamma\cdot p=(t+\lambda^2\cdot b-\mathrm{Im}\langle z,\lambda\cdot Aw\rangle,(\lambda\cdot x,z+\lambda\cdot Aw)).$$

There is the $\mathcal{N}\rtimes\mathrm{U}(n)$- invariant Riemannian metric $g^0_{\mathcal{N}}=\omega_{\mathcal{N}}\cdot\omega_{\mathcal{N}}+d\omega_{\mathcal{N}}(J\ ,\ )$ on $\mathcal{N}$. Let $\mu:\mathrm{Aut}_{CR}(\mathcal{N})=\mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+)\to\mathbb{R}^+$ be the scale factor projection. Then $g^0_{\mathcal{N}}$ has the following dilation properties by Corollary 3.2, see also §2.1: For

$\gamma \in \mathrm{Aut}_{CR}(\mathcal{N})$,

$$(3.20)\qquad \begin{aligned} (g^0_{\mathcal{N}})_{\gamma p}(\gamma_* X, \gamma_* Y) &= \mu(\gamma)^2 \cdot (g^0_{\mathcal{N}})_p(X,Y) \text{ for } X, Y \in \text{Null } (\omega_{\mathcal{N}})_p. \\ (g^0_{\mathcal{N}})_{\gamma p}(\gamma_* X, \gamma_* Y) &= \mu(\gamma)^4 \cdot (g^0_{\mathcal{N}})_p(X,Y) \text{ for } X, Y \in T_p\mathcal{R}. \end{aligned}$$

Here $\mathcal{R} = \mathbb{R}$ is the center of $\mathcal{N}$. Consider the restriction $g = g^0_{\mathcal{N}}|\mathcal{M} \times \mathbb{R}^-$. Then g is an $\mathcal{M} \rtimes \mathrm{U}(n-1)$-invariant Riemannian metric on $\mathcal{M} \times \mathbb{R}^-$ and g has the dilation property as in (3.20) where $\mathrm{Null}\, \omega_{\mathcal{N}}$ replaces $T(\mathcal{M} \times \mathbb{R}^-) \cap (\mathrm{Null}\omega_{\mathcal{N}}|\mathcal{M} \times \mathbb{R}^-)$. As $T\mathcal{R}$ is characteristic for $\omega_{\mathcal{N}}$, $d\omega_{\mathcal{N}}(T\mathcal{R}, Y) = 0$ for arbitrary $Y \in T\mathcal{N}$. If $(T\mathcal{R})^\perp$ is the orthogonal complement of $T\mathcal{R}$ with respect to g, we notice that

$$(3.21)\qquad (T\mathcal{R})^\perp = T(\mathcal{M} \times \mathbb{R}^-) \cap (\mathrm{Null}\omega_{\mathcal{N}}|\mathcal{M} \times \mathbb{R}^-).$$

As $\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$ normalizes $\mathcal{R}$ so that $T\mathcal{R}$ is invariant under $\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$. Let $T(\mathcal{M} \times \mathbb{R}^-) = T\mathcal{R} \oplus (T\mathcal{R})^\perp$ be the decomposition with respect to g such that $X = X^f \oplus X^b$, $Y = Y^f \oplus Y^b \in T(\mathcal{M} \times \mathbb{R}^-)$. Then $g(X,Y) = g(X^f, Y^f) + g(X^b, X^b)$. It follows from (3.20) that for $\gamma \in \mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$,

$$(3.22)\qquad g_{\gamma p}(\gamma_* X, \gamma_* Y) = \mu(\gamma)^4 \cdot g_p(X^f, Y^f) + \mu(\gamma)^2 \cdot g_p(X^b, Y^b).$$

In general, there is no $\mathrm{Aut}_{CR}(\mathcal{N})$-invariant Riemannian metric on $\mathcal{N}$ within the conformal class of $g^0_{\mathcal{N}}$. However, in our case, $\mathcal{M} \times \mathbb{R}^-$ is a proper affine subspace of $\mathcal{N}$, it is possible to construct an $\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$-invariant Riemannian metric on $\mathcal{M} \times \mathbb{R}^-$ in the Carnot-Carathéodory class. Let $\{\psi_s\}_{-1<s<\infty}$ be the one-parameter group on $\mathcal{M} \times \mathbb{R}^-$ defined by $\psi_s((t,(x,z))) = (t, ((1+s)x, z))$. Then it is easy to check that for any $\gamma \in \mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$,

$$(3.23)\qquad \gamma \circ \psi_s = \psi_s \circ \gamma \ \ (-1 < s < \infty).$$

Denote by V the vector field induced by the one-parameter group $\{\psi_s\}_{-1<s<\infty}$ on $\mathcal{M} \times \mathbb{R}^-$:

$$(3.24)\qquad V_p = \frac{d\psi_s(p)}{ds}|_{s=0} = x(\frac{d}{dx_1})_p \ \text{ for } p = (t,(x,z)).$$

From (3.23), V is an $\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$-invariant vector field on $\mathcal{M} \times \mathbb{R}^-$. The formula (2.3) of $\omega_{\mathcal{N}}$ shows that $\omega_{\mathcal{N}}(V_x) = -xy_1$, hence $\omega_{\mathcal{N}}(V_p) = 0$ on $\mathcal{M} \times \mathbb{R}^-$ ($y_1 = 0$ for $z_1 = x + \mathrm{i}y_1 \in \mathbb{C}$). Noting (3.21), it follows $V \in (T\mathcal{R})^\perp$. Let $||V_p|| = \sqrt{g(V_p, V_p)}$. We define a Riemannian metric $g_{\mathcal{C}}$ on $\mathcal{M} \times \mathbb{R}^-$:

$$(3.25)\qquad (g_{\mathcal{C}})_p(X,Y) = \frac{g_p(X^f, Y^f)}{||V_p||^4} + \frac{g_p(X^b, Y^b)}{||V_p||^2}.$$

By the dilation property (3.20) and (3.22), it is easily seen that $g_{\mathcal{C}}$ is an $\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$- invariant Riemannian metric on $\mathcal{M} \times \mathbb{R}^-$. By the existence of this invariant Riemannian metric $g_{\mathcal{C}}$, we remark that $\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$ acts properly on $\mathcal{M} \times \mathbb{R}^-$.

3.9. **Case (iv).** As $\rho(t)=((0,(t,0,\cdots,0)),\begin{pmatrix}1&0\\0&B_t\end{pmatrix})$, $\rho(t)(s,(z_1,\cdots,z_n))=(s-ty_1,(t+x_1+iy_1,B_t(z_2,\cdots,z_n))$. Then, $\xi=-y_1\dfrac{d}{dt}+\dfrac{d}{dx_1}+\sum_{j=2}^{k+1}b_j(x_j\dfrac{d}{dy_j}-y_j\dfrac{d}{dx_j})$. It follows that $\omega(\xi)=-2y_1+(b_2|z_2|^2+\cdots+b_n|z_n|^2)$ and so $\mathcal{W}=\{(s,(z_1,\cdots,z_n))\in\mathbb{R}\times\mathbb{C}^n \mid 2\mathrm{Im}z_1<\sum_{j=2}^{n}b_j|z_j|^2\}$. Since $B_t\neq \mathrm{I}$, note that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))=\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$ by Corollary 3.2. As $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))$ preserves S, so does $\mathcal{W}$ such that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{N})}(\rho(\mathbb{R}))=\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$. As in **(ii)**, the normalizer of $\{B_t\}$ in $\mathrm{U}(n)$ coincides with $\mathrm{U}(1)\times\mathrm{U}(\ell_1,\cdots,\ell_{m'})\times\mathrm{U}(n-k-1)$ $(\ell_1+\cdots+\ell_{m'}=k)$. We can deduce that

$$\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))=\left(\mathbb{R},\begin{pmatrix}\mathbb{R}\\0\\\mathbb{C}^{n-k-1}\end{pmatrix}\right)\rtimes\begin{pmatrix}1&0&0\\0&\mathrm{U}(\ell_1,\cdots,\ell_{m'})&0\\0&0&\mathrm{U}(n-k-1)\end{pmatrix}.$$

Denote the nilpotent Lie subgroup of $\mathcal{M}$ to be $\mathcal{M}'=\mathbb{R}\times\mathbb{C}^{n-k-1}$. As the normalizer coincides with the centralizer, we obtain that $\mathcal{H}=\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R})=(\mathcal{M}'\rtimes\mathrm{U}(n-k-1))\times\mathrm{U}(\ell_1,\cdots,\ell_{m'})$.

Suppose that $b_2\leq\cdots\leq b_\ell<0<b_{\ell+1}\leq\cdots\leq b_{k+1}$ and the remaining are zeros. Put

$$\mathbf{b}=\sqrt{-2y_1+(b_{\ell+1}|z_{\ell+1}|^2+\cdots+b_{k+1}|z_{k+1}|^2)}>0.$$

For a point $(s,x_1+iy_1,z_2,\cdots,z_n)\in\mathcal{W}$, put

$$p(z_2,\cdots,z_{k+1})=\frac{K_{-x_1}(\sqrt{-b_2}z_2,\cdots,\sqrt{-b_\ell}z_\ell,\sqrt{b_{\ell+1}}z_{\ell+1},\cdots,\sqrt{b_{k+1}}z_{k+1})}{\mathbf{b}},$$

where K is the action $K_{-x_1}=\begin{pmatrix}e^{i(-x_1)\cdot b_2}&&\\&\ddots&\\&&e^{i(-x_1)\cdot b_{k+1}}\end{pmatrix}$. Then we define a map $F:\mathcal{W}\to\mathbb{R}\times(\mathbb{B}_{\mathbb{C}}^{\ell-1}\times\mathbb{C}^{k-\ell+1})\times((\mathbb{R})\times\mathbb{C}^{n-k-1}\times\mathbb{R}^+)$ to be

$$F(p)=(s+x_1y_1,p(z_2,\cdots,z_{k+1}),x_1,\frac{(z_{k+2},\cdots,z_n)}{\mathbf{b}},\mathbf{b}).$$

Note always $\dfrac{-b_2|z_2|^2-\cdots-b_\ell|z_\ell|^2}{\mathbf{b}^2}<1$. Moreover, it is easy to check that

$$\frac{K_{-x_1}(\sqrt{b_{\ell+1}}z_{\ell+1},\cdots,\sqrt{b_{k+1}}z_{k+1})}{\mathbf{b}}\in\begin{cases}\mathbb{B}_{\mathbb{C}}^{k-\ell+1}, & y_1<0\\ S^{2(k-\ell)+1}, & y_1=0\\ (\mathbb{B}_{\mathbb{C}}^{k-\ell+1})^c, & y_1>0\end{cases}$$

Here $(\mathbb{B}_{\mathbb{C}}^{k-\ell+1})^c$ is the complement of the complex unit ball $\mathbb{B}_{\mathbb{C}}^{k-\ell+1}$. Noting that $\mathbb{C}^{k-\ell+1}=\mathbb{B}_{\mathbb{C}}^{k-\ell+1}\cup S^{2(k-\ell)+1}\cup(\mathbb{B}_{\mathbb{C}}^{k-\ell+1})^c$, it is easy to check that F is a diffeomorphism. By the the above action of $\rho(\mathbb{R})$ on $\mathcal{W}$, we define an action of $\mathbb{R}$ on $\mathbb{R}\times(\mathbb{B}_{\mathbb{C}}^{\ell-1}\times\mathbb{C}^{k-\ell+1})\times((\mathbb{R})\times\mathbb{C}^{n-k-1}\times\mathbb{R}^+)$ by

$$t\circ(u,(z_1,w_1),(v,z_2,\lambda))=(u,(z_1,w_1),(t+v,z_2,\lambda)).$$

Noting $K_{-(x_1+t)} = K_{-x_1} \cdot K_{-t}$, we can check that F is equivariant; $F(\rho(t)p) = t \circ F(p)$, $p \in \mathcal{W}$. Hence F induces a diffeomorphim of $\mathcal{W}/\rho(\mathbb{R})$ onto $\mathbb{R} \times (\mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell+1}) \times (\mathbb{C}^{n-k-1} \times \mathbb{R}^+) = \mathcal{M}' \times \mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell-1} \times \mathbb{R}^+$.
We summarize the results of this section.

