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Absence of the Efimov Effect in a Homogeneous Magnetic Field

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Abstract

A system of three identical particles in a homogeneous magnetic field is studied. It is shown that the hamiltonian of this system with short-range potentials after the separation of the center of mass motion has finite discrete spectrum for each fixed type m of the rotational ($SO(2)$) symmetry.

Mathematics Subject Classification (1991) 81V10.

1 Introduction.

It is well known that a system of three identical particles with short-range pair potentials may have infinite number of bound states. This fact was established by physicist V. Efimov in 1970 [1] and is known as the Efimov effect. Mathematical proofs of its existence were given by D. Yafaev [2], Y. Ovchinnicov and I.M. Sigal [3] and H. Tamura [4] by different methods. If the potentials are spherically symmetric, the infinite number of eigenvalues

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occurs in the Efimov effect case for at least one value of the weight of the rotational symmetry. For almost all others (exclusing finite number of the weights) the number of eigenvalues equals zero [5] .

In his first paper on the subject, Efimov also conjected that the magnetic fields destroys the Efimov effect. But the spectral theory of many particle Schrödinger operators with a magnetic field was not developed enough to give a rigorous proof of this fact at that time.

In this paper we study the Schrödinger operator of a system of three identical particles with short-range interactions in a homogeneous magnetic field. Our goal is to show that after separation of the center of mass motion this operator has finite discrete spectrum of the rotational ($SO(2)$) symmetry m for each m. This means that the Efimov effect does not exist in a magnetic field. We shall see that this phenomenon is closely related to the absence of the Efimov effect in the one dimensional case [6]. In the magnetic field case any wave-function of the whole system can be decomposed into two components. The first one corresponds to the states describing the particles on the lowest Landau level. For this states each particle moves freely only along the direction of the magnetic field. In this sense it is "one-dimensional" part of the wave-function. The system looks like a system of three one-dimensional particles. As was mentioned above, there is no Efimov effect in this case.

The other part of the wave-function can not produce the Efimov effect either, because for this part the infimum of the kinetic energy is greater then the bottom of the essential spectrum for the three particle operator. This case is analogous to the case of a three particle system (without magnetic field) possessing stable two-particle subsystems. It is known that only a finite number of discrete eigenvalues may exist in this case [7].

The method of the proof is a variational one. It is based on decompositions of the configuration space. Two different types of the decompositions are used. For the "one-dimensional" part of a trial function we use the techniques of nonsmooth decompositions, suggested by G. Zhislin and the author in [6]. For the other part standard smooth decompositions [7] are used.

2 Main Definitions and Results.

Let $Z_0 = \{1, 2, 3\}$ be a system of three identical particles with masses $M = 1$ and charges $e = -1$ in a homogeneous magnetic field B and let $r_i = (r_i^1, r_i^2, r_i^3)$ be the position vector of the i-th particle. The Schrödinger operator for the system has the form

$$\mathcal{H} = \sum_{j=1}^{3} \frac{1}{2} \left(i\nabla_j + A_j\right)^2 + \sum_{\substack{j,k=1 \\ j<k}}^{3} V(|r_{ij}|), \tag{2.1}$$

where $A_j = \frac{1}{2}B\{-r_j^2, r_j^1, 0\}$ is a vector-potential of the magnetic field, $r_{ij} = r_j - r_i$, V is a potential of the interaction between the particles. We assume that the direction of the magnetic field coincides with the direction of the third axis. We assume also that the potential V is a real function and the inequality

$$|V(r_{ij})| \leq C_0(1 + |r_{ij}|)^{-2-\delta_0} \tag{2.2}$$

holds for some constants $C_0 > 0$ and $\delta_0 > 0$. For a system of identical particles one can separate the center of mass motion. Let $r_c = \frac{1}{3}(r_1 + r_2 + r_3)$ be the radius vector of the center of mass, α - be an arbitrary pair of particles. for example $\alpha = (i, j)$. Let us introduce the Jacobi coordinates for the pair α:

$$q^\alpha = (q_1^\alpha, q_2^\alpha, q_3^\alpha) = r_j - r_i, \quad \xi^\alpha = \frac{1}{2}(r_i + r_j) - r_p,$$

where $p \neq i, j$.
Operator $\mathcal{H}$ can be rewritten in the form

$$\mathcal{H} = T + \mathcal{H}_0, \tag{2.3}$$

where T is the kinetic energy operator of center of mass motion and $\mathcal{H}_0$ is the energy operator in the center of mass system of coordinates;

$$T = \frac{1}{6}(i\nabla_{r_c} + A_c)^2, \tag{2.4}$$

$$\mathcal{H}_0 = \left(i\nabla_{q^\alpha} + A_{q^\alpha}\right)^2 + \frac{3}{4}\left(i\nabla_{\xi^\alpha} + A_{\xi^\alpha}\right)^2 + \sum_{\substack{i,j \\ i<j}} V\left(|r_{ij}|\right), \tag{2.5}$$

where

$$\begin{aligned} A_c &= \frac{3}{2}B\{-r_c^2, r_c^1, 0\}, \\ A_{q^\alpha} &= \frac{1}{4}B\{-q_2^\alpha, q_1^\alpha, 0\} \\ A_{\xi^\alpha} &= \frac{1}{3}B\{-\xi_2^\alpha, \xi_1^\alpha, 0\}, \quad B > 0. \end{aligned}$$

Our goal is to study the spectral properties of the operator $\mathcal{H}_0$.

Let $SO(2)$ be the group of rotations around the direction of the third axis (the direction of the magnetic field). By m and P^m we denote the weight of the representation of this group and the projection onto the subspace of function transformed according to the representations of the weight m; we set $\mathcal{H}_0^m = P^m\mathcal{H}_0$. Let h_α be the energy operator for the subsystem $\alpha = (i,j)$

$$h_\alpha = (i\nabla_{q^\alpha} + A_{q^\alpha})^2 + V\left(|r_{ij}|\right), \tag{2.6}$$

$$h_\alpha^m = h_\alpha P^{m'}, \quad \nu^{m'} = \inf h_\alpha^{m'} \quad \text{and}$$

$$\mu^m = \inf_{m'}\left\{\nu^{m'} + B(|m - m'| + m + m' + 1)\right\}.$$

According to the HVZ type theorem [8, 9] μ^m is the infimum of the essential spectrum of the operator $\mathcal{H}_0^m$. It was shown in [8, 9] that if $\mu^m < B(|m| - m + 2)$ the discrete spectrum of the operator $\mathcal{H}_0^m$ is finite. In this Letter we consider the operator $\mathcal{H}_0^m$ with $\mu^m = B(|m| - m + 2)$[1].

