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Abstract

We define the Moyal product on a Hida test functional space endowed with the Wick product.

1 Introduction

This work follows the work of Dito-Léandre ([5]) which was using tools of the Malliavin Calculus in order to perform the Moyal product on a Wiener space.

Let us consider a finite dimensional symplectic manifold M. It inherits from the symplectic form ω a Poisson structure $\{.,.\}$ whose matrix is the inverse of the matrix of the symplectic structure. Deformation quantization program of a Poisson structure was initiated by Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer ([1], [2], [6]). The simplest case to study is the case of $R^n \oplus R^{n*}$ endowed with its constant natural symplectic structure. This leads to the notion of Moyal product.

In the case of an Hilbert space H, Dito ([4]) considers $H \oplus H^*$ endowed with its constant canonical symplectic structure and perform the Moyal product on it. The main remark is that the constant matrix of the associated Poisson structure is still bounded.

This allows to Dito-Léandre ([5]) to define the Moyal on $W \oplus W^*$ where W is an abstract Wiener space. The constant symplectic form is the standard one on $H \oplus H^*$, the underlying Hilbert space of $W \oplus W^*$, such that the constant matrix of Poisson structure is still bounded. Dito-Léandre ([6]) consider the algebra of functional smooth in the Malliavin sense ([9], [11], [12], [14], [16])in order to define the Moyal product on $W \oplus W^*$.

We consider in this work the case where the underlying Hilbert space of the theory is a Sobolev Hilbert space of maps from the circle into R^n endowed with a constant symplectic structure. We do not consider the standard 2-form in order to do the quantization, but another form which is still interesting to consider.

The constant matrix of the involved Poisson structure is unbounded such that we cannot use the theory of Dito-Léandre. This leads to some modifications:

-) We replace the algebra of functionals of Malliavin type by a Hida test functional space ([3],[7], [8], [15]).

-)We replace the Wiener product by the Wick product ([13]).

2 Deformation quantization in white noise analysis

Let $H(S^1; R^n)$ be the Hilbert space of maps γ from the circle into R^n endowed with its canonical Hilbert structure such that

$$\int_0^1 |\gamma(s)|^2 ds + \int_0^1 |\gamma'(s)|^2 ds = \|h\|^2 < \infty \tag{1}$$

We get by Fourier expansion a orthonormal basis $h_{k,i}$ of this Hilbert space:

$$\gamma_{k,i}(s) = \sqrt{Ck^2+1}^{-1} e_i \cos(2\pi ks) \tag{2}$$

if $k \geq 0$ and if $k < 0$

$$\gamma_{k,i}(s) = \sqrt{Ck^2+1}^{-1} e_i \sin(2\pi ks) \tag{3}$$

where e_i is the canonical basis of R^n.

We consider a multiindex $I = ((k_1, i_1), .., (k_{|I|}, i_{|I|}))$. We introduce the Hida weight:

$$w_r(I) = \prod \sqrt{Ck_i^2+1}^r \tag{4}$$

associated to this mulitiindex. F^I denotes the normalized symmetric tensor product of the γ_{k_i,e_i} associated to this multiindex.

We consider the weighted Fock space $W.N_{r,C}$ of series

$$\sum a_I F^I = F \tag{5}$$

such that

$$\|F\|^2_{r,C} = \sum |a_I|^2 w_r(I) C^{|I|} < \infty \tag{6}$$

where $a_I \in C$ In order to avoid some redundances, we decide to order the multiindices by lexicographic order such that after this choice F is written in a unique way.

Definition 1 *The Hida test function* $W.N_{\infty-}$ *space is the intersection of all* $W.N_{r,C}$ *for* $r > 0$ *and* $C > 0$ *endowed with its natural topology.*

We define the Wick product $: F^{I_1} F^{I_2} :$ as the normalized symmetric tensor product of all $\gamma_{k,i}$ in the concatenation of the multiindices I_1 and I_2.

Theorem 2 *$W.N_{\infty-}$ is a commutative algebra for the Wick product.*

Proof: Let

$$F_1 = \sum a_I^1 F^I \tag{7}$$

$$F_2 = \sum a_I^2 F^I \tag{8}$$

Therefore

$$F_3 =: F_1 F_2 := \sum a_I^3 F^I \tag{9}$$

where

$$a_I^3 = \sum_{I_1, I_2} a_{I_1}^1 a_{I_2}^2 \tag{10}$$

where the sum runs over all considered multiindices I_1 and I_2 whose concatenation is I. If I_1 and I_2 are such multiindices, we have clearly

$$w_r(I) = w_r(I_1) w_r(I_2) \tag{11}$$

Moreover there are at most $C^{|I|} = C^{|I_1|} C^{|I_2|}$ terms in the sum (10) such that

$$|a_I^3|^2 \leq C \sum_{I_1, I_2} (|a_{I_1}^1|^2 C^{|I_1|})(|a_{I_2}^2|^2 C^{|I_2|}) \tag{12}$$

Therefore the result.
◊

Definition 3 *A Poisson structure on $W.N_{\infty-}$ $\{.,.\}$ is given by a C-bilinear map $\{.,.\}$ from $WN_{\infty-} \times WN_{\infty-}$ into $W.N_{\infty-}$ such that:*

(i)$\{.,.\}$ is antisymmetric, satisfies the Jacobi derivation and is a derivation with respect of the Wick product in each argument.