Theorem 3.8. *Let $\rho(\mathbb{R})$ be a closed subgroup isomorphic to $\mathbb{R}$ which fixes one point $\{\infty\}$. Then $\mathcal{W} = \mathcal{N} - S$. The Bochner flat Kähler geometry $(\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R}), \mathcal{W}/\rho(\mathbb{R}), \hat{g}, \hat{J})$ consists of the following types:*

(i) *Similarity geometry $(\mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+), \mathbb{C}^n, g_{\mathbb{C}}, J_{\mathbb{C}})$ in which $\mathbb{R}^+$ acts as similarity transformations.*

(ii) *Intransitive Kähler geometry*

(**a**) : $(\mathbb{C}^{n-k} \rtimes \mathrm{U}(n-k)) \times \mathrm{U}(\ell_1, \cdots, \ell_m), \mathbb{C}^n, \hat{g}_a, J_{\mathbb{C}})$.
(**b**) : $((\mathbb{C}^{n-k} \rtimes \mathrm{U}(n-k)) \times \mathrm{U}(\ell_1, \cdots, \ell_m), \mathbb{C}^{n-k} \times \mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell+1}, \hat{g}_b, J_{\mathbb{C}})$,

where $1 \le m \le \ell_1 + \cdots + \ell_m = k \le n$. The group acts as Kähler isometries with cohomogeneity m.

(iii) *Similarity geometry $(\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+), \mathcal{M} \times \mathbb{R}^-, \hat{g}, \hat{J})$ with homothetic transitive group. The cohomogeneity is 1 as the group of Kähler isometries $\mathcal{M} \rtimes \mathrm{U}(n-1)$.*

(iv) *Intransitive Kähler geometry*

(**a**) : $((\mathcal{M}' \rtimes \mathrm{U}(n-k-1)) \times \mathrm{U}(\ell_1, \cdots, \ell_{m'}), \mathcal{M}' \times \mathbb{B}_{\mathbb{C}}^{k} \times \mathbb{R}^+, \hat{g}, \hat{J})$.
(**b**) : $((\mathcal{M}' \rtimes \mathrm{U}(n-k-1)) \times \mathrm{U}(\ell_1, \cdots, \ell_{m'}), \mathcal{M}' \times \mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell+1} \times \mathbb{R}^+, \hat{g}, \hat{J})$,

where $1 \le m' \le \ell_1 + \cdots + \ell_{m'} = k \le n-1$. The group acts as Kähler isometries with cohomogeneity m'.

We have an application using the Riemannian metric $g_{\mathcal{C}}$ lying in the Carnot-Carathéodory class (cf. §3.8). First of all

Proposition 3.9. *There exists no compact space form except for the complex euclidean space form of case* **(i)**.

Proof. Put $G = \mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R})$ and $Y = \mathcal{W}/\rho(\mathbb{R})$. Note that Y is contractible in each case. Suppose there exists a discrete subgroup Γ of G that acts properly discontinuously on Y with compact quotient. Choose a torsionfree subgroup of finite index in G by Selberg's Lemma, we can assume Y/Γ is a compact aspherical manifold. It follows that the cohomological dimension $\mathrm{ch}\,\Gamma = \dim Y = 2n$. On the other hand, if K is a maximal compact subgroup of G, then the double coset space $\Gamma\backslash G/K$ is an aspherical manifold. Since $\mathrm{ch}\,\Gamma \le \dim G/K$, the possible case is either **(i)** or **(iii)**. We prove that **(iii)** does not occur. Let $G = \mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$ in this case. Note that the subgroup $\mathcal{S} = \mathcal{M} \rtimes \mathbb{R}^+$ is a connected normal solvable subgroup of G. Suppose that Γ is a torsionfree discrete cocompact subgroup of G. Let $q : G \to \mathrm{U}(n-1)$ be the projection with kernel $\mathcal{S}$. As Γ is discrete, the closure of the identity component $\overline{q(\Gamma)}^0$ is solvable in $\mathrm{U}(n-1)$ by Lemma 8.24 [16]. It is contained in T^{n-1} of $\mathrm{U}(n-1)$ up to conjugacy. Since $\overline{q(\Gamma)}$ is compact in $\mathrm{U}(n-1)$, $\overline{q(\Gamma)}^0$ is of finite index in $\overline{q(\Gamma)}$. Hence we can find a subgroup Γ' of finite index in Γ such that $q(\Gamma') \subset T^{n-1}$. On the other hand, G is a closed subgroup of $\mathrm{Aut}_{CR}(\mathcal{N})$

from (3.19). Let $L : \mathrm{Aut}_{CR}(\mathcal{N}) \to \mathrm{U}(n) \times \mathbb{R}^+$ be the holonomy homomorphism. Note that

$$L(\Gamma') \subset T^{n-1} \times \mathbb{R}^+. \tag{3.26}$$

Suppose that there exists an element $\gamma = (x, \lambda \cdot A) \in \Gamma'$ such that λ is nontrivial. Then the determinant of matrix $\lambda \cdot A$ is not 1 by action of (3.16). Since the intersection $\mathcal{N} \cap \Gamma'$ is discrete in $\mathcal{N}$, γ cannot preserve $\mathcal{N} \cap \Gamma'$ which must be trivial. This implies that $L : \Gamma' \to L(\Gamma')$ is an isomorphism and so Γ' is abelian by (3.26). Moreover, as $L(\gamma) = \lambda \cdot A$ has no eigenvalue 1, the same argument of proof of Proposition 3.1 shows that γ has the unique fixed point $(0, 0) \in \mathcal{N}$ up to conjugacy. Since Γ' is abelian, Γ' fixes $(0, 0)$ by uniqueness. It follows that $\Gamma' \subset \mathrm{U}(n) \times \mathbb{R}^+$. Hence, ch $\Gamma' = 1$, which is impossible because ch $\Gamma' =$ ch $\Gamma = 2n$. This contradiction implies that $L(\Gamma') \subset T^{n-1} \subset \mathrm{U}(n-1)$. As $\Gamma' \subset G$, this concludes that $\Gamma' \subset \mathcal{M} \rtimes \mathrm{U}(n-1)$. Then ch $\Gamma' \leq \dim(\mathcal{M} \rtimes \mathrm{U}(n-1))/\mathrm{U}(n-1) = 2n - 1$, a contradiction again, so case **(iii)** cannot occur. □

Proposition 3.10. *There exists no compact similarity manifold uniformizable with respect to the geometry* $(\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+), \mathcal{M} \times \mathbb{R}^-)$.

Proof. Suppose that there exists a developing map dev : $\tilde{M} \to \mathcal{M} \times \mathbb{R}^-$ which is equivariant with respect to the holonomy homomorphism $\rho : \pi_1(M) \to \mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$ where $\tilde{M}$ is the universal covering space and $\pi_1(M)$ is the fundamental group. As there is an $\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+)$-invariant Riemannian metric $g_{\mathcal{C}}$ on $\mathcal{M} \times \mathbb{R}^-$, dev : $\tilde{M} \to \mathcal{M} \times \mathbb{R}^-$ is a local isometry by the pullback metric. Since the pullback metric on $\tilde{M}$ is invariant under $\pi_1(M)$ and M is compact, the pullback metric is complete for which dev : $\tilde{M} \to \mathcal{M} \times \mathbb{R}^-$ is a covering. As $\mathcal{M} \times \mathbb{R}^-$ is contractible, dev is a diffeomorphism. This implies that $\mathcal{M} \times \mathbb{R}^-/\rho(\pi_1(M))$ is compact, which is impossible by Corollary 3.9.

□

3.10. **Non-compact holonomy with two fixed points.** When G is noncompact, another possibility is that G fixes exactly two points $\{0, \infty\}$. We have already seen in the proof of Proposition 3.1 that $\rho(\mathbb{R})$ stabilizes exactly two points $\{0, \infty\}$ if and only if $G \subset \mathrm{U}(n) \times \mathbb{R}^+$. As $\mathcal{N} - \{0\} (= S^{2n+1} - \{0, \infty\}) = S^{2n} \times \mathbb{R}^+$, the CR-structure on $S^{2n} \times \mathbb{R}^+$ is a restriction to that on $\mathcal{N}$. In addition, it induces a CR-structure on the Hopf manifold $S^{2n} \times S^1$.

Corollary 3.11. *Suppose that G stabilizes exactly two points $\{0, \infty\}$. Then, $\rho(\mathbb{R})$ is a closed subgroup isomorphic to $\mathbb{R}$ in $\mathrm{U}(n) \times \mathbb{R}^+ = \mathrm{Aut}_{CR}(S^{2n} \times \mathbb{R}^+)$ and $\rho(\mathbb{R})$ has the form $\rho(t) = (A_t, e^t)$ for $A_t \in T^n$. Moreover, if $a_{i_1} < \cdots < a_{i_m}$ are mutually distinct among $a_1, \cdots, a_k$ and the rest of $n - k$ are zeros, then*

$$\mathbf{N}_{\mathrm{Aut}_{CR}(S^{2n} \times \mathbb{R}^+)}(\rho(\mathbb{R})) = \mathbf{C}_{\mathrm{Aut}_{CR}(S^{2n} \times \mathbb{R}^+)}(\rho(\mathbb{R})) = \mathrm{U}(\ell_1, \cdots, \ell_m) \times \mathrm{U}(n-k) \times \mathbb{R}^+$$

where $\ell_1 + \cdots + \ell_m = k$.