Theorem 1 *For the operator $\mathcal{H}_0^m$ let the condition (2.2) holds. Then the discrete spectrum of $\mathcal{H}_0^m$ is finite.*

Remarks.

(1) The finiteness of the discrete spectrum also holds for the operator $\mathcal{H}_0^m$ with nonhomogeneous magnetic field if the magnetic vector potential satisfies the next condition

$$A_j = \{-r_j^2 \cdot f(\rho), r_j^1 \cdot f(\rho), 0\},$$

where $\quad \rho = [(r_j^1)^2 + (r_j^2)^2]^{1/2}, \quad f \in C(R_+^1)$

and $\quad |f(\rho)| \cdot \rho^1 \to \infty$ for $\rho \to \infty$.

[1] The situation $\mu^m > B(|m| - m + 2)$ can not exist because $\nu^{m'} \leq (|m'| - m' + 1)$.

(2) Let us consider the Schrödinger operator for a system of three nonidentical particles in a homogeneous magnetic field

$$\mathcal{H} = \sum_{j=1}^{3} \frac{1}{2} M_j^{-1} \left(i\nabla_j + A_j\right)^2 + \sum_{\substack{j,k=1 \\ j<k}}^{3} V_{ij}(|r_{ij}|),$$

where M_j is the mass of the particle with the number j, V_{ij} satisfies (2.2) and

$$A_j = \frac{1}{2} B\{-e_j r_j^2, e_j r_i^1, 0\},$$

e_j - is a charge.

Let us assume that $e_i e_j > 0$ for $i, j = 1, \dots, 3$. Then for the operator $\mathcal{H}^m = P^m \mathcal{H}$ the HVZ theorem holds and our main result also may be generalized to this operator(the discrete spectrum of the operator $\mathcal{H}^m$ is finite).

(3) Let $\mathcal{H}^m_{as}$ be the restriction of the operator $\mathcal{H}^m_0$ onto the subspace of functions which are antisymmetrical according to the permutations of particles. Finiteness of the discrete spectrum for the operator $\mathcal{H}^m_{as}$ does not follow from the same one for the operator $\mathcal{H}^m_0$ because the bottoms of the essential spectrum may be different. Nevertheless the discrete spectrum of the operator $\mathcal{H}^m_{as}$ is finite.

The proofs of all the remarks are similar to the proof of the main theorem.

3 Proof of the Theorem.

As it was mentioned above we prove the theorem for the case
$\mu^m = B(|m| - m + 2)$. The proof for $\mu^m < B(|m| - m + 2)$ was given in [9].
First, we establish two lemmas. Let

$$T_{-\alpha} = \left\{ \left(i\frac{\partial}{\partial q_1^\alpha} - \frac{1}{4} B q_2^\alpha \right)^2 + \left(i\frac{\partial}{\partial q_2^\alpha} + \frac{1}{4} B q_1^\alpha \right)^2 \right\}, \qquad (3.1)$$

$$T^{m'}_{-\alpha} = T_{-\alpha} \cdot P^{m'}, \quad \varphi_0^{m'}(q_1, q_2)$$

be an eigenfunction of the operator $T_{-\alpha}^{m'}$ corresponding to the lowest eigenvalue $\nu_0^{m'} = B(|m'| - m' + 1)$. It is known that the eigenvalue $\nu_0^{m'}$ is nondegenerated and $\varphi_0^{m'}$ is the corresponding Landau solution.

Lemma 1 *Let* $\inf h_\alpha^{m'} = \nu_0^{m'}$ *and the potential satisfies (2.2). Then for an arbitrary function*

$$\psi(q_1^\alpha, q_2^\alpha, q_3^\alpha) = \varphi_0^{m'}(q_1^\alpha, q_2^\alpha) f(q_3^\alpha) + g(q^\alpha)$$

such that

$$f(q_3^\alpha) \in C^1(R_0^1), \quad g(q^\alpha) \in C_0^2(R^3), \quad g(q^\alpha) - \varphi_0^{m'}$$

for every fixed value of q_3^α, $g(q^\alpha) = 0$ *for* $|q_3^\alpha| \geq a$
the next inequality holds

$$\begin{aligned} L_0[\psi] &= \left(\left(T_{-\alpha}^{m'} - \nu_0^{m'}\right)\psi, \psi\right) + \int_{|q_3^\alpha| \leq a} V(q^\alpha)|\psi|^2 dq^\alpha + \\ &+ \left\|\frac{\partial}{\partial q_3^\alpha} g\right\|^2 + \int_{|q_3^\alpha| \leq a} \left|\frac{\partial f}{\partial q_3^\alpha}\right|^2 dq_3^\alpha \geq \\ &\geq \left\{f^2(a) + f^2(-a)\right\} \cdot C_0 \left(1 + \delta_0\right)^{-1} |a|^{-1-\delta_0} \end{aligned} \tag{3.2}$$

Lemma 2 *For an arbitrary function* $y \in C_0^1(R^1)$, *constants* $d_2 > d_1 > 0$ *and* $i = 1, 2$

$$\begin{aligned} |y(d_i)|^2 &\leq 2(d_2 - d_1)^{-1} \int_{d_1}^{d_2} |y|^2 \, dt + \\ &+ 2(d_2 - d_1) \int_{d_1}^{d_2} \left|\frac{\partial y(t)}{\partial t}\right|^2 dt \end{aligned} \tag{3.3}$$

Proof of Lemma 1 . Let $\psi_n \quad n = 2, 3, \ldots$ be the sequence of function such that $\psi_n = \psi$ for $|q_3| \leq a$, $\quad \psi_n = \varphi_0^{m'}(q_1^\alpha, q_2^\alpha) \cdot f(a) \cdot \frac{na - q_3^\alpha}{na - a}$ for $a \leq q_3^\alpha \leq na$, $\quad \psi_n = 0$ for $|q_3^\alpha| \geq na$,

$$\psi_n = \varphi_0^{m'}(q_1^\alpha, q_2^\alpha) \cdot f(-a) \cdot \frac{na + q_3^\alpha}{na - a}$$

. For each function ψ_n the inequality

$$\left(\left(h_\alpha^{m'} - \nu_0^{m'}\right)\psi_n, \psi_n\right) \geq 0 \tag{3.4}$$

holds. The inequality (3.2) follows from (3.4) if $n \to \infty$.

