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THREE-DIMENSIONAL REAL-VALUED MAXWELL-BLOCH EQUATIONS WITH CONTROLS

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ABSTRACT. The three dimensional real-valued Maxwell-Bloch equations with controls are defined and some of their geometrical and dynamical properties are pointed out.

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1. Introduction

In the physics of self-induced transparency (see for details Allen and Eberly [1]) for the most lasers and the most atoms the so called two level lossless model is an excellent approximation and is quite adequate for an understanding of the basic physics behind many coherent transient phenomena. Self-induced transparency equations based upon this model are derived from the Maxwell-Schrödinger equations in Holm and Kovacic [6] by averaging over fast phases in the variational principle for the Maxwell-Schrödinger equations. More precisely the five-dimensional Maxwell-Bloch equations can be written on $\mathbb{C}^2 \times \mathbb{R}$ coordinatized by x, y, z in the following form:

$$(1.1) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = xz \\ \dot{z} = -\dfrac{1}{2}(\bar{x}y + x\bar{y}) \end{cases}$$

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where "$-''$ denotes the complex conjugation. Physically speaking the variables x, y and z represent the electric field, the polarization and the population inversion, respectively.

Fordy and Holm [5] discuss the phase space geometry of the solutions of the system (1.1) and show that it has three Hamiltonian structures, by using the Lax-pair representation of these equations. More recently David and Holm [3] discuss the phase-space geometry and the Hamilton-Poisson structures of the invariant subsystem of (1.1) obtained by restricting to real-valued x and y. The equations of motion (1.1) become the three-dimensional real-valued Maxwell-Bloch equations (or shorter in all that follows Maxwell-Bloch equations):

$$(1.2) \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 x_3 \\ \dot{x}_3 = -x_1 x_2 \end{cases}$$

where

$$(1.3) \qquad \begin{cases} x_1 = Re(x) \\ x_2 = Re(y) \\ x_3 = z \end{cases}$$

The aim of our paper is to present some geometrical and dynamical properties of the Maxwell-Bloch equations with a particular class of linear or quadratic control about $0x_2$ axis.

The paper is organized as follows. In section 2 we study the Maxwell-Bloch equations with a particular linear control about $0x_2$ axis. In section 3 we present following [9] the case of a particular quadratic control about $0x_2$ axis and finally in the last section we discuss the problem of controlling the homoclinic orbit of the Maxwell-Bloch equations with three independent controls via an idea of Bloch and Marsden [2].

2. Maxwell-Bloch equations with a linear control

2.1 The equations of motion and Poisson structure. The Maxwell-Bloch equations with a control about the $0x_2$ may be written as:

$$(2.1) \qquad \begin{cases} \dot{x}_1 = x_2 \\ x_2 = x_1 x_3 + u_2 \\ \dot{x}_3 = -x_1 x_2 \end{cases}$$

In all that follows we shall employ the linear feedback

$$(2.2) \qquad u_2 = \cdot - k x_1$$

where $k \in \mathbb{R}$ is the feedback gain parameter. We refer to the system (2.1), (2.2) as the controlled system.

Proposition 2.1. *For the controlled system* (2.1) (2.2) *the quantities*

$$H = \frac{1}{2}(x_2^2 + x_3^2) + \frac{k}{2}x_1^2 \tag{2.3}$$

and

$$C = x_3 + \frac{1}{2}x_1^2 \tag{2.4}$$

are constants of motion.

