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# Cohomology for a Group of Diffeomorphisms of a Manifold Preserving an Exact Form

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# COHOMOLOGY FOR A GROUP OF DIFFEOMORPHISMS OF A MANIFOLD PRESERVING AN EXACT FORM.

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Abstract. Let $M$ be a $G$-manifold and $\omega$ a $G$-invariant exact $m$-form on $M$. We indicate when these data allow us to constract a cocycle on a group $G$ with values in the trivial $G$-module $\mathbb{R}$ and when this cocycle is nontrivial.

## 1. Introduction

Let $M$ be a manifold, let $G$ be a group of diffeomorphisms of $M$, and let $\omega$ be a $G$-invariant exact $m$-form on $M$. In this paper we apply the construction of [5] to get from these data a cocycle on the group $G$ with values in the trivial $G$-module $\mathbb{R}$. We prove that this cocycle may be chosen differentiable (continuous) whenever $G$ is a subgroup of a Lie group (a topological group). Moreover, we prove that for a manifold $\mathbb{R}^n \times M$ with an exact form $\omega$ which is either of type $\omega_0 + \omega_M$ or of type $\omega_0 \wedge \omega_M$, where $\omega_0$ is a nonzero form on $\mathbb{R}^n$ with constant coefficients and $\omega_M$ is a form on $M$, and the group $\operatorname{Diff}(\mathbb{R}^n \times M, \omega)$ of diffeomorphisms of $\mathbb{R}^n \times M$ preserving the form $\omega$ the corresponding cocycle is nontrivial.

## 2. A construction of cohomology classes for a group of diffeomorphisms

Let $G$ be a group and let $A$ be a right $G$-module. Let $C^p(G, A)$ be the set of maps from $G^p$ to $A$ for $p > 0$ and let $C^0(G, A) = A$. Define the differential $D : C^p(G, A) \to C^{p+1}(G, A)$ as follows, for $f \in C^p(G, A)$ and $g_1, \dots, g_{p+1} \in G$:

$$(1) \quad (Df)(g_1, \dots, g_{p+1}) = f(g_2, \dots, g_{p+1}) + \sum_{i=1}^{p} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{p+1}) + (-1)^{p+1} f(g_1, \dots, g_p) g_{p+1}.$$

Then $C^*(G, A) = (C^p(G, A), D)_{p \geq 0}$ is the standard complex of nonhomogeneous cochains of the group $G$ with values in the right $G$-module $A$ and its cohomology $H^*(G, A) = (H^p(G, A)_{p \geq 0})$ is the cohomology of the group $G$ with values in $A$.

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Let $M$ be a smooth $n$-dimensional $G$-manifold, where $G$ is a group of diffeomorphisms of $M$. Denote by $\Omega(M) = (\Omega^p(M))_{p=1,\dots,n}$ the de Rham complex of differential forms on $M$ and consider the natural (right) action of the group $G$ on $\Omega(M)$ by pull backs. Denote by $\Omega(M)^G$ the subcomplex of $\Omega(M)$ consisting of $G$-invariant forms. Next we denote by $H_q(M)$ the $q$-dimensional real homology of $M$ and by $H^q(M)$ the $q$-dimensional real cohomology of $M$.

Let $C^*(G,\Omega(M)) = \{C^p(G,\Omega^q(M)),\delta'\}_{p,q\geq 0}$ be the standard complex of nonhomogeneous cochains of $G$ with values in the $G$-module $\Omega(M)$. We define the second differential $\delta'' : C^p(G,\Omega^q(M)) \to C^p(G,\Omega^{q+1}(M))$ by

$$(\delta''c)(g_1,\dots,g_p) = (-1)^p dc(g_1,\dots,g_p),$$

where $f \in C^p(G,\Omega^q(M))$, $g_1,\dots,g_p \in G$, and where $d$ is exterior derivative. Since $\delta'\delta''+\delta''\delta' = 0$, we have on $C^*(G,\Omega(M))$ the structure of double complex. Denote by $C^{**}(G,\Omega(M))$ the cochain complex $C^*(G,\Omega(M))$ with respect to the total differential $\delta = \delta'+\delta''$. We denote by $H(G,M,\Omega(M))$ the cohomology of the complex $C^{**}(G,\Omega(M))$.

It is easily checked that the inclusion $\Omega(M)^G \subset C^0(G,\Omega(M))$ induces an injective homomorphism of complexes $\Omega(M)^G \to C^{**}(G,\Omega(M))$ and thus also a homomorphism $H(\Omega(M)^G) \to H(G,M,\Omega(M))$ of cohomologies. We identify $\omega \in \Omega(M)^G$ with its image by the inclusion $\Omega(M)^G \subset C^{**}(G,\Omega(M))$ and denote by $h(\omega)$ the cohomology class of $\omega$ in the complex $C^{**}(G,\Omega(M))$ whenever the form $\omega$ is closed.

We shall use some standard facts on spectral sequences (see, for example, [1]). Consider the first filtration

$$F_{1,p}(G,M,\Omega(M)) := \oplus_{q\geq p} C^q(G,\Omega(M))$$

of the double complex $C^*(G,\Omega(M))$. By definition, $F_{1,p}(G,M,\Omega(M))$ is a subcomplex of the complex $C^{**}(G,\Omega(M))$ and $F_{1,0}(G,M,\Omega(M)) = C^{**}(G,\Omega(M))$. Denote by $E_{1,r} = (E^{p,q}_{1,r}, d^{pq}_{1,r})_{p,q\geq 0}$ for $r = 0,1,\dots,\infty$ the corresponding spectral sequence. Denote by $h_p$ the homomorphism of cohomologies $H(F_{1,p}(G,M,\Omega(M)) \to H(G,M,\Omega(M))$ induced by the inclusion $F_{1,p}(G,M,\Omega(M)) \subset C^{**}(G,\Omega(M))$. Then $(\operatorname{Im} h_p)_{p\geq 0}$ is a filtration of the cohomology $H(G,M,\Omega(M))$ and

$$E^{p,q}_{1,\infty} = h_p(H^{p+q}(F_{1,p}(G,M,\Omega(M))))/h_{p+1}(H^{p+q}(F_{1,p+1}(G,M,\Omega(M)))).$$

For this spectral sequence we have $E^{p,q}_{1,2} = H^p(G,H^q(M))$, where an action of the group $G$ on $H^q(M)$ is induced by its action on $\Omega(M)$. Moreover, there is a natural homomorphism $H^p(G,H^0(M)) = E^{p,0}_{1,2} \to E^{p,0}_\infty$.

