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DISCRETE SPECTRUM ASYMPTOTICS FOR THE SCHRÖDINGER OPERATOR WITH A SINGULAR POTENTIAL AND A MAGNETIC FIELD

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Abstract. Object of the study is the operator $H = H_0(h,\mu) + V$ in $L^2(\mathbb{R}^d)$, $d \ge 2$, where $H_0(h,\mu)$ is the Schrödinger operator with a magnetic field of intensity $\mu \ge 0$ and the Planck constant $h \in (0, h_0]$. The electric (real-valued) potential $V = V(x)$ is assumed to be asymptotically homogeneous of order $-\beta$, $\beta \ge 0$ as $x \to 0$. One obtains asymptotic formulae with remainder estimates as $h \to 0$, $\mu h \le C$ for the trace $\mathcal{M}_s = \operatorname{tr}\{\psi g_s(H)\}$ where $\psi \in C_0^\infty(\mathbb{R}^d)$, $g(\lambda) = \lambda_-^s, s \in [0,1]$. Due to the condition $\mu h \le C$ the leading term of $\mathcal{M}_s$ does not depend on μ. It depends on the relation between the parameters d, s and β. There are five regions, in which either leading terms or remainder estimates have different form. In one of these regions $\mathcal{M}_s$ admits a two-term asymptotics. In this case, for an asymptotically Coulomb potential the second term coincides with the well known Scott correction term.

1. Introduction

We study in $L^2(\mathbb{R}^d)$, $d \ge 2$, the Schrödinger operator

$$\left.\begin{aligned} H_{\mathbf{a},V} &= H_{\mathbf{a},V}(h,\,\mu) = H_0(h,\mu) + V,\\ H_0 &= H_0(h,\mu) = \sum_{l=1}^d (-ih\partial_l - \mu a_l)^2. \end{aligned}\right\} \tag{1.1}$$

Here $H_0 = H_{\mathbf{a},0}$ is the unperturbed operator with a magnetic real-valued vector-potential $\mathbf{a} = (a_1, a_2, \dots, a_d)$, the parameter $\mu \ge 0$ has the meaning of intensity of the field. The function V(electric potential) is real-valued. Sometimes for the sake of brevity we use the notation $\mathfrak{a} = (\mathbf{a}, V)$. We analyse the asymptotics as[2] $h \to 0$, $\mu \ge 0$, $\mu h \le C$ of traces of the form

$$\mathcal{M}_s(h,\mu) = \mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = \operatorname{tr}\{\psi g_s(H_{\mathfrak{a}})\}. \tag{1.2}$$

Here $\psi \in C_0^\infty(\mathbb{R}^d)$ and the function $g_s, s \ge 0$, is defined as follows:

$$\left.\begin{aligned} g_s(\lambda) &= |\lambda|^s, \lambda < 0,\\ g_s(\lambda) &= 0,\ \lambda \ge 0. \end{aligned}\right\}$$

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[2] Here and in what follows we denote by C and c (with or without indices) various positive constants whose precise value is of no importance.

In case $\mu = 0$ we write $H_V(h)$ (or H_V) and $\mathcal{N}_s(h;\psi,V)$ instead of $H_{\mathbf{a},V}(h,\mu)$ and $\mathcal{M}_s(h,\mu;\psi,\mathfrak{a})$ respectively. The condition $\mu h \le C$ will ensure that the leading terms in the asymptotic formulae which we obtain, do not depend on the magnetic field. In this sense the magnetic field under consideration is moderate, though the inequality $\mu h \le$C allows μ to grow as $h\to 0$.

The quantity (1.2) can be viewed as a "local version" of the sum

$$\mathcal{M}_s(h,\mu;1,\mathfrak{a}) = \sum_{k\ge 1} |\lambda_k|^s,$$

where $\lambda_k = \lambda_k(h,\mu;\mathfrak{a})$, $k\ge 1$, are negative eigenvalues of $H_{\mathfrak{a}}$ enumerated in the non-decreasing order. In particular, $\mathcal{M}_0$ is a local counterpart of the number of all negative eigenvalues. Note that due to the truncation ψ the trace (1.2) can be finite even if negative spectrum of $H_{\mathfrak{a}}$ is not discrete.

The asymptotics of $\mathcal{M}_s(h,\mu)$ has been analysed in [14] in the case $\mathbf{a}, V \in C_0^\infty(\mathbb{R}^d)$. It was shown there that for any $s\in[0,1]$, $h\to 0, 0\le\mu\le Ch^{-1}$, the trace (1.2) obeys the formula

$$\mathcal{M}_s(h,\mu;\psi,V) = \mathfrak{W}_s(h;\psi,V) + \langle\mu\rangle^{s+1} O(h^{-(d-s-1)}),\ \langle\mu\rangle = (1+\mu^2)^{\frac{1}{2}}, \tag{1.3}$$

with the standard Weylian leading term

$$\left.\begin{aligned} \mathfrak{W}_s(h) = \mathfrak{W}_s(h;\,\psi,V) = \Xi_s h^{-d}\int \psi(x)\big(V_-(x)\big)^{s+\frac{d}{2}}dx,\\ \Xi_s = \frac{|\mathbb{S}^{d-1}|}{(2\pi)^d}\int_0^1 t^{d-1}(1-t^2)^s\,dt, \end{aligned}\right\} \tag{1.4}$$

where $|\mathbb{S}^{d-1}|$ stands for the surface area of the $(d-1)$-dimensional unit sphere. Corresponding result for the case $\mu=0$ was obtained in [8], [9].

In the present paper the smoothness assumption is removed. More precisely, we assume that the function V is C^∞ outside $x=0$ and asymptotically homogeneous at $x=0$:

$$V(x)\sim W(x) = \frac{\Phi(\hat x)}{|x|^\beta},\ x\to 0;\ \hat x = \frac{x}{|x|},\ 0\le\beta<2, \tag{1.5}$$

with some $\Phi\in C^\infty(\mathbb{S}^{d-1})$. In Section 2 we shall formulate the conditions on V in a more precise form. In fact, one could have assumed that there are several points at which the potential behaves similarly to (1.5). Our results however, are stated in a form which allows one to decouple the singularities using an appropriate partition of unity and then, by means of the translation, perform in each patch the reduction to the potential satisfying (1.5).

Under condition (1.5) the formula (1.3) is not necessarily true. The answer depends on the relation between the parameters β, s and d. A natural parameter that determines the form of the asymptotics in this case is

$$\omega = \omega(\beta,s) = \frac{2\beta s}{2-\beta}. \tag{1.6}$$

The values d and $d-s-1$ (which are the orders of h in the leading and remainder term in (1.3) respectively) serve as "threshlods" – when ω crosses either of them, the asymptotics of $\mathcal{M}_s$ changes its form. Note in particular, that the condition $\omega < d$ is necessary and sufficient for $\mathfrak{W}_s$ to be finite. Let us discuss individually all possible cases. Below $h \to 0,\ 0 \le \mu \le Ch^{-1}$ and $0 \le s \le 1$.

1. $\omega > d$. The leading term of $\mathcal{M}_s$ is completely determined by the asymptotic potential W defined in (1.5). Namely,

$$\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = h^{-\omega}\biggl(\psi(0)\mathcal{N}_s(1;1,W) + o(1)\biggr). \tag{1.7}$$

Finiteness of the trace $\mathcal{N}_s(1;1,W)$ for any $d \ge 2,\ \omega > d$ will follow from the Cwickel type estimate (2.19).

2. $\omega = d$. The asymptotics is still determined by W. However, in contrast to (1.7) the leading term can be calculated effectively. Denote

$$\mathfrak{B}_s(\Phi) = \frac{2\Xi_s}{2-\beta}\int_{\mathbb{S}^{d-1}} \Phi_-(\vartheta)^{\frac{d}{2}+s} d\vartheta.$$

Then

$$\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = h^{-d}|\ln h|\psi(0)\mathfrak{B}_s(\Phi) + (|\ln h|+1)o(h^{-d}). \tag{1.8}$$

We emphasize that the remainders in (1.7), (1.8) are bounded uniformly in $\mu \le Ch^{-1}$.

3. $d-s-1 < \omega < d$. The leading order is given by the classical Weyl coefficient (1.4). The singularity of the potential gives rise to a second term, that occupies an intermediate position between the main term and the remainder in (1.3):

$$\begin{aligned}\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = \mathfrak{W}_s(h;\psi,V) \ &+ h^{-\omega}\psi(0)\Theta_s \\ &+ o(h^{-\omega}) + O(\langle\mu\rangle^{s+1}h^{s+1-d}).\end{aligned} \tag{1.9}$$

The coefficient $\Theta_s = \Theta_s(\Phi,\beta)$ depends only on W. Loosely speaking, it is defined as the difference of two infinite quantities:

$$\Theta_s(\Phi,\beta) = \mathcal{N}_s(1;1,W) - \mathfrak{W}_s(1;1,W). \tag{1.10}$$

Theorems 2.4 and 2.4′ in Sect. 2 provide two equivalent regularized versions of this definition. Presumably, in general, Θ_s does not admit any explicit representation in terms of W. However, for the particular case of a purely Coulomb potential, it does. Let $d=3, s=1, \beta=1,\ \Phi=-q, q\ge 0$. Then, using precise formulae for the eigenvalues of $H_W(1)$, one can prove that $\Theta_1 = -q^2/8$. This expession for Θ_1 has been known since a long time in connection with the so-called Scott correction to the ground state energy of a large atom. We refer to [8] for details and further references. It is worth mentioning that by virtue of the condition $d-s-1<\omega<d$, the choice $\beta=1, s=1$ implies that $d=3$. In other words, the Scott correction term is meaningful only in the three-dimensional case.

4. $\omega = d-s-1$. Then

$$\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = \mathfrak{W}_s(h;\psi,V) + O(|\ln h|h^{s+1-d}) + O(\langle\mu\rangle^{s+1}h^{s+1-d}). \tag{1.11}$$

5. $\omega < d - s - 1$. The asymptotics looks like (1.3) – the contribution from the singularity gets "absorbed" by the remainder:

$$\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = \mathfrak{W}_s(h;\psi,V) + O(\langle\mu\rangle^{s+1}h^{s+1-d}). \tag{1.12}$$

In particular, the local counting function $\mathcal{M}_0$ satisfies (1.12).

We point out that in all five cases above the leading term of $\mathcal{M}_s$ does not depend on μ. In cases (3)–(5) the asymptotics gets more precise if $\mu h \to 0$. On the contrary, if $\mu h = const$, the formulae (1.9), (1.11), (1.12) loose their asymptotic character since the remainders have the same order as $\mathfrak{W}_s$ in this situation. This observation is consistent with the well known fact that the Weyl term no more describes the behaviour of $\mathcal{M}_s(h,\mu)$ if $\mu h \geq c$. We refer to [12](see also [15]), where the asymptotics of $\mathcal{M}_1(h,\mu;1,\mathfrak{a})$ was studied for $d = 3$ and a homogeneous magnetic field for any $\mu \geq 0$. It was shown that for $\mu h \geq c$ the leading term is to be replaced by another coefficient that takes into account the magnetic field.

To conclude the introduction we sketch main steps of the proof. As in [8], we analyse separately contributions from the regions around the origin and away from it. Precisely, we split $\mathcal{M}_s(h,\mu;\psi,\mathfrak{a})$ into the sum of $\mathcal{M}_s(h,\mu;\psi_1,\mathfrak{a})$ and $\mathcal{M}_s(h,\mu;\psi_2,\mathfrak{a})$, where ψ_1 (resp. ψ_2) is supported inside (resp. outside) the ball $B(r) = \{x : |x| \leq r\}$ of radius $r \sim h^{\frac{2}{2-\beta}}$. The share of the ball depends on the interrelation between ω and d. We explain further proof in the most interesting case $d - s - 1 < \omega < d$. Inside the ball one can neglect the magnetic field and replace the potential V by its asymptotics W. This reduces the problem to the study of $\mathcal{N}_s(h;\psi_1,W)$. Using homogeneity of W one can "scale out" the parameter h, after which the definition (1.10) yields almost automatically that

$$\mathcal{M}_s(h,\mu;\psi_1,\mathfrak{a}) \sim \mathcal{N}_s(h;\psi_1,W) = \mathfrak{W}_s(h;\psi_1,W) + h^{-\omega}\Theta_s + o(h^{-\omega}). \tag{1.13}$$

The change W back to V affects only the error $o(h^{-\omega})$.

Since V is smooth outside $B(r)$, we can use for $\mathcal{M}_s(h,\mu;\psi_2,\mathfrak{a})$ the asymptotics (1.3) established in [14]. However the result of [14] does not apply directly, because by (1.5) $V(x)$ is not bounded uniformly in h for $|x| \geq r$ and, consequently, we cannot control the remainder estimate in (1.3). To avoid this difficulty, we use the so-called multiscale approach invented by V. Ivrii (see [9]–[11], [8]), that provides a good control of the remainder estimate under fairly general conditions on V. A version of this method adjusted to our purposes, is described in [14]. As a result, we obtain

$$\mathcal{M}_s(h,\mu;\psi_2,\mathfrak{a}) = \mathfrak{W}_s(h;\psi_2,V) + o(h^{-\omega}) + O(\langle\mu\rangle^{s+1}h^{s+1-d}).$$

Adding up this relation with (1.13), we arrive at (1.9). We point out again that in the exterior of $B(r)$ we need only the first term of the asymptotics with a proper remainder estimate. The second term in (1.9) is produced totally by the interior of $B(r)$ (see (1.13)).

Precise definitions of objects we shall be working with, and statements of the main results are given in Sect. 2. In Sect. 3, 4 we investigate the possibility of replacing the potentials $\mathbf{a}, V$ with $\mathbf{0}, W$ in the asymptotics of (1.2). In Sect. 5 we summarize the results from [14] on the multiscale analysis and establish existence of the limit (1.10) in a suitable regularized sense. Sect. 6 contains the proof of the asymptotic formulae discussed above.

Notation. As a rule, m stands for a $d-$tuple of non-negative integer numbers: $m = (m_1, m_2, \cdots, m_d)$, $|m| = m_1 + m_2 + \cdots + m_d$.

For any measurable function f one writes $f_\pm = (|f| \pm f)/2$.

For a domain $X \subset \mathbb{R}^d$ we denote by $\mathcal{B}^\infty(X)$ the set of functions $f \in C^\infty(X)$ bounded along with all their derivatives. This space forms a Fréchet space with the family of natural semi-norms $|||f|||_m = \sup_x |\partial_x^m f(x)|, \quad \forall |m| \ge 0$.

A constant C is said to be uniform in $f \in \mathcal{B}^\infty(X)$ (or $f \in C_0^\infty(X)$), if it depends only on the constants in the estimates $|||f|||_m \le C_m, |m| \ge 0$

A function g is said to belong to $\mathcal{B}^\infty(X)$ uniformly in $f \in \mathcal{B}^\infty(X)$ if the derivatives $|\partial_x^m g(x)|$, $|m| \ge 0$, are estimated by constants which are uniform in $f \in \mathcal{B}^\infty(X)$.

For spaces of vector-valued functions $\mathbf{a}(x) = \{a_1(x), \dots, a_d(x)\}$ we use the same notation as for scalar functions. This convention does not cause any confusion. For instance, the notation $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$ means that each component of $\mathbf{a}$ belongs to $L^2_{loc}(\mathbb{R}^d)$.

$B(z, E)$, $z \in \mathbb{R}^d$, $E > 0$, denotes the closed ball $\{x \in \mathbb{R}^d : |x - z| \le E\}$; $B(E) = B(0, E)$. Sometimes we use open balls $\mathring{B}(z, E) = \{x \in \mathbb{R}^d : |x - z| < E\}$ and $\mathring{B}(E) = \mathring{B}(0, E)$ as well.

For any self-adjoint operator T, $D(T)$ denotes its domain and $R(z; T) = (T - z)^{-1}$ – its resolvent for $z \in \mathbb{C}$ outside the spectrum of T. If T is semi-bounded, $T[\cdot, \cdot]$ and $D[T]$ stand for the associated quadratic form and its domain respectively.

Notation $\mathfrak{S}_p$, $p \ge 1$ stands for the Neumann-Schatten class of compact operators with the norm

$$\| T \|_p = \left(\operatorname{tr}\{(T^*T)^{\frac{p}{2}}\}\right)^{\frac{1}{p}}.$$

Classes $\mathfrak{S}_1, \mathfrak{S}_2$ are called the trace class and the Hilbert-Schmidt class respectively. Operators $T_1 \in \mathfrak{S}_p$, $T_2 \in \mathfrak{S}_q$ and any bounded operator T_0 satisfy the following inequalities:

$$\left.\begin{aligned} \| T_1 T_2 \|_t \le \| T_1 \|_p \ \| T_2 \|_q, \ t^{-1} = p^{-1} + q^{-1}; \\ \| T_1 T_0 \|_p \le \| T_0 \| \, \| T_1 \|_p. \end{aligned}\right\} \tag{1.14}$$

These and other properties of compact operators can be found in the book [7].

2. Main results

1. **Definition of the Schrödinger operator with a magnetic field**. Let $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$, $d \ge 2$ be a real-valued (vector-)function. We denote by $Q_l, \Pi_l = \Pi^*$, $l = 1, 2, \dots, d$, closures of the differential operators $-ih\partial_l - \mu a_l$, $-ih\partial_l$, on $C_0^\infty(\mathbb{R}^d)$. We define the unperturbed Schrödinger operator as

$$H_0 = H_{\mathbf{a},0}(h, \mu) = Q_l^* Q_l, \ H_{\mathbf{0},0} = H_{\mathbf{0},0}(h) = \Pi_l^* \Pi_l = -h^2 \Delta.$$

Here and below we assume summation over repeating indices. The operator H_0 can also be interpreted as that associated with the quadratic form $(Q_l u, Q_l u)$ (see [3]). Due to the condition $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$, the set $C_0^\infty(\mathbb{R}^d)$ is a form core for H_0.