Proof. Indeed, we can write successively:

$$\begin{aligned}
\frac{dH}{dt} &= \frac{\partial H}{\partial x_1}\dot{x}_1 + \frac{\partial H}{\partial x_2}\dot{x}_2 + \frac{\partial H}{\partial x_3}\dot{x}_3 \\
&= kx_1x_2 + x_2(x_1x_3 - kx_1) + x_3(-x_1x_2) \\
&= 0; \\
\frac{dC}{dt} &= \frac{\partial C}{\partial x_1}\dot{x}_1 + \frac{\partial C}{\partial x_2}\dot{x}_2 + \frac{\partial C}{\partial x_3}\dot{x}_3 \\
&= x_1x_2 + (-x_1x_2) \\
&= 0
\end{aligned}$$

□

Further we have:

Theorem 2.1. *The controlled system* (2.1), (2.2) *is a Hamilton-Poisson mechanical system with the phase space* $P = \mathbb{R}^3$, *the Hamiltonian* H *given by* (2.3) *and with respect to the Poisson structure* $\{\cdot, \cdot\}$ *given by:*

$$\{f, g\} = \det \begin{bmatrix} x_1 & 0 & 1 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{bmatrix} \tag{2.5}$$

for each $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

Proof. One can easily check that

$$\dot{x}_i = \{x_i, H\}; i = 1, 2, 3$$

which gives the result. □

Remark 2.1. It is not hard to see that the function C given by (2.4) is a Casimir of our configuration, i.e.

$$\{C, f\} = 0$$

for each $f \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

2.2. Stability about the $0x_3$ axis. It is easy to see that $(0,0,M)$ is an equilibrium state of our system. Consider firstly the system linearized about $(0,0,M)$. Its eigenvalues are given by:

$$\lambda[\lambda^2 + M - k] = 0.$$

Hence for $k \in (-\infty, M)$ the equilibrium state $(0,0,M)$ is unstable, but for $k \in (M,\infty)$, we have two imaginary eigenvalues and one zero eigenvalue. Is the system stable? We shall prove that it is via the energy-Casimir method [7], [8].

Theorem 2.2. *The controlled system* (2.1), (2.2) *may be nonlinear stabilized about the equilibrium* $(x_1, x_2, x_3) = (0,0,M)$, *for* $k > M$.

Proof. Recall that the energy-Casimir method (see e.g. Holm, Marsden, Ratiu and Weinstein [7]) requires finding a constant of motion for the system, H, usually the energy, and a family of constants of motion $\mathcal{C}$, such that for some C, $H + C$ has a critical point at the equilibrium of interest. (Often the C's are taken to be Casimir-functions that commute with all other functions under the Poisson bracket). Then, in finite dimensions, definiteness of $\delta^2(H+C)$ at the critical point is sufficient to prove stability.

Now here we have the modified energy-Casimir function:

$$H + \varphi(C) = \frac{1}{2}[x_2^2 + x_3^2] + \frac{k}{2}x_1^2 + \varphi(x_3 + \frac{1}{2}x_1^2),$$

where φ is an arbitrary smooth function, and then

$$\delta(H + \varphi(C)) = kx_1\delta x_1 + x_2\delta x_2 + x_3\delta x_3 + \dot{\varphi}(\delta x_3 + x_1\delta x_1).$$

At the equilibrium of interest this equals zero if and only if

$$\dot{\varphi}(M) = -M.$$

Then

$$\delta^2(H + \varphi(C)) = k(\delta x_1)^2 + (\delta x_2)^2 + (\delta x_3)^2 + \ddot{\varphi}(\delta x_3 + x_1\delta x_1)^2 + \dot{\varphi}(\delta x_1)^2.$$

At the equilibrium of interest $(x_1, x_2, x_3) = (0,0,M)$ we have:

$$\delta^2(H + \varphi(C))(0,0M) = (k - M)(\delta x_1)^2 + (\delta x_2)^2 + (\delta x_3)^2 + \ddot{\varphi}(M)(\delta x_1)^2.$$

Hence for k sufficiently large such that $k > M$ and choosing φ such that

$$\ddot{\varphi}(M) \geq 0$$

the second variation is positive definite and we have nonlinear stability as desired. $\square$

