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Gaussian decay of the magnetic eigenfunctions

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Abstract

We investigate whether the eigenfunctions of the two-dimensional magnetic Schrödinger operator have a Gaussian decay of type $\exp(-Cx^2)$ at infinity (the magnetic field is rotationally symmetric). We establish this decay if the energy (E) of the eigenfunction is below the bottom of the essential spectrum (B), and if the angular Fourier components of the external potential decay exponentially (real analyticity in the angle variable). We also demonstrate that almost the same decay is necessary. The behavior of C in the strong field limit and in the small $(B-E)$ limit is also studied.

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1 Introduction

It is one of the basic phenomena in quantum mechanics that the eigenfunctions of the usual Schrödinger operator, $-\frac{1}{2}\Delta + V$, decay exponentially in the classically forbidden region (in particular, at infinity for energy $E < \liminf_\infty V$). In the presence of a constant B_0 magnetic

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field, supposedly a stronger, Gaussian ($\sim \exp(-Cx^2)$) localization occurs in the direction orthogonal to the field. This decay can explicitly be seen in the resolvent kernel of the two-dimensional magnetic Schrödinger operator $[\frac{1}{2}(\mathbf{p}-\mathbf{A})^2]^{-1}$ (here $\mathrm{rot}\mathbf{A} = B_0$, $\mathbf{p} = -i\nabla$). Note that the usual Landau eigenfunctions have also Gaussian decay, nevertheless this is a bit misleading since, in the infinitely degenerate Landau level, one can easily choose an other eigenbasis without strong decay.

Thomas Hoffmann-Ostenhof has asked the question whether the eigenfunctions of the two-dimensional operator $\frac{1}{2}(\mathbf{p}-\mathbf{A})^2 + V$ with eigenvalues below the essential spectrum have Gaussian decay at infinity. One can fairly easily verify this decay if V is radially symmetric or if it has compact support, but these conditions are far from being necessary.

We can give a fairly complete answer in the sense that it sheds light onto the real reason behind the Gaussian versus non-Gaussian decay. We cannot prove a complete "if and only if" theorem, although our results are close to that. Nevertheless, the mechanism will, hopefully, be clear.

The methods come partly from probability theory, partly from standard functional analysis. In the next section we present the results and the necessary Feynman-Kac formula. In Sections 3 and 4 we prove that the L^2-norms and the L^∞-norms, respectively, of the angular momentum modes of the eigenfunctions decay exponentially. In Section 5 we show that each angular mode has a Gaussian decay. Combining these two results implies the main theorem. Section 6 presents a partial converse of our main theorem. Finally, two Appendices are devoted to the necessary technical tools; the proof of the Feynman-Kac formula and a large deviation estimate on a sum of Poisson processes.

2 Definitions and the result

Consider a radially symmetric magnetic field $B(x) = B(|x|) = B(r)$ in two dimensions ($x = (r, \theta)$ in radial coordinates). Let $\mathbf{A}(r, \theta) := (-a(r)\sin\theta, a(r)\cos\theta)$ be the radial gauge with $a(r) := (1/r)\int_0^r B(s)s\mathrm{d}s$. Assume that $B(r) \geq B_0 > 0$, then $a(r) \geq B_0 r/2$. For simplicity, one can think of the constant magnetic field; in our problem the phenomenon is the same.

Let V be a real potential with certain properties specified below, and we shall consider $H := \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 + V$ acting on $L^2(\mathbf{R}^2)$. It is well known that this operator is selfadjoint if $V \in L^2_{loc}(\mathbf{R}^2)$, the negative part of V is in the Kato class and $\mathbf{A} \in (L^4_{loc}(\mathbf{R}^2))^2$ (using the Leinfelder-Simander theorem, the diamagnetic inequality and basic perturbation techniques, see e.g. [CFKS]).

As we shall see, the angular momentum modes $V_m(r)$ of $V(r, \theta) = \sum_{m=-\infty}^{\infty} e^{im\theta} V_m(r)$ will play a crucial role in our problem ($V_m = V_{-m}$, since V is real). Therefore our setup is the following.