To define the perturbed operator we use the following estimates resulting from the diamagnetic inequality (see [3]):

Proposition 2.1. *Let X be multiplication by a measurable function and $\kappa > 0$. Then for any $\lambda > 0$*

$$\|XR(-\lambda; H_0)^\kappa\| \le \|XR(-\lambda; H_{\mathbf{0},0})^\kappa\| \tag{2.1}$$

and for any positive integer n

$$\big\| XR(-\lambda; H_0)^\kappa \big\|_{2n} \le \big\| XR(-\lambda; H_{\mathbf{0},0})^\kappa \big\|_{2n}. \tag{2.2}$$

It follows immediately from (2.1) with $\kappa = 1/2$ that the inequality

$$\|Xu\|^2 \le \epsilon(H_{\mathbf{0},0}u, u) + M(h)\|u\|^2, \ \forall u \in C_0^\infty(\mathbb{R}^d), \tag{2.3}$$

entails

$$\|Xu\|^2 \le \epsilon(H_0 u, u) + M(h)\|u\|^2, \ \forall u \in C_0^\infty(\mathbb{R}^d), \tag{2.4}$$

with the same positive constants ϵ and $M(h)$. Let V be a real-valued function such that the estimate (2.3) is fulfilled for the function $X = |V|^{1/2}$ with some $\epsilon < 1$ and $M(h) > 0$. Due to (2.4) the perturbed operator $H_{\mathfrak{a}} = H_{\mathbf{a},V} = H_0 + V$ is well defined in the form sense.

To study the local trace (1.2) it will be sufficient to assume that the operator is of the form (1.1) only in a neighbourhood of $\operatorname{supp}\psi$. Its behaviour outside is irrelevant. To distinguish it from the "true" Schrödinger operator $H_{\mathfrak{a}}$ we shall use for such an operator the notation $A_{\mathfrak{a}} = A_{\mathfrak{a}}(h, \mu)$ (or simply A).

Assumptions on A will be stated in terms of the quadratic form $A[\cdot,\cdot]$. Below $\mathcal{D} \subset \mathbb{R}^d$ denotes an open bounded domain.

Assumption 2.2. *The operator A is selfadjoint in $L^2(\mathbb{R}^d)$, semibounded from below and for any $\zeta \in C_0^\infty(\mathcal{D})$ the following conditions are satisfied:*

1. *For any $u \in D[A]$ one has $u\zeta \in D[A]$; there exists a function $\zeta_1 \in C_0^\infty(\mathcal{D})$ (depending on ζ) such that*
$$A[u, \zeta v] = A[\zeta_1 u, \zeta v],$$
for all $u, v \in D[A]$;
2. *There exist real-valued functions $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$ and V with $X = |V|^{1/2}$ obeying (2.3) with some $\epsilon \in (0,1), M(h) > 0$, such that for any $v \in D[A]$, $u \in D[H_{\mathfrak{a}}]$, $\mathfrak{a} = (\mathbf{a}, V)$, one has $\zeta u \in D[A]$, $\zeta v \in [H_{\mathfrak{a}}]$ and*
$$A[\zeta u, \zeta v] = H_{\mathfrak{a}}[\zeta u, \zeta v].$$

Though this assumption may look cumbersome, it is very natural in the sense that it is fulfilled for some standard special cases. For instance, the operator $A_{\mathfrak{a}}$ defined by the differential expression

$$\sum_l (-ih\partial_l - a_l)^2 + V(x)$$

with the Dirichlet condition on the sphere $\{x : |x| = R\}, R > 0$, obeys Assumption 2.2 with $\mathcal{D} = \mathring{B}(R)$.

Our basic tool in the study of the operator $A = A_{\mathfrak{a}}$ is the following resolvent identity:

Lemma 2.3. *Let the operator A be as specified above. Then for any function $\zeta \in C_0^\infty(\mathcal{D})$ one has*

$$\zeta R(z; A_{\mathfrak{a}}) = R(z; H_{\mathfrak{a}})\zeta + R(z; H_{\mathfrak{a}}) Z R(z; A_{\mathfrak{a}}), \tag{2.5}$$
$$Z = Z(\zeta) = -ih(2Q_l^* \partial_l \zeta + ih\Delta\zeta). \tag{2.6}$$

Proof. Clearly, it suffices to verify that for any $u \in D(A)$, $v \in D(H)$, $H = H_{\mathfrak{a}}$, the inequality holds:

$$(\zeta u, Hv) - (Au, \bar\zeta v) = (u, Z^*(\zeta) v). \tag{2.7}$$

To prove this notice that $u \in D[A]$, $v \in D[H]$, so that by Assumption 2.2 the l.h.s. of (2.7) equals

$$\begin{aligned} H[\zeta u, v] - A[\zeta_1 u, \bar\zeta v] &= H[\zeta u, v] - H[\zeta_1 u, \bar\zeta v] = (Q_l \zeta u, Q_l v) - (Q_l \zeta_1 u, Q_l \bar\zeta v) \\ &= \big([Q_l, \zeta]\zeta_1 u, Q_l v\big) - \big(Q_l \zeta_1 u, [Q_l, \bar\zeta] v\big) \\ &= -ih(\partial_l \zeta u, Q_l v) - ih(Q_l \zeta_1 u, \partial_l \bar\zeta v). \end{aligned}$$

Recall that $D(Q_l) \subset D(Q_l^*)$, so that one can rewrite this as

$$-ih(\partial_l \zeta u, Q_l v) - ih(\zeta_1 u, Q_l \partial_l \bar\zeta v) = -ih\big(u, (2\partial_l \bar\zeta Q_l - ih\Delta\bar\zeta) v\big),$$

which coincides with the r.h.s. of (2.7). □

In the same way one proves the identity for powers of the resolvents:

$$\zeta R(z; A)^k = R(z; H)^k \zeta + \sum_{j=1}^{k} R(z; H)^j Z(\zeta) R(z; A)^{k-j+1}, \ \forall k \in \mathbb{N}. \tag{2.8}$$

For $\operatorname{tr}\{\psi g_s(A)\}$ we keep the same notation as for $\operatorname{tr}\{\psi g_s(H)\}$, i.e. $\mathcal{M}_s(h, \mu; \psi, \mathfrak{a})$. This will not cause any confusion in what follows.

We shall use extensively the following scaling properties of the operator $A_{\mathfrak{a}}$ and the trace $\mathcal{M}_s(h, \mu; \psi, \mathfrak{a})$. Let f, ℓ be some positive numbers. and let $z \in \mathbb{R}^d$. Let the unitary dilation operator $\mathcal{U}_\ell$ and the translation operator $\mathcal{T}_z$ be defined by

$$(\mathcal{U}_\ell u)(x) = \ell^{\frac{d}{2}} u(\ell x), \ (\mathcal{T}_z u)(x) = u(x + z).$$

Denote

$$\hat V(x) = f^{-2} V(\ell x + z), \ \hat{\mathbf{a}}(x) = \ell^{-1} \mathbf{a}(\ell x + z), \ \hat\psi(x) = \psi(\ell x + z). \tag{2.9}$$

Define also two auxiliary parameters which will play the role of the Planck constant and the size of the magnetic field after the scaling:

$$\alpha = \frac{h}{f\ell}, \quad \nu = \frac{\mu \ell}{f}. \tag{2.10}$$

Since V obeys (2.3), we have

$$\| \, |\hat{V}|^{\frac{1}{2}} u\|^2 \le \epsilon\big(H_{\mathbf{0},0}(\alpha)u, u\big) + \hat{M}(\alpha)\|u\|^2, \ \hat{M}(\alpha) = f^{-2}M(h), \ \forall u \in C_0^\infty(\mathbb{R}^d). \tag{2.11}$$

It is clear that the operator

$$f^{-2}(\mathcal{U}_\ell \mathcal{T}_z)A_{\mathfrak{a}}(\mathcal{U}_\ell \mathcal{T}_z)^* \tag{2.12}$$

satisfies Assumption 2.2 with the set $\hat{\mathcal{D}} = \{x \in \mathbb{R}^d : \ell x + z \in \mathcal{D}\}$ and the operator $H_{\hat{\mathfrak{a}}}(\alpha, \nu)$, $\hat{\mathfrak{a}} = \{\hat{\mathbf{a}}, \hat{V}\}$. Therefore it is natural to denote the operator (2.12) by $A_{\hat{\mathfrak{a}}}$. By the unitary equivalence of trace,

$$\mathcal{M}_s(h, \mu; \psi, \mathfrak{a}) = f^{2s}\mathcal{M}_s(\alpha, \nu; \hat{\psi}, \hat{\mathfrak{a}}). \tag{2.13}$$

Note also the scaling property of the Weyl coefficient (1.4):

$$\mathfrak{W}_s(h; \psi, V) = f^{2s}\mathfrak{W}_s(\alpha; \hat{\psi}, \hat{V}), \tag{2.14}$$

which can be verified by direct calculation.

Notice that in the case $\mathcal{D} = \mathring{B}(z, \ell)$ the set $\hat{\mathcal{D}}$ is simply $\mathring{B}(1)$.

2. **Conditions on $\mathbf{a}, V$.** Let us specify conditions on the potential V and vector-potential $\mathbf{a}$, for which we shall obtain asymptotic formulae described in Introduction. Assume that $V, \mathbf{a} \in C^\infty(\mathbb{R}^d \setminus \{0\})$, $\mathbf{a}$ is continuous, and for all $x \ne 0$

$$\left.\begin{aligned} |\partial^m V(x)| \le C_m |x|^{-\beta-|m|}, \quad 0 \le \beta < 2, \ |m| \ge 0; \\ |\partial^m \mathbf{a}(x)| \le C_m |x|^{1-|m|}, \quad |m| \ge 1. \end{aligned}\right\} \tag{2.15}$$

Moreover,

$$V(x) = |x|^{-\beta}\big(\Phi(\hat{x}) + U(x)\big), \tag{2.16}$$

with a function $\Phi \in C^\infty(\mathbb{S}^{d-1})$ and some $U \in L^\infty(\mathbb{R}^d)$ such that

$$\text{v-sup}_{|x| \le t} |U(x)| \le U_0(t), \ U_0 \in L^\infty(0, \infty), \quad U_0(t) \to 0, \ t \to 0. \tag{2.17}$$

As a rule we use the notation $W(x) = \Phi(\hat{x})|x|^{-\beta}$. Notice that (2.15) contains no estimates on the function $\mathbf{a}$ itself, but only on its derivatives. This fact is quite natural, since the constant component of $\mathbf{a}$ can be chosen arbitrarily or eventually eliminated by a simple gauge transformation.

It is easy to check that the functions $X = |V|^{\frac{1}{2}}$ and $X = |W|^{\frac{1}{2}}$ obey (2.3) for any $\epsilon > 0$ and

$$M(h) = C(\epsilon h^2)^{-\frac{\beta}{2-\beta}}, \tag{2.18}$$

the constant C here being dependent only on C_0 in (2.15). In the sequel, we usually subsume ϵ into the constant C and omit ϵ from notation.

Unless otherwise stated all the functions denoted by $O(\,\cdot\,)$ or $o(\,\cdot\,)$ will be uniform in all C^∞-functions involved (in the sense specified in the end of Introduction) and in $\mu \in [0, Ch^{-1}]$. Moreover, they will be also uniform in the functions $V, \mathbf{a}$ satisfying (2.15), (2.16) and (2.17). In other words, they will depend only on the constants C_m

from (2.15) and the function U_0. For instance, $o(1)$ in (1.7) stands for a function which tends to zero as $h \to 0$ uniformly in $\psi \in C_0^\infty(\mathbb{R}^d)$, the functions V, W, U and $\mu \le Ch^{-1}$. The symbols "lim", "lim sup" are used in situations, when no uniformity (in the sense specified above) is claimed.

2. **Results**. We always assume that $A_\mathfrak{a}$ obeys Assumption 1.1 with $\mathcal{D} = \mathring{B}(4E)$ for some fixed $E > 0$. The function ψ is supposed to belong to $C_0^\infty(B(E/2))$. As was mentioned in Introduction, the form of the asymptotics is governed by the parameter ω defined in (1.6). For some values of ω the leading or the second term are expressed in terms of the trace $\mathcal{N}_s$ for the operator H_W. In the next Theorem we prepare some properties of this trace needed for stating our main result. Let $\mathfrak{W}_s$ be the Weyl coefficient defined in (1.4). Below we denote $\phi_\rho(x) = \phi(x\rho^{-1}),\ \rho > 0$, for any function ϕ.

Theorem 2.4. *Let $\Phi \in C^\infty(\mathbb{S}^{d-1}),\ s \in (0,1],\ \beta \in (0,2)$ and*

$$W(x) = \frac{\Phi(x)}{|x|^\beta},\ \ \Phi \in C^\infty(\mathbb{S}^{d-1}).$$

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be a function such that $\phi(x) = 1, |x| \le 1$ and let $\phi_\rho(x) = \phi(x/\rho),\ \rho > 0$.

(1) *If $\omega > d$, then*

$$\mathcal{N}_s(1;\phi_\rho, W) = \mathcal{N}_s(1;1,W) + o(1),\ \ \rho \to \infty. \tag{2.19}$$

(2) *If $d - s - 1 < \omega(\beta, s) < d$, then*

$$\mathcal{N}_s(1;\phi_\rho, W) - \mathfrak{W}_s(1;\phi_\rho, W) = \Theta_s + o(1),\quad \rho \to \infty, \tag{2.20}$$

with some real number $\Theta_s = \Theta_s(\Phi, \beta)$ independent of the function ϕ.

The relation (2.20) can be interpreted as a "regularized limit" of $\mathcal{N}_s(1;\phi_\rho, W)$. A remarkable fact is that there is another choice of regularization that yields the same value of Θ_s:

Theorem 2.4′. *Let the parameters β, s and the function W be as in Theorem 2.4, and let $d - s - 1 < \omega(\beta, s) < d$. Then*

$$\lim_{\kappa \to +0} \big(\mathcal{N}_s(1;1,W+\kappa) - \mathfrak{W}_s(1;1,W+\kappa)\big) = \Theta_s(\Phi, \beta). \tag{2.21}$$

This limit may be not uniform in the function W.

We stress again that the relations (2.19), (2.20) are uniform in W and ϕ. If one does not require any uniformity, the proof of (2.19) is fairly simple. Indeed, the finiteness of $\mathcal{N}_s$ in (2.19) can be easily obtained from the following bound for the number of negative eigenvalues of $H_{W+\kappa},\ \kappa > 0$ (see [4]). For any $V \in L^q_w(\mathbb{R}^d),\ q > d/2$,

$$\mathcal{N}_0(1;1,V+\kappa) \le C\kappa^{\frac{d}{2}-q}\|V\|^q_{L^q_w} = C\kappa^{\frac{d}{2}-q}\sup_{t>0}\big(t^q\,\mathrm{mes}\{x : V_-(x) > t\}\big). \tag{2.22}$$

This bound results from a Cwickel type estimate (see [6], [13]). Clearly, $W \in L^q_w(\mathbb{R}^d)$ with $q = d\beta^{-1} > d/2$, so that (2.22) yields

$$\mathcal{N}_0(1;1;W+\kappa) \le C\kappa^{\frac{d}{2}-\frac{d}{\beta}},\ C = C(\beta,\Phi). \tag{2.23}$$

For $\omega > d$, we have

$$\mathcal{N}_s(1;1,W) = -\int_0^\infty \kappa^s d\mathcal{N}_0(1;1,W+\kappa) = s\int_0^\infty \kappa^{s-1}\mathcal{N}_0(1;1,W+\kappa)d\kappa,$$

Now one concludes from (2.23) that $g_s(H_W) \in \mathfrak{S}_1$, i.e.

$$\mathcal{N}_s(1;1,W) < \infty,\ \omega(\beta,s) > d.$$

Note that for $d \ge 3$ (2.23) can be obtained also from the classical Rosenblum-Lieb-Cwickel estimate:

$$\mathcal{N}_0(1;1;W+\kappa) \le C\int \big(W(x)+\kappa\big)_-^{\frac{d}{2}} dx.$$

Since ϕ_ρ converges weakly to 1 as $\rho \to \infty$, and $g_s(H_W)$ is trace-class, we have

$$\lim_{\rho\to\infty} \mathcal{N}_s(1;\phi_\rho,W) = \mathcal{N}_s(1;1,W).$$

This convergence however is not a priori uniform in W. The proof of Theorem 2.4 is more complicated, but provides the uniformity in W.

Next Theorem constitutes the main result of the paper.

Theorem 2.5. *Let $s \in [0,1]$ be a fixed number and let $h \in (0,h_0]$, $\mu \ge 0, \mu h \le C$. Suppose that the operator $A_{\mathfrak{a}}$ satisfies Assumption 2.2 with $\mathcal{D} = \mathring{B}(4E)$ and $\psi \in C_0^\infty(B(E/2))$. Then the following assertions take place:*

1. *If $\omega > d$, then (1.7) holds.*
2. *If $\omega = d$, then (1.8) holds.*
3. *If $d - s - 1 < \omega < d$, then (1.9) holds.*
4. *If $\omega = d - s - 1$, then (1.11) holds.*
5. *If $\omega < d - s - 1$, then (1.12) holds.*

The remainder estimates are uniform in the functions $\mathbf{a}, V, \psi$ and may depend on E.

It is worth pointing out that no information on $A_{\mathfrak{a}}$ outside $B(4E)$ is involved in Theorem 2.5. In particular, the lower bound of $A_{\mathfrak{a}}$ is irrelevant.

Our assumption that the potential has only one singularity located at the origin, has been imposed for convenience only. One can easily obtain corresponding asymptotics for $\mathcal{M}_s$ in the case of many singular points, by reducing the problem to Theorem 2.5 with the help of an appropriate partition of unity and a translation transformation. The partition of unity argument permits also to extend Theorem 2.5 to arbitrary $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\mathcal{D} \supset \operatorname{supp}\psi$.

To conclude this section we discuss some properties of the coefficient Θ_s defined in (2.20). As was mentioned in Introduction, in general it cannot be expressed explicitly in terms of W. We can only say that it is homogeneous in Φ:

$$\Theta_s(q\Phi,\beta)=q^{\frac{2s}{2-\beta}}\Theta_s(\Phi,\beta),\ \forall q>0. \tag{2.24}$$

This relation follows from definition (2.20), homogeneity of the function W and the equalities

$$\left.\begin{aligned}\mathcal{N}_s(1;\phi_\rho,qW)&=q^{\frac{2s}{2-\beta}}\mathcal{N}_s(1;\phi_{\rho'},W),\\ \mathfrak{W}_s(1;\phi_\rho,qW)&=q^{\frac{2s}{2-\beta}}\mathfrak{W}_s(1;\phi_{\rho'},W),\end{aligned}\right\}\rho'=\rho q^{\frac{1}{2-\beta}},$$

which result from (2.13), (2.14) with

$$f=\ell^{-1},\ \ell=q^{\frac{1}{\beta-2}}.$$

Nevertheless, the limit (2.20) can be calculated explicitly for $d=3, s=1,\beta=1$ and $\Phi(\theta)=-q, q\ge 0$. In this case

$$\Theta_1(-q,1)=-\frac{q^2}{8}. \tag{2.25}$$

A proof of this result, based on the relation (2.21) and a precise formula for the eigenvalues of $H_W(1)$, was given in [8]. This proof is so simple that we reproduce it here.