Proof of Lemma 2 . Lemma 2 is one-dimensional variant of Lemma 2 [6]. The proof of it is similar and simpler then in [6].

Proof of the Theorem.

To prove the finiteness of the discrete spectrum of the operator $\mathcal{H}_0^m$ we construct a finite-dimensional subspace $\mathcal{M} \subset C_0^2$ such that

$$(\mathcal{H}_0^m \psi, \psi) \geq \mu^m ||\psi||^2 \tag{3.5}$$

for all

$$\psi - \mathcal{M}, \quad \psi \in C_0^2.$$

For this goal we will make partition of unity on the plane $(q_3^\alpha, \xi_3^\alpha)$.

Let for $a > 0$

$$S(a) = \{(q^\alpha, \xi^\alpha) | \quad |q_3^\alpha|^2 + |\xi_3^\alpha|^2 \leq a\},$$

$r_3 = (q_3^\alpha, \xi_3^\alpha), \quad U(t), V(t)$ be some functions such that

$$U(t), V(t) \in C^2(R_+^1), \quad U^2 + V^2 = 1,$$

$$U = 1 \quad \text{for} \quad t \leq a, \quad U = 0 \quad \text{for} \quad t \geq 2a,$$

$$\lim_{t \to 2a} V'^2 (1 - V^2) = 0.$$

Then, according to [7]

$$(\mathcal{H}_0^m \psi, \psi) \geq L_1 [\psi U(|r_3|)] + L_2 [\psi V(|r_3|)], \tag{3.6}$$

where

$$\begin{aligned} L_1[\varphi] &= (\mathcal{H}_0^m \varphi, \varphi) - C||\varphi||^2, \\ L_2[\varphi] &= (\mathcal{H}_0^m \varphi, \varphi) - \epsilon ||\varphi |r_3|^{-1,5}||^2 \end{aligned}$$

and the constant $\epsilon > 0$ may be chosen small if $C > 0$ is large enough. Operator $\mathcal{H}_0^m$ is a compact operator in the region $|r_3| \leq 2a$ with the Dirichlet boundary conditions at $|r_3| = 2a$ [9]. So one can fined such a finite dimensional subspace $\mathcal{M}$ that for all $\psi_{U(|r_3|)} - \mathcal{M}$

$$L_1 \left[\psi_{U(|r_3|)} \right] \geq \mu^m ||\psi_{U(|r_3|)}||^2. \tag{3.7}$$

Therefore the Theorem holds if one can fined such a constant $a > 0$ that

$$L_2\left[\psi_{V(|r_3|)}\right] \geq \mu^m ||\psi_{V(|r_3|)}||^2. \tag{3.8}$$

For an arbitrary pair of particles α and numbers $b > 0, \quad a > 0$ let

$$K(\alpha, b, a) = \{(q^\alpha, \xi^\alpha)| \quad |q_3^\alpha| \leq b|\xi_3^\alpha|, \quad |r_3| \geq a\} \tag{3.9}$$

It is easy to see [7] that one can choose such a small positive number $b_0 > 0$ that for all $b < b_0$ regions $K(\alpha, b, a)$ does not overlap for different α.

Let T_- be the operator of the kinetic energy of the system, corresponding to the motion on the plane orthogonal to the direction of the magnetic field

$$\begin{aligned} T_- \;&=\; \left\{\left(i\frac{\partial}{\partial q_1^\alpha} - \frac{1}{4}Bq_2^\alpha\right)^2 + \left(i\frac{\partial}{\partial q_2^\alpha} + \frac{1}{4}Bq_1^\alpha\right)^2\right\} + \\ &+\; \frac{3}{4}\left\{\left(i\frac{\partial}{\partial \xi_1^\alpha} - \frac{1}{3}B\xi_2^\alpha\right)^2 + \left(i\frac{\partial}{\partial \xi_2^\alpha} + \frac{1}{3}B\xi_1^\alpha\right)^2\right\} \end{aligned} \tag{3.10}$$

$$T_-^m = T_- \cdot P^m.$$

By $\mathcal{P}_0$ we denote the projection in $\mathcal{L}_2(R^6)$ onto the eigensubspace of the operator T_-^m corresponding to the lowest eigenvalue $\mu^m, \mathcal{P}_- = (I - \mathcal{P}_0)$. We will make different decomposition for the functions $\mathcal{P}_0\psi_v$ and $\mathcal{P}_-\psi_v$. For $\mathcal{P}_-\psi_v$. it will be ordinary smooth partition [7]. For $\mathcal{P}_0\psi_v$. nonsmooth partition will be made similar to [6].