We decompose the Hilbert space $L^2(\mathbf{R}^2)$ according to the angular momentum sectors:

$$L^2(\mathbf{R}^2) \cong \mathcal{H} := \bigoplus_{m=-\infty}^{\infty} L^2(\mathbf{R}, 2\pi r\mathrm{d}r), \tag{1}$$

$$f \in L^2(\mathbf{R}^2) \mapsto (\ldots, f_{-1}, f_0, f_1, \ldots) \in \mathcal{H},$$

where $f(r, \theta) = \sum_m e^{im\theta} f_m(r)$. For simplicity, we continue to denote the elements of $\mathcal{H}$ by f. In general, $\|\cdot\|_2$ will stand for the norm in $L^2(\mathbf{R}^2)$ or in $L^2(\mathbf{R}, 2\pi r\mathrm{d}r)$, depending whether the function has one or two variables. For radial functions the two norms coincide.

Let $U_m(r) := \frac{1}{2}(m/r - a(r))^2$ for each $m \in \mathbf{Z}$, and assume that V and its angular modes $V_m(r)$ satisfy the following properties:

- $\|V\|_\infty \leq U$ (assume $U \geq 1$);
- $V_m(r)$ is continuous;

- $|V_m(r)| \leq a_m \equiv a_{-m}$ for $m \neq 0$ (let $a_0 := U$), such that $M := \sum_{m\geq 1} m a_m < \infty$;
- The negative part, $|V_0(r)|_-$, of the function $V_0(r)$ goes to zero at infinity;
- $W(r) := (\sum_{m\neq 0} |V_m(r)|^2)^{1/2} \to 0$ for $r \to \infty$.

The continuity of V_m and the boundedness of V_0 are not essential, rather technically simplifying conditions. The decay of V at infinity and the decay of the a_m's are important (see later). Note that the positive part of V_0 is not required to decay at infinity.

For any $\varepsilon > 0$, we define

$$R(\varepsilon) := \inf\{r_0 \geq 100 \,:\, W(r) + |V_0(r)|_- \leq \varepsilon U \quad \text{for all} \quad r \geq r_0/2\}; \tag{2}$$

this quantity measures the radial decay of the potential (clearly, $R(\varepsilon)$ is finite by the assumptions above).

The operator H, under the isomophism (1), acts on $\mathcal{H}$ as follows

$$(Hf)_n(r) := -\frac{1}{2}\Delta f_n(r) + U_n(r) f_n(r) + \sum_{m=-\infty}^{\infty} V_{-m}(r) f_{n+m}(r). \tag{3}$$

Here $-\Delta = -\partial_r^2 - (1/r)\partial_r$ denotes the usual Laplacian on the half line with the domain of the usual two-dimensional Laplacian restricted to the radial functions (in our problem it is convenient to think of Δf_n as the radial part of the radially symmetric two-dimensional function $\Delta \tilde{f}_n$ with $\tilde{f}_n(x) := f_n(|x|)$). The infinite sum converges for almost all r and the map

$$H_2 : f = (f_n)_{n\in\mathbf{Z}} \mapsto \left(\sum_{m\neq 0} V_m f_{n+m} \right)_{n\in\mathbf{Z}} \tag{4}$$

is bounded on $\mathcal{H}$.

Our main Theorem is the following (for notational simplicity, $(const)$ will always denote positive universal constants):

Theorem 2.1 *Assume that*

$$a_m \leq D\delta^m \tag{5}$$

for some constants $D > 0$, $0 < \delta < 1$ and assume that the rotationally symmetric magnetic field is bounded from below by $B_0 > 0$. Consider an eigenfunction f with eigenvalue $E < B := B_0/2$, $Hf = Ef$, $\|f\|_2 = 1$, then there exist two effective constants C_1 and C_2, depending on the functions $B(r)$, $R(\varepsilon)$ and on the constants U, D, δ and E such that for any $r > 0$

$$|f(r,\theta)| \leq C_1 \cdot \exp(-C_2 r^2). \tag{6}$$

If the magnetic field is constant ($B(r) = B_0$), the following bounds hold:

Case I. (small $B - E$): There exist constants C_1, C_3, C_4 and $E_0 < B = B_0/2$, depending on U, D, δ and B_0, such that (6) is satisfied for $r \geq (const)\varepsilon^{-1/4}\sqrt{R(\varepsilon)}$, where $\varepsilon := C_3(B - E)^2|\log(B-E)|^{-2}$, with a constant

$$C_2 = \frac{C_4}{|\log(B-E)|} \tag{7}$$

if $E > E_0$.