Due to (2.24) one can assume that $q=1$. The eigenvalues of $H_W(1)$ are $-(4n^2)^{-1}$, $n\ge 1$, each of them having a multiplicity n^2 (see e.g. [5]). Consequently,

$$\mathcal{M}_1(\kappa)=\mathcal{M}_1(1;1,W+\kappa)=\sum_{n=1}^{m}\left(\frac{1}{4n^2}-\kappa\right)n^2,\ \forall\kappa>0. \tag{2.26}$$

Here m denotes the entire part of $a=(2\sqrt{\kappa})^{-1}$. On the other hand,

$$\begin{aligned}\mathfrak{W}_1(1;1,W+\kappa)&=\frac{1}{15\pi^2}\int\left(\kappa-|x|^{-1}\right)_-^{\frac{5}{2}}dx=\frac{4}{15\pi\sqrt{\kappa}}\int_0^1(1-t)^{\frac{5}{2}}t^{-\frac{1}{2}}dt\\ &=\frac{4}{15\pi\sqrt{\kappa}}B(1/2,7/2)=\frac{1}{12\sqrt{\kappa}},\end{aligned} \tag{2.27}$$

where $B(\,\cdot\,,\,\cdot\,)$ denotes the beta function. It follows from (2.26) that

$$\begin{aligned}\mathcal{M}_1(\kappa)&=\frac{m}{4}-\kappa\sum_{n=1}^{m}n^2\\ &=\frac{m}{4}-\frac{\kappa}{12}m(2m+1)(2m+2)=\frac{m}{4}-\frac{\kappa m}{6}(2m^2+3m+1).\end{aligned}$$

Representing m as $a+v$ with some $v\in(-1,0]$, we obtain that

$$\mathcal{M}_1(\kappa)=\frac{a}{4}-\frac{\kappa a^3}{3}-\frac{\kappa a^2}{2}-\kappa a^2v+\frac{v}{4}+O(\sqrt{\kappa}).$$

Taking into account that $a=(2\sqrt{\kappa})^{-1}$, this leads to

$$\mathcal{M}_1(\kappa)=\frac{1}{12\sqrt{\kappa}}-\frac{1}{8}+O(\sqrt{\kappa}).$$

Comparing this with (2.27) and using (2.21), we arrive at (2.25).

3. Reduction to $H_{\mathfrak{a}}$

The purpose of this and the next section is to show that in the asymptotics of $\operatorname{tr}\{\psi g(A_{\mathfrak{a}})\}$ one can replace $A_{\mathfrak{a}}$ by the "asymptotic" operator H_W with the potential (1.5). We shall do this in two steps. Firstly, in this section we justify the change $A_{\mathfrak{a}} \to H_{\mathfrak{a}}$. Further reduction to H_W will be done in Section 4. All the bounds to be obtained do not depend on the magnetic potential, so that without loss of generality one can set $\mu = 1$.

1. First we study the resolvent $R(z, H_{\mathfrak{a}})$. We always assume that $\mathbf{a} \in L^2_{loc}(\mathbb{R}^d)$ and the function $X = |V|^{\frac{1}{2}}$ satisfies (2.3) with some $\epsilon < 1$, so that the operator $H_{\mathfrak{a}}$ is well-defined. Furthermore, it follows from (2.4) that

$$\inf \sigma(H_{\mathfrak{a}}) \geq -M(h). \tag{3.1}$$

Sometimes for shortness we write simply H and M instead of $H_{\mathfrak{a}}$ and $M(h)$ respectively. It will be convenient to assume that $M \geq 1$. We denote $d_M(z) = \operatorname{dist}\{z, [-M, \infty)\}$. Everywhere below $z \in \mathbb{C} \setminus \{[-M, \infty)\}$, so that $R(z, H)$ exists and is bounded. The integer number l(with or without indices) takes the values $1, 2, \dots, d$ and the numbers $m, m_1, m_2, \dots$ equal either 0 or 1.

We begin with some straightforward estimates. Let X denote an arbitrary function satisfying (2.4) with the same ϵ, M as the function $|V|^{\frac{1}{2}}$. It follows from the definition of H_0 that

$$\|Q_l^m R(-\lambda; H_0)^{\frac{1}{2}}\| \leq \lambda^{\frac{m-1}{2}},\ m = 0, 1, \forall \lambda > 0. \tag{3.2}$$

Furthermore, (2.4) entails that

$$\left.\begin{aligned} \|X R(-\lambda; H_0)^{\frac{1}{2}}\| &\leq \epsilon^{\frac{1}{2}}, \\ \|(H_0 + \lambda)^{\frac{1}{2}} R(-\lambda; H)^{\frac{1}{2}}\ \| &\leq (1-\epsilon)^{-\frac{1}{2}}, \end{aligned}\right\} \forall \lambda \geq \epsilon^{-1} M. \tag{3.3}$$

Further, due to the resolvent identity, for any $\lambda > M$ we have the equality:

$$R(z; H) = R(-\lambda; H_0)^{\frac{1}{2}} S(\lambda, z) R(-\lambda; H_0)^{\frac{1}{2}}, \tag{3.4}$$

$$S(\lambda, z) = (H_0 + \lambda)^{\frac{1}{2}} R(-\lambda; H)^{\frac{1}{2}} \big(I + (\lambda + z) R(z; H)\big) R(-\lambda; H)^{\frac{1}{2}} (H_0 + \lambda)^{\frac{1}{2}}.$$

By the second inequality in (3.3),

$$\|S(\lambda, z)\| \leq 2(1-\epsilon)^{-1} \frac{\lambda + |z|}{d_M(z)}, \lambda \geq \epsilon^{-1} M. \tag{3.5}$$

As a rule, in what follows we omit the dependence of the coefficients on ϵ.

Lemma 3.1. *Let X satisfy (2.4) and let $m_1, m_2 = 0, 1$ be such that $m_1 + m_2 \leq 1$. Then*

$$\|X^{m_1} Q_l^{m_2} R(-\lambda; H)^{\frac{1}{2}}\| \leq C \lambda^{\frac{m_1 + m_2 - 1}{2}},\ \forall \lambda \geq \epsilon^{-1} M, \tag{3.6}$$

$$\|X^{m_1} Q_l^{m_2} R(z; H)\| \leq C \frac{(|z| + M)^{\frac{m_1 + m_2}{2}}}{d_M(z)}, \tag{3.7}$$

Proof. The estimate (3.6) is a consequence of (3.2) and (3.3). Let us prove (3.7). By (3.4), (3.5) and (3.6) with $\lambda = |z| + \epsilon^{-1}M$, we have

$$\|X^{m_1} Q_l^{m_2} R(z;H)\| \le \|X^{m_1} Q_l^{m_2} R(-\lambda;H_0)^{\frac12}\| \, \|S(\lambda,z)\| \, \|R(-\lambda;H_0)^{\frac12}\| \le C\lambda^{\frac{m_1+m_2-1}{2}} \frac{\lambda+|z|}{d_M(z)} \lambda^{-\frac12}.$$

This provides (3.7). □

Let us proceed to estimates of $R(z,H)$ in the classes of compact operators. For the resolvent of the operator $H_{\mathbf{0},0}$ necessary bounds can be found in [14, Lemma 3.3]. Their proof is based on a simple criterion for the operators of the form $a(x)b(-ih\partial)$ (see, e.g. [13]). It provides for any $f \in L^p(\mathbb{R}^d)$, $\kappa > 0$ and $\lambda > 0$ the bound

$$\big\| fR(-\lambda;H_{\mathbf{0},0})^\kappa \big\|_p \le C_p \|f\|_{L^p} \lambda^{-\kappa+\frac{d}{2p}} h^{-\frac{d}{p}}, \ \forall p \ge 2, p > d(2\kappa)^{-1}.$$

In combination with Proposition 2.1 this yields for $\kappa > 0$ and any integer $n \ge 1$:

$$\big\| fR(-\lambda;H_0)^\kappa \big\|_{2n} \le C_n \|f\|_{L^{2n}} \lambda^{-\kappa+\frac{d}{4n}} h^{-\frac{d}{2n}}, 2n > d(2\kappa)^{-1}. \tag{3.8}$$

Moreover, the following Lemma holds.

Lemma 3.2. *Let $f \in L^{2n}(\mathbb{R}^d)$ for some $n > d/2$. Then*

$$\big\| fR(z;H) \big\|_{2n} \le C\|f\|_{L^{2n}} h^{-\frac{d}{2n}} \frac{(M+|z|)^{\frac{d}{4n}}}{d_M(z)}.$$

Proof. By (3.4) for $\lambda = |z| + \epsilon^{-1}M$, and (1.14),

$$\big\| fR(z;H) \big\|_{2n} \le \big\| fR(-\lambda;H_0)^{\frac12} \big\|_{2n} \|S(\lambda,z)\| \, \|R(-\lambda;H_0)^{\frac12}\|.$$

It remains to apply (2.2), (3.8) for $\kappa = 1/2$ and (3.5). □

2. Now we shall study the properties of the resolvent "sandwiched" between two functions with disjoint supports. Until the end of this Section the functions χ and ϕ will be supposed to satisfy the conditions

$$\left.\begin{aligned} \chi \in C_0^\infty(B(\rho)); \ |\chi| \le 1; \\ \phi \in \mathcal{B}^\infty(\mathbb{R}^d), \ \operatorname{supp}\phi \subset \mathbb{R}^d \setminus B(\nu\rho), \ \ |\phi| \le 1, \end{aligned}\right\} \tag{3.9}$$

with some $\rho > 0$ and $\nu > 1$. All the constants in Theorems below do not depend on ρ, χ, ϕ but may depend on ν.

Lemma 3.3. *Let χ and ϕ obey (3.9). Let $m, m_1, m_2 = 0, 1$ be such that $m + m_1 \le 1$. Then for any $N \ge 0$*

$$\|X^m \chi Q_{l_1}^{m_1} R(z;H) \ (Q_{l_2}^*)^{m_2} \phi\| \le C_{N,\nu} \frac{(M+|z|)^{\frac{m+m_1+m_2}{2}}}{d_M(z)} \left[\frac{(M+|z|)h^2}{\rho^2 d_M(z)^2}\right]^N. \tag{3.10}$$

Let n and k be two integers such that $n > d/2$ and $k \le 2n$. Then for $p = 2n/k$ and any $N \ge 1$

$$\Big\| X^m \chi\, Q_{l_1}^{m_1} R(z;H)(Q_{l_2}^*)^{m_2}\phi \Big\|_p \le C_{N,\nu} \frac{(M+|z|)^{\frac{m+m_1+m_2}{2}}}{d_M(z)} \left[\frac{\rho(M+|z|)^{\frac12}}{h}\right]^{\frac{d}{p}} \left[\frac{(M+|z|)^{\frac12} h}{\rho d_M(z)}\right]^{Nk}. \tag{3.11}$$

In particular, for any $N > d/2$

$$\Big\| X^m \chi\, Q_{l_1}^{m_1} R(z;H)(Q_{l_2}^*)^{m_2}\phi \Big\|_1 \le C_{N,\nu} \frac{(M+|z|)^{\frac{m+m_1+m_2}{2}}}{d_M(z)} \left[\frac{\rho(M+|z|)^{\frac12}}{h}\right]^{d} \left[\frac{(M+|z|)^{\frac12} h}{\rho d_M(z)}\right]^{2N}. \tag{3.12}$$

Proof. We prove first (3.11). It suffices to do that for $N = 1$. The result for all N will follow if one replaces n by nN and k by kN. We start with the following simple observation: Let $\eta \in C^\infty(\mathbb{R})$ be a function such that $0 \le \eta \le 1$ and $\eta(t) = 1, \quad t \le 1/3; \quad \eta(t) = 0, \quad t \ge 2/3$. Let us define the following family of functions $\chi^{(j)} \in C_0^\infty(\mathbb{R}^d),\ j = 1, 2, \dots, k$:

$$\chi^{(j)}(x) = \eta\left[\frac{k}{\rho(\nu-1)}\left(|x| - \rho - \frac{\rho(\nu-1)(j-1)}{k}\right)\right]. \tag{3.13}$$

It is clear that $\chi = \chi\chi^{(1)}$ and $\chi^{(j)} = \chi^{(j)}\chi^{(j+1)}, \quad \chi^{(j)}\phi = 0$, so that

$$\begin{aligned}\chi^{(j)} R(z;H)(Q_{l_2}^*)^{m_2}\phi &= -\,[R(z;H), \chi^{(j)}](Q_{l_2}^*)^{m_2}\phi\\ &= R(z;H)[H_0, \chi^{(j)}]\chi^{(j+1)} R(z;H)(Q_{l_2}^*)^{m_2}\phi.\end{aligned}$$

Therefore the representation holds:

$$\begin{aligned}&X^m \chi Q_{l_1}^{m_1} R(z;H)(Q_{l_2}^*)^{m_2}\phi\\ &\qquad = X^m \chi Q_{l_1}^{m_1} \prod_{j=1}^{k}\Big\{R(z;H)[H,\chi^{(j)}]\Big\} R(z;H)(Q_{l_2}^*)^{m_2}\phi.\end{aligned}$$

According to (3.4) for any $\lambda > M$ one can write

$$R(z;H)[H,\chi^{(j)}] = R(-\lambda;H_0)^{\frac12} S T_j (H_0+\lambda)^{\frac12}, \quad S = S(\lambda, z),$$

where

$$T_j = R(-\lambda;H_0)^{\frac12}[H_0, \chi^{(j)}] R(-\lambda;H_0)^{\frac12}.$$

Therefore

$$\begin{aligned}X^m \chi Q_{l_1}^{m_1} R(z;H)(Q_{l_2}^*)^{m_2}\phi &= X^m \chi Q_{l_1}^{m_1} R(-\lambda;H_0)^{\frac12}\\ &\quad\times\Big\{\prod_{j=1}^{k} S T_j\Big\} S R(-\lambda;H_0)^{\frac12}(Q_{l_2}^*)^{m_2}\phi,\end{aligned} \tag{3.14}$$

Noting that

$$[H_0,\zeta] = -ih(Q_l^*\partial_l\zeta + \partial_l\zeta Q_l), \quad \forall\zeta\in\mathcal{B}^\infty(\mathbb{R}^d),$$

we conclude that

$$T_j = -ih(Z_j + Z_j^*),$$
$$Z_j = \big[R(-\lambda;H_0)^{\frac12}Q_l^*\big]\big[\partial_l\chi^{(j)}R(-\lambda;H_0)^{\frac12}\big].$$

Consequently, (3.2), (3.8) and (3.13) yield that

$$\| Z_j\|_{2n} \le C\|\partial\chi^{(j)}\|_{L^{2n}}\lambda^{-\frac12+\frac{d}{4n}}h^{-\frac{d}{2n}} \le C_{k,\nu}\rho^{\frac{d}{2n}-1}\lambda^{-\frac12+\frac{d}{4n}}h^{-\frac{d}{2n}}.$$

Let $\lambda = \epsilon^{-1}M + |z|$. Then by (1.14) and (3.5),

$$\Big\|\prod_{j=1}^k\{ST_j\}\Big\|_{\frac{2n}{k}} \le \|S\|^k\prod_{j=1}^k\| T_j\|_{2n} \le Ch^k\left[\frac{M+|z|}{d_M(z)}\right]^k\rho^{\frac{dk}{2n}-k}\lambda^{-\frac{k}{2}+\frac{dk}{4n}}h^{-\frac{dk}{2n}}$$
$$\le C\left[\frac{\rho(M+|z|)^{\frac12}}{h}\right]^{\frac{dk}{2n}}\left[\frac{(M+|z|)^{\frac12}h}{\rho d_M(z)}\right]^k.$$

Now we get from (3.14):

$$\| X^m\chi Q_{l_1}^{m_1}R(z;H)(Q_{l_2}^*)^{m_2}\phi\|_{\frac{2n}{k}} \le \|X^mQ_{l_1}^{m_1}R(-\lambda;H_0)^{\frac12}\|$$
$$\Big\|\prod_{j=1}^k\{ST_j\}\Big\|_{\frac{2n}{k}}\|S\|\|R(-\lambda;H_0)^{\frac12}(Q_{l_2}^*)^{m_2}\|.$$

Using (3.2), (3.3), (3.5), this yields (3.11). The estimate (3.12) is a direct consequence of (3.11) for $k = 2n$.

The bound (3.10) can be proven analogously. □

Corollary 3.4. *Let the functions χ,ϕ satisfy (3.9) and the numbers m, m_1, m_2 be as in Lemma 3.3. Then for any $k\ge 1$ and $N > d/2$*

$$\| X^m\chi\, Q_{l_1}^{m_1}R(z;H)^k(Q_{l_2}^*)^{m_2}\phi\|_1$$
$$\le C_{k,N}\,\frac{(M+|z|)^{\frac{m+m_1+m_2}{2}}}{d_M(z)^k}\left[\frac{\rho(M+|z|)^{\frac12}}{h}\right]^d\left[\frac{(M+|z|)^{\frac12}h}{d_M(z)\rho}\right]^{2N}. \tag{3.15}$$

Proof is by induction. Corollary is already proved for $k = 1$. Assume that (3.15) is true for some k. We shall deduce from here that it is also true for $k+1$. Let $\chi_1\in C_0^\infty\big(B((1+2\nu)\rho/3)\big)$ be a function such that $\chi_1(x) = 1$, $|x|\le(2+\nu)\rho/3$ and $\phi_1 = 1-\chi_1$. Then by (1.14)

$$\| X^m\chi Q_{l_1}^{m_1}R(z;H)^{k+1}\,(Q_{l_2}^*)^{m_2}\phi\|_1$$
$$\le \|X^m\chi Q_{l_1}^{m_1}R(z;H)^k\|\,\| \chi_1R(z;H)(Q_{l_2}^*)^{m_2}\phi\|_1$$
$$+\| X^m\chi Q_{l_1}^{m_1}R(z;H)^k\phi_1\|_1\,\|R(z;H)(Q_{l_2}^*)^{m_2}\phi\|.$$

Since $\operatorname{supp}\phi_1\subset\mathbb{R}^d\setminus B\big((2+\nu)\rho/3\big)$, the pairs of functions χ_1,ϕ and χ,ϕ_1 satisfy the conditions of Lemma 3.3. The desired estimate for the first summand in the r.h.s. follows from Lemmas 3.3 and 3.1. The second summand obeys the same estimate due to the inductive assumption and Lemma 3.1. □

In the next Lemma we assume that $\rho\ge C$.