Let

$$U_1(t), V_1(t) \in C^2(R_+^1), \quad U_1^2 + V_1^2 = 1, \quad U_1 = 1$$

$$\text{for} \quad t \leq \frac{1}{3}b_0, \qquad U_1 = 0$$

$$\text{for} \quad t \geq \frac{2}{3}b_0, \qquad 0 < U_1 < 1$$

$$\text{for} \quad t \in \left[\frac{1}{3}b_0, \frac{2}{3}b_0\right], \qquad U_t'^2(1 - U^2) \to 0$$

if $\quad t \to 1/3b_0$ and let $U_1^\alpha = U_1(|q_3^\alpha| \cdot |\xi_3^\alpha|^{-1}), \quad V_1^\alpha = V_1(|q_3^\alpha| \cdot |\xi_3^\alpha|^{-1}),$

$V = (1 - \sum_\alpha (U_1^\alpha)^2)^{1/2}$, χ_α be the characteristic function of the support of the function U_1^α,

$$\tilde{\chi} = 1 - \sum_\alpha \chi_\alpha.$$

We denote by ψ_α and $\tilde{\psi}$ the functions

$$\psi_\alpha = (\mathcal{P}_- \psi_V)\, U_1^\alpha + (\mathcal{P}_0 \psi_V)\, \chi_\alpha \tag{3.11}$$

and

$$\tilde{\psi} = (\mathcal{P}_- \psi_V)\, V + (\mathcal{P}_0 \psi_V)\, \tilde{\chi}.$$

It is clear that $\mathcal{P}_0 \psi_V \chi_\alpha$ is orthogonal to $\mathcal{P}_- \psi_V U_1^\alpha$ and $\mathcal{P}_0 \psi_V \tilde{\chi}$ is orthogonal to $\mathcal{P}_- \psi_V V$, so

$$\sum_\alpha ||\psi_\alpha||^2 + ||\tilde{\psi}||^2 = ||\mathcal{P}_0 \psi_V||^2 + ||\mathcal{P}_- \psi_V||^2 = ||\psi_V||^2 \tag{3.12}$$

Now let us begin to estimate

$$L_2[\psi_V] \equiv (\mathcal{H}_0^m \psi_V, \psi_V) - \epsilon ||\psi_V |r_3|^{-1,5}||^2. \tag{3.13}$$

Because of the same reasons as in (3.12)

$$||\psi_V |r_3|^{-1,5}||^2 = \sum_\alpha ||\psi_\alpha |r_3|^{-1,5}||^2 + ||\tilde{\psi} |r_3|^{-1,5}||^2. \tag{3.14}$$

Further

$$\begin{aligned} (\mathcal{H}_0^m \psi_V, \psi_V) \ &\equiv\ (T_- \psi_V, \psi_V) + \left\| \frac{\partial}{\partial q_3^\alpha} \psi_V \right\|^2 + \frac{3}{4} \left\| \frac{\partial}{\partial \xi_3^\alpha} \psi_V \right\|^2 \\ &+\ \sum_{(i,j)} (V(|r_{ij}|) \psi_V, \psi_V). \end{aligned} \tag{3.15}$$

The decompositions with the functions $U_1^\alpha, V_1^\alpha, \chi_\alpha$ do not depend on variables that are orthogonal to the direction of the magnetic field, so

$$(T_- \psi_V, \psi_V) = \sum_\alpha (T_- \psi_\alpha, \psi_\alpha) + (T_- \tilde{\psi}, \tilde{\psi}). \tag{3.16}$$

Let us note, that

$$|\psi_V|^2 \neq \sum |\psi_\alpha|^2 + |\tilde{\psi}|^2$$

only for such r_3 that

$$1/3b_0|\xi_3^\alpha| \quad \leq |q_3^\alpha| \quad \leq 2/3b_0|\xi_3^\alpha|$$

for at least one pair α . For all these points and all pairs $\alpha = (i,j)$ the inequality

$$|V(r_{ij})| \leq C|r_3|^{-2-\delta_0} \tag{3.17}$$

holds for some $C > 0$. So for all pairs (i,j)

$$\begin{aligned}(V_{ij}\psi_V, \psi_V) - \sum_\alpha (V_{ij}\psi_\alpha, \psi_\alpha) - \left(V_{ij}\tilde{\psi}, \tilde{\psi}\right) \geq \\ \geq -C\{||\psi_V|r_3|^{-1-\delta_0/2}||^2 + \sum_\alpha ||\psi_\alpha|r_3|^{-1-\delta_0/2}||^2 + \\ +||\tilde{\psi}|r_3|^{-1-\delta_0/2}||^2\} \geq -2C\{\sum_\alpha ||\psi_\alpha|r_3|^{-1-\delta_0/2}||^2 + \\ +||\tilde{\psi}|r_3|^{-1-\delta_0/2}||^2\}\end{aligned} \tag{3.18}$$

Now let us estimate the terms

$$\left\|\frac{\partial}{\partial q_3^\alpha}\psi_V\right\|^2 \quad \text{and} \quad \frac{3}{4}\left\|\frac{\partial}{\partial \xi_3^\alpha}\psi_V\right\|^2.$$

Their estimates are similar one to another and we will do it, for $\left\|\frac{\partial}{\partial q_3^\alpha}\psi_V\right\|^2$. First let us note that

$$\left\|\frac{\partial}{\partial q_3^\alpha}\psi_V\right\|^2 = \left\|\frac{\partial}{\partial q_3^\alpha}\mathcal{P}_0\psi_V\right\|^2 + \left\|\frac{\partial}{\partial q_3^\alpha}\mathcal{P}_-\psi_V\right\|^2. \tag{3.19}$$

For the component $\mathcal{P}_0\psi_V$ we have

$$\left\|\frac{\partial}{\partial q_3^\alpha}\mathcal{P}_0\psi_V\right\|^2 = \sum_\beta \int \left|\frac{\partial}{\partial q_3^\alpha}\mathcal{P}_0\psi_V\right|^2 \cdot \chi_\beta d\Omega + \int \left|\frac{\partial}{\partial q_3^\alpha}\mathcal{P}_0\psi_V\right|^2 \cdot \tilde{\chi} d\Omega. \tag{3.20}$$