Case II. (large field): Introduce large parameters $b \geq 1$ in the magnetic field and $v \geq 1$ in the potential, i.e. consider $H(b,v) := \frac{1}{2}(\mathbf{p} - b\mathbf{A}(x))^2 + vV(x)$, where $rot\mathbf{A} = 2$ and $\|V\|_\infty \leq 1$ (so the constant magnetic field is $2b$ and $\inf SpecH(0,b) = b$). Assume that $b \geq v \log v$ and $E \leq b - v^{3/2}(\log b)/b$, then (6) is true for $r \geq C_3'\sqrt{R(C_3' b^{-2})}$ with

$$C_2 = C_4' \frac{b}{\log b} \tag{8}$$

for large enough b (C_1, C_3' and C_4' depend only on D and δ).

Remark 1. For small r all we can say is that $f(r)$ is bounded by the trivial bound $(const)\sqrt{U+E}$ (using the diamagnetic inequality in the usual magnetic Feynman-Kac formula).

Remark 2. Condition (5) is equivalent to the uniform real analyticity of $\theta \to V(r, \theta)$ for all r. Notice that in the case of a constant magnetic field, the origin can be chosen arbitrarily.

Remark 3. In Cases I. and II. notice that C_2, the lower bound on the coefficient of the decay rate, depend on the potential only via D, δ and $\|V\|_\infty$. The radial decay of V (characterized by the function R) appears only in the threshold for r.

Remark 4. The result of Case I. should be compared with the well-known theorem on the nonmagnetic case, namely that the rate of the exponential falloff at infinity is basically proportional to the square root of the difference between the energy and the bottom of the essential spectrum.

Remark 5. The motivation behind Case II. is the question whether the coefficient of the Gaussian decay is proportional with the magnetic field (as in the $\exp(-(const)Br^2)$ decay of the free resolvent kernel). To see this effect better, we have rescaled both the magnetic field and the potential. The result shows that almost the same coefficient remains valid in the presence of an external potential as well.

A partial converse of Theorem 2.1 is the following

Theorem 2.2 *Given a positive sequence a_m with $M := \sum_{m\geq 1} m a_m < \infty$ and*

$$\frac{|\log a_m|}{m} \to 0. \tag{9}$$

Let $a_{-m} := a_m$ and

$$V(r,\theta) := \left(-3M + \sum_{m\in \mathbf{Z}\setminus\{0\}} a_m e^{im\theta} \right) h(r), \tag{10}$$

where $1 > h(r) > 0$, it goes to 0 at infinity, but slower than an exponential (i.e. $|\log h(r)| = o(r)$ at infinity). Then the ground state, $f(r,\theta)$, of $H = \frac{1}{2}(\mathbf{p} - \mathbf{A})^2 + V$, with a constant magnetic field B_0 (i.e. $a(r) = B_0 r/2$), decays slower than a Gaussian, namely

$$\limsup_{r\to\infty} \frac{\log \sup_\theta |f(r,\theta)|}{r^2} \geq 0. \tag{11}$$

Remark. Theorem 2.2 states that in general the exponential decay rate of a_m's is necessary for the Gaussian decay of the eigenfunctions (at least for the ground state).

Our basic tool is the following generalized version of the Feynman-Kac formula, which is proven in Appendix A.