Lemma 3.5. *Let χ obey (3.9) with $\rho \ge C$ and let k, n be two integers such that $n > d/2, k \le 2n$. Then*

$$\| \chi R(\pm iM; H)^k \|_p \le C M^{-k} \left[\frac{\rho M^{\frac12}}{h} \right]^{\frac{d}{p}}, \quad p = \frac{2n}{k}. \tag{3.16}$$

Proof. By Lemma 3.2 the bound (3.16) is true for $k = 1$. Further proof is by induction: assuming that (3.16) is true for some k, we shall deduce (3.16) for $k+1$. Let $\chi_1 \in C_0^\infty(B(3\rho))$ be a function such that $\chi_1(x) = 1,\ |x| \le 2\rho; |\chi_1| \le 1$, and $\phi = 1 - \chi_1$. Then $\operatorname{supp}\phi \subset \mathbb{R}^d \setminus B(2\rho)$. By (1.14)

$$\begin{aligned} \| \chi R(\pm iM; H)^{k+1} \|_{\frac{2n}{k+1}} &\le \| \chi R(\pm iM; H)\phi \|_{\frac{2n}{k+1}} \|R(\pm iM; H)^k\| \\ &+ \| \chi R(\pm iM; H) \|_{2n} \| \chi_1 R(\pm iM; H)^k \|_{\frac{2n}{k}}. \end{aligned}$$

Since χ, ϕ obey the conditions of Lemma 3.3, by (3.11) with $N = 1$ the first summand is bounded by

$$C \frac{1}{d_M(\pm iM)^{k+1}} \left[\frac{\rho (2M)^{\frac12}}{h} \right]^{\frac{d(k+1)}{2n}} \left[\frac{(2M)^{\frac12} h}{d_M(\pm iM)\rho} \right]^{k+1} \le C M^{-k-1} \left[\frac{\rho M^{\frac12}}{h} \right]^{\frac{d(k+1)}{2n}}.$$

Here we used the bounds $M \ge 1, \rho \ge C$. Further, by Lemma 3.2 and the inductive assumption the second term does not exceed

$$C \left\{ \|\chi\|_{L^{2n}} h^{-\frac{d}{2n}} (2M)^{\frac{d}{4n}} M^{-1} \right\} \left\{ M^{-k} \left[\frac{\rho M^{\frac12}}{h} \right]^{\frac{dk}{2n}} \right\} \le C M^{-k-1} \left[\frac{\rho M^{\frac12}}{h} \right]^{\frac{d(k+1)}{2n}}.$$

This and preceding estimate provide (3.16) with $k+1$. Proof is completed. □

Corollary 3.6. *Let χ be as in Lemma 3.5. Let $g = g(\lambda)$ be a function such that $g(\lambda) = 0,\ \lambda \ge \lambda_0$ and $|g(\lambda)| \le C|\lambda|^s,\ s \ge 0$. Then*

$$\| \chi g(H) \|_1 \le C M^s \left[\frac{\rho M^{\frac12}}{h} \right]^d.$$

Proof. It follows from (3.16) with $2n = k > d$ that

$$\begin{aligned} \| \chi g(H) \|_1 &\le \| \chi R(iM; H)^k \|_1 \|(H - iM)^k g(H)\| \\ &\le C M^{-k} \left[\frac{\rho M^{\frac12}}{h} \right]^d \|(H - iM)^k g(H)\|. \end{aligned}$$

For $H \ge -M$, the last factor is bounded by CM^{s+k}. □

Corollary 3.7. *Let χ be as in Lemma 3.5 and let $m_1, m_2 = 0, 1$ be such integers that $m_1 + m_2 \le 1$. Then for any $k > d$*

$$\left\| X^{m_1} \chi Q_l^{m_2} R(\pm iM; H)^{k+1} \right\|_1 \le C M^{\frac{m_1+m_2}{2} - k - 1} \left[\frac{\rho M^{\frac{1}{2}}}{h} \right]^d.$$

Proof. Let χ_1 be a function introduced in the proof of Lemma 3.5. Then

$$\left\| X^{m_1} \chi Q_l^{m_2} R(\pm iM; H)^{k+1} \right\|_1 \le \left\| X^{m_1} \chi Q_l^{m_2} R(\pm iM; H) \phi \right\|_1 \| R(\pm iM; H)^k \| + \| X^{m_1} \chi Q_l^{m_2} R(\pm iM; H) \| \left\| \chi_1 R(\pm iM; H)^k \right\|_1.$$

It remains to apply Lemma 3.1, (3.16) and (3.12). □

3. Until now the operator under consideration was supposed to have the form $H_{\mathfrak{a}}$ in the entire space. As was explained in Sect. 2, we may assume that A coincides with some $H = H_{\mathfrak{a}}$ on some open set only, for we are dealing with local traces of the form (1.2). ¿From now on we work with an operator A satisfying Assumption 2.2 with $\mathcal{D} = \mathring{B}(4\rho), \rho > 0$. Our goal is to show that in the asymptotics of $\operatorname{tr}\{\chi g(A)\}$, $\chi \in C_0^\infty(B(4\rho))$ one can replace the operator A by $H = H_{\mathfrak{a}}$. To this end we use the identities (2.5), (2.8).

Lemma 3.8. *Let the operator A obey Assumption 2.2 with $\mathcal{D} = \mathring{B}(4\rho)$, where $\rho \ge C$. Then for any χ satisfying (3.9), any integers $k \ge 1$ and $N > d/2$, the bound holds:*

$$\left\| \chi \left[R(z; A)^k - R(z; H)^k \right] \right\|_1 \le C_N \left[\frac{\rho (M + |z|)^{\frac{1}{2}}}{h} \right]^d \left[\frac{h (M + |z|)^{\frac{1}{2}}}{\rho d_M(z)} \right]^{2N+1} \frac{1}{|\operatorname{Im} z|^k}, \tag{3.17}$$

The constant C_N does not depend on χ, V and ρ.

Proof. Let $\eta \in C_0^\infty(\mathbb{R})$ be a function such that $\eta(t) = 1$, $|t| \le 2$. Denote $\chi_1(x) = \eta(|x|\rho^{-1})$, $\phi = 1 - \chi_1$. Then $\chi_1 \chi = \chi$ and the pair ϕ, χ satisfies conditions of Lemma 3.3. Due to the obvious identity

$$\chi \left[R(z; A)^k - R(z; H)^k \right] = \chi \left[\chi_1 R(z; A)^k - R(z; H)^k \chi_1 \right] - \chi R(z; H)^k \phi$$

the problem amounts to proving the bound (3.17) for the operators

$$T_1 = \chi \left[\chi_1 R(z; A)^k - R(z; H)^k \chi_1 \right],$$
$$T_2 = \chi R(z; H)^k \phi.$$

By (3.15), the estimate (3.17) is obviously satisfied for T_2. Further, by virtue of (2.8), to prove (3.17) for T_1 it suffices to establish (3.17) for each of the operators

$$T_1^{(j)} = \chi R(z; H)^j Z(\chi_1) R(z; A)^{k-j+1},\ j = 1, \dots, k$$

(See (2.6) for definition of $Z(\cdot)$). Now it is clear that

$$\| T_1^{(j)} \|_1 \le h \Big[2 \| \chi R(z;H)^j Q_l \partial_l \chi_1 \|_1 + h \| \chi R(z;H)^j \Delta \chi_1 \|_1 \Big] \| R(z;A)^{k-j+1} \|.$$

By definition $\operatorname{supp} \partial\chi_1$ and $\operatorname{supp} \chi$ obey the conditions of Corollary 3.4. Taking into account that $|\partial\chi_1| \le C\rho^{-1}$ and $|\Delta\chi_1| \le C\rho^{-2}$, estimating the terms in the brackets by means of (3.15), and the last factor by $|\operatorname{Im} z|^{-k+j-1}$, we get (3.17). □

Corollary 3.9. *Let the operator A and the function χ be as in Lemma 3.8. Then for any $g \in C_0^\infty(\mathbb{R})$ one has*

$$\| \chi g(A) \|_1 \le C \left[\frac{\rho M^{\frac12}}{h} \right]^d.$$

The constant C depends on g only.

Proof. For $z = iM$ and $k > d$

$$\begin{aligned} \| \chi g(A) \|_1 &\le \| \chi R(z;A)^k \|_1 \| (A-z)^k g(A) \| \\ &\le \| (A-z)^k g(A) \| \Big(\| \chi R(z;H)^k \|_1 + \| \chi [R(z;A)^k - R(z;H)^k] \|_1 \Big). \end{aligned}$$

The first factor is bounded by CM^k, since $g \in C_0^\infty$. By (3.16) and (3.17) the second factor does not exceed

$$CM^{-k} \left[\frac{\rho M^{\frac12}}{h} \right]^d.$$

This provides the desired estimate. □

4. When studying the difference $g(A) - g(H)$ we use the following representation for a function of a selfadjoint operator in terms of its resolvent (see [2]):

Proposition 3.10. *Let $g \in C_0^\infty(\mathbb{R})$. Then for any selfadjoint operator B the relation holds:*

$$\begin{aligned} g(B) &= \sum_{j=0}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j g(\lambda) \operatorname{Im}[i^j R(\lambda + i; B)] d\lambda \\ &+ \frac{1}{\pi(n-1)!} \int_0^1 \tau^{n-1} d\tau \int_{\mathbb{R}} \partial^n g(\lambda) \operatorname{Im}[i^n R(\lambda + i\tau; B)] d\lambda, \ \forall n \ge 2. \end{aligned} \tag{3.18}$$

Combining Lemma 3.8 and this representation, we obtain

Theorem 3.11. *Suppose that A satisfies Assumption 2.2 with $\mathcal{D} = \mathring{B}(4\rho)$, $\rho \ge C$, and χ obeys (3.9). Let $g \in C^\infty(\mathbb{R})$ be a function such that $g(\lambda) = 0$ if $\lambda \ge \lambda_0$ and for some $s \ge 0$, $L \ge L_0 > 0$*

$$|\partial^n g(\lambda)| \le C_n L^n \langle \lambda \rangle^s, \ \forall n \in \mathbb{N}. \tag{3.19}$$

Then for any $N > (d+1)/2 + s$

$$\| \chi[g(A) - g(H)]\|_1 \le C_N L^{2N+3} M^{s+1} \left[\frac{\rho M^{\frac{1}{2}}}{h}\right]^d \left[\frac{h M^{\frac{1}{2}}}{\rho}\right]^{2N+1}, \tag{3.20}$$

where the constant C_N *does not depend on* $\mathbf{a}, V, \chi$ *and* ρ.

Proof. Let $\lambda \in \mathbb{R}$ and $0 < |\tau| \le 1$. Denote

$$\delta(\lambda, \tau) = R(\lambda + i\tau; A) - R(\lambda + i\tau; H).$$

Then (3.17) with $k = 1$ yields for any $N > d/2$:

$$\| \chi\delta(\lambda, \tau)\|_1 \le C_N \left[\frac{h}{\rho}\right]^{2N+1-d} M^{N+\frac{d+1}{2}} |\tau|^{-2N-2}, \ -2M \le \lambda \le \lambda_0, \tag{3.21}$$

$$\| \chi\delta(\lambda, \tau)\|_1 \le C_N \left[\frac{h}{\rho}\right]^{2N+1-d} |\lambda|^{-N+\frac{d-1}{2}} |\tau|^{-1}, \ \lambda \le -2M. \tag{3.22}$$

The representation (3.18) does not apply to the function g since it is allowed to grow as $\lambda \to -\infty$. Instead of g we use its modification. Let $\zeta \in C^\infty(\mathbb{R})$ be a function such that

$$\zeta(t) = 0, \quad t \le -2;$$
$$\zeta(t) = 1, \quad t \ge -3/2.$$

Let $\tilde{M} = \max\{(\inf A)_-, M\}$. Define the function $\tilde{g} \in C_0^\infty(\mathbb{R})$ by the equality $\tilde{g}(\lambda) = \zeta(\lambda/\tilde{M}) g(\lambda)$. Then because of (3.1) we have $\tilde{g}(A) = g(A), \ \ \tilde{g}(H) = g(H)$. By virtue of (3.19)

$$|\partial^n \tilde{g}(\lambda)| \le C_n L^n \langle\lambda\rangle^s, \ \forall n \in \mathbb{N}, \tag{3.23}$$

with some constants C_n independent of A, H. According to (3.18)

$$\tilde{g}(A) - \tilde{g}(H) = I_1^{(n)} + I_2^{(n)}, \ \forall n \in \mathbb{N};$$

$$I_1^{(n)} = \sum_{j=0}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j \tilde{g}(\lambda) \operatorname{Im}\{i^j \delta(\lambda, 1)\} d\lambda,$$

$$I_2^{(n)} = \frac{1}{\pi(n-1)!} \int_0^1 \tau^{n-1} d\tau \int_{\mathbb{R}} \partial^n \tilde{g}(\lambda) \operatorname{Im}\{i^n \delta(\lambda, \tau)\} d\lambda.$$

Let us estimate first the integral $I_2^{(n)}$. To that end set $n = 2N+3$. Then, according to (3.23) and (3.21), (3.22)

$$\| \chi I_2^{(n)}\|_1 \le C_N \ L^{2N+3} \left[\frac{h}{\rho}\right]^{2N+1-d} \left\{ \int_0^1 d\tau \int_{-2M \le \lambda \le \lambda_0} M^{N+\frac{d+1}{2}} \langle\lambda\rangle^s d\lambda \right.$$
$$\left. + \int_0^1 \tau^{2N+1} d\tau \int_{\lambda \le -2M} |\lambda|^{-N+\frac{d-1}{2}+s} d\lambda \right\}.$$

Assuming that $N > (d+1)/2 + s$ (so that the second integral exists), we obtain

$$\| \chi I_2^{(n)} \|_1 \le C_N L^{2N+3} \left[\frac{h}{\rho}\right]^{2N+1-d} M^{N+s+\frac{d+3}{2}}. \tag{3.24}$$

Now, in view of (3.21), (3.22) and (3.23),

$$\begin{aligned} \| \chi I_1^{(n)} \|_1 \le & C_N L^{2N+2} \left[\frac{h}{\rho}\right]^{2N+1-d} \biggl\{ \int_{-2M\le\lambda\le\lambda_0} M^{N+\frac{d+1}{2}} \langle\lambda\rangle^s d\lambda \\ & + \int_{\lambda\le -2M} |\lambda|^{-N+\frac{d-1}{2}+s} d\lambda \biggr\} \\ \le & C_N L^{2N+2} \left[\frac{h}{\rho}\right]^{2N+1-d} M^{N+s+\frac{d+3}{2}}. \end{aligned}$$

Combining this bound with (3.24), we obtain from here (3.20). □

4. Reduction to asymptotic potential

Here we continue the analysis of the previous section and show that under suitable conditions one can replace $H_{\mathfrak{a}}$ in the trace $\operatorname{tr}\{\psi g(H_{\mathfrak{a}})\}$ by the operator H_W without any magnetic potential and with the asymptotic electric potential (1.5). Actually, our argument does not use the precise form of W. The following general conditions will be sufficient. Firstly, we assume that both $X = |V|^{\frac12}$ and $X = |W|^{\frac12}$ satisfy the inequality (2.3) for some $M = M(h)$ and a fixed $\epsilon < 1$. Hence both $H_{\mathfrak{a}}$ and H_W are semibounded from below and

$$H_{\mathfrak{a}} \ge -M(h), \quad H_W \ge -M(h). \tag{4.1}$$

Recall that $M(h) \ge 1$. Secondly, instead of (2.16) we impose the condition

$$W(x) = Y(x)\Psi(x)Y(x), \quad V(x) = Y(x)(\Psi(x) + F(x))Y(x), \ x \in B(\rho), \tag{4.2}$$

where Y, Ψ, F are some real-valued functions such that Y obeys (2.3) and $\Psi \in L^\infty(\mathbb{R}^d)$, $F \in L^\infty_{loc}(\mathbb{R}^d)$. Moreover, throughout this section we suppose that $\mathbf{a} \in L^\infty_{loc}(\mathbb{R}^d)$.

1. Let us study resolvents of the operators $H_{\mathfrak{a}}$ and H_W. As in the previous section we rely upon an appropriate version of the resolvent identity. Namely, for any $\phi \in \mathcal{B}^\infty(\mathbb{R}^d)$ we have

$$\left.\begin{aligned} \phi R(z; H_{\mathfrak{a}}) = R(z; H_W)\phi + R(z; H_W) Z_1 R(z; H_{\mathfrak{a}}), \\ Z_1 = Z_1(\phi) = -ih\bigl(\Pi_l \partial_l\phi + \partial_l\phi Q_l\bigr) \ + a_l\phi Q_l + \Pi_l \phi a_l - \phi F Y^2. \end{aligned}\right\} \tag{4.3}$$

Note also the identity for the difference of powers of the resolvents, similar to (2.8):

$$\phi R(z; H_{\mathfrak{a}})^k = R(z; H_W)^k \phi + \sum_{j=1}^k R(z; H_W)^j Z_1(\phi) R(z; H_{\mathfrak{a}})^{k-j+1}, \ \forall k \in \mathbb{N} \tag{4.4}$$

Denote $\|f\|_\rho = \text{v-sup}_{|x|\le\rho} |f(x)|$ for $f \in L^\infty_{loc}(\mathbb{R}^d)$ and

$$K(z) = K(z, \rho) = (M + |z|)^{\frac12}\bigl(\|\mathbf{a}\|_\rho + \|F\|_\rho (M + |z|)^{\frac12}\bigr). \tag{4.5}$$

As in Sect. 3 we begin with estimating difference of the resolvents. Throughout this section we assume as a rule that $\chi \in C_0^\infty\bigl(B(\rho/2)\bigr)$ and $|\chi(x)| \le 1$.

Lemma 4.1. *Let the functions V, W be as specified above and let $\rho \ge C$ be a number from (4.2). Then for any $\chi \in C_0^\infty(B(\rho/2))$ and $N \ge 0$*

$$\|\chi\big[R(z;H_{\mathfrak{a}}) - R(z;H_W)\big]\| \le C_N \frac{1}{d_M(z)} \times \left[\left(\frac{(M+|z|)^{\frac12} h}{\rho d_M(z)}\right)^{2N+1} + \frac{K(z,\rho)}{d_M(z)}\right]. \tag{4.6}$$

If $k > 2d+1$ then for any $N > d/2$

$$\big\|\chi\big[R(\pm iM;H_{\mathfrak{a}})^k - R(\pm iM;H_W)^k\big]\big\|_1 \le C_N \frac{1}{M^k}\left[\frac{\rho M^{\frac12}}{h}\right]^d \left[\left(\frac{h}{M^{\frac12}\rho}\right)^{2N+1} + \frac{K(0,\rho)}{M}\right], \tag{4.7}$$

The constant C_N does not depend on V, W, χ and ρ.