As $S^{2n+1} - \{0, \infty\} = \mathcal{N} - \{0\}$, the CR-structure is a restriction to that on $\mathcal{N}$. As $S^{2n+1} - \{0, \infty\}$ is identified with $S^{2n} \times \mathbb{R}^+$, it induces a CR-structure on the Hopf manifold $S^{2n} \times S^1$.

3.11. **Hopf case $\mathcal{W} = S^{2n} \times \mathbb{R}^+ - S$.** We have the contact form on $S^{2n} \times \mathbb{R}^+$, which is a restriction of the contact form $\omega = \omega_{\mathcal{N}}$ to $\mathcal{N} - \{0\} = \mathbb{R} \times \mathbb{C}^n - \{0\}$. Let $\rho(t) = (A_t, e^t)$ be as in Corollary 3.11 where $A_t = (e^{it\cdot a_1}, \cdots, e^{it\cdot a_n})$. The action of $\rho(\mathbb{R})$ has the form $\rho(t)(\theta, z) = (e^{2t}\cdot\theta, e^t\cdot A_t z)$ $((\theta, z) \in \mathbb{R}\times\mathbb{C}^n - \{0\})$. A calculation shows that

$$\xi = 2t\frac{d}{dt} + \sum_{j=1}^{n}((x_j - a_j y_j)\frac{d}{dx_j} + (y_j + a_j x_j)\frac{d}{dy_j}). \tag{3.27}$$
$$\omega(\xi) = 2t + (a_1|z_1|^2 + \cdots + a_n|z_n|^2).$$

Since t is arbitrary, we note that the singular set $S \neq \emptyset$ for which $\mathcal{W} = \{(t, z) \in \mathbb{R}\times\mathbb{C}^n - \{0\} \mid -2t < a_1|z_1|^2 + \cdots + a_n|z_n|^2\}$. It is easy to check that

$$\rho(t)^*\omega = e^{2t}\cdot\omega. \tag{3.28}$$

Then $\rho(\mathbb{R})$ leaves S invariant so that $\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R})) = \mathbf{N}_{\mathrm{Aut}_{CR}(S^{2n}\times\mathbb{R}^+)}(\rho(\mathbb{R}))$.

Suppose that $a_1 \leq a_2 \cdots \leq a_{\ell-1} < 0 < a_\ell \leq \cdots \leq a_k$ and the remaining $n-k$ are all zero. Then it follows that $0 \leq -(a_1|z_1|^2 + \cdots + a_{\ell-1}|z_{\ell-1}|^2) < 2t + a_\ell|z_\ell|^2 + \cdots + a_k|z_k|^2$. Put

$$\mathbf{b} = \mathbf{b}_t = \sqrt{2t + a_\ell|z_\ell|^2 + \cdots + a_k|z_k|^2} > 0. \tag{3.29}$$

We define a diffeomorphism G from $\mathcal{W}$ onto $(\mathbb{R}^+) \times \mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell+1} \times \mathbb{C}^{n-k}$ by setting:

$$G(t, z) = (\mathbf{b}, \frac{K_{-\log\mathbf{b}}(\sqrt{-a_1}z_1, \cdots, \sqrt{-a_{\ell-1}}z_{\ell-1}, \sqrt{a_\ell}z_\ell, \cdots, \sqrt{a_k}z_k}{\mathbf{b}}, \frac{z_{k+1}, \cdots, z_n}{\mathbf{b}}).$$

Here K_θ is defined as $K_\theta(z_1, \cdots, z_k) = (e^{i\theta\cdot a_1}\cdot z_1, \cdots, e^{i\theta\cdot a_k}\cdot z_k)$. Note that it is always true that

$$\frac{-a_1|z_1|^2 - \cdots - a_{\ell-1}|z_{\ell-1}|^2}{\mathbf{b}^2} < 1.$$

As before we can check

$$\frac{K_{-\log\mathbf{b}}(\sqrt{a_\ell}z_\ell, \cdots, \sqrt{a_k}z_k\ell)}{\mathbf{b}} \in \begin{cases} \mathbb{B}_{\mathbb{C}}^{k-\ell+1} & t > 0 \\ S^{2(k-\ell)+1} & t = 0 \\ (\mathbb{B}_{\mathbb{C}}^{k-\ell+1})^c & t < 0 \end{cases}$$

where $\mathbb{B}_{\mathbb{C}}^{k-\ell+1} \cup S^{2(k-\ell)+1} \cup (\mathbb{B}_{\mathbb{C}}^{k-\ell+1})^c = \mathbb{C}^{k-\ell+1}$. Define an action of $\mathbb{R}^+$ on the space $(\mathbb{R}^+) \times \mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell+1} \times \mathbb{C}^{n-k}$ to be

$$e^\theta \circ (a, (w_1, w_2, w_3)) = (e^\theta\cdot a, (w_1, w_2, w_3)),$$

where $w_1 \in \mathbb{B}_{\mathbb{C}}^{\ell-1}, w_2 \in \mathbb{C}^{k-\ell+1}, w_3 \in \mathbb{C}^{n-k}$. Obviously the orbit space is $\mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell+1} \times \mathbb{C}^{n-k}$. In this case we note that

$$\rho(\theta)(t, z) = (e^{2\theta}\cdot t, e^\theta\cdot(K_\theta(z_1, \cdots, z_k), (z_{k+1}, \cdots, z_n)),$$
$$\mathbf{b}_{e^{2\theta}\cdot t} = e^\theta\cdot\mathbf{b}_t,\ K_{-\log e^\theta\cdot\mathbf{b}} = K_{-\theta}\circ K_{-\log\mathbf{b}} \quad \text{(cf. (3.29))}.$$

Using these, we see that G is equivariant, $G(\rho(\theta)(t, z)) = e^\theta \circ G(t, z)$. Therefore G induces a diffeomorphism from $\mathcal{W}/\rho(\mathbb{R})$ onto $\mathbb{B}_{\mathbb{C}}^{\ell-1} \times \mathbb{C}^{k-\ell+1} \times \mathbb{C}^{n-k}$. Let $c = (a_1, \cdots, a_{\ell-1} < 0 < a_\ell, \cdots, a_k)$ and put the induced Kähler metric $\hat{g} = \hat{g}_c$ on $\mathcal{W}$. We have shown the following.

Theorem 3.12. *Let $\rho(\mathbb{R})$ be a closed subgroup isomorphic to $\mathbb{R}$ which fixes two points $\{0,\infty\}$. Then $\mathcal{W} = S^{2n}\times\mathbb{R}^+ - S$ with $S\neq\emptyset$. The Bochner flat Kähler geometry $(\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R}), \mathcal{W}/\rho(\mathbb{R}), \hat{g}, \hat{J})$ is the intransitive geometry:*

$$(\mathrm{U}(\ell_1,\cdots,\ell_m)\times\mathrm{U}(n-k), \mathbb{B}_{\mathbb{C}}^{\ell-1}\times\mathbb{C}^{k-\ell+1}\times\mathbb{C}^{n-k}, \hat{g}_c, \hat{J}).$$

$(0\leq m\leq \ell_1+\cdots+\ell_m = k\leq n)$. The group acts as Kähler isometries with cohomogeneity $m+1$.

4. Classification to the compact holonomy case

When the closure G is compact, it is a compact connected abelian Lie subgroup which is cojugate to a subgroup of the maximal torus T^{n+1} of $\mathrm{U}(n+1)$ in $\mathrm{PU}(n+1,1)$. We can write

$$\rho(t) = (e^{\mathrm{i}t\cdot a_1},\cdots,e^{\mathrm{i}t\cdot a_{n+1}}) \tag{4.1}$$

In particular, if all a_i's are rational, $\rho(\mathbb{R}) = S^1$ and if one of a_i's is irrational, $\rho(\mathbb{R})$ is non-closed as usual. Note that $\mathrm{Aut}_{CR}(S^{2n+1}-S^{2m-1}) = P(\mathrm{U}(m,1)\times\mathrm{U}(n-m+1))$ such that $G\subset\mathrm{U}(n-m+1)$ as in §1.2, §1.3. We obtain a finer classification of G according to the fixed point set S^{2m-1} $(m=0,1,\cdots,n)$ in $X = S^{2n+1}$.

Proposition 4.1. *G belongs to the $(n-m+1)$-dimensional torus $T^{n-m+1}\subset P(\mathcal{Z}\mathrm{U}(m,1)\times\mathrm{U}(n-m+1))$ up to conjugacy, $m=0,1,\cdots,n$. (Here $\mathcal{Z}\mathrm{U}(0,1) = \mathrm{U}(0,1) = S^1$.) The element of $\rho(\mathbb{R})$ has the form $\rho(t) = (1,(e^{\mathrm{i}t\cdot a_1},\cdots,e^{\mathrm{i}t\cdot a_{n-m+1}}))$ for some $a_1,\cdots,a_{n-m+1}\in\mathbb{R}^*$.*

1. *When $m=0$, $G\subset T^{n+1}$. In particular, if $\rho(\mathbb{R})$ is the center $\mathcal{Z}\mathrm{U}(n+1) = S^1$ (i.e. all a_i are equal), then $\mathbf{N}_{\mathrm{Aut}_{CR}(S^{2n+1})}(\rho(\mathbb{R})) = \mathbf{C}_{\mathrm{Aut}_{CR}(S^{2n+1})}(\rho(\mathbb{R})) = \mathrm{U}(n+1)$.*
2. *When $m=n$, $\rho(\mathbb{R})$ is necessarily closed so that $\rho(\mathbb{R}) = G = P(\mathcal{Z}\mathrm{U}(n,1)\times\mathrm{U}(1)) = \mathcal{Z}\mathrm{U}(n,1) = S^1$ in which*

$$\mathbf{N}_{\mathrm{Aut}_{CR}(S^{2n+1}-S^{2n-1})}(\rho(\mathbb{R})) = \mathbf{C}_{\mathrm{Aut}_{CR}(S^{2n+1}-S^{2n-1})}(\rho(\mathbb{R})) = \mathrm{U}(n,1).$$

In general, if $a_{i_1},\cdots,a_{i_k}$ $(1\leq i_1<\cdots<i_k\leq n-m+1)$ are mutually distinct among a_i's, then

$$\begin{aligned}&\mathbf{N}_{\mathrm{Aut}_{CR}(S^{2n+1}-S^{2m-1})}(\rho(\mathbb{R})) = \mathbf{C}_{\mathrm{Aut}_{CR}(S^{2n+1}-S^{2m-1})}(\rho(\mathbb{R}))\\ &= P(\mathrm{U}(m,1)\times\mathrm{U}(\ell_1,\cdots,\ell_k)) \quad (\ell_1+\cdots+\ell_k = n-m+1).\end{aligned} \tag{4.2}$$