**Proposition 2.1.** *Let $\omega \in \Omega(M)^G$ be an exact $m$-form and let $p$ be the maximal integer such that $h(\omega) \in h_{p+1}(H^m(F_{1,p+1}(G,M,\Omega(M))))$. Then the image of $h(\omega)$ under the natural homomorphism $\operatorname{Im} h_{p+1} \to \operatorname{Im} h_{p+1}/\operatorname{Im} h_{p+2}$ belongs to $E^{p+1,m-p-1}_{1,m-p+1}$. In particular, if $m = p+1$ and the manifold $M$ is connected, the above image of $h(\omega)$ is an $m$-dimensional cohomology class of the group $G$ with values in the trivial $G$-module $\mathbb{R}$.*

*Proof.* By assumption the image of $h(\omega)$ under the homomorphism

$$\operatorname{Im} h_{p+1} \to \operatorname{Im} h_{p+1}/\operatorname{Im} h_{p+2}$$

belongs to $E_{1,\infty}^{p+1,m-p-1}$. Since $d_{1,r}^{p,q} : E_{1,r}^{p,q} \to E_{1,r}^{p+r,q-r+1}$ vanishes whenever $r > q+1$, we have $E_{1,\infty}^{p+1,m-p-1} = E_{1,m-p+1}^{p+1,m-p-1}$. If $m = p+1$ and the manifold $M$ is connected we have $E_{1,\infty}^{p+1,m-p-1} = E_2^{m,0} = H^m(G, H^0(M)) = H^m(G, \mathbb{R})$. □

**Theorem 2.2.** *Let $\omega$ be a $G$-invariant exact $m$-form on $M$ and let $H^{m-p}(M) = \cdots = H^{m-1}(M) = 0$ and $H^{m-p-1}(M) \neq 0$ for some $1 \leq p \leq m-1$. Then $h(\omega) \in \operatorname{Im} h_{p+1}$ and $h(\omega)$ defines a unique $(p+1)$-dimensional cohomology class $c(\omega)$ of the group $G$ with values in the natural $G$-module $H^{m-p-1}(M)$.*

*Proof.* By assumption there is a form $\varphi_{0,m-1} \in C^0(G, \Omega^{m-1}(M))$ such that $\omega = -d\varphi_{0,m-1} = -\delta''\varphi_{0,m-1}$. Then we have $\omega + \delta\varphi_{0,m-1} = \delta'\varphi_{0,m-1}$.

Since $H^{m-1}(M) = 0$ and $\delta''\delta'\varphi_{0,m-1} = -\delta'\delta''\varphi_{0,m-1} = \delta'\omega = 0$, there is a cochain $\varphi_{1,m-2} \in C^1(G, \Omega^{m-2}(M))$ such that $\delta'\varphi_{0,m-1} = -\delta''\varphi_{1,m-2}$. Thus we have $\omega + \delta(\varphi_{0,m-1} + \varphi_{1,m-2}) = \delta'\varphi_{1,m-2}$.

Using the conditions $H^{m-2}(M) = \cdots = H^{m-p}(M) = 0$ and proceeding in the same way we get for $i = 1, \ldots, p$ the cochains $\varphi_{i,m-i-1} \in C^i(G, \Omega^{m-i-1}(M))$ such that

$$\delta'\varphi_{i-1,m-i} + \delta''\varphi_{i,m-i-1} = 0 \tag{2}$$

and so

$$\omega + \delta(\varphi_{0,m-1} + \cdots + \varphi_{p,m-p-1}) = \delta'\varphi_{p,m-p-1} \in C^{p+1}(G, \Omega^{m-p-1}(M)).$$

Moreover, we have

$$d\delta'\varphi_{p,m-p-1} = -\delta''\delta'\varphi_{p,m-p-1} = \delta''\delta''\varphi_{p-1,m-p} = 0.$$

Consider $H^{m-p-1}(M)$ as a $G$-module with respect to the natural action of $G$ on $H^{m-p-1}(M)$. Then the cochain $\delta'\varphi_{p,m-p-1}$ defines a cocycle on $G$ of degree $p+1$ with values in the $G$-module $H^{m-p-1}(M)$. We claim that the cohomology class of this cocycle depends only on the cohomology class of $\omega$ in the complex $\Omega(M)^G$.

If we replace the form $\omega$ by a form $\omega + d\omega_1$, where $\omega_1 \in \Omega^{m-1}(M) \cap \Omega(M)^G$, one can replace the sequence $\varphi_{0,m-1}, \ldots, \varphi_{p,m-p-1}$ by the sequence $\varphi_{0,m-1} - \omega_1, \varphi_{1,m-2}, \ldots, \varphi_{p,m-p-1}$ and obtain the same cochain $\varphi_{p,m-p-1}$ at the end.

Consider another sequence $\bar{\varphi}_{0,m-1}, \ldots, \bar{\varphi}_{p,m-p-1}$ $(i = 0, \ldots, p)$ such that $\omega = -d\bar{\varphi}_{0,m-1}$ and $\delta'\bar{\varphi}_{i-1,m-i} + \delta''\bar{\varphi}_{i,m-i-1} = 0$ for $i = 1, \ldots, p$. Since $H^{m-1}(M) = 0$ we have

$$\bar{\varphi}_{0,m-1} = \varphi_{0,m-1} + \delta''\psi_{0,m-2},$$

where $\psi_{0,m-2} \in C^0(G, \Omega^{m-2}(M))$. If $p = 1$ we have $\delta'\bar{\varphi}_{0,m-1} = \delta'\varphi_{0,m-1} - \delta''\delta'\psi_{0,m-2}$ and we are done. If $p > 1$ we have

$$\delta'\bar{\varphi}_{0,m-1} = \delta'\varphi_{0,m-1} - \delta''\delta'\psi_{0,m-2} = -\delta''(\varphi_{1,m-2} + \delta'\psi_{0,m-2}) = -\delta''\bar{\varphi}_{1,m-2}.$$

Since $H^{m-2}(M) = 0$ there is a cochain $\psi_{1,m-3} \in C^1(G, \Omega^{m-3}(M))$ such that $\bar{\varphi}_{1,m-2} = \varphi_{1,p-2} + \delta'\psi_{0,m-2} + \delta''\psi_{1,m-3}$. For $i = 2, \dots, p-1$ proceeding in the same way we get the cochains $\psi_{i,m-i-2} \in C^i(G, \Omega^{m-i-2}(M))$ such that

$$\bar{\varphi}_{i,m-i-1} = \varphi_{i,m-i-1} + \delta'\psi_{i-1,m-i-1} + \delta''\psi_{i,m-i-2}.$$