(ii) If $F_1 = 1$, $\{F_1, F_2\} = 0$.

(iii)For all r and C, there exists r_1 and C_1 such that

$$\|\{F_1, F_2\}\|_{r,C} \leq K \|F_1\|_{r_1, C_1} \|F_2\|_{r_1, C_1} \tag{13}$$

We can carry easily in this formalism the notion of deformation quantization of [1], [2], [6].

We consider the set of formal series $W.N_{\infty-}[[h]]$ with values in the Hida test functional space.

Definition 4 *A star-product on $W.N_{\infty-}[[h]]$ is a $C[[h]]$-bilinear product $*_h$ on $WN_{\infty-}[[h]] \times W.N_{\infty-}[[h]]$ with values in $W.N_{\infty-}[[h]]$ given by*

$$F_1 *_h F_2 = \sum_{l \geq 0} h^l P_l(F_1, F_2) \tag{14}$$

for F_1 and F_2 belonging to the Hida test functional space. The star-product is extended by $C[[h]]$-bilinearity to $W.N_{\infty-}[[h]]$and satisfy if F_1, F_2, F_3 belong to $W.N_\infty$:

(i) $P_0(F_1, F_2) =: F_1 F_2 :$
(ii)$P_1(F_1, F_2) - P_1(F_2, F_1) = 2\{F_1, F_2\}$
(iii)For all r, C, l, *there exists* r_1, C_1 *such that*

$$\|P_l(F_1, F_2)\|_{r,C} \leq K\|F_1\|_{r_1,C_1}\|F_2\|_{r_2,C_2} \tag{15}$$

and P_l *vanishes on constants.*
(iv)$F_1 *_h (F_2 *_h F_3) = (F_1 *_h F_2) *_h F_3$

3 An example

Let $\omega = \omega_{i,j}$ be a non degenerated antisymmetric bilinear form on R^n. Without to loose some generalities, we can suppose that

$$\omega_{2i,2i+1} = -\omega_{2i+1,2i} = 1 \tag{16}$$

and that $\omega_{i,j} = 0$ elsewhere. We introduce the antisymmetric bilinear form on $H(S^1; R^n)$

$$\Omega(\gamma^1, \gamma^2) = \int_0^1 \omega(\gamma^1(s), \gamma^2(s))ds \tag{17}$$

$\Omega = \Omega_{(k_1,i_1),(k_2,i_2)}$ where

$$\Omega_{(k,2i),(k,2i+1)} = (Ck^2 + 1)^{-1} = -\Omega_{(k,2i+1),(k,2i)} \tag{18}$$

and where the component of Ω elsewhere vanish.

Let $a_{(k,i)}$ the annihilation operator on the symmetric Fock space associated to $\gamma_{(k,i)}$. We have that:

$$a_{(k_1,i_1)}..a_{(k_l,i_l)}F^I = C(I_1, (k_1, i_1), ..(k_l, i_l))F^{I_1} \tag{19}$$

where in I_1 we have removed $(k_1, i_1), ..(k_l, i_l)$ if it is possible (in the other case the expression vanishes) and where we have the bound

$$|C(I_1, (k_1, i_1), .., (k_l, c_l))| \leq C^{|I|} \tag{20}$$

We consider finite sum $F_1 = \sum a_I^1 F^I$ and $F_2 = \sum a_I^2 F^I$. As tradidional, we can put

$$\{F_1, F_2\} = \sum \Omega^{(k_1,i_1),(k_2,i_2)} : a_{(k_1,i_1)}F_1 a_{(k_2,a_2)}F_2 : \tag{21}$$

(We consider Wick products). In the previous formula the sum in finite. But we can extend it to series and we have:

Theorem 5 $\{.,\}$ *defines a Poisson structure in the sense of Definition 3 on* $W.N_{\infty-}$.