Functions U_1^β $\beta = (1,2), (1,3), (2,3)$ and V make smooth decomposition of the support of the function ψ_V in the plane $(q_3^\alpha, \xi_3^\alpha)$. Using the estimate of the localization error from [7] one can fined that

$$
\begin{aligned}
\left\| \frac{\partial}{\partial q_3^\alpha} \mathcal{P}_- \psi_V \right\|^2 \geq \sum_\beta \left\| \frac{\partial}{\partial q_3^\alpha} \mathcal{P}_- \psi_V U_1^\beta \right\|^2 + \\
+ \left\| \frac{\partial}{\partial q_3^\alpha} \mathcal{P}_- \psi_V V \right\|^2 - \epsilon \sum_\beta \int \left| \mathcal{P}_- \psi_V U_1^\beta \right|^2 |r_3|^{-2} \, d\Omega - \\
-C \int |\mathcal{P}_- \psi_V V|^2 |r_3|^{-2} \, d\Omega
\end{aligned}
\tag{3.21}
$$

where the constant $\epsilon > 0$ may be chosen small if the number $C > 0$ is large enough. It follows from (3.13)–(3.16), (3.18)–(3.21) that

$$
L_2[\psi_V] \geq \sum_\alpha L_3[\psi_\alpha] + L_4[\tilde{\psi}], \tag{3.22}
$$

where

$$
\begin{aligned}
L_3[\psi_\alpha] \; &= \; (T_- \psi_\alpha, \psi_\alpha) + \sum_{(i,j)} (V_{ij} \psi_\alpha, \psi_\alpha) + \\
&+ \; \int \left| \frac{\partial}{\partial q_3^\alpha} \mathcal{P}_0 \psi_\alpha \right|^2 \chi_\alpha d\Omega + \frac{3}{4} \int \left| \frac{\partial}{\partial \xi_3^\alpha} \mathcal{P}_0 \psi_\alpha \right|^2 \chi_\alpha d\Omega + \\
&+ \; \left\| \frac{\partial}{\partial q_3^\alpha} \mathcal{P}_- \psi_\alpha \right\|^2 + \frac{3}{4} \left\| \frac{\partial}{\partial \xi_3^\alpha} \mathcal{P}_- \psi_\alpha \right\|^2 - \\
&- \; \epsilon \int |\mathcal{P}_- \psi_\alpha|^2 |r_3|^{-2} d\Omega - C_1 ||\psi_\alpha |r_3|^{-1-1/2\delta_1}||^2,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
L_4[\tilde{\psi}] \; &= \; \left(T_- \tilde{\psi}, \tilde{\psi} \right) + \sum_{(i,j)} \left(V_{ij} \tilde{\psi}, \tilde{\psi} \right) + \\
&+ \; \left\| \frac{\partial}{\partial q_3^\alpha} \mathcal{P}_- \tilde{\psi} \right\|^2 + \frac{3}{4} \left\| \frac{\partial}{\partial \xi_3^\alpha} \mathcal{P}_- \tilde{\psi} \right\|^2 + \\
&+ \; \int \left\{ \left| \frac{\partial}{\partial q_3^\alpha} \mathcal{P}_0 \tilde{\psi} \right|^2 + \frac{3}{4} \left| \frac{\partial}{\partial \xi_3^\alpha} \mathcal{P}_0 \tilde{\psi} \right|^2 \right\} \tilde{\chi} d\Omega - \\
&- \; C \int \left| \mathcal{P}_- \tilde{\psi} \right|^2 |r_3|^{-2} d\Omega - C_2 \left\| \tilde{\psi} |r_3|^{-1-1/2\delta_1} \right\|^2,
\end{aligned}
\tag{3.24}
$$

$\delta_1 = \min\{\delta_0, 1/2\}, \quad C, C_1, C_2$ are some constants. Formula (3.22) gives the estimate of the functional $L_2[\psi_V]$ (and the quadratic form $(\mathcal{H}_0^m \psi, \psi)$) with the estimates of some other functionals on functions having the supports in the regions where all the distances among the particles are large or the distance from one particle to two others is large.

Let us begin to estimate the functional $L_3[\psi_\alpha]$. We assume, for example, that $\alpha=(1,2)$. On the support of $\psi_{1,2}$ the particle (3) is far from two others. If $(i,j)\neq\alpha$ we have $|V_{ij}|\leq C|r_3|^{-2-\delta_0}$ on the support of ψ_α and so

$$\sum_{i,j}(V_{ij}\psi_\alpha,\psi_\alpha)\geq(V_\alpha\psi_\alpha,\psi_\alpha)-C||\psi_\alpha|r_3|^{-1-1/2\delta_0}||^2. \tag{3.25}$$

Let $P^{m_1}_{q^\alpha}(P^{m_2}_{\xi^\alpha})$ be the projection onto the subspace of functions having as functions of variables $q^\alpha\quad(\xi^\alpha)\quad SO(2)$ symmetry of the weight $m_1\quad(m_2)$. For any fixed $\bar m$ such that $\mathcal{P}_0P^{\bar m}_{q_\alpha}P^{m-\bar m}_{\xi_\alpha}\not\equiv 0$ by $G^{\bar m}_\alpha$ we denote the function

$$G^{\bar m}_\alpha(q^\alpha_3,\xi^\alpha_3)=\left(\psi_\alpha,\varphi^{\bar m}_0(q^\alpha_1,q^\alpha_2)\tilde\varphi^{m-\bar m}_0(\xi^\alpha_1,\xi^\alpha_2)\right), \tag{3.26}$$

where $\varphi^{\bar m}_0\tilde\varphi^{m-\bar m}_0$ is a normalized eigenfuntion of the operator $P^{\bar m}_{q^\alpha}P^{m-\bar m}_{\xi^\alpha}T_-$ corresponding to the eigenvalue μ^m. If $\mathcal{P}_0P^{\bar m}_{q_\alpha}P^{m-\bar m}_{\xi_\alpha}=0$ we suppose that $G^{\bar m}_\alpha=0$. Let us note that the inequality $\mathcal{P}_0P^{\bar m}_{q_\alpha}P^{m-\bar m}_{\xi_\alpha}\not\equiv 0$ holds only for finite number of $\bar m$.