Put $\omega = \omega_{\mathcal{S}}$. By the formula $\rho(t)$, it is easy to check that

$$\xi = \sum_{j=1}^{n-m+1} a_j(x_j\frac{d}{dy_j} - y_j\frac{d}{dx_j}) \text{ on } S^{2n+1}-S^{2m-1} \tag{4.3}$$

so that

$$\omega(\xi) = \sum_{j=1}^{n-m+1} a_j|z_j|^2. \tag{4.4}$$

4.1. **Hyperbolic case** $\mathcal{W} = S^{2n+1} - S^{2n-1} = V_{-1}^{2n+1}$**.** From (2.4) recall that $\rho(\mathbb{R})(= \mathcal{Z}\mathrm{U}(n,1)) = S^1$ acts freely and properly on $S^{2n+1} - S^{2n-1}$ whose orbit space is the complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^n$. In this case, the fixed point set is S^{2n-1} which is the subset of S^{2n+1} with $z_1 \neq 0$. Since $\omega_{\mathcal{S}}(\xi) = a_1|z_1|^2$ by (4.4), it follows that $\omega(\xi) \neq 0$. We may assume $a_1 > 0$ so that $S = \emptyset$ and $\mathcal{W} = S^{2n+1} - S^{2n-1}$. Using (2) of Proposition 4.1, we have that

Theorem 4.2. *Suppose that the fixed point set of $\rho(\mathbb{R})$ is S^{2n-1}. Then the Bochner flat Kähler geometry $(\mathcal{H}, \mathcal{W}/\rho(\mathbb{R}), \hat{g}, \hat{J})$ is the complex hyperbolic geometry* $(\mathrm{PU}(n,1), \mathbb{H}_{\mathbb{C}}^n, g_{\mathbb{H}}, J_{\mathbb{H}})$.

4.2. **Hyperbolic-Sphere case** $S^{2n+1} - S^{2m-1} - E - S$ **with** $\rho(\mathbb{R}) = S^1$**.** In general, S^1 has the exceptional orbits E. If $\rho(\mathbb{R})$ is the center $S^1 = P(\mathcal{Z}\mathrm{U}(m,1) \times \mathcal{Z}\mathrm{U}(n-m+1))$, then all a_j's are equal. We may assume $a_1 \geq 0$ so that $S = \emptyset$. Then $\mathcal{W} = X = S^{2n+1} - S^{2m-1}$. When one of a_j's is negative, (4.4) shows $S \neq \emptyset$ so that $\mathcal{W} = S^{2n+1} - S^{2m-1} - E - S$ on which S^1 acts freely (cf. 1.3). Noting §2.2, we obtain that

Theorem 4.3. *Suppose that the fixed point set of $\rho(\mathbb{R})$ is S^{2m-1}, $m = 0, \cdots, n-1$. The Bochner flat Kähler geometry $(\mathbf{N}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))/\rho(\mathbb{R}), \mathcal{W}/\rho(\mathbb{R}), \hat{g}, \hat{J})$ is the following type:*

$$(\mathrm{PU}(m,1) \times P(\mathrm{U}(\ell_1) \times \cdots \times \mathrm{U}(\ell_k)), (S^{2n+1} - S^{2m-1} - E - S)^0/S^1, \hat{g}, \hat{J}).$$

$(1 \leq k \leq \ell_1 + \cdots + \ell_k = n - m + 1 \leq n + 1.)$ The group acts as Kähler isometries with cohomogeneity $k-1$. In particular, if $\rho(\mathbb{R})$ is the center $S^1 = P(\mathcal{Z}\mathrm{U}(m,1) \times \mathcal{Z}\mathrm{U}(n-m+1))$, then the orbit space $S^{2n+1} - S^{2m-1}/S^1$ is holomorphically isometric to the product of complex hyperbolic space and projective space $\mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}$, and the above geometry becomes:

$$(\mathrm{PU}(m,1) \times \mathrm{PU}(n-m+1), \mathbb{H}_{\mathbb{C}}^m \times \mathbb{CP}^{n-m}, g_{\mathbb{H}} \times g_{\mathbb{CP}}, J_{\mathbb{H}} \times J_{\mathbb{CP}}).$$

We note that $\omega_{\mathcal{S}}(\xi) > 0$ whenever $0 < a_1 \leq a_2 \leq \cdots \leq a_{n-m+1}$. So $\mathcal{W} = S^{2n+1} - S^{2m-1} - E$. If we relax the domain $\mathcal{W}$ to be $S^{2n+1} - S^{2m-1}$, i.e. S^1 acts properly but not necessarily freely so that the quotient is an orbifold. Let $\mathbb{CP}^{n-m}(a_1, \cdots, a_{n-m+1}) = S^{2(n-m)+1}/\langle e^{\mathrm{i}t\cdot a_1}, \cdots, e^{\mathrm{i}t\cdot a_{n-m+1}}\rangle$ be the weighted complex projective space. We obtain the following.

Corollary 4.4. *Suppose that all a_i are mutually distinct positive rational numbers. Then a compact orbifold $\mathbb{H}_{\mathbb{C}}^m/\Gamma \times \mathbb{CP}^{n-m}(a_1, \cdots, a_{n-m+1})$ admits a Bochner flat Kähler (singular) metric.*

The following is an affirmative answer to the uniqueness of Remark 4.30 [3].

Corollary 4.5. *The weighted complex projective space $\mathbb{CP}^n(a_1, \cdots, a_{n+1})$ supports a unique Bochner flat Kähler metric.*

Proof. Let $\mathbb{CP}^{n-m}(a_1, \cdots, a_{n+1}) = S^{2n+1}/\langle e^{\mathrm{i}t\cdot a_1}, \cdots, e^{\mathrm{i}t\cdot a_{n+1}}\rangle$ for the action $S^1 = \langle e^{\mathrm{i}t\cdot a_1}, \cdots, e^{\mathrm{i}t\cdot a_{n+1}}\rangle$. If E is the set of exceptional orbits of S^1, then S^1 acts freely on $S^{2n+1} - E$ such that the orbit space $S^{2n+1} - E/S^1 = \mathbb{CP}^n(a_1, \cdots, a_{n+1}) - E^*$ topologically. Here E^* is the image of E in $\mathbb{CP}^n(a_1, \cdots, a_{n+1})$.

Given a Bochner flat Kähler (singular) metric on the underlying complex orbifold $\mathbb{CP}^n(a_1,\cdots,a_{n+1})$, the contactization (locally if necessary) implies that $S^{2n+1}-E$ admits a spherical CR structure for which S^1 acts freely as CR-transformations. By Theorem 1.3 and the classification of $\mathcal{W}$ (§3.2, §3.11, §4.1, §4.2), the orbit space $S^{2n+1}-E/S^1$ is $\mathbb{CP}^n(a_1,\cdots,a_{n+1})-E^*$ geometrically, i.e. S^{2n+1}/S^1 is holomorphically isometric to the weighted complex projective orbifold $\mathbb{CP}^n(a_1,\cdots,a_{n+1})$. □

5. Classification of complete case

The most tractable is the case when $\mathcal{W}$ is complete to determine $\mathcal{W}$. The classification of complete Bochner flat Kähler manifolds is completely determined by R. Bryant [3]. We reprove this with our method.

5.1. **Sasakian metric on** $X-S$**.** As in Remark 1.2, we have a Riemannian metric (Sasakian metric) on $(X-S)^0$ defined by $\bar{g}(X,Y)=\eta(X)\cdot\eta(Y)+d\eta(J_0X,Y)$, $X,Y\in T((X-S)^0)$. As $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ preserves η, $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))$ is a subgroup of isometries of $\bar{g}|\mathcal{W}$. By the construction of $\hat{g}$,
$\nu:(\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R})),\mathcal{W},\bar{g})\rightarrow(\mathcal{H},\mathcal{W}/\rho(\mathbb{R}),\hat{g})$ is an equivariant Riemannian submersion. (Similarly for $(\mathcal{W}/\rho(\triangle),\hat{g})$.)

5.2. **Another metric** g' **on** X**.** Any local CR-diffeomorphism of S^{2n+1} extends to a unique global element of $\mathrm{Aut}_{CR}(S^{2n+1})$ and since $\overline{\mathcal{W}}=X\subset S^{2n+1}$, note that $\mathrm{Aut}_{CR}(\mathcal{W})\subset\mathrm{Aut}_{CR}(X)$. If $X=\mathcal{N},S^{2n+1}-S^{2m-1}$, recall that $\mathrm{Psh}(X)=\mathcal{N}\rtimes\mathrm{U}(n)\subset\mathrm{Aut}_{CR}(X)=\mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+)$, $\mathrm{Psh}(X)=P(\mathrm{U}(m,1)\times\mathrm{U}(n-m+1))=\mathrm{Aut}_{CR}(X)$ respectively, see §1.3. Since $\mathrm{Psh}(X)$ acts transitively on X with compact stabilizer, we choose a homogeneous Riemannian metric g' on X in this case. If $X=\mathcal{N}-\{0\}\approx S^{2n}\times\mathbb{R}^+$, then $\mathrm{Aut}_{CR}(X)=\mathrm{U}(n)\times\mathbb{R}^+$ which acts properly on X (but not transitive). We choose a left invariant Riemannian metric g' on X so that **(i)** $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))\subset\mathrm{U}(n)\times\mathbb{R}^+\subset\mathrm{Iso}(X,g')$. If $X=S^{2n+1}-S^{2m-1}$, it follows that **(ii)** $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))\subset\mathrm{Psh}(X)$. When X is $\mathcal{N}$, by the action of $\mathrm{U}(n)\times\mathbb{R}^+\subset\mathrm{Aut}_{CR}(\mathcal{N})$ (cf. §2.1), $\mathbb{R}^+$ does not belong to the centralizer $\mathbf{C}_{\mathrm{Aut}_{CR}(X)}(\rho(\mathbb{R}))$ so that **(iii)** $\mathbf{C}_{\mathrm{Aut}_{CR}(\mathcal{W})}(\rho(\mathbb{R}))\subset\mathrm{Psh}(X)$.