2.3. Alternative Poisson structures. Let $\mathbb{R}^3_{cd}$ be the Euclidean space $\mathbb{R}^3$ with the Lie algebra bracket

$$[u,v]_{cd} = \begin{bmatrix} -d-ck & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -c \end{bmatrix} \cdot (u \times v) \tag{2.6}$$

where $c, d \in \mathbb{R}$, "$\cdot$" is the usual inner product on $\mathbb{R}^3$ and "$\times$" is the cross-product on $\mathbb{R}^3$. Then the minus-Lie-Poisson structure on $(\mathbb{R}^3_{cd})^* \simeq \mathbb{R}^3$ is given by the matrix:

$$[\{x_i, x_j\}^-_{cd}] = \begin{bmatrix} 0 & cx_3 & -cx_2 \\ -cx_3 & 0 & (d+ck)x_1 \\ cx_2 & -(d+ck)x_1 & 0 \end{bmatrix} \tag{2.7}$$

or equivalent

$$\{f,g\}^-_{cd} = \det \begin{bmatrix} (d+ck)x_1 & cx_2 & cx_3 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{bmatrix}$$

for each $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$, and we can prove:

Theorem 2.3. *The controlled system (2.1), (2.2) is a Hamilton-Poisson mechanical system with the phase space $P = (\mathbb{R}^3_{cd})^* \simeq \mathbb{R}^3$, the Hamiltonian H' given by:*

$$H' = \frac{a}{2}(x_2^2 + x_3^2) + \frac{ak}{2}x_1^2 + bx_3 + \frac{b}{2}x_1^2 \tag{2.8}$$

where $a, b \in \mathbb{R}, ad - bc = 1$, and with respect to the Poisson structure $\{\cdot,\cdot\}$ given by:

$$\{f,g\} = \{f,g\}^-_{cd} + d\left(\frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}\right) \tag{2.9}$$

for each $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$.

Proof. Indeed, we have successively:

$$\begin{aligned} \{x_1, H'\} &= \det \begin{bmatrix} (d+ck)x_1 & cx_2 & cx_3 + d \\ 1 & 0 & 0 \\ (ak+b)x_1 & ax_2 & ax_3 + b \end{bmatrix} \\ &= acx_2x_3 + adx_2 - acx_2x_3 - bcx_2 \\ &= (ad - b.c.)x_2 \\ &= x_2 \\ &= \dot{x}_1 \end{aligned}$$

and similarly $\{x_2, H'\} = \dot{x}_2; \{x_3, H'\} = \dot{x}_3$, as required. $\square$

Remark 2.2. It is not hard to see that the function C' given by:

$$C' = \frac{c}{2}(x_2^2 + x_3^2) + \frac{ck}{2}x_1^2 + dx_3 + \frac{d}{2}x_1^2 \tag{2.10}$$

is a Casimir of our configuration.

The above theorem tells us in fact that the equations (2.1), (2.2) are unchanged (so the trajectories of motion in $\mathbb{R}^3$ remain the same) when the conserved quantities H and C are replaced by the $Sl(2, \mathbb{R})$ combinations H' and C'. Geometrically, the invariance of the trajectories in $\mathbb{R}^3$ means that while the level surfaces of H' and C' may be radically different from those of H and C, their intersections are exactly the same.

2.4. The structure of the Lie algebra $\mathbb{R}^3_{cd}$. Let $\hat{e} = \{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ be the canonical basis of $\mathbb{R}^3$. Then the commutators are found to be

$$[e_1, e_2]_{cd} = -ce_3; [e_2, e_3]_{cd} = -(d + ck)e_1; [e_1, e_3]_{cd} = ce_2.$$

It is clear that the Lie algebra $\mathbb{R}^3_{cd}$ spanned by $\{e_1, e_2, e_3\}$ depends on c and d. Thus four cases can arise:

Case 1: $c = 0; d \neq 0$.

Let us define:

$$f_1 = -de_1; f_2 = e_2; f_3 = e_3$$

Then the structure of the algebra $\mathbb{R}^3_{cd}$ is:

$$[f_1, f_2]_{cd} = 0; [f_2, f_3]_{cd} = f_1; [f_2, f_1]_{cd} = 0.$$

This algebra is of type II in Bianchi's classification (see for details Dubrovin, Fomenko and Novikov [4]). It is solvable and it is identified with the well known Heisenberg algebra.