Proof. Let $\eta \in C_0^\infty(\mathbb{R})$ be a function such that $|\eta| \le 1$ and $\eta(t) = 1$, $|t| \le 2/3$; $\eta(t) = 0$, $|t| \ge 1$. Denote $\chi_1(x) = \eta(|x|/\rho)$ and $\phi = 1 - \chi_1$. Due to the obvious identity

$$\chi\big[R(z;H_{\mathfrak{a}})^k - R(z;H_W)^k\big] = \chi\big[\chi_1 R(z;H_{\mathfrak{a}})^k - R(z;H_W)^k\chi_1\big] - \chi R(z;H_W)^k\phi,$$

the problem amounts to proving the desired bounds for the operators

$$\left.\begin{aligned} T_1 &= \chi\big[\chi_1 R(z;\,H_{\mathfrak{a}})^k - R(z;H_W)^k\chi_1\big], \\ T_2 &= \chi R(z;H_W)^k\phi \end{aligned}\right\} \tag{4.8}$$

($k = 1$ for (4.6)).

Let us prove first (4.7). For T_2 the estimate (4.7) is fulfilled due to Corollary 3.4. Further, by (4.4)

$$T_1 = \sum_{j=1}^k T_1^{(j)},\ T_1^{(j)} = \chi R(z;H_W)^j Z_1 R(z;H_{\mathfrak{a}})^{k-j+1},\ Z_1 = Z_1(\chi_1).$$

Thus it suffices to obtain (4.7) for each $T_1^{(j)}$ individually. We analyse first the terms with $j \le k/2$, so that $k - j > d$. It is clear that $Z_1 = Z_1\chi_2$ for $\chi_2(x) = \eta\big(|x|/(2\rho)\big)$. Therefore for $z = \pm iM$

$$\begin{aligned} \big\|T_1^{(j)}\big\|_1 \le{}& h\,\big\|\chi R(z;H_W)^j \Pi_l\partial_l\chi_1\big\|_1\ \|\chi_2 R(z;H_{\mathfrak{a}})^{k-j+1}\| \\ &+ h\,\big\|\chi R(z;H_W)^j\partial_l\chi_1\big\|_1\ \|\chi_2 Q_l R(z;H_{\mathfrak{a}})^{k-j+1}\| \\ &+ \|\chi R(z;H_W)^j\|\ \big\|\chi_1 a_l Q_l R(z;H_{\mathfrak{a}})^{k-j+1}\big\|_1 \\ &+ \|\chi R(z;H_W)^j\Pi_l\|\ \big\|\chi_1 a_l R(z;H_{\mathfrak{a}})^{k-j+1}\big\|_1 \\ &+ \|\chi R(z;H_W)^j Y\chi_1 F\|\ \big\|Y\chi_2 R(z;H_{\mathfrak{a}})^{k-j+1}\big\|_1. \end{aligned} \tag{4.9}$$

Since $\operatorname{supp}\chi$ and $\operatorname{supp}\partial\chi_1$ are separated from each other and $|\partial\chi_1| \le C\rho^{-1}$, by Corollary 3.4 and Lemma 3.1 the first and the second terms are bounded by

$$CM^{-k}\left[\frac{h}{M^{\frac12}\rho}\right]^{2N+1-d}, \ \forall N > d/2.$$

Further, by Lemma 3.1 and Corollary 3.7 the third and the forth terms do not exceed

$$C\|\mathbf{a}\|_\rho M^{-k-\frac12}\left[\frac{\rho M^{\frac12}}{h}\right]^d.$$

And, finally, the fifth term is bounded by

$$C\|F\|_\rho M^{-k}\left[\frac{\rho M^{\frac12}}{h}\right]^d.$$

Combining the bounds above, we obtain for all $j \le k/2$ the estimate

$$\left\| T_1^{(k)} \right\|_1 \le CM^{-k}\left[\frac{\rho M^{\frac12}}{h}\right]^d\left[\left(\frac{h}{M^{\frac12}\rho}\right)^{2N+1} + \frac{K(0,\rho)}{M}\right].$$

In case $j > k/2$ the proof is the same except that the last three terms in (4.9) should be replaced with

$$\begin{aligned} &\left\| \chi R(z;H_W)^j \right\|_1 \|\chi_1 a_l Q_l R(z;H_{\mathfrak{a}})^{k-j+1}\| \\ &+ \left\| \chi R(z;H_W)^j \Pi_l \right\|_1 \|\chi_1 a_l R(z;H_{\mathfrak{a}})^{k-j+1}\| \\ &+ \left\| \chi R(z;H_W)^j Y\chi_1 F \right\|_1 \|Y\chi_2 R(z;H_{\mathfrak{a}})^{k-j+1}\|. \end{aligned}$$

The trace norms here are finite since $j > d+1$. Further proof goes as before. Summing the bounds for $T_1^{(j)}$ over j, one obtains (4.7).

Proof of (4.6). Let $k = 1$ in (4.8). The desired bound for T_2 follows from(3.10). Furthermore, as in the proof of (4.7), by (4.3)

$$\begin{aligned} \|T_1\| \le\ & h\|\chi R(z;H_W)\Pi_l\partial_l\chi_1\| \ \|\chi_2 R(z;H_{\mathfrak{a}})\| \\ &+ h\|\chi R(z;H_W)\partial_l\chi_1\| \ \|\chi_2 Q_l R(z;H_{\mathfrak{a}})\| \\ &+ \|\chi R(z;H_W)\| \ \|\chi_1 a_l Q_l R(z;H_{\mathfrak{a}})\| \\ &+ \|\chi R(z;H_W)\Pi_l\| \ \|\chi_1 a_l R(z;H_{\mathfrak{a}})\| \\ &+ \|\chi R(z;H_W)Y\chi_1 F\| \ \|Y\chi_2 R(z;H_{\mathfrak{a}})\|. \end{aligned}$$

By Lemmas 3.2, 3.4 first two terms are bounded by

$$C\frac{1}{d_M(z)}\left(\frac{(M+|z|)^{\frac12}h}{\rho d_M(z)}\right)^{2N+1}, \ \forall N \ge 0.$$

In view of Lemma 3.1 the remaining terms do not exceed

$$\|\mathbf{a}\|_\rho\frac{(M+|z|)^{\frac12}}{d_M(z)^2} + \|F\|_\rho\frac{M+|z|}{d_M(z)^2}.$$

Combining the bounds above, we arrive at (4.6). □

2. The next step is to study the functions of operators $H_{\mathfrak{a}}$ and H_W. Throughout the rest of the Section the conditions of Lemma 4.1 are always assumed to be fulfilled. For functions of selfadjoint operators we use the representation (3.18).

Theorem 4.2. *Let $g \in C^\infty(\mathbb{R})$ be a function satisfying the conditions of Theorem 3.11 and $\chi \in C_0^\infty(B(\rho/2))$. Then for any $N \geq 0$*

$$\|\chi[g(H_{\mathfrak{a}}) - g(H_W)]\| \leq C_N L^{2N+3} M^{s+1} \big[(hM^{\frac{1}{2}}\rho^{-1})^{2N+1} + K(0,\rho)\big], \tag{4.10}$$

where the function $K(z,\rho)$ is defined in (4.5) and the constant C_N depends only on constants C_n in (3.19) and the function χ.

Proof mimicks the proof of Theorem 3.11. Let $\lambda \in \mathbb{R}$ and $0 < |\tau| \leq 1$. Denote

$$\delta(\lambda,\tau) = R(\lambda + i\tau; H_{\mathfrak{a}}) - R(\lambda + i\tau; H_W).$$

Then (4.6) yields for any $N \geq 0$:

$$\|\chi\delta(\lambda,\tau)\| \leq \frac{1}{\tau}\bigg[\bigg(\frac{(M+|\lambda|)^{\frac{1}{2}}h}{\rho\tau}\bigg)^{2N+1} + \frac{K(\lambda)}{\tau}\bigg], \ \forall \lambda \in \mathbb{R}. \tag{4.11}$$

In order to apply the representation (3.18), instead of g we use its modification. Let $\zeta \in C^\infty(\mathbb{R})$ be the function introduced in the proof of Theorem 3.11. Define $\tilde g \in C_0^\infty(\mathbb{R})$ by the equality $\tilde g(\lambda) = \zeta(\lambda/M) g(\lambda)$. Then because of (4.1) we have $\tilde g(H_{\mathfrak{a}}) = g(H_{\mathfrak{a}}), \tilde g(H_W) = g(H_W)$. By virtue of (3.19)

$$|\partial^n \tilde g(\lambda)| \leq C_n L^n \langle\lambda\rangle^s, \ \forall n \in \mathbb{N}, \tag{4.12}$$

with constants C_n independent of $H_{\mathfrak{a}}, H_W$. According to (3.18)

$$\tilde g(H_{\mathfrak{a}}) - \tilde g(H_W) = I_1^{(n)} + I_2^{(n)}, \ \forall n \in \mathbb{N};$$

$$I_1^{(n)} = \sum_{j=0}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j \tilde g(\lambda) \operatorname{Im}\big\{i^j \delta(\lambda,1)\big\} d\lambda,$$

$$I_2^{(n)} = \frac{1}{\pi(n-1)!} \int_0^1 \tau^{n-1} d\tau \int_{\mathbb{R}} \partial^n \tilde g(\lambda) \operatorname{Im}\big\{i^n \delta(\lambda,\tau)\big\} d\lambda. \tag{4.13}$$

Set $n = 2N+3$. In view of (4.12) and (4.11)

$$\begin{aligned}\|\chi I_1^{(n)}\| \leq & C L^{2N+2} M^s \int_{-2M\leq\lambda\leq\lambda_0} \big((hM^{1/2}\rho^{-1})^{2N+1} + K(0)\big) d\lambda \\ \leq & C L^{2N+2} M^{s+1} \big((hM^{1/2}\rho^{-1})^{2N+1} + K(0)\big).\end{aligned}$$

Let us estimate the integral (4.13). According to (4.12) and (4.11)

$$\begin{aligned}\|\chi I_2^{(n)}\| \leq C L^{2N+3} M^s \bigg\{ & \int_0^1 d\tau \int_{-2M\leq\lambda\leq\lambda_0} (hM^{\frac{1}{2}}\rho^{-1})^{2N+1} d\lambda \\ & + \int_0^1 \tau^{2N+1} d\tau \int_{-2M\leq\lambda\leq\lambda_0} K(\lambda) d\lambda \bigg\} \\ \leq & \ C L^{2N+3} M^{s+1} \big((hM^{\frac{1}{2}}\rho^{-1})^{2N+1} + K(0)\big).\end{aligned}$$

Along with the bound for $I_1^{(n)}$, this yields (4.10). □

Now we estimate the trace class norm of the difference $g(H_{\mathfrak{a}}) - g(H_W)$:

Theorem 4.3. *Let the functions $g \in C^\infty(\mathbb{R})$ and $\chi \in C_0^\infty(B(\rho/2))$ be as in Theorem 4.2. Then for any $N > d/2$*

$$\begin{aligned} &\| \chi \big[g(H_{\mathfrak{a}}) - g(H_W)\big] \chi \|_1 \\ &\qquad \le C_N L^{2N+3} M^{s+1} \left[\frac{\rho M^{\frac12}}{h}\right]^d \left[\left(\frac{M^{\frac12} h}{\rho}\right)^{2N+1} + K(0,\rho)\right]. \end{aligned} \tag{4.14}$$

The constant C_N depends on the function χ and does not depend on V, W, ρ, h.

Proof. Denote

$$\tilde g(\lambda) = (\lambda - iM)^k g(\lambda), \ k > 2d+1.$$

Then with $z = iM$

$$\begin{aligned} g(H_{\mathfrak{a}}) - g(H_W) = &\ R(z; H_{\mathfrak{a}})^k \tilde g(H_{\mathfrak{a}}) - R(z; H_W)^k \tilde g(H_W) \\ = &\ \big[R(z; H_{\mathfrak{a}})^k - R(z; H_W)^k\big] \tilde g(H_{\mathfrak{a}}) \\ + &\ R(z; H_W)^k \big[\tilde g(H_{\mathfrak{a}}) - \tilde g(H_W)\big]. \end{aligned}$$

Therefore

$$\begin{aligned} \| \chi \big(g(H_{\mathfrak{a}}) - g(H_W)\big) \chi \|_1 \le &\ \| \chi \big[R(z; H_{\mathfrak{a}})^k - R(z; H_W)^k\big] \|_1 \ \|\tilde g(H_{\mathfrak{a}}) \chi\| \\ + &\ \| \chi R(z; H_W)^k \|_1 \ \| \big[\tilde g(H_{\mathfrak{a}}) - \tilde g(H_W)\big] \chi \|. \end{aligned} \tag{4.15}$$

Let us estimate each factor individually. Note that $K(z, \rho) \le C K(0, \rho)$. Therefore, according to (4.7)

$$\begin{aligned} &\| \chi \big[R(z; H_{\mathfrak{a}})^k - R(z; H_W)^k\big] \|_1 \\ &\qquad \le C M^{-k} \left[\frac{\rho M^{\frac12}}{h}\right]^d \left[\left(\frac{h}{\rho M^{\frac12}}\right)^{2N+1} + \frac{K(0,\rho)}{M}\right], \ \forall N > d/2. \end{aligned} \tag{4.16}$$

Further, due to (3.16)

$$\| \chi R(z; H_W)^k \|_1 \le C M^{-k} \left[\frac{\rho M^{\frac12}}{h}\right]^d. \tag{4.17}$$

Since the function $\tilde g$ satisfies (3.19) with $s_1 = s + k$, we have in view of Theorem 4.2

$$\begin{aligned} &\| \big(\tilde g(H_{\mathfrak{a}}) - \tilde g(H_W)\big) \chi \| \\ &\le C_N L^{2N+3} M^{s+k+1} \big[(h M^{\frac12} \rho^{-1})^{2N+1} + K(0,\rho)\big], \ \forall N \ge 0. \end{aligned} \tag{4.18}$$

To estimate $\chi \tilde g(H_{\mathfrak{a}})$ recall that $H_{\mathfrak{a}} \ge -M$, so that

$$\|\tilde g(H_{\mathfrak{a}})\| \le C M^{s+k}.$$

Combining this bound with (4.16), (4.17) and (4.18), we obtain from (4.15):

$$\| \chi\big[g(H_{\mathfrak{a}}) - g(H_W)\big]\chi\|_1 \le C_N\ M^s \left[\frac{\rho M^{\frac12}}{h}\right]^d \left[\left(\frac{h}{M^{\frac12}\rho}\right)^{2N+1} + \frac{K(0,\rho)}{M}\right] + C_N L^{2N+3} M^{s+1}\ \left[\frac{\rho M^{\frac12}}{h}\right]^d \left[\left(\frac{hM^{\frac12}}{\rho}\right)^{2N+1} + K(0,\rho)\right].$$

This provides (4.14). □

3. Now we combine the results of this section and Sect. 3. Precisely, let V, W be functions introduced in the beginning of the section and let the operator A obey Assumption 2.2 with the functions $\mathbf{a} \in L^\infty_{loc}(\mathbb{R}^d)$, V and $\mathcal{D} = \mathring{B}(4\rho)$. We shall compare the traces $\operatorname{tr}\{\chi g_s(A)\}$ and $\operatorname{tr}\{\chi g_s(H_W)\}$ for $\chi \in C_0^\infty(B(r/2))$ with $r \le \rho$.

Theorem 4.4. *Let the operators $A, H_{\mathfrak{a}}, H_W$ be as above, and $\chi \in C_0^\infty(B(r/2))$ with some r, $\rho \ge r \ge C$. Then for any $L \ge L_0 > 0$ and $N > (d+1)/2 + s, s \ge 0$*

$$\| \chi\big[g_s(A) - g_s(H_W)\big]\ \chi\|_1 \le C_N L^{2N+3} M^{s+1} \left[\frac{\rho M^{\frac12}}{h}\right]^d \times\left[\ \left(\frac{M^{\frac12} h}{\rho}\right)^{2N+1} + K(0,\rho)\right] + CL^{-s}\left[\frac{M^{\frac12} r}{h}\right]^d. \tag{4.19}$$

Proof. Let $\zeta \in C_0^\infty(\mathbb{R})$ be a non-negative function such that $\zeta(\lambda) = 1,\ |\lambda| \le 1/2$, and $\zeta(\lambda) = 0,\ |\lambda| \ge 1$. Denote $g^{(1)}(\lambda) = g_s(\lambda)\zeta(L\lambda)$ and $g^{(2)}(\lambda) = g_s(\lambda)(1-\zeta(L\lambda))$. Obviously, $g^{(2)} \in C^\infty$ and satisfies (3.19). Since $\chi \in C_0^\infty(B(r/2)) \subset C_0^\infty(B(\rho/2))$, by Theorems 3.11 and 4.3 we have:

$$\| \chi\big[g^{(2)}(A) - g^{(2)}(H_W)\big]\chi\|_1 \le C_N L^{2N+3} M^{s+1} \left[\frac{\rho M^{\frac12}}{h}\right]^d \times \left[\left(\frac{M^{\frac12} h}{\rho}\right)^{2N+1} + K(0,\rho)\right], \tag{4.20}$$

as $N > (d+1)/2 + s$. On the other hand,

$$\| \chi g^{(1)}(A)\chi\|_1 \le CL^{-s} \| \chi\zeta(A)\|_1.$$

Therefore by Corollary 3.9

$$\| \chi\big(g^{(1)}(A) - g^{(1)}(H_W)\big)\chi\|_1 \le CL^{-s}\big(\| \chi\zeta(A)\|_1 + \| \chi\zeta(H_W)\|_1\big) \le CL^{-s}\left[\frac{M^{\frac12} r}{h}\right]^d.$$

Combining this estimate with (4.20), we arrive at (4.19). □

5. Asymptotics in case of smooth potentials. Proof of Theorems 2.4, 2.4′

1. As mentioned in Introduction, one of the basic ingredients of our method is the asymptotics of $\mathcal{M}_s(h,\mu;\psi,\mathfrak{a})$ for smooth functions $\mathbf{a}, V$ obtained in [14]. Before stating the result from [14] we specify conditions on the operator $A = A_{\mathfrak{a}}$. Let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded open domain.