In particular, we have

$$\bar{\varphi}_{p,m-p-1} = \varphi_{p,m-p-1} + \delta'\psi_{p-1,m-p-1} + \delta''\psi_{p,m-p-2}$$

and $\delta'\bar{\varphi}_{p,m-p-1} = \delta'\varphi_{p,m-p-1} - \delta''\psi_{p,m-p-2}$. Thus the cochains $\delta'\bar{\varphi}_{p,m-p-1}$ and $\delta'\varphi_{p,m-p-1}$ define the same cohomology class of $H^{p+1}(G, H^{m-p-1}(M))$. □

Suppose that the conditions of theorem 2.2 are satisfied for an exact $m$-form $\omega \in \Omega(M)^G$. Let $\alpha$ be a singular smooth cycle of $M$ of dimension $m-p-1$ whose homology class $a$ is invariant under the natural action of the group $G$ on $H_{m-p-1}(M)$. Put

$$c_a(\omega)(g_1, \dots, g_{p+1}) = \int_\alpha (\delta'\varphi_{p,m-p-1})(g_1, \dots, g_{p+1}). \tag{3}$$

By definition, $c_a(\omega)$ is a $(p+1)$-cocycle on the group $G$ with values in the trivial $G$-module $\mathbb{R}$ which is independent of a choice of the cycle $\alpha$ in the homology class $a$.

Let $p = 0$. Evidently, the cocycle $c_a(\omega)$ is nontrivial if and only if it does not vanish. From now on we assume $p > 0$.

*Remark* 2.3. Let the assumptions of theorem 2.2 be satisfied for an exact $m$-form $\omega \in \Omega(M)^G$. If either the manifold is connected and $m = p+1$, or $G$ is a connected topological group, the action of $G$ on the $H_{m-p-1}(M)$ is trivial. If the homology class $a$ of the cycle $\alpha$ is not invariant under the action of $G$, consider the vector subspace $H_a$ of $H_{m-p-1}(M)$ generated by the orbit of $a$. Then (3) defines a $(p+1)$-cocycle on the group $G$ with values in the $G$-module $H_a$.

Consider the partial case when $M = G$ is a connected Lie group and the group $G$ acts on $M$ by left translations. It is clear that the complex $\Omega(G)^G$ is isomorphic to the complex $C^*(\mathfrak{g}, \mathbb{R})$ of standard cochains of the Lie algebra $\mathfrak{g}$ of the group $G$ with values in the trivial $\mathfrak{g}$-module $\mathbb{R}$. Consider the second filtration

$$F_{2,p}C^{**}(G, G, \Omega(G)) := \oplus_{q \ge p} C^*(G, \Omega^q(G))$$

of the double complex $C^{**}(G, \Omega(G))$ and the corresponding spectral sequence $E_{2,r} = (E_{2,r}^{p,q}, d_r^{p,q})_{p,q \ge 0}$ for $r = 0, 1, \dots, \infty$. It is easily seen that $E_{2,1}^{p,q} = H^p(G, \Omega^q(G))$.

**Lemma 2.4.** *The inclusion $\Omega(M)^G \subset C^{**}(G, G, \Omega(G))$ induces an isomorphism of cohomologies.*

*Proof.* We prove that for each $q \ge 0$ we have

$$H^p(G, \Omega^q(G)) = 0 \quad \text{for } p > 0 \text{ and } H^0(G, \Omega^q(G)) = \Omega^q(G)^G.$$

First we consider the case when $q = 0$. We use the standard operator $B : C^p(G, \Omega^0(G)) \to C^{p-1}(G, \Omega^0(G))$ defined as follows. For $p > 0$, put

$$(Bc)(g_1, \dots, g_{p-1})(g) = (-1)^p c(g_1, \dots, g_{p-1}, g)(e),$$

where $c \in C^p(G, \Omega^0(G))$, $g, g_1, \dots, g_{p-1} \in G$, and $e$ is the identity element of $G$. For $c \in C^0(G, \Omega^0(G))$, put $Bc = 0$. It is easy to check that $B$ is a homotopy operator between the identity isomorphism of $C^*(G, \Omega^0(G))$ and the map of $C^*(G, \Omega^0(G))$ into itself which vanishes on $C^p(G, \Omega^0(G))$ for $p > 0$ and takes $c \in C^0(G, \Omega^0))$ to $c(e)$. This proves our statement for $p = 0$.

To prove our statement for $p > 0$ we note that $\Omega^q(G) = \Omega^q(G)^G \otimes \Omega^0(G)$, where $\Omega^p(G)^G$ is the space of left invariant $q$-forms on $G$. Since $G$ acts trivially on $\Omega^p(G)^G$, its action on $\Omega^q(G)$ is induced by its action on $\Omega^0(G)$. Then we have $H^p(G, \Omega^q(G)) = 0$ for $p > 0$ and $H^0(G, \Omega^q(G)) = \Omega^q(G)^G$.

The above statement implies that $E^{p,q}_{2,1} = 0$ when $p > 0$ and $E^{0,q}_{2,1} = \Omega^q(G)^G$. Then $E_{2,1} = \Omega(G)^G$ and evidently the differential $d^{0,q}_{2,1}$ equals the exterior derivative $d$ on $\Omega(G)^G$ up to sign. Therefore we have $E^{p,q}_{2,2} = 0$ when $p > 0$ and $E^{0,q}_{2,2} = H^q(\Omega(G)^G)$. Thus implies that $E^{p,q}_{2,\infty} = E^{p,q}_{2,2}$ and therefore the inclusion $\Omega(G)^G \subset C^{**}(G, \Omega(G))$ induces an isomorphism of cohomologies. □

**Proposition 2.5.** *Let $\omega \in \Omega(G)^G$ be an exact $m$-form whose cohomology class in the complex $\Omega(G)^G$ is nontrivial and let*

$$\varphi_{i,m-i-1} \in C^i(G, \Omega^{m-i-1}(G)) \quad (i = 0, \dots, m-1)$$

*be a sequence of cochains such that $\delta(\omega + \varphi_{0,m-1} + \cdots + \varphi_{m-1,0}) = \delta' \varphi_{m-1,0}$. Then, for a point $x \in G$, a cocycle $\delta' \varphi_{m-1,0}(g)(x)$ of the complex $C^*(G, \mathbb{R})$ is nontrivial.*

*Proof.* By lemma 2.4 the cohomology class of $\omega$ in the complex $C^{**}(G, \Omega(G))$ is nontrivial and then by assumption $\omega$ defines a nontrivial element of $E^{m,0}_{2,\infty}$. Since

$$H^m(G, \mathbb{R}) = E^{m,0}_{2,2} = E^{m,0}_{2,\infty},$$

the cocycle $\delta' \varphi_{m-1,0}(g)(x)$ of the complex $C^*(G, \mathbb{R})$ is nontrivial. □

## 3. The map $f_\gamma$ and its properties

Let $G$ be a finite dimensional Lie group. For $X \in T_e(M)$ denote by $X^r$ the right invariant vector field on $G$ such that $X^r(e) = X$ and by $\tilde{X}$ the fundamental vector field on $M$ corresponding to $X$ for the action of $G$ on $M$. We denote the action by $\varphi : G \times M \to M$ and write $gx = \varphi(g, x) = \varphi^x(g) = \varphi_g(x)$. By definition, $T(\varphi^x)X^r(g) = \tilde{X}(gx)$ and for each $g \in G$ we have $\varphi^x \circ \mathrm{L}_g = \varphi_g \circ \varphi^x$, where $\mathrm{L}_g$ is left translation on $G$.