Proposition 5.1. *Let $(\tilde{M},g)$ be a simply connected Bochner flat Kähler manifold. If $\tilde{M}$ is complete with respect to g, then the non-closed holonomy case does not occur.*

Proof. Suppose that there exists an isometric immersion $D:\tilde{M}\longrightarrow\mathcal{W}/\rho(\triangle)$ such that $D^*\hat{g}=g$ by Theorem 1.3. As g is complete, $D:\tilde{M}\longrightarrow\mathcal{W}/\rho(\triangle)$ is a covering map and so $(\mathcal{W}/\rho(\triangle),\hat{g})$ is complete. Since $\rho(\triangle)$ acts properly on $\mathcal{W}$, $(\mathcal{W},\bar{g})$ is also complete. As $\rho(\triangle)$ cannot act properly on the entire space $(X-S)^0$, $\partial\mathcal{W}=\bar{\mathcal{W}}-\mathcal{W}$ is not empty in $(X-S)^0$. Let x be a point of $\partial\mathcal{W}$. Choose a sequence $\{x_i\}\in\mathcal{W}$ such that $\lim x_i=x\in\partial W$. As $\bar{g}$ is defined on $(X-S)^0$, the sequence $\{x_i\}$ becomes Cauchy in $(\mathcal{W},\bar{g})$. The completeness implies that $x\in\mathcal{W}$, a contradiction. □

Suppose that $X=\mathcal{N}$ or $\mathcal{N}-\{0\}\approx S^{2n}\times\mathbb{R}^+$. When the holonomy $\rho(\mathbb{R})$ is closed, it acts properly and freely on X (cf. §1.3). In particular, when $\mathcal{W}$ is complete, the proof of Proposition 5.1 implies that the holonomy $\rho(\mathbb{R})$ is closed. Moreover when $X = S^{2n+1} - S^{2m-1}$, $m = 0, \cdots, n$, we have

Lemma 5.2. *Let* $\mathcal{W} = (S^{2n+1} - S^{2m-1} - E - S)^0$. *If* $(\mathcal{W}/\rho(\mathbb{R}), \hat{g})$ *is complete, then* $E = \emptyset$, *i.e.* $\rho(\mathbb{R}) = S^1$ *acts freely on* $S^{2n+1} - S^{2m-1}$.

Proof. Note that $\rho(\mathbb{R})$ is closed, $\rho(\mathbb{R}) = S^1$. Recall from §5.1 that there is an equivariant Riemannian submersion: $(\mathcal{W}, \bar{g}) \stackrel{\nu}{\longrightarrow} (\mathcal{W}/S^1, \hat{g})$ where $\bar{g}$ is defined on $(S^{2n+1} - S^{2m-1} - S)^0 \supset \mathcal{W}$. As $(\mathcal{W}/S^1, \hat{g})$ is complete, $(\mathcal{W}, \bar{g})$ is also complete. It follows from Proposition 4.1 that the S^1-action has the form $\rho(t) = (1, e^{it\cdot a_1}, \cdots, e^{it\cdot a_{n-m+1}}) \in P(\mathcal{Z}\mathrm{U}(m,1) \times T^{(n-m+1)})$ for which $a_1, \cdots, a_{n-m+1}$ $(m = 0, \cdots, n)$ are all rational numbers in this case. Moreover we have from (4.4), (4.3) that $(*)$ $\xi = \displaystyle\sum_{j=1}^{n-m+1} a_j(x_j\frac{d}{dy_j} - y_j\frac{d}{dx_j})$ and $(**)$ $\omega_{\mathcal{S}}(\xi) = \displaystyle\sum_{j=1}^{n-m+1} a_j|z_j|^2$. We can assume that all $a_j \in \mathbb{Z}$ up to change of parameter. First we show that each integer $a_j = \pm 1$ after normalized. Since $2 \leq n - m + 1$ by the condition $0 \leq m \leq n - 1$, for instance suppose that a_1 $(\neq \pm 1)$ is distinct from one of the rest of others a_j. Then the point $x = (z_1, 0 \cdots, 0) \in S^{2n+1} - S^{2m-1}$ belongs to E because $S^1_x = \dfrac{1}{a_1}2\mathbb{Z}/2\mathbb{Z} = \mathbb{Z}/a_1\mathbb{Z}$ is a nontrivial cyclic group. By $(**)$, $\omega_{\mathcal{S}}(\xi) = a_1|z_1|^2 = a_1 \neq 0$, hence $x \in (S^{2n+1} - S^{2m-1} - S)^0$. Choose a sequence $\{x_i\} \in \mathcal{W} = (S^{2n+1} - S^{2m-1} - E - S)^0$ such that $\lim x_i = x$. As $\bar{g}$ is defined on $(S^{2n+1} - S^{2m-1} - S)^0$, the sequence $\{x_i\}$ is Cauchy in $\mathcal{W}$ so that the completeness shows that $x = \lim x_i \in \mathcal{W}$, and so $x \notin E$, a contradiction. Therefore, all $|a_j|$ $(j = 1, \cdots n - m + 1)$ are equal. Normalized, we have that $a_j = \pm 1$, i.e. $E = \emptyset$. In particular $\rho(\mathbb{R}) = S^1$ acts properly and freely on $S^{2n+1} - S^{2m-1}$. □

Remark 5.3. *From* $(**)$, $\omega(\xi) > 0$ *on* $X = S^{2n+1} - S^{2m-1}$ *if and only if all* $a_j = 1$, *i.e.* $\rho(\mathbb{R})$ *belongs to the center* $S^1 = \mathcal{Z}\mathrm{U}(n-m+1) = P(\mathcal{Z}\mathrm{U}(m,1) \times \mathcal{Z}\mathrm{U}(n-m+1))$.

Let X be one of the domains $\mathcal{N}$, $\mathcal{N} - \{0\}$, $S^{2n+1} - S^{2m-1}$ or S^{2n+1} as before.

Corollary 5.4. *If* $(\mathcal{W}/\rho(\mathbb{R}), \hat{g})$ *is complete, then* $\rho(\mathbb{R})$ *acts freely on* X.

Note that the principal bundle $\rho(\mathbb{R}) \to X \longrightarrow X/\rho(\mathbb{R})$ which restricts the Riemannian submersion (cf. (1.3)): $\rho(\mathbb{R}) \to (\mathcal{W}, g_{\mathcal{W}}) \stackrel{\nu}{\longrightarrow} (\mathcal{W}/\rho(\mathbb{R}), \hat{g})$ where $g_{\mathcal{W}} = \eta \cdot \eta + d\eta(J\ ,\ )$. We need to prove the following.

Proposition 5.5. *If* $\hat{g}$ *is complete on* $\mathcal{W}/\rho(\mathbb{R})$, *then* $X = \mathcal{W}$, *that is* $S = \emptyset$.

Proof. Let $\hat{d}$ be the distance function induced from $\hat{g}$. For a curve σ between x and y in $\mathcal{W}$ such that $\dot{\sigma}(t) \in \mathrm{Null}\,\omega$, it follows that $d\eta(J\dot{\sigma}(t), \dot{\sigma}(t)) = g_{\mathcal{W}}(\dot{\sigma}(t), \dot{\sigma}(t)) = \hat{g}(\nu_*\dot{\sigma}(t), \nu_*\dot{\sigma}(t))$. We see that (cf. (3.5))

$$\hat{d}(\nu(x), \nu(y)) \leq d_\eta(x, y) \quad (x, y \in \mathcal{W}). \tag{5.1}$$

Let $B^\eta(p, r) = \{x \in \mathcal{W} \mid d_\eta(p, x) \leq r\}$ be the closed metric ball of radius r about a point p in $\mathcal{W} = (X - S)^0$. The space X is the domain of S^{2n+1} with boundary ∂X $(= \{\emptyset\}, \{\infty\}, \{0, \infty\}$, or $S^{2m-1})$. Note that ∂X is the fixed point subset of the group $\rho(\mathbb{R})$. Take the closure $\overline{B^\eta(p, r)}$ of $B^\eta(p, r)$ in S^{2n+1}. Obviously it is

compact. We show that $\overline{B^\eta(p,r)}$ still lies in $\mathcal{W}$. By the definition, $B^\eta(p,r)$ lies outside the fixed point set $\partial X \neq \emptyset$. As $\rho(\mathbb{R})$ acts freely and (properly) on X, there exists an element $\rho(t) \in \rho(\mathbb{R})$ which separates $B^\eta(p,r)$ far from $\partial X \neq \emptyset$. In particular, $\partial X \cap \overline{\rho(t)\cdot B^\eta(p,r)} = \emptyset$. Noting that $\rho(t)$ is a homeomorphism of S^{2n+1}, we conclude that $\partial X \cap \overline{B^\eta(p,r)} = \emptyset$, i.e. $\overline{B^\eta(p,r)} \subset X$.
Let $\{x_i\}$ be a sequence of points of $B^\eta(p,r)$ such that $\lim_{i\to\infty} x_i = x \in \overline{B^\eta(p,r)}\backslash B^\eta(p,r)$. From (5.1), $\hat{d}(\nu(p),\nu(x_i)) \leq r$, it follows that $\{\nu(x_i)\} \in \hat{B}(\nu(p),r)$, which is the closed metric ball in $\mathcal{W}/\rho(\mathbb{R})$. If $\hat{g}$ is complete on $\mathcal{W}/\rho(\mathbb{R})$, then $\hat{B}(\nu(p),r)$ is compact (cf. [21]). There is a point $\hat{q} \in \mathcal{W}/\rho(\mathbb{R})$ such that $\lim_{i\to\infty} \nu(x_i) = \hat{q}$. Choose $q \in \mathcal{W}$ with $\nu(q) = \hat{q}$. Using a local section of the above principal bundle : $\rho(\mathbb{R})\to X \xrightarrow{\nu} X/\rho(\mathbb{R})$, we can find a compact subset K in X such that $\nu : K\to\nu(K)$ is diffeomorphic and $\nu(K)$ is a (compact) neighborhood of $\hat{q}$. For sufficiently large i, $\nu(x_i) \in \nu(K)$. Thus there exists a sequence $\{t_i\} \in \mathbb{R}$ such that $\rho(t_i)\cdot x_i \in K$ with $\lim_{i\to\infty} \rho(t_i)\cdot x_i = q$. Since $\rho(\mathbb{R})$ acts properly on X with $\lim_{i\to\infty} x_i = x$, the sequence $\{\rho(t_i)\}$ converges to an element $\rho(t) \in \rho(\mathbb{R})$. Then, $q = \lim_{i\to\infty} \rho(t_i)\cdot x_i = \rho(t)\cdot x$. If we note that $\mathcal{W}$ is invariant under $\rho(\mathbb{R})$, then $x \in \mathcal{W}$. Therefore, $\overline{B^\eta(p,r)} \subset B^\eta(p,r)$, which implies that every closed metric ball of $(\mathcal{W}, d_\eta)$ is compact.