Now the function $\mathcal{P}_0\psi_\alpha$ may be rewritten in the form

$$\mathcal{P}_0\psi_\alpha=\sum_{\bar m}G^{\bar m}_\alpha(q^\alpha_3,\xi^\alpha_3)\varphi^{\bar m}_0\tilde\varphi^{m-\bar m} \tag{3.27}$$

Due to (3.23) and (3.25)

$$L_3[\psi_\alpha]\geq\sum_{\bar m}L_5[\psi^{\bar m}_\alpha], \tag{3.28}$$

where

$$\begin{aligned}L_5[\psi^{\bar m}_\alpha] &= (T_-\psi^{\bar m}_\alpha,\psi^{\bar m}_\alpha)+(V_\alpha\psi^{\bar m}_\alpha,\psi^{\bar m}_\alpha)+\\ &+ \int\left\{\left|\frac{\partial}{\partial q^\alpha_3}G^{\bar m}_\alpha\right|^2+\frac{3}{4}\left|\frac{\partial}{\partial\xi^\alpha_3}G^{\bar m}_\alpha\right|^2\right\}\chi_\alpha dq^\alpha_3d\xi^\alpha_3+\\ &+ \left\|\frac{\partial}{\partial q^\alpha_3}\mathcal{P}_-\psi^{\bar m}_\alpha\right\|^2+\frac{3}{4}\left\|\frac{\partial}{\partial\xi^\alpha_3}\mathcal{P}_-\psi^{\bar m}_\alpha\right\|^2-\epsilon\left\|\mathcal{P}_-\psi^{\bar m}_\alpha|r_3|^{-1}\right\|^2-\\ &- C_3\left\|\mathcal{P}_-\psi^{\bar m}_\alpha|r_3|^{-1-1/2\delta_1}\right\|^2-C_3\left\|\mathcal{P}_0\psi^{\bar m}_\alpha|r_3|^{-1-1/2\delta_1}\right\|^2,\\ &\\ & C_3=C_1+C.\end{aligned}$$

The function $\mathcal{P}_-\psi^{\bar m}_\alpha \equiv 0$ for $|\xi^\alpha_3| \le a_0(1+4/9b_0^2)^{1/2}$. Using the Hardy inequality one can get for $\epsilon < 1/8$ that

$$\left\|\frac{\partial}{\partial \xi^\alpha_3}\mathcal{P}_-\psi^{\bar m}_\alpha\right\|^2 - \epsilon\left\|\mathcal{P}^m_-|r_3|^{-1}\right\|^2 - C_3\left\|\mathcal{P}_-\psi^{\bar m}_\alpha|r_3|^{-1-1/2\delta_1}\right\|^2 \ge 0 \qquad (3.29)$$

Further, applying the lemma 1 for fixed ξ^α_3 we get

$$\begin{aligned}(T_-\psi^{\bar m}_\alpha,\psi^{\bar m}_\alpha) &+ (V_\alpha\psi^{\bar m}_\alpha,\psi^{\bar m}_\alpha)+\\ +\int\left|\frac{\partial}{\partial q^\alpha_3}G^{\bar m}_\alpha\right|^2 dq^\alpha_3 d\xi^\alpha_3 &+ \left\|\frac{\partial}{\partial q^\alpha_3}\mathcal{P}_-\psi^{\bar m}_\alpha\right\|^2 \ge \\ &\ge \mu^m||\psi^{\bar m}_\alpha||^2 - J^{\bar m}_\alpha,\end{aligned} \qquad (3.30)$$

where

$$\begin{aligned}J^{\bar m}_\alpha &= C_0(1+\delta_0)^{-1}(2/3b_0)^{-1}\int|\xi^\alpha_3|^{-1-\delta_0}\{|G^{\bar m}_\alpha(2/3b_0|\xi^\alpha_3|,\xi^\alpha_3)|^2+\\ &+ |G^{\bar m}_\alpha(-2/3b_0|\xi^\alpha_3|,\xi^\alpha_3)|^2\}d\xi^\alpha_3.\end{aligned} \qquad (3.31)$$

Let us define the function $\tilde G^m_\alpha(q^\alpha_3,\xi^\alpha_3)$ by the formula

$$\tilde G^{\bar m}_\alpha(q^\alpha_3,\xi^\alpha_3) = G^{\bar m}_\alpha(q^\alpha_3,\xi^\alpha_3) - G^{\bar m}_\alpha(q^\alpha_3, 3/2b_0^{-1}|q^\alpha_3|\,\mathrm{sign}\,(\xi^\alpha_3)).$$

It is clear that $\tilde G^{\bar m}_\alpha(q^\alpha_3,\xi^\alpha_3) \in C^1_0\left(K(\alpha,2/3b_0,a_0)\right)$, because $G^{\bar m}_\alpha = 0$ for $|\xi^\alpha_3| \le a_0(1+4/9b_0^2)^{1/2}$.
Furthermore,

$$\left|\frac{\partial}{\partial \xi^\alpha_3}\tilde G^{\bar m}\right| = \left|\frac{\partial}{\partial \xi^\alpha_3}G^{\bar m}_\alpha\right|$$

and

$$\begin{aligned}&||\mathcal{P}_0\psi^{\bar m}_\alpha|r_3|^{-1-1/2\delta_1}||^2 = ||G^{\bar m}_\alpha|r_3|^{-1-1/2\delta_1}||^2 \le\\ &\le \{||\tilde G^{\bar m}_\alpha|r_3|^{-1-1/2\delta_1}||^2 + \int\{|G^{\bar m}_\alpha(q^\alpha_3, 3/2b_0^{-1}|q^\alpha_3|)|^2+\\ &+|G^{\bar m}_\alpha(q^\alpha_3,-3/2b_0^{-1}|q^\alpha_3|)|^2\}(1+\delta_1)^{-1}|q^\alpha_3|^{-1-\delta_1}dq^\alpha_3\}.\end{aligned} \qquad (3.32)$$

The function $\tilde G^{\bar m}_\alpha \in C^1_0(K(\alpha,2/3b_0,a_0))$ and one can apply the Hardy inequality to it. For sufficiently large a_0 we have

$$\int\left\{|\frac{\partial \tilde G^{\bar m}_\alpha}{\partial \xi^\alpha_3}|^2 - C_3|\tilde G^{\bar m}_\alpha|\,|\xi^\alpha_3|^{-2-\delta_1}\right\}\chi_\alpha d\xi^\alpha_3 dq^\alpha_3 \ge 0. \qquad (3.33)$$