Let $\gamma$ be a singular smooth cycle of dimension $q$ on $M$. Define a map $f_\gamma : \Omega(M) \to \Omega(G)$ as follows. Let $\omega \in \Omega(M)$. If $\deg \omega < q$ put $f_\gamma(\omega) = 0$. If

$\deg\omega = p+q$ with $p\geq 0$ put

$$f_\gamma(\omega)(X_1^r,\dots,X_p^r)(g) = \int_\gamma \varphi_g^*\Big(i(\tilde X_p)\dots i(\tilde X_1)\omega\Big) = \int_{g\gamma} i(\tilde X_p)\dots i(\tilde X_1)\omega, \tag{4}$$

where $X_1,\dots,X_p\in T_e(G)$ and $g\in G$. Clearly $\omega\to f_\gamma(\omega)$ is a linear map from $\Omega(M)$ to $\Omega(G)$ decreasing degrees to $q$.

Consider the action of the group $G$ on itself by left translations.

**Lemma 3.1.** *The map $f_\gamma$ is $G$-equivariant.*

*Proof.* It suffices to consider the case when $\deg\omega\geq q$. Let $\omega\in\Omega^{p+q}(M)$, $X\in T_e(G)$, and $g,\tilde g\in G$. It is easy to check that

$$T(\mathrm{L}_{\tilde g})\circ X^r = (\operatorname{ad}\tilde g(X))^r\circ L_{\tilde g}: G\to TG, \tag{5}$$

$$T(\varphi_{\tilde g})\circ\tilde X = \widetilde{\operatorname{ad}\tilde g(X)}\circ\varphi_{\tilde g}: M\to TM \tag{6}$$

From this we get

$$\begin{aligned}
L_{\tilde g}^* f_\gamma(\omega))(X_1^r,\dots,X_p^r)(g) &= \int_{\tilde g g\gamma} i(T\varphi_{\tilde g}.\tilde X_p)\dots i(T\varphi_{\tilde g}.\tilde X_1)\omega\\
&= \int_{g(\gamma)}\varphi_g^*\Big(i(T\varphi_{\tilde g}\tilde X_p)\dots i(T\varphi_{\tilde g}.\tilde X_1))\omega\Big) = \int_{g\gamma} i(\tilde X_p)\dots i(\tilde X_1)\varphi_g^*\omega = f_\gamma(\varphi_g^*\omega).
\end{aligned}$$

□

**Lemma 3.2.** *$d_G\circ f_\gamma = f_\gamma\circ d$, where $d_G$ is the exterior derivative in $\Omega(G)$.*

*Proof.* Let $\omega\in\Omega^m(M)$. If $m<q$, by definition we have $d_G\circ f_\gamma(\omega) = f_\gamma\circ d(\omega) = 0$.

Let $\omega\in\Omega^{p+q}(M)$, where $p\geq 0$, and $X_1,\dots,X_p\in T_e(G)$. Then we have

$$\begin{aligned}
(d_G f_\gamma(\omega))(X_1^r,\dots,X_p^r)(g) &= \sum_{i=1}^p(-1)^{i-1}X_i^r(g)f_\gamma(\omega)(X_1^r,\dots,\widehat{X_i^r},\dots,X_p^r)\\
&+\sum_{i<j}(-1)^{i+j}f_\gamma(\omega)([X_i^r,X_j^r],X_1^r,\dots,\widehat{X_i^r},\dots,\widehat{X_j^r}\dots,\dots,X_p^r)(g)\\
&= \sum_{i=1}^p(-1)^{i-1}X_i^r(g)\int_\gamma g^*\Big(i(\tilde X_p)\dots\widehat{i(\tilde X_i)}\dots i(\tilde X_1)\omega\Big)\\
&+\sum_{i<j}(-1)^{i+j}\int_\gamma g^*\Big(i(\tilde X_p)\dots\widehat{i(\tilde X_j)}\dots\widehat{i(\tilde X_i)}\dots i(\tilde X_1)i([\tilde X_i,\tilde X_j])\omega\Big)\\
&= \sum_{i=1}^p(-1)^{i-1}\int_\gamma g^*\Big(\mathrm{L}_{\tilde X_i}(i(\tilde X_p)\dots\widehat{i(\tilde X_i)}\dots i(\tilde X_1)\omega\Big)\\
&+\sum_{i<j}(-1)^{i+j}\int_\gamma g^*\Big(i(\tilde X_p)\dots\widehat{i(\tilde X_j)}\dots\widehat{i(\tilde X_i)}\dots i(\tilde X_1)i([\tilde X_i,\tilde X_j])\omega\Big),
\end{aligned}\tag{7}$$

where $\mathrm{L}_X$ denote the Lie derivative with respect to a vector field $X$ and, as usual, $\widehat{i(\tilde{X})}$ means that the term $i(\tilde{X})$ is omitted.

Using the formula $[\mathrm{L}_X, i(Y)] = i([X,Y])$ it is easy to check by induction over $p$ that for any manifold $M$ and vector fields $X_1, \dots, X_p$ on $M$ the following formula is true.

$$\sum_{i=1}^{p}(-1)^{i-1}\,\mathrm{L}_{X_i}\, i(X_p)\dots\widehat{i(X_i)}\dots i(X_1) + \sum_{i<j}(-1)^{i+j} i(X_p)\dots\widehat{i(X_j)}\dots\widehat{i(X_i)}\dots i(X_1) i([X_i, X_j] = i(X_q)\dots i(X_1)d + (-1)^{p-1} d i(X_q)\dots i(X_1).$$

Applying this formula in (7) we get

$$(d_G f_\gamma(\omega_1))(X_1^r, \dots, X_p^r)(g) = \int_\gamma g^* i(\tilde{X}_p)\dots i(\tilde{X}_1) d\omega = f_\gamma(d\omega)(X_1^r, \dots, X_p^r)(g).$$

□

Consider the double complex $(C^*(G, \Omega(G)), \delta'_G, \delta''_G)$ for the action of the group $G$ on $G$ by left translations. Define the map $F_\gamma : (C^{**}(M, \Omega(M))(C^{**}(G, \Omega(G))$ as follows: for a cochain $c \in C^p(G, \Omega^q(M))$ put $F_\gamma(c) = f_\gamma \circ c$.