Suppose that $S \neq \emptyset$. Choose a point $p \in \mathcal{W}$ and a sequence $\{r_i\}$ with $r_i < r_{i+1}$, $\lim_{i\to\infty} r_i = \infty$. By the above fact, each $B^\eta(p,r_i)$ is compact. Then $f(x) = \dfrac{1}{\omega(\xi_x)}$ has a minimum $m_i > 0$ on $B^\eta(p,r_i)$. As $B^\eta(p,r_i) \subset B^\eta(p,r_{i+1})$, it follows that $m_i \geq m_{i+1} \geq \cdots > 0$. We can assume that $\lim_{i\to\infty} m_i = m \geq 0$. From (3.5), $d\eta(J\dot{\sigma}(t),\dot{\sigma}(t)) = f(\sigma(t))\cdot d\omega(J\dot{\sigma}(t),\dot{\sigma}(t)) \geq m_i\cdot d\omega(J\dot{\sigma}(t),\dot{\sigma}(t))$ on $B^\eta(p,r_i)$. By the definition, we obtain that

$$d_\eta(p,x) \geq m_i\cdot d_\omega(p,x) \quad (x \in B^\eta(p,r_i)). \tag{5.2}$$

In particular, $B^\eta(p,r_i) \subset B^\omega(p,\dfrac{r_i}{m_i})$. Noting $B^\eta(p,r_i) \subset \mathcal{W} = (X-S)^0$, if $r_i\to\infty$, then the boundary of $B^\eta(p,r_i)$ approaches to the set S, and thus the boundary of $B^\omega(p,\dfrac{r_i}{m_i})$ also approaches to S. If we note that d_ω has been defined on the whole space X, i.e. $d_\omega(p,S) < \infty$, then the sequence of radii $\{\dfrac{r_i}{m_i}\}$ converges to $d_\omega(p,S)$. Then,

$$r_i = m_i\cdot\frac{r_i}{m_i}\longrightarrow m\cdot d_\omega(p,S) < \infty,$$

which contradicts the choice of $\{r_i\}$. Therefore, if $\hat{g}$ is complete, then $S = \emptyset$. □

Remark 5.6. *By this proposition, the completeness imposes a restraint that* $(*)$ $\omega(\xi) > 0$ *on* X. *By Remark 5.3, the holonomy* $\rho(\mathbb{R})$ *turns out to be the center* $S^1 = \mathcal{Z}\mathrm{U}(n-m+1)$.

Suppose that $(\mathcal{H}, \mathcal{W}/\rho(\mathbb{R}), \hat{g}, \hat{J})$ is complete. If $\rho(\mathbb{R})$ is a closed subgroup isomorphic to $\mathbb{R}$, then it occurs as (1) Complex euclidean geometry $(\mathbb{C}^n \rtimes \mathrm{U}(n), \mathbb{C}^n, g_{\mathbb{C}}, J_{\mathbb{C}})$, (2) Intransitive Kähler geometry $(\mathbb{C}^{n-k} \rtimes \mathrm{U}(n-k)) \times \mathrm{U}(\ell_1,\cdots,\ell_m), \mathbb{C}^n, \hat{g}_a, J_{\mathbb{C}})$, $k \geq 1$, by Theorem 3.8. For other cases of $\rho(\mathbb{R}) = \mathbb{R}$, note that $S \neq \emptyset$. If $\rho(\mathbb{R}) = S^1$, it occurs as (3) Product of complex hyperbolic and projective geometry

$(\mathrm{PU}(m,1)\times \mathrm{PU}(n-m+1), \mathbb{H}_{\mathbb{C}}^m\times \mathbb{CP}^{n-m}, g_{\mathbb{H}}\times g_{\mathbb{CP}}, J_{\mathbb{H}}\times J_{\mathbb{CP}})$, $m=0,\cdots,n$ by Remark 5.3, Theorem 4.3 and Theorem 4.2. The following result was obtained by Bryant [3]. We learned that the original proof is based on the Lie's Third Theorem combined with the Abreu-Guillemin formalism for toric Kähler metrics.

Theorem 5.7. *The complete Bochner flat Kähler geometry* $(\mathcal{H}, X/\rho(\mathbb{R}))$ *consists of the following:*

(1) $(\mathbb{C}^n\rtimes \mathrm{U}(n), \mathbb{C}^n, g_{\mathbb{C}}, J_{\mathbb{C}})$.
(2) $(\mathbb{C}^{n-k}\rtimes \mathrm{U}(n-k)\times \mathrm{U}(\ell_1,\cdots,\ell_m), \mathbb{C}^n, \hat{g}_a, J_{\mathbb{C}})$, $k\geq 1$.
(3) $(\mathrm{PU}(m,1)\times \mathrm{PU}(n-m+1), \mathbb{H}_{\mathbb{C}}^m\times \mathbb{CP}^{n-m}, g_{\mathbb{H}}\times g_{\mathbb{CP}}, J_{\mathbb{H}}\times J_{\mathbb{CP}})$, $m=0,\cdots,n$.

If M is a complete Bochner flat Kähler manifold, then M is holomorphically isometric to the orbit space of the model manifold $X/\rho(\mathbb{R})$ by a discrete subgroup $\Gamma\subset\mathcal{H}$ acting freely on $X/\rho(\mathbb{R})$. The compact manifold is only the complex euclidean space form $\mathbb{C}^n/\Gamma$ or the locally product space $\mathbb{H}_{\mathbb{C}}^m\underset{\Gamma}{\times}\mathbb{CP}^{n-m}$, $m=0,\cdots,n$.

Proof. By completeness and simply connectedness of $Y=X/\rho(\mathbb{R})$, $D:\tilde{M}\to Y$ is a diffeomorphism by using Theorem 1.3. Then D induces a holomorphic isometry of M onto $Y/\Psi(\pi_1(M))$ where $\Psi(\pi_1(M))\subset\mathcal{H}$. The compact case for $Y/\Psi(\pi_1(M))$ follows from Proposition 3.9, Theorem 4.3 and Remark 5.6. □

6. Nonclosed Holonomy group

Given a collection of Kähler manifolds $\{U_\alpha,\Omega_\alpha\}_{\alpha\in\Lambda}$ on M, we have constructed the (trivial) principal bundle $(M_\alpha,p_\alpha,U_\alpha)$ of the spherical CR-manifold for each α. As dev_α is an immersion, there exists a maximal open interval $I_\alpha=(-\varepsilon_\alpha,\varepsilon_\alpha)$ of $\mathbb{R}$ such that

$$\mathrm{dev}_\alpha: I_\alpha\times U_\alpha\to \mathrm{dev}_\alpha(I_\alpha\times U_\alpha)=\rho_\alpha(I_\alpha)\cdot \mathrm{dev}_\alpha(U_\alpha)$$

is equivariantly diffeomorphic. Let ξ be the vector field induced by the holonomy group $\rho_\alpha(\mathbb{R})$ as before. Notice that $\omega_{\mathcal{S}}(\xi)>0$ on $\mathrm{dev}_\alpha(I_\alpha\times U_\alpha)$. Recall from Proposition 4.1 that the $(n-m+1)$-torus T^{n-m+1} acts properly on $S^{2n+1}-S^{2m-1}$. In our case, $\rho_\alpha(\mathbb{R})$ is not closed in T^{n-m+1}. For a sufficiently small open interval $\triangle=(-\varepsilon,\varepsilon)\subset I_\alpha$, the image $\rho_\alpha(\triangle)$ is viewed as a local additive group of T^{n-m+1} acting properly and freely on a neighborhood containing $\mathrm{dev}_\alpha(I_\alpha\times U_\alpha)$.

Definition 6.1. *Let W_α be a maximal connected domain of $S^{2n+1}-S^{2m-1}$ containing* $\mathrm{dev}_\alpha(I_\alpha\times U_\alpha)$ *such that:*

(1) *Every local 1-parameter group $\rho_\alpha(\triangle)$ for a small interval $\triangle=(-\varepsilon,\varepsilon)\subset I_\alpha$ acts properly and freely.*
(2) $\omega_{\mathcal{S}}(\xi)>0$.

For such intervals $\triangle'\subset\triangle$ of I_α, note that the orbit spaces coincide; $W_\alpha/\rho_\alpha(\triangle')=W_\alpha/\rho_\alpha(\triangle)$. The map dev_α induces an immersion $\widehat{\mathrm{dev}}_\alpha: U_\alpha\longrightarrow W_\alpha/\rho_\alpha(\triangle)$ with the commutative diagram:

$$
\begin{array}{ccc}
I_\alpha \times U_\alpha & \xrightarrow{\mathrm{dev}_\alpha} & W_\alpha \\
{\scriptstyle p_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \nu_\alpha} \\
U_\alpha & \xrightarrow{\widehat{\mathrm{dev}}_\alpha} & W_\alpha/\rho_\alpha(\triangle).
\end{array} \tag{6.1}
$$

Fix some α and then set $W_\alpha = W,\ \rho_\alpha(\mathbb{R}) = \rho$ once for all.

Remark 6.2. *In §4.2, if $\rho(\mathbb{R})$ is S^1, $\mathcal{W} = (S^{2n+1} - S^{2m-1} - E - S)^0$ is the maximal domain with respect to that S^1 acts freely and $\omega_{\mathcal{S}}(\xi) > 0$. In view of the above definition 6.1, we justify the definition of maximal domain by showing that W and $\mathcal{W}$ coincide when $\rho(\mathbb{R}) = S^1$. Let $\rho(s) \in \rho(\mathbb{R}) = S^1$ $(s \in \mathbb{R})$. For sufficiently large $k \in \mathbb{Z}$, $\dfrac{s}{k} \in \triangle$. If $x \in (S^{2n+1} - S^{2m-1} - S)^0$, then $\rho(s) \cdot x = \rho(\dfrac{s}{k})^k \cdot x = \rho(\dfrac{s}{k}) \cdot \rho(\dfrac{s}{k}) \cdot \cdots \cdot \rho(\dfrac{s}{k}) \cdot x$, so that the quotients coincide:*

$$
(S^{2n+1} - S^{2m-1} - S)^0/\rho(\triangle) = (S^{2n+1} - S^{2m-1} - S)^0/\rho(\mathbb{R}). \tag{6.2}
$$

However we show that

Lemma 6.3. *$\rho(\triangle)$ does not act freely on $(S^{2n+1} - S^{2m-1} - S)^0$ whenever $E \neq \emptyset$ for the set E of exceptional orbits of S^1.*

Proof. Note that if $\rho(\triangle)$ acts freely, then the orbit space $(S^{2n+1} - S^{2m-1} - S)^0/\rho(\triangle)$ will be a smooth manifold. Suppose that $E \neq \emptyset$. Let $x \in E$ so that $S^1_x = \langle\gamma\rangle$ is a finite cyclic subgroup. If we recall the *slice theorem of compact Lie groups* (cf. [2]), then there exists a S^1_x-invariant slice V in a tubular neighborhood of the orbit $S^1 \cdot x$ in $(S^{2n+1} - S^{2m-1} - S)^0$ for which $\langle\gamma\rangle$ *acts linearly on V centered at the origin x*. Noting (6.2), let $[x] \in (S^{2n+1} - S^{2m-1} - S)^0/\rho(\triangle) = (S^{2n+1} - S^{2m-1} - S)^0/\rho(\mathbb{R})$, so the neighborhood around the point $[x]$ in the quotient space becomes $V/\langle\gamma\rangle$, which does not admit a smooth structure by the above remark of the slice theorem at least $2n > 2$. Hence $\rho(\triangle)$ cannot act freely on $(S^{2n+1} - S^{2m-1} - S)^0$. □

In particular, $\rho(\triangle)$ cannot act freely on the neighborhood around a point of E. As $\omega_{\mathcal{S}}(\xi) > 0$ on W, $W \subset (S^{2n+1} - S^{2m-1} - S)^0$. By Definition 6.1, W is a maximal domain on which $\rho(\triangle)$ acts freely, so W misses E. Hence, $W \subset (S^{2n+1} - S^{2m-1} - E - S)^0$. As $\rho(\triangle)$ acts freely on $(S^{2n+1} - S^{2m-1} - E - S)^0$, by maximality, $W = (S^{2n+1} - S^{2m-1} - E - S)^0$.