Putting together (3.28)–(3.33) we obtain that

$$L_3[\psi_\alpha] \geq \mu^m ||\psi_\alpha||^2 - \sum_{\bar{m}} \{J_\alpha^{\bar{m}} + F_\alpha^{\bar{m}}\},$$

where

$$\begin{aligned} F_\alpha^{\bar{m}} &= C_3(1+\delta_1)^{-1} b_0^{-1-\delta_1} \int_{-\infty}^{\infty} \{|G_\alpha^{\bar{m}}(q_3^\alpha, 3/2b_0^{-1}|q_3^\alpha|)|^2 + \\ &+ |G_\alpha^{\bar{m}}(q_3^\alpha, -3/2b_0^{-1}|q_3^\alpha|)|^2\}|q_3^\alpha|^{-1-\delta_1} dq_3^\alpha. \end{aligned} \tag{3.34}$$

Now let us begin to estimate the functional $L_4[\tilde{\psi}]$. On the support of $\tilde{\psi}$ all the distances among the particles on the plane $(q_3^\alpha, \xi_3^\alpha)$ are large and for all (i,j) and some $C_4 > 0$

$$|V_{ij}| \leq C_4 |r_3|^{-2-\delta_0}.$$

So we have

$$\begin{aligned} &\sum_{i,j} \left(V_{ij}\tilde{\psi}, \tilde{\psi}\right) \geq -3C_4 ||\tilde{\psi}|r_3|^{-1-1/2\delta_0}||^2 \geq \\ &\geq -3C_4||\mathcal{P}_0\tilde{\psi}|r_3|^{-1-1/2\delta_0}||^2 - 3C_4||\mathcal{P}_-\tilde{\psi}|r_3|^{-1-1/2\delta_0}||^2. \end{aligned}$$

It follows from the inequality $\delta_1 \leq \delta_0$ and (3.24) that

$$\begin{aligned} L_4[\tilde{\psi}] &\geq (T_-\tilde{\psi}, \tilde{\psi}) + \int \{|\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 - \\ &- (C_2 + 3C_5)|\mathcal{P}_0\tilde{\psi}|^2 |r_3|^{-2-\delta_1}\} \tilde{\chi} d\Omega - \\ &- (C_2 + 3C_4 + C)\|\mathcal{P}_-\tilde{\psi}|r_3|^{-1}\|^2, \end{aligned} \tag{3.35}$$

where

$$\nabla_3 = \left(\frac{\partial}{\partial q_3^\alpha}, \frac{\sqrt{3}}{2}\frac{\partial}{\partial \xi_3^\alpha}\right).$$

Further,

$$(T_-\tilde{\psi}, \tilde{\psi}) = (T_-\mathcal{P}_-\tilde{\psi}, \mathcal{P}_-\tilde{\psi}) + (T_-\mathcal{P}_0\tilde{\psi}, \mathcal{P}_0\tilde{\psi}) \geq \mu^m ||\tilde{\psi}||^2 + \kappa ||\mathcal{P}_-\tilde{\psi}||^2,$$

where $\kappa > 0$ is the distance between μ^m and the next point of the spectrum of T_-. Let us take $a_0 > \kappa^{-1/2}(C_2 + 3C_4 + C)^{1/2}$, than

$$\kappa ||\mathcal{P}_-\tilde{\psi}||^2 \geq (C_2 + 2C_4 + C)||\mathcal{P}_-\tilde{\psi}|r_3|^{-1}||^2$$

and

$$L_4[\tilde{\psi}] \geq \mu^m ||\tilde{\psi}||^2 + \int |\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 \tilde{\chi} d\Omega - C_5 ||\mathcal{P}_0 \tilde{\psi} |r_3|^{-1-1/2\delta_1}||^2, \qquad (3.36)$$

where $C_5 = C_2 + 3C_4$. Let us note that $\mathcal{P}_0 \tilde{\psi} = 0$ for $|r_3| \leq a_0$, so using the polar coordinates on the plane $(q_3^\alpha, \xi_3^\alpha)$ and the inequality [10]

$$\int_1^\infty U^2(t) t^{-1} \ln t^{-2} dt \leq 4 \int_1^\infty t U'^2 dt \qquad (3.37)$$

which holds for any function $U(t) \in C_0^1(R_+^1), \quad U(1) = 0$, one can show that if the number $a_0 > 0$ is large enough

$$\frac{1}{2} \int |\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 \tilde{\chi} d\Omega - C_6 ||\mathcal{P}_0 \tilde{\psi} |r_3|^{-1-1/2\delta_1}||^2 > 0.$$

So

$$L_4[\tilde{\psi}] \geq \mu^m ||\tilde{\psi}||^2 + \frac{1}{2} \int |\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 \tilde{\chi} d\Omega. \qquad (3.38)$$

Due to (3.34) and (3.38) to prove the theorem it is enough to show that

$$\frac{1}{2} \int |\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 \tilde{\chi} d\Omega - \sum_\alpha \sum_{\bar{m}} \{J_\alpha^{\bar{m}} + F_\alpha^{\bar{m}}\} \geq 0 \qquad (3.39)$$

for large $a_0 > 0$.

For an arbitrary pair α let us define the region

$$\Omega_\alpha = K(\alpha, b_0, a_0) \backslash K(\alpha, 2/3 b_0, a_0).$$

It is clear that the regions Ω_α do not intersect for different α and $\Omega_\alpha \subset \sup \tilde{\chi}$. So we have

$$\int |\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 \tilde{\chi} d\Omega \geq \sum_\alpha \int_{\Omega_\alpha} |\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 d\Omega \qquad (3.40)$$

For $(q_3^\alpha, \xi_3^\alpha) \in \Omega_\alpha$ let

$$\hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha) = \Big((\tilde{\psi}, \varphi_0^{\bar{m}}(q_1^\alpha, q_2^\alpha) \tilde{\varphi}_0^{m-\bar{m}}(\xi_1^\alpha, \xi_2^\alpha) \Big) \qquad (3.41)$$

According to the formula (3.26)

$$\hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha) = G_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha) \quad \text{for} \quad |q_3^\alpha| = 2/3 b_0 |\xi_3^\alpha|.$$