**Lemma 3.3.** $\delta'_G \circ F_\gamma = F_\gamma \circ \delta'$.

*Proof.* Let $c \in C^s(G, \Omega^{p+q}(M)$ and $g, g_1, \dots, g_{s+1} \in G$. By definition we have

(8) $$(\delta'_G \circ F_\gamma)(c))(g_1, \dots, g_{s+1})(g) = F_\gamma(c)(g_2, \dots, g_{s+1})(g) + \sum_{i=1}^{s}(-1)^i F_\gamma(c)(g_1, \dots, g_i g_{i+1}, \dots, g_{s+1})(g) + (-1)^{s+1}\,\mathrm{L}^*_{g_{s+1}} F_\gamma(c)(g_1, \dots, g_s)(g).$$

For $X_1, \dots, X_p \in T_e(G)$ and $g \in G$ by (5) and (6) we get

(9) $$\mathrm{L}^*_{g_{s+1}} F_\gamma(c)(g_1, \dots, g_s)(X_1^r, \dots, X_p^s)(g) = \int_\gamma (g_{s+1}g)^* \Big(i(\mathrm{ad}\,\widetilde{g_{s+1}(X_p)})\dots i(\mathrm{ad}\,\widetilde{g_{s+1}(X_1)}) c(g_1, \dots g_s)\Big) = \int_\gamma g^* \Big(i(\tilde{X}_p)\dots i(\tilde{X}_1) c(g_1, \dots, g_s)\Big) = F_\gamma(g^*_{p+1} c(g_1, \dots, g_s))(X_1^r, \dots, X_p^s)(g).$$

Replacing the last summand in (8) by formula (9) we get

$$(\delta'_G \circ F_\gamma)(c)(g_1, \dots, g_{s+1})(g) = (F_\gamma \circ \delta')(c)(g_1, \dots, g_{s+1})(g).$$

□

Lemmas 3.1,3.2, and (8) imply the following

**Theorem 3.4.** *The map $F_\gamma : C^*(G, \Omega(M)) \to C^*(G, \Omega(G))$ is a homomorphism of double complexes decreasing the second degree to $q$.*

Suppose that the conditions of theorem 2.2 for an exact $m$-form $\omega$ are satisfied. Let $\alpha$ be a singular smooth cycle of $M$ of dimension $m-p-1$ whose homology class $a$ is invariant under the natural action of the group $G$ on $H_{m-p-1}(M)$. Consider the sequence of cochains $\varphi_{i,m-i-1}$ $(i = 0, \dots, p)$ constructed in the proof of theorem 2.2. By theorem 3.4 $F_\alpha \circ \omega$ is a left invariant $(p+1)$-form on $G$, i.e., a $(p+1)$-cocycle of the complex $C^*(\mathfrak{g}, \mathbb{R})$. Moreover, we have

$$\delta'_G F_\alpha \circ \varphi_{i-1,m-i} + \delta''_G F_\alpha \circ \varphi_{i,m-i-1} = 0 \quad (i = 1, \dots, p).$$

Since $d_G(\delta'_G \circ F_\alpha \circ \varphi_{p,m-p-1}) = 0$, for any $g \in G$ we have

$$c_a(\omega) = \int_\gamma \delta' \varphi_{p,m-p-1} = (F_\alpha \circ \delta' \circ \varphi_{p,m-p-1})(e) = (\delta' \circ F_\alpha \circ \varphi_{p,m-p-1})(e).$$

Consider the complex $(C^*(G, \mathbb{R}), D)$ of nonhomogeneous cochains on the group $G$ with values in the trivial $G$-module $\mathbb{R}$. Define a cochain $b \in C^p(G, \mathbb{R})$ as follows:

$$b(g_1, \dots, g_p) = \int_\alpha \varphi_{p,m-p-1}(g_1, \dots, g_p).$$

By the definitions of the cocycle $c_a(\omega)$ and the map $f_\gamma$ and by (1) we have

$$\begin{aligned} c_a(g_1, \dots, g_p, g) = (-1)^{p+1} &\left(f_\alpha \circ \varphi_{p,m-p-1}(g_1, \dots, g_p)(g) - b(g_1, \dots, g_p)\right) \\ &+ (Db)(g_1, \dots, g_p, g). \end{aligned} \tag{10}$$

**Proposition 3.5.** *Let the cycle $\gamma$ be $G$-invariant. Then the cocycle $c_a(\omega)$ is trivial.*

*Proof.* By assumption we have

$$(F_\alpha \circ \varphi_{p,m-p-1})(g_1, \dots, g_p)(g) = \int_{g\alpha} \varphi_{p,m-p-1}(g_1, \dots, g_p) = b(g_1, \dots, g_p)$$

Then by (10) we have $c_a(\omega) = Db$. □

Denote by $H$ the subgroup of $G$ consisting of all elements $g \in G$ preserving the cycle $\gamma$. Consider the natural action of the group $G$ on the homogeneous space $G/H$. The projection $p_H : G \to G/H$ induces a homomorphism $\tilde{p}_H : C^*(G, \Omega(G/H)) \to C^*(g, \Omega(G))$ of double complexes.

**Proposition 3.6.** *There is a unique homomorphism of double complexes $F_{\gamma,H} : C^*(G, \Omega(M)) \to C^*(G, \Omega(G/H))$ such that $F_\gamma = \tilde{p}_H \circ F_{\gamma,H}$.*

*Proof.* Note that formula (4) implies that the form $f_\gamma(\omega)$ is $H$-invariant. Moreover, by assumption for each $X \in T_e(H)$ the fundamental vector field $\tilde{X}$ preserves the cycle $\gamma$. Thus the form $i(X)\omega$ vanishes on the cycle $\gamma$. This implies that $f_\gamma(\omega)(X_1^r, \dots, X_p^r) = 0$ whenever one of the vectors $X_1, \dots, X_p \in T_e(G)$ belongs to $T_e(H)$. Thus the form $f_\gamma(\omega)$ lies in the image of the map $p^* : \Omega(G/H) \to \Omega(G)$. □

We point out the following sufficient condition of nontriviality of the cocycle $c_a(\omega)$.