Case 2: $c \neq 0, d = 0$.

Then two subcases arise:

(i) $k > 0$ Then, let us define:

$$f_1 = -\frac{1}{c}e_1; f_2 = -\frac{1}{c\sqrt{k}}e_2; f_3 = -\frac{1}{c\sqrt{k}}e_3.$$

then the structure of the algebra is:

$$[f_1, f_2] = f_3; [f_2, f_3] = f_1; [f_3, f_1] = f_2.$$

This algebra is of type IX in Bianchi's classification and is isomorphic to $so(3)$.

(ii) $k < 0$. Then let us define:

$$f_1 = -\frac{1}{c}e_1; f_2 = -\frac{1}{c\sqrt{-k}}e_2; f_3 = -\frac{1}{c\sqrt{-k}}e_3.$$

Then the structure of the algebra is:

$$[f_1, f_2] = f_3; [f_2, f_3] = f_1; [f_3, f_1] = f_2.$$

This algebra is of type IX in Bianchi's classification and is isomorphic to $so(3)$.

Case 3: $c \neq 0, d \neq 0, c(d+ck) > 0$**.**

Let us define:

$$f_1 = -\frac{1}{c}e_1; f_2 = -\frac{1}{\sqrt{c(d+ck)}}e_2; f_3 = -\frac{1}{\sqrt{c(d+ck)}}e_3.$$

then the structure of the algebra is:

$$[f_1, f_2] = f_3; [f_2, f_3] = f_1; [f_3, f_1] = f_2.$$

This algebra is of type IX in Bianchi's classification and it is isomorphic to $so(3)$.

Case 4: $c \neq 0, d \neq 0, c(d+ck) < 0$**.**

Let us define:

$$f_1 = -\frac{1}{c}e_1; f_2 = -\frac{1}{\sqrt{-c(d+ck)}}e_2; f_3 = -\frac{1}{\sqrt{-c(d+ck)}}e_3.$$

Then the structure of the algebra is:

$$[f_1, f_2] = -f_3; [f_2, f_3] = f_1; [f_3, f_1] = -f_2.$$

This algebra is of type VIII in Bianchi's classification and it is isomorphic to $so(2,1)$.

2.5. The controlled system (2.1),(2.2)**and the perturbated Duffing oscillator.** It is clear that in each case presented above the controlled system (2.1),(2.2) may be reduced to a 2-dimensional mechanical system on the level set of the corresponding Casimir function C'. For instance if we take $b = 0$ in Case 1 this yields to the Duffing oscillator dynamics with one control. More precisely we have:

Theorem 2.4. *Under an explicit change of coordinates the dynamics of the controlled system* (2.1),(2.2) *reduces to the Duffing oscillator dynamics with one linear control.*

Proof. Let us take in the above Case 1, $b = 0, a = 1/d$. Then the Casimir C' and the Hamiltonian H' are respectively given by:

$$C' = d(x_3 + \frac{1}{2}x_1^2) \tag{2.11}$$

$$H' = \frac{1}{2d}(x_2^2 + x_3^2 + kx_1^2) \tag{2.12}$$

Define now the new coordinates (x, y, z) by

$$\begin{cases} x = x_1 \\ y = x_2 \\ z = x_3 + \frac{1}{2}x_1^2 \end{cases} \tag{2.13}$$

It follows that

$$
\left\{
\begin{aligned}
x_1 &= x \\
x_2 &= y \\
x_3 &= z - \frac{1}{2}x^2
\end{aligned}
\right.
\tag{2.14}
$$

With respect to these new variables the equations of motion of our controlled system (2.1),(2.2) become:

$$
\left\{
\begin{aligned}
\dot{x} &= y \\
\dot{y} &= xz - \frac{x^3}{2} - kx \\
z &= constant
\end{aligned}
\right.
\tag{2.15}
$$

or equivalent on z=constant:

$$
\ddot{x} = xz - \frac{x^3}{2} - kx
\tag{2.16}
$$

as required. $\square$

Let us observe also that our system possesses the following three equilibrium states:

$$
(0,0); (\pm\sqrt{\alpha(z-k)}, 0)
\tag{2.17}
$$

The first of these is unstable and the other two are stable centers (see Fig. 1)