Further
$$\int_{\Omega_\alpha} |\nabla_3 \mathcal{P}_0 \tilde{\psi}|^2 d\Omega = \sum_{\bar{m}} \int_{\Omega_\alpha} |\nabla_3 \hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)|^2 dq_3^\alpha d\xi_3^\alpha. \qquad (3.42)$$
Let us show that for large $a_0 > 0$
$$\frac{1}{2}\int_{\Omega_\alpha} |\nabla_3 \hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)|^2 dq_3^\alpha d\xi_3^\alpha - J_\alpha^{\bar{m}} - F_\alpha^{\bar{m}} \geq 0. \qquad (3.43)$$
Functionals $J_\alpha^{\bar{m}}$ and $F_\alpha^{\bar{m}}$ give the estimates for the localization error of the nonsmooth decomposition and they may be estimated similar to [6].

Really, using the lemma 2 with $d_1 = 2/3b_0|\xi_3^\alpha|, \quad d_2 = b_0|\xi_3^\alpha|$ and function $y(q_3^\alpha) \equiv \hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)$ we obtain that
$$\begin{aligned} |G_\alpha^{\bar{m}}(\frac{2}{3}b_0|\xi_3^\alpha|, \xi_3^\alpha)|^2 \;&\leq\; 6b_0^{-1}|\xi_3^\alpha|^{-1}\int_{2/3b_0|\xi_3^\alpha|}^{b_0|\xi_3^\alpha|} \left|\hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)\right|^2 dq_3^\alpha + \\ &+\; \frac{2}{3}b_0|\xi_3^\alpha| \int_{2/3b_0|\xi_3^\alpha|}^{b_0|\xi_3^\alpha|} \left|\frac{\partial \hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)}{\partial q_3^\alpha}\right|^2 dq_3^\alpha \end{aligned} \qquad (3.44)$$
It follows from (3.44) that the term in $J_\alpha^{\bar{m}}$ containing $G_\alpha^{\bar{m}}(2/3b_0|\xi_3^\alpha|, \xi_3^\alpha)$ is greater then
$$\begin{aligned} &C_1 \int_{\Omega_\alpha} |\xi_3^\alpha|^{-2-\delta_0} |\hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)|^2 dq_3^\alpha d\xi_3^\alpha + \\ &+C_2 \int_{\Omega_\alpha} \left|\frac{\partial \hat{G}_\alpha^{\bar{m}}}{\partial q_3^\alpha}\right|^2 |\xi_3^\alpha|^{-\delta_0} dq_3^\alpha d\xi_3^\alpha, \end{aligned} \qquad (3.45)$$
where C_1 and C_2 are some constants. The similar estimate may be obtained for the term, containing $G_\alpha^{\bar{m}}(-2/3b_0|\xi_3^\alpha|, \xi_3^\alpha)$. So
$$\begin{aligned} J_\alpha^{\bar{m}} \;&\leq 2C_1 \int_{\Omega_\alpha} |\xi_3^\alpha|^{-2-\delta_0} \left|\hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)\right|^2 dq_3^\alpha d\xi_3^\alpha + \\ &+2C_2 \int_{\Omega_\alpha} \left|\frac{\partial \hat{G}_\alpha^{\bar{m}}}{\partial q_3^\alpha}\right|^2 |\xi_3^\alpha|^{-\delta_0} dq_3^\alpha d\xi_3^\alpha, \end{aligned} \qquad (3.46)$$
Analogously, applying lemma 2 to the function $\hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)$ for fixed q_3^α one can get that
$$\begin{aligned} F_\alpha^{\bar{m}} \;&\leq C_3 \int_{\Omega_\alpha} |q_3^\alpha|^{-2-\delta_1} |\hat{G}_\alpha^{\bar{m}}(q_3^\alpha, \xi_3^\alpha)|^2 dq_3^\alpha d\xi_3^\alpha + \\ &+C_4 \int_{\Omega_\alpha} \left|\frac{\partial \hat{G}_\alpha^{\bar{m}}}{\partial \xi_3^\alpha}\right|^2 |q_3^\alpha|^{-\delta_1} dq_3^\alpha d\xi_3^\alpha \end{aligned} \qquad (3.47)$$

with some constants C_3 and C_4. For large $a_0 > 0$

$$C_4|q_3^\alpha|^{-\delta_1} + 2C_2|\xi_3^\alpha|^{-\delta_0} < \frac{1}{4}.$$

Furthermore, for some $C_5 > 0$

$$|q_3^\alpha|^{-2-\delta_1} \geq C_5|r_3|^{-2-\delta_1}$$

and

$$|\xi_3^\alpha|^{-2-\delta_0} \geq C_5|r_3|^{-2-\delta_1}$$

So

$$\begin{aligned} F_\alpha^{\bar m} + J_\alpha^{\bar m} &\leq \frac{1}{4}\int_{\Omega_\alpha} |\nabla_3 \hat{G}_\alpha^{\bar m}|^2 dq_3^\alpha d\xi_3^\alpha + \\ &+ C_5 \int_{\Omega_\alpha} |r_3|^{-2-\delta} |\hat{G}_\alpha^{\bar m}|^2 dq_3^\alpha d\xi_3^\alpha \end{aligned} \tag{3.48}$$

¿From (3.48) it follows that the inequality (3.43) holds if

$$\frac{1}{4}\int_{\Omega_\alpha} |\nabla_3 \hat{G}_\alpha^{\bar m}(q_3^\alpha, \xi_3^\alpha)|^2 dq_3^\alpha d\xi_3^\alpha - C_5 \int_{\Omega_\alpha} |r_3|^{-2-\delta_1} |\hat{G}_\alpha^{\bar m}|^2 dq_3^\alpha d\xi_3^\alpha \geq 0 \tag{3.49}$$

The function $\hat{G}_\alpha^{\bar m}$ equals zero for $|r_3| \leq a_0$. Using the polar coordinates on the plane (q_3, ξ_3) and the inequality (3.37) one can get (3.49) for large a_0.

The theorem is proved.

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