Proof: Let us show first (iii) in Definition 3. We have

$$\{F_1, F_2\} = \sum a_I^3 F^I \tag{22}$$

where

$$a_I^3 = \sum (Ck^2+1) a^1_{I_1 \cup (k,2i)} a^2_{I_2 \cup (k,2i+1)} C(I_1, (k,2i)) C(I_2, (k,2i+1)) + A \tag{23}$$

where A is a similar term and in where we sum over all k, i, I_1, I_2 such that the concatenation $I_1 \cup I_2$ of I_1, I_2 is equal to I. We apply Cauchy-Schwartz inequality in (k,i), we use the bound of $C(I_1,(k,2i)), C(I_2,(k,2i+1))$ given in (20) in order to see that

$$\omega_r(I)|a_I^3|^2 \le K \sum (\omega_{r_1}(I_1 \cup (k,2i)) |a^1_{I_1\cup(k,2i)}|^2 C_1^{|I_1|+1}) \\ (\omega_{r_1}(I_2 \cup (k,2i+1)) |a^2_{I_2\cup(k,2i+1)}|^2 C_1^{|I_2|+1}) \tag{24}$$

for a big r_1 and a big C_1, where we sum on the same set as in the sum appearing in (23). We do the same for A in (23). We use for that that

$$\sum (Ck^2+1)^{-r'} < \infty \tag{25}$$

if $r' > 1$. We deduce from the previous inequality that

$$\|\{F_1, F_2\}\|_{r,C} \le K(\sum \omega_{r_1}(I)|I| C_1^{|I|} |a_I^2|^2)(\sum \omega_{r_1}(I)|I| C_1^{|I|} |a_I^2|^2) \tag{26}$$

Therefore we deduce that

$$\|\{F_1, F_2\}\|_{r,C} \le K \|F_1\|_{r_1, C_1'} \|F_2\|_{r_1, C_1'} \tag{27}$$

for a big r_1 and a big C_1'. This shows (iii).

The algebraic properties of the Poisson product arises from the fact that the family of annihilation operators commute and from that a annihilation operator is a derivation for the Wick product.

◇

If F_1 and F_2 are finite sum, we can define as usual by using the Wick product:

$$F_1 *_h F_2 = \sum_{l \ge 0} (-h/2)^l l!^{-1} \sum \Omega^{(k_1,i_1),(k_1',i_1')} .. \Omega^{(k_l,i_l),(k_l',i_l')} \\ : a_{k_1,i_1)} .. a_{(k_l,i_l)} F_1 a_{(k_1',i_1')} .. a_{(k_l',i_l')} F_2 : \tag{28}$$

The sum is in fact finite since F_1 and F_2 are finite sums.

Theorem 6 *Formula (28) can be extended in a star-product in the sense of Definition 4. It is called the Moyal product in Hida sense associated to the symplectic structure given by* Ω.

proof: The algebra is the same than in the classical case ([6]). Only the analysis is different. We put

$$P_l(F_1, F_2) = \sum a_I^3 F^I \tag{29}$$

where a_I^3 is a sum of a bounded terms of the following type:

$$\begin{aligned} A = \sum (Ck_1^2 + 1)..(Ck_l^2 + 1) a^1_{I_1 \cup (k_1, 2i_1)..\cup (k_l, 2i_l)} \\ C(I_1, (k_1, 2i_1), .., (k_l, 2i_l)) a^2_{I_2 \cup (k'_1, 2i'_1 + 1)..\cup (k_l, 2i'_l + 1)} \\ C(I_2, (k'_1, 2i'_1 + 1), .., (k'_l, 2i'_l + 1)) \end{aligned} \tag{30}$$

where we sum on all k_i, i_l, i'_l and all multiindices I_1 and I_2 such that their concatenation $I_1 \cup I_2$ equals I. By doing as before and using the estimates (20), we deduce that

$$\begin{aligned} \omega_r(I)|A|^2 \leq K \sum (\omega_{r_1}(I_1 \cup (k_1, 2i_1) \cup .. \cup (k_l, 2i_l))) C_1^{|I_1| + l} \\ |a^1_{I_1 \cup (k_1, 2i_1)..\cup (k_l, 2i_l)}|^2)(\omega_{r_1}(I_2 \cup (k'_1, 2i'_1 + 1) \cup .. \cup (k'_l, 2i'_l + 1) \\ C_1^{|I_2| + l} |a^2_{I_2 \cup (k'_1, 2i'_1)..\cup (k'_l, 2i'_l + 1)}|^2) \end{aligned} \tag{31}$$

where we sum on the same set than in (30).

We deduce that:

$$\begin{aligned} \|P_l(F_1, F_2)\|^2_{r,C} \leq K (\sum \omega_{r_1}(I) C_1^{|I|} |I|^l \; |a_I^1|^2) \\ (\sum \omega_{r_1}(I) C_1^{|I|} |I|^l |a_I^2|^2) \end{aligned} \tag{32}$$

for r_1 and C_1 big enough such that

$$\|P_l(F_1, F_2)\|_{r,C} \leq K \|F_1\|_{r_1, C'_1} \|F_2\|_{r_1, C'_1} \tag{33}$$

Therefore the result.
♢

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