**Theorem 3.7.** *Let $\omega \in \Omega(M)^G$ be an exact $m$-form such that the conditions of theorem 2.2 are satisfied and let $\alpha$ be a singular smooth $(m-p-1)$-cycle on $M$ whose homology class $a$ is $G$-invariant. If the cohomology class of the closed left invariant form $f_\alpha \circ \omega$ in the complex $\Omega(G)^G$ is nontrivial, the cocycle $c_a(\omega)$ of the complex $C^*(G, \mathbb{R})$ is nontrivial as well.*

*Proof.* It is easy to check that the form $f_a \circ \omega$ satisfies the conditions of proposition 2.5 and the cocycle $c_a(\omega)$ equals the cocycle $\delta'_G \circ f_\alpha \circ \varphi_{p,m-p-1}(e)$. Thus the cocycle $c_a(\omega)$ is nontrivial. □

Let $H$ be a subgroup $G$ preserving the cycle $\gamma$. By proposition 3.6 the condition of theorem 3.7 can be satisfied only if $\dim G/H \geq m$.

## 4. Continuous and differentiable cocycles

Let $G$ be a topological group (or a Lie group which may be infinite-dimensional), $E$ a Frechét space, and $\rho : G \to \mathrm{GL}(E)$ a representation of $G$ in $E$. A cochain $f \in C^p(G, E)$ is continuous (differentiable) if it is a continuous (differentiable of class $C^\infty$) map from $G^p$ to $E$. Let $C^p_{\mathrm{c}}(G, E)$ and $C^p_{\mathrm{diff}}(G, E)$ denote the subspaces of continuous and differentiable cochains of the space $C^p(G, E)$, respectively. Denote by $H^*_{\mathrm{c}}(G, E)$ and $H^*_{\mathrm{diff}}(G, E)$ the cohomology of the complex $C^*_{\mathrm{c}}(G, E) = (C^p_{\mathrm{c}}(G, E), \delta')_{p\geq 0}$ and of $C^*_{\mathrm{c}}(G, E) = (C^p_{\mathrm{diff}}(G, E), \delta')_{p\geq 0}$, respectively. It is known (see [2] and [3]) that the inclusion $C^*_{\mathrm{diff}}(G, E) \subset C^*_{\mathrm{c}}(G, E)$ induces an isomorphism $H^*_{\mathrm{c}}(G, E) = H^*_{\mathrm{diff}}(G, E)$ whenever $G$ is a finite dimensional Lie group.

Later we apply these notions to $\Omega(M)$ as a topological vector space with the $C^\infty$-topology. Evidently both $C^{**}_{\mathrm{c}}(G, \Omega(M)) = (C^p_{\mathrm{c}}(G, \Omega^q(M)))$ and $C^{**}_{\mathrm{diff}}(G, \Omega(M)) = (C^p_{\mathrm{diff}}(G, \Omega^q(M)))$ are subcomplexes of $C^{**}(G, \Omega(M))$.

Let the conditions of theorem 2.2 be satisfied for an exact $m$-form $\omega \in \Omega(M)^G$. Assume that $G$ is a topological group or a Lie group. We investigate whether we can construct a sequence $\varphi_{i,m-i-1}$ for $i = 1, \dots, p$ as above which consists of continuous or differentiable cochains. For such a sequence $c_a(\omega)$ is a continuous or differentiable cocycle.

**Theorem 4.1.** *Let $M$ be a connected manifold with a countable base. Then for each $p > 0$ we have the following decomposition in the category of topological vector spaces*

$$\Omega^p(M) = d\Omega^{p-1}(M) \oplus H^p(M) \oplus \Omega^p(M)/Z^p(M),$$

*where $Z^p(M)$ is the space of closed $p$-forms. If $H^p(M) = 0$, $d\Omega^{p-1}(M) = Z^p(M)$ and $\Omega^p(M)/Z^p(M)$ are Frechét spaces.*

*Proof.* For compact $M$ the statement follows from the Hodge decomposition for the identity operator 1 on $\Omega^p(M)$ $1 = d \circ \delta \circ G \oplus H^p(M) \oplus \delta \circ d \circ G$ (see, for example, [9]).

For noncompact $M$ the statement follows from Palamodov's theorem (see [8], Proposition 5.4). □

**Corollary 4.2.** *Let the conditions of theorem 2.2 be satisfied for an exact $m$-form $\omega \in \Omega(M)^G$ and let $G$ be a topological group (a Lie group). Then one can construct a sequence $\varphi_{i,m-i-1}$ for $i = 1, \dots, p$ consisting of continuous (differentiable) cochains and thus for a singular smooth $(m-p-1)$-dimensional cycle $\alpha$ whose homology class $a$ is $G$-invariant the corresponding cocycle $c_a(\omega)$ is continuous (differentiable).*

*Proof.* The sequence $\varphi_{i,m-i-1}$ $(i = 1, \dots, p)$ is constructed successively by means of the equation $\delta' \varphi_{i-1,m-i} + (-1)^i d\varphi_{i,m-i-1} = 0$. By theorem 4.1 for each of the above cases this equation has a continuous (differentiable) solution $\varphi_{i,m-i-1} = L_{m-i} \circ \delta' \varphi_{i_1,m-i}$ whenever the cochain $\varphi_{i_1,m-i}$ is continuous (differentiable). □

## 5. Conditions of triviality of a differentiable cocycle $c_a(\omega)$

In this section we study the conditions of triviality of the cocycle $c_a(\omega)$ in the complex $C^*_{\text{diff}}(G, \mathbb{R})$ whenever $G$ is a Lie group and for the exact $m$-form $\omega$ we choose a sequence of cochains $\varphi_{i,m-i-1}$ $(i = 1, \dots, p)$ consisting of differentiable cochains.

**Theorem 5.1.** *Let $M$ be a $G$-manifold, where $G$ is a Lie group preserving an exact $m$-form $\omega$, let the conditions of theorem 2.2 be satisfied, and for $i = 1, \dots, p$ let $\varphi_{i,m-i-1} \in C^i_{\text{diff}}(G, \Omega^{m-i-1}(M))$. Then, if the cocycle $c_a(\omega)$ is trivial, there is a cochain $f \in C^{p-1}_{\text{diff}}(G, \Omega^0(G))$ such that $\delta'_G(f_\gamma \circ \varphi_{p-1,m-p} - d_G f) = 0$. If the group $G$ is connected, this condition implies the triviality of the cocycle $c_a(\omega)$.*

*Proof.* Let the cocycle $c_a(\omega)$ be trivial. By (10) there is a cochain $\bar{f} \in C^p_{\text{diff}}(G, \mathbb{R})$ such that for any $g, g_1, \dots, g_p \in G$ we have

$$(11) \quad (-1)^{p+1} (f_a \circ \varphi_{p,m-p-1}(g_1, \dots, g_p)(g) - b(g_1, \dots, g_p)) = D\bar{f}(g_1, \dots, g_p, g).$$