Assumption 5.1.

(1) *A is selfadjoint and semibounded from below;*
(2) *There exist real-valued functions* $V, \mathbf{a} \in C^\infty(\bar{\mathcal{D}})$ *such that* $C_0^\infty(\mathcal{D}) \in D(A)$ *and* $u = H_{\mathfrak{a}} u$ *for any* $u \in C_0^\infty(\mathcal{D})$.

Let $f \in C(\bar{\mathcal{D}}), \ell \in C^1(\bar{\mathcal{D}})$ be two functions such that

$$f(x) > 0, \quad \ell(x) > 0, \quad x \in \bar{\mathcal{D}};$$
$$|\partial_x \ell(x)| \le \varrho < 1, \quad x \in \mathcal{D}; \tag{5.1}$$
$$cf(y) \le f(x) \le Cf(y), \quad \forall x \in \mathcal{D} \cap B(y, \ell(y)), \ y \in \mathcal{D}; \tag{5.2}$$

We are interested in the asymptotics of $\mathcal{M}_s$ for an operator $A_{\mathfrak{a}}$, satisfying Assumption 5.1 with the domain $\mathcal{D}$ and some functions $V, \mathbf{a} \in C^\infty(\bar{\mathcal{D}}),\ \psi \in C_0^\infty(\mathcal{D})$ which obey the bounds

$$\left.\begin{aligned} |\partial_x^m \mathbf{a}(x)| \le C_m \ell(x)^{1-|m|}, \quad |m| \ge 1; \\ |\partial_x^m V(x)| \le C_m f(x)^2 \ell(x)^{-|m|}, \ |\partial_x^m \psi(x)| \le C_m \ell(x)^{-|m|}, \quad |m| \ge 0, \end{aligned}\right\} x \in \mathcal{D}. \tag{5.3}$$

One can think of $f(x)^2$ as a measure of the size of $V(x)$, while $\ell(x)$ characterizes the behaviour of $V(x), \mathbf{a}(x)$ and $\psi(x)$ under differentiation. Emphasize that the functions $f(x), \ell(x)$ are allowed to depend on h, μ. We require only that

$$f(x)\ell(x) \ge ch; \quad f(x)^2 \ge c\mu h, \quad x \in \mathcal{D}. \tag{5.4}$$

We also need the following condition on $\operatorname{supp}\psi$:

$$\bigcup B\big(x, 8\ell(x)\big) \subset \mathcal{D}, \tag{5.5}$$

where the union is taken over those $x \in \mathcal{D}$, for which $B(x,\ell(x)) \cap \operatorname{supp}\psi \ne \emptyset$.

The next Proposition results from [14]:

Proposition 5.2. *Let the operator A obey Assumption 5.1 for an open set* $\mathcal{D}$ *with the functions* $V, \mathbf{a}$ *and* ψ *satisfying conditions (5.1)–(5.5) for some* $\varrho < 1/8$. *Then*

$$\big|\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) - \mathfrak{W}_s(h;\psi,V)\big| \le C\mathcal{R}(h,\mu),$$

where

$$\mathcal{R}(h,\mu) = \int_{\mathcal{D}} f(x)^{2s} g\left(\frac{h}{\ell(x)f(x)}, \frac{\mu\ell(x)}{f(x)}\right)\ell(x)^{-d}dx, \ g(a,b) = (1+b^{s+1})a^{s+1-d}.$$

The constant C is uniform in the functions $\mathbf{a}, V, f, \ell, \psi$ satisfying (5.1)–(5.5).

Using the explicit form of the function $g(h,\mu)$, one can rewrite $\mathcal{R}(h,\mu)$:

$$\mathcal{R}(h,\mu) = h^{s+1-d} I_1(h,\mu) + \mu^{s+1} h^{s+1-d} I_2(h,\mu),$$

with

$$I_1 = I_1(h,\mu) = \int_{\mathcal{D}} f(x)^{d+s-1} \ell(x)^{-s-1} dx, \quad I_2 = I_2(h,\mu) = \int_{\mathcal{D}} f(x)^{d-2} dx. \tag{5.6}$$

2. **Proofs of Theorems 2.4, 2.4′** rely on the following Lemma.

Lemma 5.3. *Let V obey (2.15) and let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a function such that* $\operatorname{supp}\psi \subset \{x : r \le |x| \le \rho\}$, $\rho \ge r \ge 1$ *and*

$$|\partial^m \psi(x)| \le C_m |x|^{-|m|}, \ \forall |m| \ge 0, \forall x \ne 0. \tag{5.7}$$

If $\omega > d - s - 1$, then for any $\kappa \in [0,1]$ and $\rho \in [r, C\kappa^{-\frac{1}{\beta}}]$ one has

$$|\mathcal{N}_s(1;\psi, V+\kappa) - \mathfrak{W}_s(1;\psi, V+\kappa)| \le C r^{-\sigma}, \tag{5.8}$$

with

$$\sigma = \frac{2-\beta}{2}(\omega - d + 1 + s) > 0,$$

uniformly in ψ and V.

Proof. We apply Proposition 5.2 in the particular case $\mathbf{a} = 0$, $\mu = 0$, $h = 1$. It is clear that the conditions (5.1), (5.2), (5.5) are satisfied for

$$f(x) = |x|^{-\frac{\beta}{2}}, \ \kappa \le 1; \quad \ell(x) = \varrho|x|, \ \varrho < 1/16,$$
$$\mathcal{D} = \{x \in \mathbb{R}^d : r/4 < |x| < 2\rho\}.$$

The condition (5.3) for $V + \kappa$ and ψ is obviously fulfilled due to the inequality $\rho \le C\kappa^{-\frac{1}{\beta}}$ and conditions (2.15) and (5.7). Moreover, since $\beta < 2$, (5.4) is also fulfilled (recall that $h = 1$). Therefore, according to Proposition 5.2

$$\big|\mathcal{N}_s(1;\psi, V+\kappa) - \mathfrak{W}_s(1;\psi, V+\kappa)\big| \le C \int_{\mathcal{D}} \big(|x|^{-\frac{\beta}{2}}\big)^{d+s-1} |x|^{-s-1} dx \le C \int_{r/4}^{2\rho} t^{-\sigma-1} dt$$

with σ defined above. For $\omega > d - s - 1$, the number σ is positive and therefore the integral is bounded by $Cr^{-\sigma}$, which implies (5.8). $\square$

3. **Proof of Theorem 2.4**. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be such that $\phi(x) = 1, |x| \le 1$. The function $\psi(x) = \phi_\rho(x) - \phi_r(x)$ obeys (5.7) (recall that $\phi_\rho(x) = \phi(x\rho^{-1})$). Thus the relation (5.8) with $\kappa = 0$ and $V = W$ reads:

$$|\mathcal{N}_s(1;\psi, W) - \mathfrak{W}_s(1;\psi, W)| \le C r^{-\sigma}. \tag{5.9}$$

This guarantees convergence of $\mathcal{N}_s(1;\phi_\rho, W) - \mathfrak{W}_s(1;\phi_\rho, W)$ as $\rho \to \infty$, which immediately entails (2.20) as $d - s - 1 < \omega < d$. Independence of the limit of the function ϕ also follows from (5.9).

To prove Theorem 2.4 for $\omega > d$, note that

$$\mathfrak{W}_s(1, \psi, W) \le C \int_r^{2\rho} t^{-\delta-1} dt, \ \delta = \frac{2-\beta}{2}(\omega - d) > 0.$$

In combination with (5.9) this gives:

$$|\mathcal{N}_s(1;\psi, W)| \le C(r^{-\sigma} + r^{-\delta}),$$

which implies the convergence of $\mathcal{N}_s(1;\phi_\rho, W)$ as $\rho \to \infty$. Furthermore, since the operator $\phi_\rho g_s(H_W)$ converges weakly to $g_s(H_W)$ as $\rho \to \infty$, the latter operator is trace class and

$$\operatorname{tr}\{\phi_\rho g_s(H_W)\} = \operatorname{tr} g_s(H_W) + o(1), \quad \rho \to \infty,$$

which is equivalent to (2.19). □

4. **Proof of Theorem 2.4$'$**. We remind that in Theorem 2.4$'$ no uniformity in W is claimed, so that we use the symbols " lim ", " lim sup " in accordance with our notational convention made in Sect. 2.

Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be as in Theorem 2.4 and let $\rho = C\kappa^{-1/\beta}$, where the constant C is such that $4|W(x)| \le \kappa, \ |x| \ge \rho/2$. For $r \in [1, \rho]$ we split $\mathcal{N}_s(1;1, W+\kappa)$ and $\mathfrak{W}_s(1;1, W+\kappa)$ as follows:

$$\left.\begin{aligned} \mathcal{N}_s(1;1,W+\kappa) = {} & \mathcal{N}_s(1;\phi_r, W+\kappa) \\ & + \mathcal{N}_s(1;\phi_\rho - \phi_r, W+\kappa) + \mathcal{N}_s(1;1-\phi_\rho, W+\kappa), \\ \mathfrak{W}_s(1;1,W+\kappa) = {} & \mathfrak{W}_s(1;\phi_r, W+\kappa) + \mathfrak{W}_s(1;\phi_\rho - \phi_r, W+\kappa). \end{aligned}\right\} \tag{5.10}$$

We shall infer Theorem 2.4$'$ from the following

Lemma 5.4. *Suppose that*

$$\mathcal{N}_s(1;1-\phi_\rho, W+\kappa) \to 0, \ \rho = C\kappa^{-\frac{1}{\beta}}, \ \kappa \to 0, \tag{5.11}$$

and

$$\lim_{r\to\infty} \limsup_{\kappa\to+0} |\mathcal{N}_s(1;\phi_r, W+\kappa) - \mathcal{N}_s(1;\phi_r, W)| = 0. \tag{5.12}$$

Then (2.21) holds.

Proof. Indeed, it follows from (5.10), (5.11) and Lemma 5.3 with $\psi = \phi_\rho - \phi_r$ that

$$\begin{aligned} \limsup_{\kappa\to+0} & |\mathcal{N}_s(1;1,W+\kappa) - \mathfrak{W}_s(1;1,W+\kappa) - \Theta_s| \\ & \le \limsup_{r\to\infty} \limsup_{\kappa\to+0} |\mathcal{N}_s(1;\phi_r, W+\kappa) - \mathfrak{W}_s(1;\phi_r, W+\kappa) - \Theta_s|. \end{aligned} \tag{5.13}$$

Further, (5.12) along with the equality

$$\lim_{\kappa\to+0}\mathfrak{W}_s(1;\phi_r,W+\kappa)=\mathfrak{W}_s(1;\phi_r,W),\forall r>0,$$

and Theorem 2.4, yield

$$\lim_{r\to\infty}\limsup_{\kappa\to+0}\big|\mathcal{N}_s(1;\phi_r,W+\kappa)-\mathfrak{W}_s(1;\phi_r,W+\kappa)-\Theta_s\big|=0.$$

Now (5.13) provides (2.21). □

Thus it remains to establish (5.11), (5.12).

Proof of (5.11). Let u_k,λ_k be normalized eigenfunctions and associated eigenvalues of the operator H_W. Denote $\psi_\rho=1-\phi_\rho$. Then

$$\mathcal{N}_s(1;\psi_\rho,W+\kappa)=\sum_{\lambda_k\le-\kappa}|\lambda_k+\kappa|^s\langle\psi_\rho u_k,u_k\rangle,$$

Recall that $\rho=C\kappa^{-1/\beta}$ was chosen in such a way that $4|W(x)|\le\kappa,\ |x|\ge\rho/2$, so that by Lemma A1 from Appendix,

$$|\langle\psi_\rho u_k,u_k\rangle|\le C\frac{1}{|\lambda_k|}\exp\Big\{-c\sqrt{|\lambda_k|}\rho\Big\},\ \forall k:\lambda_k<-\kappa.$$

Hence

$$\begin{aligned}\mathcal{N}_s(1;\psi_\rho,W+\kappa)&\le C\sum_{\lambda_k\le-\kappa}|\lambda_k+\kappa|^s\frac{1}{|\lambda_k|}\exp\big\{-c\sqrt{|\lambda_k|}\rho\big\}\\&\le C\mathcal{N}_0(1;1,W+\kappa)\kappa^{s-1}\exp\big\{-c\kappa^{\frac12-\frac1\beta}\big\}.\end{aligned}$$

Since $\beta<2$, this bound in combination with (2.23) provides (5.11).

Proof of (5.12). The potentials W and $W+\kappa$ can be presented in the form (4.2) with

$$\Psi(x)=\Phi(\hat x),\ Y(x)=|x|^{-\frac\beta2},\ F(x)=\kappa|x|^\beta,\ \forall x\in\mathbb{R}^d.$$

Since $\beta<2$, the functions $Y,|W+\kappa|^{1/2},|W|^{1/2}$ obey (2.3) for $h=1$, any prescribed $\epsilon<1$ and $M=M(\epsilon)$. Therefore the quantity $K(0,r)$ defined in (4.5) is bounded by

$$K(0,r)\le C\|F\|_r=C\kappa r^\beta.$$

According to Theorem 4.4 with $A=H_{W+\kappa}$ and H_W, we have for any $L\ge L_0,\ \rho\ge r$

$$|\mathcal{N}_s(1;\phi_r,W+\kappa)-\mathcal{N}_s(1;\phi_r,W)|\le C_NL^{2N+3}\rho^d\big(\rho^{-2N-1}+\kappa\rho^\beta\big)+L^{-s}r^d.$$

Set $L=r^{d/s+\varepsilon}$ (recall that $s>0$ in Theorem 2.4′), $\rho=L^{1+\varepsilon},\ \varepsilon>0$. Then for sufficiently large N the equality (5.12) follows.

Now Theorem 2.4′ follows from Lemma 5.4.

6. Proof of Theorem 2.5

Throughout this section we assume that $V, \mathbf{a}$ obey (2.15), (2.16), (2.17) and all the other conditions of Theorem 2.5 are fulfilled. It will be convenient to choose for the vector-potential such a gauge that $\mathbf{a}(0) = 0$, so that

$$|\mathbf{a}(x)| \le C|x|, \ x \in \mathbb{R}^d. \tag{6.1}$$

Recall that the functions $X = |V|^{1/2}$ and $X = |W|^{1/2}$ obey (2.3) for any $\epsilon \in (0,1)$ and $M(h)$ defined in (2.18).

1. First of all we study the asymptotics of $\mathcal{M}_s(h, \mu; \psi, \mathfrak{a})$, where ψ is supported in a small neighbourhood of the origin, which depends on h. Precisely, let

$$\phi(x) = \chi^2(x), \quad \begin{cases} \chi \in C_0^\infty(B(1)); \\ \chi(x) = 1, \ |x| \le 1/2. \end{cases} \tag{6.2}$$

Our aim is to study $\mathcal{M}_s(h, \mu; \phi_r, \mathfrak{a})$, $\phi_r(x) = \phi(xr^{-1})$, with

$$r = r_\theta(h) = \left(\frac{h}{\theta}\right)^{\frac{2}{2-\beta}}, \ 0 < \theta \le 1. \tag{6.3}$$

Here $\theta \in (0,1]$ plays the role of an additional parameter. Denote

$$\varkappa = \varkappa(\beta) = \frac{2+\beta}{2-\beta}. \tag{6.4}$$

We shall find the asymptotics of $\mathcal{M}_s(h, \mu; \phi_r, \mathfrak{a})$ as $h \to 0, \mu h^\varkappa \to 0$ and $\theta \to 0$. Note that due to the inequality $\varkappa > 1$ the condition $\mu h^\varkappa \to 0$ is less restrictive than our usual condition $\mu h \le C$.

To distinguish asymptotics in h and in θ we introduce the notation $\epsilon(t)$ for an arbitrary function of a parameter $t \in [0, C]$, which possesses the following two properties uniformly in the functions $U, \mathbf{a}$ and the parameters $h \in (0, h_0], \mu h^\varkappa \le C$:

(1) $\epsilon(t)$ is bounded on $[0, C]$;
(2) $\epsilon(t) \to 0$ as $t \to 0$.

Lemma 6.1. *Let the function $\mathbf{a}, V$ and ϕ be as specified above, $s \ge 0$, $\beta \in [0,2)$, and let $h \in (0, h_0]$, $\mu h^\varkappa \le C$. Let $r = r_\theta$ be defined in (6.3). Suppose that the operator $A_\mathfrak{a}$ satisfies Assumption 2.2 with $\mathcal{D} = \mathring{B}(4E)$ and functions $\mathbf{a}, V$, which obey (2.15)-(2.17). Then*

$$\mathcal{M}_s(h, \mu; \phi_r, \mathfrak{a}) \le C\theta^{-\sigma} h^{-\omega}, \ \forall \theta \in (0,1], \ C = C_{\beta,s}, \ \sigma = \sigma(\beta, s) > 0. \tag{6.5}$$

Moreover, there exists such $\theta_0 = \theta_0(h, \mu) = \epsilon(h + \mu h^\varkappa)$ that for any $\theta \in [\theta_0, 1]$ the following relations hold:

$$\mathcal{M}_s(h, \mu; \phi_r, \mathfrak{a}) = h^{-\omega}\big(\mathcal{N}_s(1; 1, W) + \epsilon(\theta)\big), \quad \omega(\beta, s) > d, \tag{6.6}$$

$$\mathcal{M}_s(h, \mu; \phi_r, \mathfrak{a}) = \mathfrak{W}_s(h; \phi_r, V) + h^{-\omega}\big(\Theta_s + \epsilon(\theta)\big), \\ d - s - 1 < \omega(\beta, s) < d. \tag{6.7}$$

Proof. We reduce the problem to that in the ball $B(1)$ by performing a scaling transformation. Let $\hat{V}, \hat{\mathbf{a}}$ be defined by (2.9) with $\ell = r$, $f = \ell^{-\beta/2}$, and α, ν – by (2.10). Then

$$\alpha = hr^{\frac{\beta}{2}-1} = \theta, \quad \nu = \mu r^{\frac{\beta}{2}+1} = \mu h^{\varkappa}\theta^{-\varkappa}, \tag{6.8}$$

and in view of (2.16),

$$\left.\begin{aligned} \hat{V}(x) = \ell^{\beta}V(\ell x) = |x|^{-\beta}\big(\Phi(\hat{x}) + U(\ell x)\big), \\ \hat{\mathbf{a}}(x) = \ell^{-1}\mathbf{a}(\ell x), \quad \hat{\phi}(x) = \phi(\ell x). \end{aligned}\right\} \tag{6.9}$$

By virtue of (2.11) and (2.18), the functions $X = |\hat{V}|^{\frac{1}{2}}$ and $X = |W|^{\frac{1}{2}}$ obey (2.3) with the Planck constant α and the constant

$$M = M(\alpha) = C\alpha^{-\frac{\beta}{2-\beta}}. \tag{6.10}$$

Let $A_{\hat{\mathfrak{a}}}$ be the operator defined in (2.12). Since $r^{-\beta s} = h^{-\omega}\alpha^{\omega}$, the relations (2.13), (2.14) yield that

$$\left.\begin{aligned} \mathcal{M}_s(h, \mu; \phi_r, \mathfrak{a}) = h^{-\omega}\alpha^{\omega}\mathcal{M}_s(\alpha, \nu; \phi, \hat{\mathfrak{a}}), \\ \mathfrak{W}_s(h; \phi_r, V) = h^{-\omega}\alpha^{\omega}\mathfrak{W}_s(\alpha; \phi, \hat{V}). \end{aligned}\right\} \tag{6.11}$$

By (6.11) the estimate (6.5) is equivalent to

$$\mathcal{M}_s(\alpha, \nu; \phi, \hat{\mathfrak{a}}) \le C\alpha^{-\sigma}, \ \forall \alpha \in (0, 1], \ \nu\alpha^{\varkappa} \le C. \tag{6.12}$$

Furthermore, the relations (6.6) and (6.7) amount to proving that there exists $\theta_0 = \epsilon(h + \mu h^{\varkappa})$ such that for any $\alpha \in [\theta_0, 1]$ and for ν defined in (6.8), one has

$$\mathcal{M}_s(\alpha, \nu; \phi, \hat{\mathfrak{a}}) = \alpha^{-\omega}\big(\mathcal{N}_s(1; 1, W) + \epsilon(\alpha)\big), \quad \omega > d; \tag{6.13}$$

$$\mathcal{M}_s(\alpha, \nu; \phi, \hat{\mathfrak{a}}) = \mathfrak{W}_s(\alpha; \phi, \hat{V}) + \alpha^{-\omega}\big(\Theta_s + \epsilon(\alpha)\big), \quad d - 1 - s < \omega < d, \tag{6.14}$$

respectively.