FIGURE 1:. The phase portrait in the case $c = 0, d \neq 0$

3. Maxwell-Bloch equations with a quadratic control

3.1. The equations of motion and Poisson structure. Maxwell-Bloch equations with one control about $0x_2$ axis may be written in the form (2.1). In all that follows we shall employ the feedback

$$
u = kx_1x_3
\tag{3.1}
$$

where $k \in \mathbb{R}$ is the feedback gain parameter. We refer to the system (2.1),(3.1) as the controlled system. It is now easy to prove:

Proposition 3.1. *For the controlled system* (2.1),(3.1) *the quantities*

$$H = \frac{1}{2}(x_2^2 + x_3^2) + \frac{k}{2}x_3^2 \tag{3.2}$$

and

$$C = x_3 + \frac{1}{2}x_1^2 \tag{3.3}$$

are constants of motion.

Further we have:

Theorem 3.1. *The controlled system* (2.1),(3.1) *is a Hamilton-Poisson mechanical system with the phase space $P = \mathbb{R}^3$, the Hamiltonian H given by* (3.2) *and with respect to the Poisson structure* (2.5).

Proof. Indeed, one readily checks that

$$\dot{x}_i = \{x_i, H\}, i = 1, 2, 3$$

which gives the result. □

Remark 3.1. The function C given by (3.3) is a Casimir of our configuration.

3.2. Stability about $0x_1$ axis. For k sufficiently large, can we stabilize the system about $0x_1$, axis, i.e. about the equilibrium state $(x_1, x_2, x_3) = (M, 0, 0)$? Consider firstly the system linearized about $(M, 0, 0)$. Its eigenvalues are given by the solutions of

$$\lambda[\lambda^2 + M^2(k + 1)] = 0 \tag{3.4}$$

Hence for $k < -1$ the equilibrium state $(m, 0, 0)$ is unstable, but for $k > -1$, we have two imaginary eigenvalues and one zero eigenvalue. Is the system stable? We shall prove that it is via the energy-Casimir method.

Theorem 3.2. *The controlled system* (2.1),(3.1) *may be nonlinear stabilized about the equilibrium state $(x_1, x_2, x_3) = (M, 0, 0)$ for k sufficiently large such that $k > -1$.*

Proof. Consider the energy-Casimir function

$$H + \varphi(C) = \frac{1}{2}(x_2^2 + x_3^2) + \frac{k}{2}x_3^2 + \varphi(x_3 + \frac{1}{2}x_1^2) \tag{3.5}$$

where φ is a smooth scalar function. Now the first variation of $H + \varphi(C)$ is given by:

$$\delta(H + \varphi(C)) = x_2\delta x_2 + (k + 1)x_3\delta x_3 + \dot{\varphi}(\delta x_3 + x_1\delta x_1)$$

This equals zero at the equilibrium of interest if and only if

$$\dot{\varphi}(\frac{1}{2}M^2) = 0 \tag{3.6}$$

Then

$$\delta^2(H+\varphi(C)) = (\delta x_2)^2 + (k+1)(\delta x_3)^2 + \ddot{\varphi}(\delta x_3 + x_1\delta x_1)^2 + \dot{\varphi}(\delta x_1)^2$$

At the equilibrium $(x_1, x_2, x_3) = (M, 0, 0)$ we have via (3.6):

$$\delta^2(H+\varphi(C))(M,0,0) = (\delta x_2)^2 + (k+1)(\delta x_3)^2 + \ddot{\varphi}(\frac{1}{2}M^2)(\delta x_3 + M\delta x_1)^2$$

Hence for k sufficiently large such that $k+1>0$ and choosing φ such that

$$\ddot{\varphi}(\frac{1}{2}M^2) > 0,$$

the second variation is positive definite and we have nonlinear stability. $\square$