We examine $W/\rho(\triangle)$. Especially we consider the case **(1)** of Proposition 4.1. Let $\rho(t) = (e^{\mathrm{i}t\cdot a_1}, \cdots, e^{\mathrm{i}t\cdot a_{n+1}})$ be the representation on S^{2n+1} from (4.1) of §4. Suppose that $a_{i_1}, \cdots, a_{i_k}$ $(1 \leq i_1 < \cdots < i_k \leq n+1)$ are mutually distinct, then $\mathbf{N}_{\mathrm{Aut}_{CR}(S^{2n+1})}(\rho(\mathbb{R})) = \mathrm{U}(\ell_1) \times \cdots \times \mathrm{U}(\ell_k)$, $\ell_1 + \cdots + \ell_k = n+1$ (cf. Proposition 4.1). As $\rho(\mathbb{R})$ is not closed in T^{n+1}, we may suppose that $a_1, \cdots, a_r$ are irrational for some $r \leq k$. Put $H = \mathrm{U}(\ell_{r+1}) \times \cdots \times \mathrm{U}(\ell_k)$ and

$$
K = \mathrm{U}(\ell_1) \times \cdots \times \mathrm{U}(\ell_k) = \mathrm{U}(\ell_1) \times \cdots \times \mathrm{U}(\ell_r) \times H.
$$

Let K_p be the stabilizer at a generic point p of $\mathrm{dev}_\alpha(I_\alpha \times U_\alpha)$. By the slice theorem, there exists a K_p-invariant slice V such that $\mathrm{dev}_\alpha(I_\alpha \times U_\alpha) \subset W \subset K \underset{K_p}{\times} V$ in S^{2n+1}. Since $\mathcal{Z}\mathrm{U}(\ell) = \{(e^{\mathrm{i}t\cdot a_\ell}, \cdots, e^{\mathrm{i}t\cdot a_\ell})\}$ is the center S^1 of $\mathrm{U}(\ell)$, we put $\mathrm{PU}(\ell) = \mathrm{U}(\ell)/S^1$ as before. Let $A(a_j, \varepsilon) = \{e^{\mathrm{i}t\cdot a_j} \mid |t| < \varepsilon\}$. Then

$$\rho(I_\alpha) \subset (A(a_1, \varepsilon) \cdot \mathrm{PU}(\ell_1)) \times \cdots \times (A(a_m, \varepsilon) \cdot \mathrm{PU}(\ell_r)) \times H.$$

In particular, we can derive that

$$\mathbf{N}_{\mathrm{Aut}_{CR}(W)}(\rho(\triangle)) = (A(a_1, \varepsilon) \cdot \mathrm{PU}(\ell_1)) \times \cdots \times (A(a_r, \varepsilon) \cdot \mathrm{PU}(\ell_r)) \times H$$
$$W = A(a_1, \varepsilon) \cdot \mathrm{PU}(\ell_1)) \times \cdots \times (A(a_r, \varepsilon) \cdot \mathrm{PU}(\ell_r)) \times H \underset{K_p}{\times} V.$$

The similar result holds for the case that $S^{2n+1} - S^{2m-1}$ of (**2**) of Proposition 4.1. We have proved that (cf. Proposition 5.1)

Theorem 6.4. *Suppose that $\rho(\mathbb{R})$ is not closed in $T^{n-m+1} \subset \mathrm{U}(n+1)$, $m = 0, \cdots, n-1$. Then the Bochner flat Kähler geometry $(\mathcal{H}, W/\rho(\triangle), \hat{g}, \hat{J})$ is an incomplete local geometry consisting of*

$$(\mathrm{PU}(\ell_1) \times \cdots \times \mathrm{PU}(\ell_r) \times H,\ \mathrm{PU}(\ell_1) \times \cdots \times \mathrm{PU}(\ell_r) \times H \underset{K_p}{\times} V, \hat{g}, \hat{J}),$$

$\ell_1 + \cdots + \ell_k = n + 1$, *or*

$$(\mathrm{PU}(m, 1) \times \mathrm{PU}(\ell_1) \times \cdots \times \mathrm{PU}(\ell_r) \times H,$$
$$\mathbb{H}^m_{\mathbb{C}} \times \mathrm{PU}(\ell_1) \times \cdots \times \mathrm{PU}(\ell_r) \times H \underset{K_p}{\times} V),$$

$\ell_1 + \cdots + \ell_k = n - m + 1$.

7. Application to locally conformal Kähler manifolds

When Hermitian manifolds are taken into account in the framework of conformal geometry, it is conceivable whether Bochner curvature tensor would be a conformal invariant of a Kähler manifold among the Hermitian manifolds. But it was not true. However, Tricerri and Vanhecke [17] have found a *Bochner component* on Hermitian manifolds. The curvature tensor of this component coincides with the original Bochner curvature tensor when a Hermitian manifold is Kähler. They observed that the Bochner curvature tensor of this component is a *conformal invariant* on Hermitian metrics on a Hermitian manifold. Here the complex structure is fixed. A locally conformal Kähler manifold is a Hermitian manifold (M, g, J) whose Hermitian metric g is locally conformal to a Kähler metric. *That is*, the restriction of g to each neighborhood $U(x)$ is conformal to a Kähler metric locally defined on $U(x)$. As a consequence, when a Hermitian manifold is a locally conformal Kähler manifold, the Bochner curvature tensor of Bochner component has the same formula as the (original) Bochner curvature tensor. A *Bochner flat locally conformal Kähler manifold* is defined to be a locally conformal Kähler manifold whose Bochner curvature tensor vanishes identically.

There is no way to find a Kähler metric within the conformal class of a locally conformal Kähler metric, however there exists a canonical Kähler metric on the universal covering (cf. [6]). Let (M, g) be a locally conformal Kähler metric with fundamental 2-form Ω. By definition, Ω satisfies that $d\Omega = \theta \wedge \Omega$ for some closed

1-form θ. Let $\tilde{M}$ be the universal covering space and $p : \tilde{M} \to M$ the covering projection. For the lift $\tilde{\theta}$ of θ, choose a function $\tau : \tilde{M} \to \mathbb{R}$ such that $\tilde{\theta} = d\tau$. We can define a 2-form $\bar{\Omega}$ on $\tilde{M}$ unique up to a constant multiple:

$$\bar{\Omega} = e^{-\tau} \cdot p^*\Omega. \tag{7.1}$$

Then it is easy to see that $d\bar{\Omega} = 0$. The form $\bar{\Omega}$ (respectively $\bar{g}(X, Y) = \bar{\Omega}(\bar{J}X, Y)$) is called the canonical Kähler form (respectively canonical Kähler metric) on $(\tilde{M}, \bar{J})$ where $\bar{J}$ is a lift of J. If g is a Bochner flat locally conformal Kähler metric on M, then so is its lift to $\tilde{M}$. By (7.1), $\bar{g}$ is a Bochner flat Kähler metric on $\tilde{M}$. If the fundamemtal group $\pi_1(M)$ acts as Kähler isometries of $\tilde{M}$ with respect to $\bar{g}$, we observe that $\bar{g}$ induces also a Kähler metric g' on M for which g' is globally conformal to g by using the equality (7.1). As a consequence, a locally conformal Kähler but not globally conformal Kähler manifold (M, g) is obtained as the orbit space of $(\tilde{M}, \bar{g})$ by nontrivial holomorphic homothetic transformations of $\pi_1(M)$. (Note that any conformal holomorphic transformation of the Kähler manifold must be homothetic provided that $\dim M > 2$.) Fix a simply connected Kähler manifold $(\tilde{M}, \bar{g}, \bar{J})$. Denote by $\mathrm{Hoth}(\tilde{M}, \bar{g})$ the group of all homothetic holomorphic transformations of $\tilde{M}$. In this setup, we shall mean a locally conformal Kähler manifold by the quotient $\tilde{M}/\Gamma$ where $\Gamma \subset \mathcal{G}$ acting properly discontinuously and freely. We don't treat a specific locally conformal Kähler metric but are interested in the conformal class of a locally conformal Kähler metric. Since there exist properly discontinuous free actions of nontrivial homothetic transformations, this fact causes the existence of a non-Kähler class of Bochner flat locally conformal Kähler manifolds. By a non-Kähler locally conformal Kähler manifold M we mean that M does not admits any Kähler structure. Given a Bochner flat locally conformal Kähler manifold (M, g), we have the simply connected Bochner flat Kähler manifold $(\tilde{M}, \bar{g})$. By Theorem 1.3, there exists a holomorphically isometric immersion $D : \tilde{M} \longrightarrow \mathcal{W}/\rho(\mathbb{R})$ (or $\mathcal{W}/\rho(\triangle)$). Let $\mathcal{G}$ be the group of all homothetic holomorphic transformations of $\mathcal{W}/\rho(\mathbb{R})$ (or $\mathcal{W}/\rho(\triangle)$). Using the properties of D of Theorem 1.3, the analytic continuation implies that the holonomy (continuous) homomorphism $\Psi : \mathrm{Iso}(\tilde{M}, \bar{g}, \bar{J}) \to \mathcal{H}$ extends to a continuous homomorphism $\Psi : \mathrm{Hoth}(\tilde{M}, \bar{g}) \to \mathcal{G}$. (We use the same notation Ψ.)