Define a cochain $f \in C^{p-1}_{\text{diff}}(G, \Omega^0(G))$ as follows

$$f(g_1, \dots, g_{p-1})(g) = \bar{f}(g_1, \dots, g_{p-1}, g).$$

By lemma 3.2 we have

$$\begin{aligned}(12) \quad & d_G((-1)^{p+1} ((f_a \circ \varphi_{p,m-p-1})(g_1, \dots, g_p)(g) - b(g_1, \dots, g_p)) \\ &= (-1)^{p+1} (f_\alpha \circ d\varphi_{p,m-p-1})(g_1, \dots, g_p)(g) \\ &= \delta'_G(f_\alpha \circ \varphi_{p-1,m-p})(g_1, \dots, g_p)(g).\end{aligned}$$

On the other hand, it is easy to check that

$$(13) \qquad D\bar{f}(g_1, \dots, g_p, g) = (\delta'_G f)(g_1, \dots, g_p)(g) + (-1)^{p+1} \bar{f}(g_1, \dots, g_p).$$

By (12) and (13), equation (11) implies

$$\delta'_G(f_\alpha \circ \varphi_{p-1,m-p})(g_1,\dots,g_p)(g) - (d_G\delta'_G f)(g_1,\dots,g_p)(g)) \\ = \delta'_G\left((f_\alpha \circ \varphi_{p-1,m-p} - d_G f)\right)(g_1,\dots,g_p)(g)) = 0.$$

Now suppose that the condition of the theorem is satisfied. We may assume that for any $g_1,\dots,g_p \in G$ we have $f(g_1,\dots,g_p)(e) = 0$. The above condition is equivalent to the following one

$$\delta''_G\left((-1)^{p+1} f_\alpha \circ \varphi_{p,m-p-1} - \delta'_G f\right)(g_1,\dots,g_p)(g) = 0.$$

Since the group $G$ is connected and $f(g_1,\dots,g_{p-1})(e) = 0$ we have

$$\left((-1)^{p+1} f_\alpha \circ \varphi_{p,m-p-1} - \delta'_G f\right)(g_1,\dots,g_p)(g) \\ = \left((-1)^{p+1} f_\alpha \circ \varphi_{p,m-p-1} - \delta'_G f\right)(g_1,\dots,g_p)(e) \\ = (-1)^{p+1}(b - \bar{f})(g_1,\dots,g_p).$$

Using (10) and (13) we get $(c_a(\omega) - D\bar{f})(g_1,\dots,g_p,g) = 0$. This concludes the proof. □

**Corollary 5.2.** *Let the conditions of theorem 5.1 be satisfied for $m = 2$ and $p = 1$. Then, if the 2-cocycle $c_a(\omega)$ is trivial, there is a smooth function $f$ on $G$ such that the 1-form $f_\alpha \circ (\varphi_{0,1}) - d_G f$ on $G$ is left invariant. In particular, the cohomology class of the form $f_\alpha \circ \omega$ in the complex $\Omega(G)^G$ is trivial. If the group $G$ is connected, the above condition implies the triviality of the cocycle $c_a(\omega)$.*

*Example* 5.3. Consider the abelian group $G = \mathbb{R}^n$ acting on itself by translations. Evidently an $m$-form

$$\omega(x) = \sum_{i_1<\dots<i_m} \omega_{i_1\dots i_m}(x)dx_{i_1} \wedge \dots \wedge dx_{i_m},$$

where $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, is $G$-invariant if and only if the coefficients $\omega_{i_1\dots i_p}$ are constant. Then the differential of the complex $\Omega(\mathbb{R}^n)^G$ is trivial.

Evidently the conditions of theorem 2.2 are satisfied for each nonzero $m$-form $\omega$ with constant coefficients on $\mathbb{R}^n$ for $p = m-1$. It is easy to check that the sequence of cochains $\varphi_{j,m-j-1} \in C^j_{\mathrm{diff}}(G, \Omega^{m-j-1}(\mathbb{R}^n))$ $(j = 0,\dots,m-1)$ corresponding to $\omega$ can be defined as follows.

$$\varphi_{j,m-j-1}(a_1,\dots,a_j) = \frac{(-1)^{j-1}}{m(m-1)\dots(m-j)} i(x)i(a_j)i(a_{j-1})\dots i(a_1)\omega,$$

where $a_k = (a_{k,1},\dots,a_{k,n}) \in \mathbb{R}^n$ $(k = 1,\dots,j)$ and on the right hand side we consider each $a_k$ as a constant vector field on $\mathbb{R}^n$ and $x$ as identical vector field on $\mathbb{R}^n$. Then we have

$$\delta'\varphi_{m-1,0}(a_1,\dots,a_m) = \frac{1}{m!} i(a_m)i(a_{m-1})\dots i(a_1)\omega,$$

where $a_1,\dots,a_m \in \mathbb{R}^n$.

Take the point $0 \in \mathbb{R}^n$ as the cycle $\alpha$. Then $F_\alpha : \Omega(\mathbb{R}^n)^G \to \Omega(G)$ is the identity map. By proposition 3.7 the cocycle $c_a(\omega)$ is nontrivial in the complex $C^*(G, \mathbb{R})$.

## 6. Cocycles on groups of diffeomorphisms

In this section we indicate nontrival cocycles for groups of diffeomorphisms of a manifold preserving a family of exact forms.

Let $(\omega_i)_{i\in I}$ be a family of smooth differential forms on a manifold $M$. Denote by $\operatorname{Diff}(M, (\omega_i))$ the group of diffeomorphisms of $M$ preserving all forms $\omega_i$. We consider $\operatorname{Diff}(M, (\omega_i))$ as a topological group with respect to $C^\infty$-topology or as a infinite-dimensional Lie group if such a structure on $\operatorname{Diff}(M, (\omega_i))$ exists. By proposition 4.2 one can suppose that the cocycle $c_a(\omega_i)$ is a continuous or differentiable cocycle.

Let $(\omega_i)_{i\in I}$ be a family of nonzero differential forms on $\mathbb{R}^n$ with constant coefficients. Put $G = \operatorname{Diff}(\mathbb{R}^n, (\omega_i))$.