3.3. Alternative Poisson structures and the dynamic. Let $\mathbb{R}^3_{cd}$ be the Euclidean space $\mathbb{R}^3$ with the Lie algebra structure given by:

$$[u,v]_{cd} = \begin{bmatrix} -d & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & -c-ck \end{bmatrix} \cdot (u \times v)$$

Then we can prove

Theorem 3.3. *The controlled system* (2.1), (3.1) *is a Hamiltonian-Poisson mechanical system with the phase space* $(\mathbb{R}^3_{cd})^* \simeq \mathbb{R}^3$, *the Hamiltonian* H'' *given by*

$$H'' = \frac{a}{2}(x_2^2 + x_3^2) + \frac{ak}{2}x_3^2 + bx_3 + \frac{b}{2}x_1^2 \tag{3.7}$$

where $a, b \in \mathbb{R}$, $ad - bc = 1$, *and with the Poisson structure* $\{\cdot,\cdot\}$ *given by:*

$$\{f,g\} = \det \begin{bmatrix} dx_1 & cx_2 & cx_3 + ckx_3 + d \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \frac{\partial g}{\partial x_3} \end{bmatrix} \tag{3.8}$$

for each $f, g \in C^\infty(\mathbb{R}^3, \mathbb{R})$,

Proof. Indeed, we have successively:

$$\begin{aligned} \{x_1, H''\} &= \det \begin{bmatrix} dx_1 & cx_2 & cx_3 + ckx_3 + d \\ 1 & 0 & 0 \\ bx_1 & ax_2 & ax_3 + akx_3 + b \end{bmatrix} \\ &= acx_2x_3 + ackx_3 + adx_2 - acx_2x_3 - ackx_2x_3 \\ &- bcx_2 \\ &= (ad - bc)x_2 \\ &= x_2 \\ &= \dot{x}_1 \end{aligned}$$

and similarly

$$\{x_2, H''\} = \dot{x}_2; \{x_3, H''\} = \dot{x}_3$$

as required. $\square$

Remark 3.2. It is not hard to see that the function C'' given by

$$C'' = \frac{c}{2}(x_2^2 + x_3^2) + \frac{ck}{2}x_3^2 + dx_3 + \frac{d}{2}x_1^2 \tag{3.9}$$

is a Casimir of our configuration.

In fact the Theorem 3.3 tells us that the equations of motion of our controlled system (2.1), (3.1) are unchanged, so the trajectories of motion in $\mathbb{R}^3$ remain unchanged when the energy $H$and the Casimir C are replaced by the $Sl(2,\mathbb{R})$ linear combinations of H and C. For example, one may choose to eliminate some terms in each expression of H'' and C'', by choosing linear combination given by:

$$b = 0; c = 0; d \neq 0; a = 1/d \tag{3.10}$$

so that

$$C'' = d(x_3 + \frac{1}{2}x_1^2) \tag{3.11}$$

$$H'' = \frac{1}{2d}[x_2^2 + (1+k)x_3^2] \tag{3.12}$$

If $k > -1$, then with choice (3.10), the orbits for the controlled system (2.1),(3.1) are realized as motion along intersections of two cylinders, one elliptic cylinder - a level surface of H'' with its translation axis along $0x_1$ (where $H'' = 0$) and the other (a paraboloid cylinder along the Ox_2 axis), a level surface of C''.

Now, we shall prove that the controlled system (2.1), (3.1) with $k > -1$ can be reduced to the Duffing oscillator with a particular polynomial control. Indeed, let us introduce new coordinates

$$\begin{cases} x = x_1 \\ y = x_2 \\ z = x_3 + \dfrac{1}{2}x_1^2 \end{cases} \tag{3.13}$$

In terms of these new coordinates, the functions H'' and C'' become respectively

$$C'' = dz \tag{3.14}$$

$$H'' = \frac{d}{2}[y^2 + (k+1)(z - \frac{1}{2}x^2)^2] \tag{3.15}$$

and therefore the equations of motion reduce to

$$\begin{cases} \dot{x} = y \\ \dot{y} = (z - \dfrac{1}{2}x^2)x(k+1) \\ z = \text{ constant} \end{cases} \tag{3.16}$$

as announced. Thus we have proved:

Theorem 3.4. *The dynamics of the controlled system* (2.1), (3.1), *with* $k > -1$, *reduces to the Duffing oscillator dynamics with control*

$$u = kx(z - \frac{1}{2}x^2) \tag{3.17}$$

on the level surface of C''.