Suppose that there exists an element $h \in \mathcal{G}$ but not an isometry of $\mathcal{W}/\rho(\mathbb{R})$ such that $h^*\hat{\Omega} = \lambda \cdot \hat{\Omega}$ for some constant λ. As in the proof of Theorem 1.3 (cf. §1.6), if $h : U_\alpha \to U_\beta$ locally, then $\lambda \cdot \theta_\alpha - h^*\theta_\beta = d\chi$ for some smooth map $\chi : U_\alpha \to \mathbb{R}$. The map $\tilde{h} : \mathbb{R} \times U_\alpha \to \mathbb{R} \times U_\beta$ defined by $\tilde{h}(t, x) = (\lambda \cdot t + \chi(x), h(x))$ is a CR-diffeomorphim satisfying that $\tilde{h}^*\omega_\beta = \lambda \cdot \omega_\alpha$. After applying the developing map, we obtain a CR-transformation $\bar{h}$ of $\mathcal{W}$ onto itself. From the above formula, $\bar{h}$ is equivariant with respect to the holonomy map, i.e. $\bar{h}(\rho(t)x) = \rho(\lambda \cdot t)\bar{h}(x)$, $x \in \mathcal{W}$. Hence $\bar{h} \cdot \rho(t) \cdot \bar{h}^{-1} = \rho(\lambda \cdot t)$, we obtain that $\bar{h} \in \mathbf{N}_{\mathrm{Aut}_{CR}(W)}(\rho(\mathbb{R}))) \backslash \mathbf{C}_{\mathrm{Aut}_{CR}(W)}(\rho(\mathbb{R}))$. (Similarly for $\mathbf{N}_{\mathrm{Aut}_{CR}(W)}(\rho(\triangle))$.) Since ξ is the vector field which generates $\rho(\mathbb{R})$, note also that $h_*\xi = \lambda \cdot \xi$. Therefore it suffices to check which Bochner flat Kähler geometry has the nontrivial normalizer $\mathbf{N}_{\mathrm{Aut}_{CR}(W)}(\rho(\mathbb{R}))$ but not the centralizer. Then it turns out that they are the cases **(i)**, **(iii)** when the holonomy $\rho(\mathbb{R}) = \mathbb{R}$ is closed with one fixed point. In fact, noting **(II)** of Corollary 3.2, if $h = ((0, 0), \lambda \cdot \mathrm{I}) \in$

$\mathbb{R}^+$, then $h^*g_{\mathbb{C}} = \lambda^2 \cdot g_{\mathbb{C}}$ and $h^*\hat{g} = \lambda \cdot \hat{g}$ respectively. It follows that (cf. Theorem 3.8, see §3 also.)

Theorem 7.1. *Suppose that M is a non-Kähler Bochner flat locally conformal Kähler manifold. Then it can be uniformized with respect to the geometries* **(i)'** $(\mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+), \mathbb{C}^n, g_{\mathbb{C}}, J_{\mathbb{C}})$ *or* **(iii)'** $(\mathcal{M} \rtimes (\mathrm{U}(n-1) \times \mathbb{R}^+), \mathcal{M} \times \mathbb{R}^-, \hat{g}, \hat{J})$. *More precisely, there exist a holomorphically isometric immersion $D : \tilde{M} \longrightarrow \mathbb{C}^n$ (respectively $\mathcal{M} \times \mathbb{R}^-$) and a holonomy homomorphism $\Psi : \mathrm{Hoth}(\tilde{M}, \bar{g}) \to \mathcal{G}$ such that $D^*\hat{g} = \bar{g}$ and $D \circ \gamma = \Psi(\gamma) \circ D$, $\forall\, \gamma \in \mathrm{Hoth}(\tilde{M}, \bar{g})$.*

Corollary 7.2. *If M is a non-Kähler Bochner flat locally conformal Kähler manifold, then the canonical Kähler metric $\bar{g}$ is incomplete on $\tilde{M}$.*

Proof. Let $Y = \mathbb{C}^n$ or $\mathcal{M} \times \mathbb{R}^-$ which is contractible. If $\bar{g}$ is complete, then the immersion $D : \tilde{M} \longrightarrow Y$ is an isometry. Since Y is complete, Y must be $\mathbb{C}^n$ by Proposition 5.5. Put $\Gamma = \Psi(\pi_1(M)) \subset \mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+)$ which acts freely on $\mathbb{C}^n$. As M is non-Kähler, there exists an element $\gamma \in \Gamma$ whose $\mathbb{R}^+$-summand is nontrivial and so the affine transformation γ has a fixed point in $\mathbb{C}^n$, a contradiction. (See the proof of Proposition 3.1.) □

Recall that a manifold M uniformized with respect to the geometry $(\mathbb{C}^n \rtimes (\mathrm{U}(n) \times \mathbb{R}^+), \mathbb{C}^n)$ is called an affine similarity manifold. By the result of Fried [4], if M is a compact similarity manifold, then M is diffeomorphic to the complex euclidean space form $\mathbb{C}^n/\Gamma$ $(\Gamma \subset \mathbb{C}^n \rtimes \mathrm{U}(n))$ or a quotient of the Hopf manifold $S^{2n-1} \times S^1$ by a finite group $F \subset \mathrm{U}(n) \times S^1$. Here the developing image for the Hopf manifold is $\mathbb{C}^n - \{0\} = S^{2n-1} \times \mathbb{R}^+$. In this case, the globally conformal Kähler metric $\bar{g}_H$ of the universal cover $S^{2n-1} \times \mathbb{R}^+$ is obtained by the reverse procedure of (7.1):

$$\bar{g}_H = e^t \cdot g_{\mathbb{C}}. \tag{7.2}$$

Here e^t is the coordinate of $\mathbb{R}^+$. Then the group $\mathrm{U}(n) \times \mathbb{R}^+$ acts as isometries of $\bar{g}_H$. Taking an infinite cyclic subgroup $\mathbb{Z}^+$ from $\mathrm{U}(n) \times \mathbb{R}^+$ such that $S^{2n-1} \underset{\mathbb{Z}^+}{\times} \mathbb{R}^+$ is diffeomorphic to the Hopf manifold $S^{2n-1} \times S^1$, $\bar{g}_H$ induces a locally conformal Kähler metric g_H on $S^{2n-1} \times S^1$. We use the same metric g_H on its finite quotient of $S^{2n-1} \times S^1$

Corollary 7.3. *Suppose that (M, g) is a compact non-Kähler Bochner flat locally conformal Kähler manifold. Then (M, g) is holomorphically locally conformal to a virtually Hopf manifold $(S^{2n-1} \underset{F}{\times} S^1, g_H)$.*

Proof. There are two possibilities **(i)'**,**(iii)'** from Theorem 7.1. Case **(iii)'** cannot occur by Proposition 3.10. Let $D : (\tilde{M}, \bar{g}) \longrightarrow (\mathbb{C}^n, g_{\mathbb{C}})$ be a holomorphic isometric immersion. By the result of [4], if M is compact, the developing image misses the unique point which we may assume the origin of $\mathbb{C}^n$. So the developing pair reduces to $(\Psi, D) : (\pi_1(M), \tilde{M}) \longrightarrow (\mathrm{U}(n) \times \mathbb{R}^+, \mathbb{C}^n - \{0\})$. Then $D : (\tilde{M}, \bar{g}') \to (\mathbb{C}^n - \{0\}, \bar{g}_H)$ is an isometry with respect to the pullback metric $\bar{g}' = D^*\bar{g}_H$. As M is compact, D induces an isometry $\hat{D} : (M, g') \to (S^{2n-1} \underset{F}{\times} S^1, g_H)$. Using (7.2) and $\bar{g} = D^*g_{\mathbb{C}}$, $\bar{g}'$ is conformal to $\bar{g}$ on $\tilde{M}$. Hence g' is locally conformal to g on M. □

References

[1] S. Bochner, "Curvature and Betti numbers. II," *Ann. of Math.*, vol. 50, pp.77–93, 1949.
[2] G. Bredon, 'Introduction to compact transformation groups,' *Academic Press, New York*, 1972.
[3] R. Bryant, "Bochner-Kähler metrics," *Jour. of A.M.S.*, vol. 14(3), pp. 623–715, 2001.
[4] D. Fried, "Closed similarity manifolds," *Comment. Math. Helv.*, vol. 55, pp.576–582, 1980.
[5] S. S. Chern and J. Moser, "Real hypersurfaces in complex manifolds," *Acta Math.*, vol. 133, pp. 48–69, 1974.
[6] S. Dragomir and L. 'Ornea, Locally conformal Kähler geometry', Progr. Math. **155**, Birkhäuser 1998.
[7] W. Goldman, 'Complex hyperbolic geometry,' *Oxford Mathematical Monogrphs, 1999*.
[8] Y. Kamishima, "Standard pseudo-Hermitian structure and Seifert fibration on CR manifold," *Annals of Global Analysis and Geometry,* vol. 12 (1994) 261–289.
[9] Y. Kamishima, "Uniformization of Kähler manifolds with vanishing Bochner tensor," *Acta Mathematica,* vol. 172:2, pp. 299–308, 1994.
[10] Y. Kamishima and T. Tsuboi, "CR-structures on Seifert manifolds," *Invent. Math.*, vol. 104 (1991) 149–163.
[11] S. Kobayashi, 'Transformation Groups in Differential Geometry,' *Springer-Verlag, Ergebnisse Math.,* vol. 70, 1970.
[12] S. Kobayashi and K. Nomizu, 'Foundations of differential geometry I,II,' *Interscience John Wiley & Sons, New York*, 1969.
[13] R. Kulkarni, "On the principle of uniformization," *J. Diff. Geom.,* vol. 13, pp. 109–138, 1978.
[14] J. Mitchell, "On Carnot-Carathéodory metrics," *J. Diff. Geom.,* vol. 21, pp. 35–45, 1985.
[15] B. O'Neill, 'Semi-Riemannian geometry with application to relativity', *Academic press, New York*, 1983.
[16] M. S. Raghunathan, 'Discrete subgroups of Lie groups', *Springer Verlag*, 1972.
[17] F. Tricerri and L. Vanhecke, "Curvature tensors on almost Hermitian manifolds," *Trans. Amer. Math. Soc.*, vol. 267, pp. 365–398, 1981.
[18] S. Webster, "Pseudo-Hermitian structures on a real hypersurface," *J. Diff. Geom.*, vol. 13, pp. 25–41, 1978.
[19] S. Webster, "On the pseudo conformal geometry of a Kähler manifold," *Jour. Math. Z.*, vol. 157, pp. 265–270, 1977.
[20] H. Weyl, 'Classical groups, their invariants and representations,' *Princeton Univ. Press.*, 1946.
[21] J. Wolf, 'Spaces of constant curvature,' *McGraw-Hill, Inc.*, 1967.

Department of Mathematics, Tokyo Metropolitan University, Minami-Ohsawa 1-1, Hachioji, Tokyo 192-0397, Japan

E-mail address: kami@comp.metro-u.ac.jp