**Theorem 6.1.** *Let $(\omega_i)_{i\in I}$ be a family of nonzero differential forms on $\mathbb{R}^n$ with constant coefficients such that $\deg \omega_i = d_i$. Then, for each $i \in I$ and for $0 \in \mathbb{R}^n$ as a zero dimensional cycle $\alpha$, the cocycle $c_a(\omega_i)$ is defined and nontrivial in the complex $C^*(G, \mathbb{R})$.*

*Proof.* Evidently the conditions of theorem 2.2 are satisfied for each form $\omega_i$ and $p = m_i - 1$ and then the cocycle $c_a(\omega_i)$ of the complex $C^*(G, \mathbb{R})$ is defined. Obviously the group $G$ contains the group $\mathbb{R}^n$ acting on $\mathbb{R}^n$ by translations. Consider the restriction of the cocycle $c_a(\omega)$ to the subgroup $\mathbb{R}^n$. By example 5.3 this restriction is nontrivial. Then the cocycle $c_a(\omega_i)$ is nontrivial as well. □

**Corollary 6.2.** *Let $M$ be a connected manifold such that*

$$H^1(M, \mathbb{R}) = \cdots = H^{m-1}(M, \mathbb{R}) = 0,$$

*let $\omega_0$ be a nonzero $m$-form on $\mathbb{R}^n$ with constant coefficients, and let $\omega_M$ be an exact $m$-form on $M$. Consider $\omega = \omega_0 + \omega_M$ as an $m$-form on $\mathbb{R}^n \times M$. Then for the group $G = \operatorname{Diff}(\mathbb{R}^n \times M, \omega)$ and a point $x \in \mathbb{R}^n \times M$ as a zero dimensional cycle $\alpha$, the cocycle $c_a(\omega)$ is defined and nontrivial in the complex $C^*(R^n \times M, \mathbb{R}))$.*

*Proof.* Evidently the conditions of theorem 2.2 for the form $\omega$ are satisfied and then the cocycle $c_a(\omega)$ of the complex $C^*(G, \mathbb{R})$ is defined. Consider the group $\operatorname{Diff}(\mathbb{R}^n, \omega_0)$ acting on the first factor of $\mathbb{R}^n \times M$ as a subgroup of $G$ and the restriction of the cocycle $c_a(\omega)$ to this subgroup. Since the subgroup $\operatorname{Diff}(\mathbb{R}^n, \omega_0)$ preserves the form $\omega_M$ as a form on $\mathbb{R}^n \times M$ by theorem 6.1, this restriction is a nontrivial cocycle. Thus $c_a(\omega)$ is a nontrivial cocycle of the complex $C^*(G, \mathbb{R})$ as well. □

We indicate the following partial case of corollary 6.2 (see also [6]).

**Corollary 6.3.** *Let $\omega_0$ be the standard symplectic 2-form on $\mathbb{R}^{2n}$ and let $m = 1, \dots, n$. Let $M$ be a connected manifold such that $2m \leq \dim M$,*

$$H^1(M, \mathbb{R}) = \cdots = H^{2m-1}(M, \mathbb{R}) = 0,$$

*and $\omega_M$ an exact $2m$-form on a manifold $M$. Consider the $2m$-form $\omega = \omega_0^m + \omega_M$ on $\mathbb{R}^{2n} \times M$. Then for the group $G = \mathrm{Diff}(\mathbb{R}^{2n} \times M, \omega)$ and a point $x \in \mathbb{R}^{2n} \times M$ as a zero dimensional cycle $\alpha$, the cocycle $c_a(\omega)$ is defined and nontrivial in the complex $C^*(R^{2n} \times M, \mathbb{R}))$.*

Let $M$ be a connected compact oriented manifold with a volume form $\upsilon_M$ such that $\int_M \upsilon_M = 1$.

**Theorem 6.4.** *Let $(\omega_i)_{i \in I}$ be a family of nonzero differential forms on $\mathbb{R}^n$ with constant coefficients such that $\deg \omega_i = d_i$. Consider the family $\{\omega_i \wedge \upsilon_M\}_{i \in I}$ of forms on $\mathbb{R}^n \times M$. For a cycle $\alpha = 0 \times M$ of the homology class $a$ on $\mathbb{R}^n \times M$ and each $i \in I$ the cocycle $c_a(\omega_i \wedge \upsilon_M)$ on the group $G = \mathrm{Diff}(\mathbb{R}^n \times M, (\omega_i \wedge \upsilon_M))$ with values in the trivial $G$-module $\mathbb{R}$ is defined and nontrivial.*

*Proof.* Evidently the conditions of theorem 2.2 for each form $\omega_i \wedge \upsilon_M$ and $p = n-1$ are satisfied. Then the cocycle $c_a(\omega_i \wedge \upsilon_M)$ of the complex $C^*(G, \mathbb{R})$ is defined.

Consider the group $\mathrm{Diff}(\mathbb{R}^n, (\omega_i))$ as a subgroup of the group $G$ and the restriction of the cocycle $c_a(\omega)$ to this subgroup. Since the subgroup $\mathrm{Diff}(\mathbb{R}^n, (\omega_i))$ preserves the form $\upsilon$ as a form on $\mathbb{R}^n \times M$, as in the proof of corollary 6.2 we can show that the above restriction is a nontrivial cocycle. Thus the cocycle $c_a(\omega)$ is nontrivial as well. □

We indicate the following partial case of theorem 6.4.

**Corollary 6.5.** *Let $\upsilon_0$ be the standard volume form on $\mathbb{R}^n$. Then for the cycle $a = 0 \times M$ the cocycle $c_a(\upsilon_0 \wedge \upsilon_M)$ on the group $G = \mathrm{Diff}(\mathbb{R}^n \times M, \upsilon_0 \wedge \upsilon_M)$ is defined and nontrivial in the complex $C^*(G, \upsilon_0 \wedge \upsilon_M), \mathbb{R})$.*

We consider the space $\mathbb{C}^{2n}$ and a skew-symmetric bilinear form $\omega$ of rank 2 on it. Let $\tilde{\omega}$ be the differential 2-form corresponding to $\omega$ on $\mathbb{C}^{2n}$ as a complex manifold. By definition the form $\tilde{\omega}$ has constant coefficients. Then $\tilde{\omega} = \tilde{\omega}_1 + i\tilde{\omega}_2$, where $i = \sqrt{-1}$, $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are real differential 2-forms with constant coefficients on $\mathbb{C}^{2n} = \mathbb{R}^{4n}$ as a real $4n$-dimensional manifold.

Similarly, consider the space $\mathbb{C}^n$, a sqew-symmetric $n$-form $\upsilon$ of maximal rank on it, and the corresponding differential $n$-form $\tilde{\upsilon}$ on $\mathbb{C}^n$. Consider the real differentiable $n$-forms $\tilde{\upsilon}_1$ and $\tilde{\upsilon}_2$ on $\mathbb{C}^n = \mathbb{R}^{2n}$ as a real $2n$-dimensional manifold defined by an equality $\tilde{\upsilon} = \tilde{\upsilon}_1 + i\tilde{\upsilon}_2$.

In both cases above we can apply theorem 6.1 to the forms $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\upsilon}_1$ and $\tilde{\upsilon}_2$. We leave to the reader to define the corresponding cocycles on the group of diffeomorphisms preserving the forms $\tilde{\omega}$ and $\tilde{\upsilon}$ and to formulate the statement similar to those of corollaries 6.2 and 6.4.

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