If we choose now:

$$a = 0; d = 0; c \neq; b = -1/c \tag{3.18}$$

then it follows that:

$$C'' = \frac{c}{2}[x_2^2 + (1+k)x_3^2] \tag{3.19}$$

and

$$H'' = -\frac{1}{c}(x_3 + \frac{1}{2}x_1^2) \tag{3.20}$$

Hence the level sets of the Hamiltonian H'' *are parabolic cylinders along the* $0x_2$ *axis and the level sets of the Casimir* C'' *are elliptic cylinders about the* $0x_3$ *axis. Therefore the orbits of the motion for our controlled system* (2.1), (3.1), *with* $k > -1$ *are intersections of these cylinders. They are periodic, except in the limit when the parabolic cylinder becomes tangent with the interior of the elliptic cylinder. Moreover we have:*

Theorem 3.5. *The dynamics of the controlled system* (2.1), (3.1) *with* $k > -1$ *reduces to the pendulum dynamics.*

Proof. Let us introduce new coordinates:

$$\begin{cases} x_1 = z \\ x_2 = \cos\Theta \\ x_3 = \dfrac{1}{\sqrt{k+1}}\sin\Theta \end{cases} \tag{3.21}$$

In terms of these coordinates the function H'' and C'' become:

$$\begin{cases} H'' = -\dfrac{1}{c}\left(\dfrac{1}{\sqrt{k+1}}\sin\Theta + \dfrac{1}{2}z^2\right) \\ C'' = \dfrac{c}{2} \end{cases} \tag{3.22}$$

and the equations of motion reduce to:

$$\begin{cases} \dot{z} = \cos\Theta \\ \dot{\Theta} = \dfrac{z}{\sqrt{k+1}} \end{cases} \tag{3.23}$$

FIGURE 1:. The phase portrait of the pendulum.

or equivalently:

$$\ddot{\Theta} + \frac{1}{\sqrt{k+1}} \cos\Theta = 0 \tag{3.24}$$

Thus in these coordinates, the motion on the reduce phase cylinder reduces to pendulum motion (see Fig. 2) □

4. Controlling the homoclinic orbit of the Maxwell-Bloch equations with three controls

4.1. The control systems with homoclinic orbits. Recently in a very nice paper Bloch and Marsden [2] have showed how to control the period of a system possessing a homoclinic orbit under the assumption that there is no noise in the system and that our controls have infinite accuracy. We shall prove that their result can be applied to the Maxwell-Bloch equations with three independent controls. First let us remind ourselves of their results.

Theorem 4.1. [2] *Consider the $C^r(r \geq 2)$ affine nonlinear control system given by:*

$$\dot{z} = f(z) + \sum_{i=1}^{m} u_i(t) g_i(z), \quad z \in \mathbb{R}^n, \tag{4.1}$$

where the u_i are piecewise continuous scalar functions and f and the g_i are C^r functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that the free system

$$\dot{z} = f(z) \tag{4.2}$$

has a hyperbolic fixed point at $z = z_0$ and that z_0 has a homoclinic orbit connecting z_0 to itself. Let $L_0(z)$ be given by

$$L_0(z) = Span\{g_i, [f, g_i], [f, [f, g_i]], \ldots, i = 1, \ldots, m\}. \tag{4.3}$$

If $\dim L_0(z_0) = n$, *then a control* u *may be found such that the system spends an arbitrary long time in a neighbourhood* U *of the fixed point* z_0 *after the control force is removed.*

Sketch of the Proof. Consider first the free system (4.2) and suppose that $Df(z_0)$ has s eigenvalues with negative real part and p with positive real part. The system may be transformed to the system

$$\begin{cases} \dot{x} = Ax + f_1(x,y) \\ \dot{y} = By + f_2(x,y) \end{cases} \tag{4.4}$$

where $z = (x,y) \in \mathbb{R}^s \times \mathbb{R}^p$, A is an $s \times s$ Jordan block with all diagonal entries having negative real parts and B is an $p \times p$ with diagonal having positive real parts.