Proof of (6.12). Let $g \in C^{\infty}(\mathbb{R})$ be a function such that $g(\lambda) = 0, \lambda \ge 1$ and $g_s(\lambda) \le g(\lambda) \le g_s(\lambda - 1)$. Then by Corollary 3.6 and Theorem 3.11

$$\begin{aligned} \mathcal{M}_s(\alpha, \nu; \phi, \hat{\mathfrak{a}}) &\le \| \phi g(A_{\hat{\mathfrak{a}}}) \|_1 \\ &\le \| \phi\big[g(A_{\hat{\mathfrak{a}}}) - g(H_{\hat{\mathfrak{a}}})\big] \|_1 + \| \phi g(H_{\hat{\mathfrak{a}}}) \|_1 \\ &\le CM(\alpha)^{s+1+\frac{d+2N+1}{2}}\alpha^{-d+2N+1} + C'M(\alpha)^{s+\frac{d}{2}}\alpha^{-d}, \end{aligned}$$

for any $N > (d + 1)/2 + s$. In view of (6.10), for any fixed N the r.h.s. does not exceed $C\alpha^{-\sigma}$ with some $\sigma > 0$, which proves (6.12).

Proof of (6.13) and (6.14) breaks into two steps.

Step 1. We may assume $s > 0$ (otherwise $\omega = 0$). First of all we shall find such $\theta_0 = \epsilon(h + \mu h^{\varkappa})$ that

$$|\mathcal{M}_s(\alpha, \nu; \phi, \hat{\mathfrak{a}}) - \mathcal{N}_s(\alpha; \phi, W)| \le C, \ \forall \alpha \in [\theta_0, 1]. \tag{6.15}$$

By cyclicity of trace

$$\mathcal{M}_s(\alpha,\nu;\phi,\hat{\mathbf{a}}) = \operatorname{tr}\{\chi g_s\big(H_{\hat{\mathbf{a}}}(\alpha,\nu)\big)\chi\}.$$

The operator $A_{\hat{\mathbf{a}}}$ obeys Assumption 2.2 with $\mathcal{D} = \mathring{B}(4\rho)$ for any $\rho \le Er_\theta^{-1}$. Further, by (6.9) the functions W and $\hat{V}$ have the form (4.2) with $\Psi(x) = \Phi(\hat{x})$, $F(x) = U(\ell x)$, $Y(x) = |x|^{-\beta}$. Thus to justify (6.15) we can use Theorem 4.4 with $r = 2$. By (4.19) and (6.10), for any $L \ge L_0 \ge C$, $\rho \le Er_\theta^{-1}$, we have

$$\begin{aligned}\big\| \chi\big[g_s\big(H_{\hat{\mathbf{a}}}(\alpha,\nu)\big) - g_s\ \big(H_W(\alpha)\big)\big]\chi\big\|_1 \\ \le C_N L^{2N+3}\ \rho^d\alpha^{-\sigma_1}\left[\left(\frac{\alpha^{-\sigma_2}}{\rho}\right)^{2N+1} + K(0,\rho)\right] + CL^{-s}\alpha^{-\sigma_3},\end{aligned} \tag{6.16}$$

$$K(0,\rho) = M^{\frac12}\big(\nu\|\hat{\mathbf{a}}\|_\rho + \|F\|_\rho M^{\frac12}\big). \tag{6.17}$$

Here and below by $\sigma_j, j = 1,\dots,$ we denote positive exponents depending only on s, d. Their precise values are of no importance. Next, we pick up the parameters ρ, L, θ_0, N so as to guarantee boundedness of the r.h.s. of (6.16) uniformly in $\alpha \in [\theta_0, 1]$. To that end set $L = \alpha^{-\sigma_3/s}$, so that the second summand in (6.16) is bounded. Then the r.h.s. can be estimated by

$$C_1\big(\alpha^{-\frac{\sigma_3}{s}-\sigma_2}\rho^{-1}\big)^{2N+1}\rho^d\alpha^{-\sigma_4} + C_2\alpha^{-(2N+3)\frac{\sigma_3}{s}}\rho^d\alpha^{-\sigma_1}K(0,\rho) + C_3. \tag{6.18}$$

Now choose $\rho = \alpha^{-\gamma_1}$, $\gamma_1 > \sigma_3/s + \sigma_2$ and fix a sufficiently big N in such a way that the first term in (6.18) is uniformly bounded. To make sure that $\rho \le Er_\theta^{-1}$, it suffices to assume that

$$\alpha \ge h^{\gamma_2},\ 0 < \gamma_2 < (1 + \gamma_1(2-\beta)/2)^{-1}.$$

Actually, under this condition we have more:

$$\rho r_\theta \le h^{\gamma_3},\ \gamma_3 = \frac{2}{2-\beta}\left[1 - \gamma_2\big(1 + \gamma_1\frac{2-\beta}{2}\big)\right] > 0. \tag{6.19}$$

By (6.10), (6.17) the second term in (6.18) does not exceed

$$\mathcal{K}(\alpha,\nu) = C\alpha^{-\sigma_5}K(0,\rho) = C\alpha^{-\sigma_6}\big(\nu\|\hat{\mathbf{a}}\|_\rho + \alpha^{-\sigma_7}\|F\|_\rho\big).$$

Here, in view of (2.17), $\|F\|_\rho \le U_0(r\rho)$. Further, according to (6.1), $|\hat{\mathbf{a}}(x)| \le C|x|$. Therefore by (6.8),

$$\nu\|\hat{\mathbf{a}}\|_\rho \le \mu h^{\varkappa}\alpha^{-\varkappa}\rho = \mu h^{\varkappa}\alpha^{-\varkappa-\gamma_1}.$$

Thus, in view of (6.19), we have

$$\mathcal{K}(\alpha,\nu) \le C\alpha^{-\sigma_6}\big(\mu h^{\varkappa}\alpha^{-\varkappa-\gamma_1} + \alpha^{-\sigma_7}U_0(h^{\gamma_3})\big)$$

Choosing

$$\theta_0 = (\mu h^{\varkappa})^{\frac{1}{\sigma_8}} + \Big[U_0(h^{\gamma_3})\Big]^{\frac{1}{\sigma_6+\sigma_7}} + h^{\gamma_2} \tag{6.20}$$

with $\sigma_8 = \sigma_6 + \varkappa + \gamma_1$, one can guarantee boundedness of $\mathcal{K}$ uniformly in α. This completes the proof of (6.15).

Step 2. Study of $\mathcal{N}_s(\alpha;\phi,W)$. According to (2.13), (2.14) with $\ell = \alpha^{\frac{2}{2-\beta}}$, we have

$$\left.\begin{aligned} \mathcal{N}_s(\alpha;\phi,W) &= \alpha^{-\omega}\mathcal{N}_s(1;\phi_\rho,W),\\ \mathfrak{W}_s(\alpha;\phi,W) &= \alpha^{-\omega}\mathfrak{W}_s(1;\phi_\rho,W),\end{aligned}\right\} \ \rho = \ell^{-1}. \tag{6.21}$$

Let us consider separately two cases: $\omega > d$ and $\omega < d$.

Let $\omega > d$. According to Theorem 2.4,

$$\mathcal{N}_s(1;\phi_\rho,W) = \mathcal{N}_s(1;1,W) + \epsilon(\alpha),\ \ \alpha\to 0.$$

Taking into account (6.21), (6.15), we get from here (6.13), and, consequently (6.6).

Let $d-s-1<\omega<d$. According to Theorem 2.4

$$\mathcal{N}_s(1;\phi_\rho,W) = \mathfrak{W}_s(1;\phi_\rho,W) + \Theta_s + \epsilon(\alpha),\ \ \alpha\to 0.$$

Along with (6.21) this yields that

$$\mathcal{N}_s(\alpha;\phi,W) = \mathfrak{W}_s(\alpha;\phi,W) + \alpha^{-\omega}\big(\Theta_s + \epsilon(\alpha)\big),\ \ \alpha\to 0. \tag{6.22}$$

Let us replace here $\mathfrak{W}_s(\alpha;\phi,W)$ with $\mathfrak{W}_s(\alpha;\phi,\hat V)$. One can check directly that

$$\begin{aligned}|\mathfrak{W}_s(\alpha;\phi,W) - \mathfrak{W}_s(\alpha;\phi,\hat V)| &\le \mathfrak{W}_s\big(\alpha;\phi,|x|^{-\beta}\big)\|F\|_1\\ &\le C\alpha^{-d}\|F\|_1 \le C\alpha^{-d}U_0(h^{\gamma_3}).\end{aligned}$$

Adding, if necessary, $U_0(h^{\gamma_3})^{\frac{1}{d}}$ to θ_0 defined in (6.20), we obtain

$$|\mathfrak{W}_s(\alpha;\phi,W) - \mathfrak{W}_s(\alpha;\phi,\hat V)| \le C,\ \forall\alpha\in[\theta_0,1].$$

Thus (6.22) transforms into

$$\mathcal{N}_s(\alpha;\phi,W) = \mathfrak{W}_s(\alpha;\phi,\hat V) + \alpha^{-\omega}\big(\Theta_s + \epsilon(\alpha)\big).$$

Combining this with (6.15), we obtain (6.14) and, consequently, (6.7). $\square$

2. **Proof of Theorem 2.5**. We break up the trace $\mathcal{M}_s(h,\mu)$ into two parts as follows. Let the function $\phi\in C_0^\infty(B(1))$ and the parameter $r = r_\theta$ be defined by (6.2), (6.3) respectively. Denote

$$\psi_1(x) = \psi(x)\phi_r(x),\quad \psi_2(x) = \psi(x)\big(1-\phi_r(x)\big). \tag{6.23}$$

Then, obviously,

$$\mathcal{M}_s(h,\mu) = \sum_{k=1}^{2}\mathcal{M}_s(h,\mu;\psi_k).$$

We study these two summands separately. Recall that $\mu h\le C$ and by definition (6.4) $\varkappa > 1$. Consequently, the parameter θ_0 defined in Lemma 6.1, is actually $\epsilon(h)$.

Step 1. Asymptotics of $\mathcal{M}_s(h,\mu;\psi_1)$. We claim that for $\theta\in[\theta_0,1]$

$$\mathcal{M}_s(h,\mu;\psi_1,\mathfrak{a}) = \psi(0)h^{-\omega}\mathcal{N}_s(1;1,W) + h^{-\omega}\big(\epsilon(\theta)+\epsilon(h)\big),\ \ \omega>d; \tag{6.24}$$

$$\mathcal{M}_s(h,\mu;\psi_1,\mathfrak{a}) = \mathfrak{W}_s(h;\psi_1,V) + h^{-\omega}\big(\psi(0)\Theta_s+\epsilon(\theta)+\epsilon(h)\big), \quad d-s-1<\omega<d. \tag{6.25}$$

Moreover,

$$\mathcal{M}_s(h,\mu;\psi_1,\mathfrak{a})\le Ch^{-\omega},\ \ \theta=1,\ \forall\omega\ge 0. \tag{6.26}$$

The latter bound follows from (6.5). To prove (6.24) and (6.25) notice that for $\theta\in[\theta_0,1]$

$$\mathcal{M}_s(h,\mu;\psi_1,\mathfrak{a}) = \psi(0)\mathcal{M}_s(h,\mu;\phi_r,\mathfrak{a}) + h^{-\omega}\epsilon(h),\ \omega>0; \tag{6.27}$$

$$\mathfrak{W}_s(h;\psi_1,V) = \psi(0)\mathfrak{W}_s(h;\phi_r,V) + h^{-\omega}\epsilon(h),\ \ \omega<d. \tag{6.28}$$

Indeed,

$$\mathcal{M}_s(h,\mu;\psi_1) = \psi(0)\mathcal{M}_s(h,\mu;\phi_r) + \mathcal{M}_s(h,\mu;\psi_3\phi_r),$$

where $\psi_3(x)=\psi(x)-\psi(0)$. Since $|\partial\psi(x)|\le C$, one has $|\psi_3(x)|\le Cr$ for $x\in\operatorname{supp}\phi_r$. Thus, by (6.5)

$$|\mathcal{M}_s(h,\mu;\psi_3\phi_r)|\le Cr\mathcal{M}_s(h,\mu;\phi_r)\le Cr\theta^{-\sigma}h^{-\omega} = Ch^{-\omega}h^{\frac{2}{2-\beta}}\theta^{-\sigma_1}.$$

Increasing (if necessary) θ_0 defined in Lemma 6.1, we prove that

$$\mathcal{M}_s(h,\mu;\psi_3\phi_r) = h^{-\omega}\epsilon(h),\ \ \theta\in[\theta_0,1],$$

which yields (6.27). The bound (6.28) can be proven similarly. Now (6.6) along with (6.27) provide (6.24). The relation (6.25) follows from (6.7) and (6.27), (6.28).

Step 2. Asymptotics of $\mathcal{M}_s(h,\mu;\psi_2)$. Here we use the multiscale method described in Sect. 5. First we introduce functions $f(x)$ and $\ell(x)$. Let $\varrho\in(0,1/32)$ be some number and let

$$f(x)=|x|^{-\frac{\beta}{2}},\ \ \ell(x)=\varrho|x|.$$

(Recall that the same f and ℓ were used in the proof of Lemma 5.3). Obviously, the functions $\ell(x)$ and $f(x)$ satisfy (5.1), (5.2) on the open set

$$\mathcal{D}=\{x\in\mathbb{R}^d: r/4<|x|<4E\}.$$

Conditions (5.3) are fulfilled for V and $\mathbf{a}$ due to (2.15). To check (5.4) notice that due to the inequality $\beta<2$ we have for $x\in\mathcal{D}$

$$f(x)\ell(x)=\varrho|x|^{\frac{2-\beta}{2}}\ge\varrho h\theta^{-1}\ge\varrho h,\ \forall\theta\in(0,1].$$

The lower bound $f(x)^2\ge c\mu h$ in (5.4) is trivially fulfilled since $f(x)\ge c,\ x\in\mathcal{D}$ and $\mu h\le C$. Further, by definitions (6.2), (6.23),

$$\operatorname{supp}\psi_2\subset\{x\in\mathbb{R}^d: r/2<|x|<E/2\},$$

so that for any $\varrho \in (0, 1/32)$ the condition (5.5) is satisfied. Furthermore, it is clear that ψ_2 obeys (5.3). Thus the conditions of Proposition 5.2 are satisfied and therefore

$$\left.\begin{aligned}\big|\mathcal{M}_s(h,\mu;\psi_2,\mathfrak{a}) - \mathfrak{W}_s(h;\psi_2,V)\big| \le C\mathcal{R}(h,\mu),\\ \mathcal{R}(h,\mu) = h^{s+1-d}I_1 + \mu^{s+1}h^{s+1-d}I_2,\end{aligned}\right\} \tag{6.29}$$

with the integrals I_1, I_2 defined in (5.6). Let us estimate I_1:

$$\begin{aligned} I_1 &= \int_{\mathcal{D}} f(x)^{d+s-1}\ell(x)^{-s-1}dx \\ &\le C\int_{r/4<|x|<4E} |x|^{-\frac{\beta}{2}(d+s-1)}|x|^{-s-1}dx \le C\int_{r/4}^{4E} t^{\sigma-1}dt, \end{aligned}$$

where

$$\sigma = \frac{2-\beta}{2}(d-1-s-\omega).$$

Hence

$$I_1 \le \begin{cases} Cr^{\sigma} \le C\big(h\theta^{-1}\big)^{d-1-s-\omega}, & \omega > d-s-1; \\ C(|\ln r|+1) \le C\big(|\ln h| + |\ln\theta| + 1\big), & \omega = d-s-1; \\ C, \ \omega < d-s-1. \end{cases}$$

The integral I_2 is always bounded uniformly in h, μ irrespectively of the values of $\beta < 2,\ s$:

$$I_2 \le C\int_{|x|<4E} |x|^{-\frac{\beta}{2}(d-2)}dx \le C.$$

Therefore

$$\mathcal{R}(h,\mu) \le \begin{cases} C_1 h^{-\omega}\theta^{\omega-d+1+s} + C_2\mu^{s+1}h^{s+1-d}, & \omega > d-s-1; \\ C(|\ln h| + |\ln\theta| + \langle\mu\rangle^{s+1})h^{s+1-d}, & \omega = d-s-1; \\ C\langle\mu\rangle^{s+1}h^{s+1-d}, \ \omega < d-s-1. \end{cases} \tag{6.30}$$

Step 3. End of the proof. Let us combine the results of the two previous steps.