Consider then the following neighbourhood of the origin:

$$N = \{(x,y) \in \mathbb{R}^s \times \mathbb{R}^p | ||x|| < \varepsilon, ||y|| < \varepsilon\} \tag{4.5}$$

whose boundary is given by the closures of the following sets:

$$\begin{aligned} C_\varepsilon^s &= \{(x,y) \in \mathbb{R}^s \times \mathbb{R}^p | ||x|| = \varepsilon, ||y|| < \varepsilon\} \\ C_\varepsilon^p &= \{(x,y) \in \mathbb{R}^s \times \mathbb{R}^p | ||x|| < \varepsilon, ||y|| = \varepsilon\} \end{aligned} \tag{4.6}$$

We denote by $\mathcal{S}_\varepsilon^s$ and $\mathcal{S}_\varepsilon^p$ the intersections of the stable manifold with C_ε^s and the intersection of the unstable manifold with C_ε^p respectively. For our purposes, there are two key results (see Wiggins [10]) that we need. The first is that near the origin the system behaves to within an error $0(\varepsilon^2)$ like its linear approximation. Secondly, the time the system spends in the neighbourhood N goes to $+\infty$ as $||y_0|| \to 0$.

Returning to the controlled system (4.1), by virtue of the condition on $L_0(z_0)$, we know the linearized system at z_0 is controlable. Hence we may find (explicitly for the linear system) a control that takes the system to a point on $C_\varepsilon^s \setminus S_\varepsilon^s$ as described above, choosing the point such that y_0 is as close to zero as we wish. We then remove the control and the theorem follows. $\square$

4.2. Nonlinear pendulum with control. The dynamics of the nonlinear pendulum with a control u is given by:

$$\ddot{\Theta} + g \sin \Theta = u(t) \tag{4.7}$$

or equivalent

$$\begin{cases} \dot{x} = y \\ \dot{y} = -g \sin x + u, \end{cases} \tag{4.8}$$

where

$$\begin{cases} x = \Theta \\ y = \dot{\Theta} \end{cases}$$

The dynamics of the corresponding free system

$$(4.9) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -g \sin x \end{cases}$$

has an homoclinic orbit at the hyperbolic equilibrium point $(\pi, 0)$. The orbits inside the homoclinic loop are periodic (see Fig. 2) Further, the linearized system at $\Theta = \pi$ i.e. the system

$$(4.10) \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

is controllable. Indeed, its controllability matrix Cm is given by

$$Cm + [B, AB]$$

where

$$A = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

and moreover

$$rank(Cm) = rank \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2,$$

as required. It follows that in the particular case of the nonlinear pendulum with control then all hypothesis of Theorem 4.1 are satisfied, hence it works well.

4.3. Maxwell-Bloch equations with three independent controls. The dynamics of the Maxwell-Bloch equations with three independent controls is given by:

$$(4.11) \qquad \begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = x_1 x_3 + u_2 \\ \dot{x}_3 = -x_1 x_2 + u_3 \end{cases}$$

The free system is in fact the Maxwell-Bloch system which is equivalent with the nonlinear pendulum via an explicit change of coordinates (see for details David and Holm [3], or our section 3). Therefore, in view of the above considerations the all hypothesis of the Theorem 4.1 are satisfied, hence it works well.

Let us observe also that in this case the controllability of the linearized system can be obtained easier if we observe that the matrix B is in fact the identity matrix.

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