Let first $\omega > d$. We show that the region outside the singularity does not contribute to the leading term of the asymptotics. Indeed, for $\mu h \le C$, we have by (6.30)

$$\mathcal{R}(h,\mu) \le Ch^{-\omega}(\epsilon(\theta) + \epsilon(h)),\ \theta \in [\theta_0, 1].$$

Further, by virtue of (2.15),

$$\begin{aligned} \mathfrak{W}_s(h;\psi_2,V) &\le Ch^{-d}\int_{r/2\le|x|\le E/2} |x|^{-\beta\left(\frac{d}{2}+s\right)}dx \\ &\le Ch^{-d}\int_{r/2}^{E/2} t^{\delta-1}dt,\ \delta = \frac{2-\beta}{2}(d-\omega). \end{aligned}$$

Thus by the definition (6.3)

$$\mathfrak{W}_s(h;\psi_2,V) \le Ch^{-d}r^{\delta} \le Ch^{-\omega}\theta^{\omega-d}.$$

Consequently, in view of (6.29),

$$\mathcal{M}_s(h,\mu;\psi_2) \le Ch^{-\omega}\big(\epsilon(\theta)+\epsilon(h)\big).$$

Combining this with (6.24), we arrive at

$$\mathcal{M}_s(h,\mu;\psi) = h^{-\omega}\big(\psi(0)\mathcal{N}_s(1;1,W)+\epsilon(\theta)+\epsilon(h)\big).$$

For $\theta=\theta_0$ this yields (1.7).

Let $\omega=d$. We fix $\theta=1$. In view of (6.26) and (6.29), (6.30),

$$\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = \mathfrak{W}_s(h;\psi_2,V)+O(h^{-d}). \tag{6.31}$$

To calculate $\mathfrak{W}_s(h;\psi_2,V)$ we present it for arbitrary $\rho\in[r,E/2]$ as follows:

$$\begin{aligned}\Xi_s^{-1}h^d\mathfrak{W}_s(h;\psi_2,V) = &\int_{r/2\le|x|\le r}\psi_2(x)\big(V_-(x)\big)^{s+\frac{d}{2}}dx\\ &+\int_{r\le|x|\le E/2}(\psi(x)-\psi(0))\big(V_-(x)\big)^{s+\frac{d}{2}}dx\\ &+\psi(0)\int_{r\le|x|\le\rho}\big(V_-(x)\big)^{s+\frac{d}{2}}dx\\ &+\psi(0)\int_{\rho\le|x|\le E/2}\big(V_-(x)\big)^{s+\frac{d}{2}}dx.\end{aligned}$$

The condition $\omega=d$ implies that $|V(x)|^{s+\frac{d}{2}}\le C|x|^{-d}$. Taking also into account that $|\psi(x)-\psi(0)|\le C|x|$, one sees that first two integrals are uniformly bounded and the last one does not exceed $C(|\ln\rho|+1)$. Hence

$$\left.\begin{aligned}\big|\mathfrak{W}_s(h;\psi_2,V) - h^{-d}\psi(0)\mathcal{I}(h,V)\big| &\le Ch^{-d}(|\ln\rho|+1),\\ \mathcal{I}(h,V) = \Xi_s\int_{r\le|x|\le\rho}&\big(V_-(x)\big)^{s+\frac{d}{2}}dx.\end{aligned}\right\} \tag{6.32}$$

Let us analyze $\mathcal{I}=\mathcal{I}(h,V)$. By (2.17)

$$\left|\mathcal{I}-\Xi_s\int_{r\le|x|\le\rho}\big(W_-(x)\big)^{s+\frac{d}{2}}dx\right| \le CU_0(\rho)\int_{r\le|x|\le\rho}|x|^{-d}dx.$$

The integral in the l.h.s. equals

$$\Xi_s\int_{\mathbb{S}^{d-1}}\Phi_-(\vartheta)^{\frac{d}{2}+s}d\vartheta\int_r^\rho t^{-1}dt = \mathfrak{B}_s(\Phi)|\ln h| + O(|\ln\rho|),$$

and the one in the r.h.s. equals

$$C(|\ln r|-|\ln\rho|)U_0(\rho)\le C'|\ln h|U_0(\rho).$$

Therefore

$$\big|\mathcal{I} - |\ln h|\mathfrak{B}_s(\Phi)\big| \le C\big(1 + |\ln\rho| + |\ln h|U_0(\rho)\big).$$

In combination with (6.32), (6.31), this yields

$$\begin{aligned}\big|\mathcal{M}_s(h,\mu,\psi,\mathfrak{a}) - &\ h^{-d}\psi(0)|\ln h|\mathfrak{B}_s(\Phi)\big| \\ &\le Ch^{-d}\big[1 + |\ln h|U_0(\rho) + |\ln\rho|\big].\end{aligned}$$

Pick $\rho = C(|\ln h| + 1))^{-1}$. For $U_0(\rho) \to 0$, $\rho \to 0$, the r.h.s. will be of order $h^{-d}|\ln h|\epsilon(h)$. This leads to (1.8).

Let $d - s - 1 < \omega < d$. According to (6.29), (6.30)

$$\mathcal{M}_s(h,\mu;\psi_2,\mathfrak{a}) = \mathfrak{W}_s(h;\psi_2,V) + h^{-\omega}\epsilon(\theta) + O(\mu^{s+1}h^{s+1-d}).$$

Adding (6.25), we get

$$\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) = \mathfrak{W}_s(h;\psi,V) + h^{-\omega}\big(\psi(0)\Theta_s + \epsilon(\theta)\big) + O(\mu^{s+1}h^{s+1-d}).$$

If $\theta = \theta_0$, this provides (1.9).

Let $\omega \le d - s - 1$. We fix $\theta = 1$. The region around the singularity does not contribute into the asymptotics. Indeed, in view of (6.26),

$$\mathcal{M}_s(h,\mu;\psi_1,\mathfrak{a}) \le Ch^{-\omega}.$$

Besides, $\mathfrak{W}_s(h;\psi_1,V) \le Ch^{-\omega}$, so that (6.29) entails

$$\begin{aligned}\mathcal{M}_s(h,\mu;\psi,\mathfrak{a}) &= \mathcal{M}_s(h,\mu;\psi_2,\mathfrak{a}) + O(h^{-\omega}) \\ &= \mathfrak{W}_s(h;\psi_2,V) + O\big(\mathcal{R}(h,\mu)\big) \\ &= \mathfrak{W}_s(h;\psi,V) + O\big(\mathcal{R}(h,\mu)\big).\end{aligned}$$

Using the second (for $\omega = d-s-1$) or the third (for $\omega < d-s-1$) inequality from (6.30), we obtain (1.11) or (1.12).

Appendix. An eigenfunction estimate for H_V

An important ingredient in the proof of Theorem 2.4′ is an estimate for the eigenfunctions of the Schrödinger operator $H_V(h)$ for large x. We shall use the method which is essentially the Agmon's method from [1]. Lemma A1 below will be valid for any (real-valued) potential V as long as the operator $H_V = -h^2\Delta + V$ is defined as a form-sum on the form domain of Δ. Besides, we assume that the function V is semibounded from below for $|x| \ge \rho_0 > 0$ and denote

$$V_1(r) = \text{v-sup}_{|x|\ge r} V_-(x), r \ge \rho_0.$$

Lemma A1. *Let the potential V be as specified above. Let $\rho \geq \rho_0 + 1$ and $u(x)$ be a normalized eigenfunction of H_V corresponding to an eigenvalue $\lambda < 0$, $|\lambda| \geq 4V_1(\rho - 1)$. Then*

$$\int_{|x|\geq 2\rho} |u(x)|^2 dx \leq C\frac{h^2}{|\lambda|} \exp\left\{-c\frac{\sqrt{|\lambda|}}{h}\rho\right\},\ \forall \rho \geq \rho_0 + 1, \tag{A1}$$

the constants c and C being dependent only on.

Proof. Let $\rho \geq \rho_0 + 1, \zeta \in C^1(\mathbb{R}^d)$ be a non-negative function such that

$$\zeta(x) = \begin{cases} 0, & |x| \leq \rho - 1, \\ 1, & |x| \geq \rho, \end{cases} \tag{A2}$$

and let $g \in \mathcal{B}^\infty(\mathbb{R}^d \setminus B(\rho_0))$ be a function $g(x) = g(r),\ r = |x|$, such that

$$\left.\begin{aligned} & g(r_1) \leq g(r_2),\ \rho_0 \leq r_1 \leq r_2, \\ & \sup_x |\partial g(x)| \leq 1, \\ & g(x) = const,\ |x| \geq \tilde{\rho} \geq \rho, \end{aligned}\right\} \tag{A3}$$

with some $\tilde{\rho}$. Furthermore, denote

$$v(x) = \phi(x)u(x), \quad \phi(x) = \zeta(x)e^{\frac{\delta}{h}g(x)},\ \delta > 0.$$

More precise choice of the functions ζ, g and the parameter δ will be made later on. Note that the function v belongs to the form-domain of H_V, since $\phi(x) \in C^1$ and $\phi(x) = const$ for sufficiently large x. Since u is the eigenfunction, we have

$$\int (-ih\partial u)(x)\overline{(-ih\partial v)(x)}dx + \int V(x)|u(x)|^2\phi(x)dx \\ + \eta^2 \int |u(x)|^2\phi(x)dx = 0,\ \lambda = -\eta^2. \tag{A4}$$

For

$$(-ih\partial v)(x) = (-ih\partial u)(x)\phi(x) - ihu(x)\partial\phi(x),$$

the first term in (A4) equals

$$\left.\begin{aligned} \int (-ih\partial u)(x)\overline{(-ih\partial v)(x)}dx &= \int |h\partial u(x)|^2\phi(x)dx + \mathcal{S}, \\ \mathcal{S} &= h\int h\partial u(x)\ \overline{u(x)}\partial\phi(x)dx. \end{aligned}\right\} \tag{A5}$$

In view of (A4), (A5),

$$\int |h\partial u(x)|^2\phi(x)dx + \int \big(V(x) + \eta^2\big)|u(x)|^2\phi(x)dx \leq |\mathcal{S}|. \tag{A6}$$

Let us estimate $\mathcal{S}$. Since

$$\partial e^{\frac{\delta}{h}g(x)} = \frac{\delta}{h}\partial g(x) e^{\frac{\delta}{h}g(x)},$$

and $|\partial g(x)| \leq 1$, we have

$$\begin{aligned} |\mathcal{S}| \leq\ & h\int |h\partial u(x)||u(x)|\Big(|\partial e^{\frac{\delta}{h}g(x)}|\zeta(x) + e^{\frac{\delta}{h}g(x)}|\partial\zeta(x)|\Big)dx \\ \leq\ & \delta\int |h\partial u(x)||u(x)|\phi(x)dx + h\int |h\partial u(x)||u(x)|e^{\frac{\delta}{h}g(x)}|\partial\zeta(x)|dx. \end{aligned}$$

Using the Younge inequality for the first and the second terms separately, we get for any $\epsilon > 0$

$$\begin{aligned} |\mathcal{S}| \leq\ & \frac{\epsilon}{2}\int |h\partial u(x)|^2\phi(x)dx + \frac{\delta^2}{2\epsilon}\int |u(x)|^2\phi(x)dx \\ +\ & \frac{\epsilon}{2}\int |h\partial u(x)|^2 e^{\frac{\delta}{h}g(x)}|\partial\zeta(x)|^2dx + \frac{h^2}{2\epsilon}\int_{\text{supp}\,\partial\zeta}|u(x)|^2e^{\frac{\delta}{h}g(x)}dx. \end{aligned} \quad (A7)$$

Estimating the r.h.s. of (A6) by (A7) and rearranging the terms, we obtain

$$\begin{aligned} &\int |h\partial u(x)|^2 e^{\frac{\delta}{h}g(x)}\left[\left(1-\frac{\epsilon}{2}\right)\zeta(x) - \frac{\epsilon}{2}|\partial\zeta(x)|^2\right]dx \\ +&\int\left(V(x)+\eta^2-\frac{\delta^2}{2\epsilon}\right)|u(x)|^2\phi(x)dx \leq \frac{h^2}{2\epsilon}\int_{\text{supp}\,\partial\zeta}|u(x)|^2e^{\frac{\delta}{h}g(x)}dx. \end{aligned} \quad (A8)$$

Let us specify now the choice of the function ζ. We shall choose ζ in such a way that the function

$$K(x) = \left(1-\frac{\epsilon}{2}\right)\zeta(x) - \frac{\epsilon}{2}|\partial\zeta(x)|^2$$

in the first integral in the l.h.s. of (A8) is non-negative for all $x \in \mathbb{R}^d$. Let $\zeta(x) = \xi(r-\rho+1),\ r = |x|$, where the function $\xi \in C^1(\mathbb{R}_+)$ is defined by

$$\begin{aligned} &\xi(r) = r^2,\ 0 \leq r \leq \frac{1}{2}; \\ &\xi(r) \geq \frac{1}{4},\ \frac{1}{2} \leq r \leq 1, \\ &\xi(r) = 1,\ r \geq 1. \end{aligned}$$

Obviously, ζ satisfies (A2). Then for the function $K(x)$ we have the relations

$$\begin{aligned} &K(x) = 0,\ 0 \leq |x| \leq \rho - 1; \\ &K(x) = \big(1-5\epsilon/2\big)(|x|-\rho+1)^2,\ \rho-1 \leq |x| \leq \rho - 1/2; \\ &K(x) \geq \big(1-\epsilon/2\big)\frac{1}{4} - \frac{\epsilon}{2}\max_r|\partial_r\xi(r)|^2,\ \rho - \frac{1}{2} < |x| \leq \rho; \\ &K(x) = (1-\epsilon/2),\ |x| > \rho. \end{aligned}$$

Choosing ϵ small enough one can guarantee that $K(x) \geq 0$ for all $x \in \mathbb{R}^d$. Therefore the first integral in the l.h.s. of (A8) is non-negative. Thus (A8) provides the estimate

$$\int \left(V(x) + \eta^2 - \frac{\delta^2}{2\epsilon} \right) |u(x)|^2 \phi(x) dx \leq \frac{h^2}{2\epsilon} \int_{\operatorname{supp} \partial\zeta} |u(x)|^2 e^{\frac{\delta}{h} g(x)} dx.$$

Taking into account also that $\operatorname{supp} \partial\zeta \in \{x : \rho - 1 \leq |x| \leq \rho\}$, and that g is a monotone function (see (A3)), we get from here:

$$\int \left(V(x) + \eta^2 - \frac{\delta^2}{2\epsilon} \right) |u(x)|^2 \phi(x) dx \leq \frac{h^2}{2\epsilon} e^{\frac{\delta}{h} g(\rho)} \int_{\operatorname{supp} \partial\zeta} |u(x)|^2 dx \leq \frac{h^2}{2\epsilon} e^{\frac{\delta}{h} g(\rho)}. \tag{A9}$$

The last inequality holds because u is normalized. Define $\delta = \epsilon^{1/2}\eta$ so that $\eta^2 - \delta^2/(2\epsilon) = \eta^2/2$. Since $\eta^2 \geq 4V_1(\rho - 1)$, we have:

$$V(x) + \frac{\eta^2}{2} \geq \frac{\eta^2}{4}, \ x \in \operatorname{supp} \zeta.$$

Therefore, by monotonicity of g, the l.h.s. of (A9) has the lower bound

$$\frac{\eta^2}{4} \int |u(x)|^2 \zeta(x) e^{\frac{\delta}{h} g(x)} dx \geq \frac{\eta^2}{4} e^{\frac{\delta}{h} g(2\rho)} \int_{|x| \geq 2\rho} |u(x)|^2 dx.$$

Now it follows from here and (A9) that

$$\frac{\eta^2}{4} e^{\frac{\delta}{h} g(2\rho)} \int_{|x| \geq 2\rho} |u(x)|^2 dx \leq \frac{h^2}{2\epsilon} e^{\frac{\delta}{h} g(\rho)},$$

which leads to

$$\int_{|x| \geq 2\rho} |u(x)|^2 dx \leq \frac{2h^2}{\eta^2 \epsilon} e^{\frac{\delta}{h} \left(g(\rho) - g(2\rho) \right)}. \tag{A10}$$

To obtain from here (A1) we specify the choice of the function g, satisfying (A3). Let $\xi_1 \in \mathcal{B}^\infty(\mathbb{R})$ be a non-negative function such that $\partial_r \xi_1(r) \geq 0$ and

$$\xi_1(r) = \begin{cases} 0, & r \leq -\frac{1}{2}, \\ 1, & r \geq \frac{1}{2}. \end{cases}$$

Denote $\xi_2 = 1 - \xi_1$ and define

$$g(r) = \gamma \left[r \xi_2 \left(\frac{r - 4\rho}{4\rho} \right) + 6\rho \xi_1 \left(\frac{r - 4\rho}{4\rho} \right) \right], \ \gamma > 0.$$

Clearly, $g(r) = \gamma r,\ r \le 2\rho$ and $g(r) = 6\gamma\rho,\ r \ge 6\rho$. In particular, the third property (A3) is fulfilled for $\tilde\rho = 6\rho$.

To prove monotonicity of g we calculate its derivative:

$$\begin{aligned}\gamma^{-1}\partial_r g(r) &= \xi_2\left(\frac{r-4\rho}{4\rho}\right) + \frac{r}{4\rho}\partial_r\xi_2\left(\frac{r-4\rho}{4\rho}\right) + \frac{3}{2}\partial_r\xi_1\left(\frac{r-4\rho}{4\rho}\right)\\ &= \xi_2\left(\frac{r-4\rho}{4\rho}\right) + \frac{1}{4\rho}(6\rho - r)\partial_r\xi_1\left(\frac{r-4\rho}{4\rho}\right).\end{aligned} \tag{A11}$$

The r.h.s. is non-negative, since $\xi_2 \ge 0, \partial_r\xi_1 \ge 0$ and $r \le 6\rho$ on the support of $\partial_r\xi_1$. This implies that g is monotone.

Notice that the r.h.s. of (A11) is bounded from above by

$$1 + \frac{3}{2}\max_r \partial_r\xi_1(r).$$

Let the number γ^{-1} be equal to this bound, so that $|\partial_r g| \le 1$. This completes the proof of (A3) for g.

To get (A1) from (A10) it remains to notice that $g(\rho) - g(2\rho) = -\gamma\rho$ and $\delta = c\eta = c\sqrt{|\lambda|}$. $\square$

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