Proposition 2.3 *The heat kernel of H in the $\mathcal{H}$-space picture has the following probabilistic representation (as a convention $\log 0 = -\infty$)*

$$\left(e^{-tH}f\right)_n(r) = \mathbf{E}_\varrho \mathbf{E}_N f_{n+N_t}(\varrho_r(t)) \exp\left[-\int_0^t \left\{U_{n+N_s}(\varrho_r(s)) + V_0(\varrho_r(s)) - \sum_{m\in\mathbf{Z}\setminus\{0\}} a_m\right\} \mathrm{d}s + \right.$$

$$\left. + \sum_{m\in\mathbf{Z}\setminus\{0\}} \int_0^t \log\left(a_m^{-1} V_{-m}(\varrho_r(s))\right) \mathrm{d}N_s^m\right], \tag{12}$$

where

$$N_s := \sum_{m\in\mathbf{Z}\setminus\{0\}} m N_s^m, \tag{13}$$

$N_s^m \in \mathbf{N}$ are independent Poisson processes with rate a_m, and $\varrho_r(s)$ is an independent Bessel process (absolute value of the two-dimensional Wiener process) starting from r ($\mathbf{E}_\varrho$ and $\mathbf{E}_N$ denote the corresponding expectations).

Remark. Equation (12) is understood pointwise for countinous and bounded functions f, and in L^2-sense for $f \in L^2(\mathbf{R}^2)$ (see Appendix A).

Before starting the proofs, we summarize the intuitive ideas. Assume here, for simplicity, that the magnetic field is constant B_0. First we show that $\|f_n\|_2$ is exponentially small. Since, for $n > 0$, $U_n(r)$ grows fast (at least quadratically) away from its zero (which is at $r_n := \sqrt{2n/B_0}$), f_n is strongly localized around this point ($n < 0$ case is even better, since then U_n is big everywhere). But for large n, the negative part of the potential is small in this region (since it goes to zero at infinity), so this region is "classically forbidden" for an eigenfunction with energy smaller than the bottom of the unperturbed spectrum ($B := B_0/2$). This implies the decay of $\|f_n\|_2$. Then we show that $\|f_n\|_\infty$ decays exponentially as well, which is not suprising, for instance, by the $L^2 \to L^\infty$ boundedness of the heat kernel. Finally, we prove that f_n has a Gaussian decay with center $\sqrt{n/B}$, i.e. $|f_n(r)| \leq \|f_n\|_\infty \exp(-c(r-\sqrt{n/B})^2)$ ($n<0$ case is even simpler). Then

$$|f(r,\theta)| \leq 2\sum_{n\geq 0}|f_n(r)| \leq 2\sum_{n\geq 0}(const)e^{-c^* n}e^{-c(r-\sqrt{n/B})^2}, \tag{14}$$

and this sum has a Gaussian decay in r with a coefficient $(const)\min(Bc^*, c)$ (divide the sum into two parts: $n \geq Br^2/4$ and $n \leq Br^2/4$).

3 L^2-estimates

Proposition 3.1 *Let f be an eigenfunction of H with energy $E < B = B_0/2$, $\|f\|_2 = 1$. Then there exist two positive constants, c_1 and c_2 and an integer N_0 (depending on the magnetic field and the external potential) such that*

$$\|f_n\|_2 \leq c_1 e^{-c_2|n|} \tag{15}$$

for any $|n| \geq N_0$.

In the case of a constant magnetic field $B(r) = B_0 = 2B$, the following effective bound holds (naturally, $E \geq \inf SpecH \geq B - U$):

$$\|f_n\|_2^2 \leq c_1\exp(-c_2|n|) := \exp\left[-(const)|n|\left|\frac{\log\delta}{\log\varepsilon}\right|\right] \tag{16}$$

for $|n| \geq N_0 := 40R/\sqrt{\varepsilon}$, where $R := R(\varepsilon)$ (i.e. $W(r) + |V_0(r)|_- \leq \varepsilon U$ for all $r \geq R/2 \geq 50$), and ε is given by

$$\varepsilon := (const)\cdot\min\left\{\delta,\ \frac{(1-\delta)^3}{D},\ \frac{1}{B^2},\ \Lambda\left(\frac{(B-E)(1-\delta)^{3/2}}{U\sqrt{1+D}}\right),\ \Lambda\left(\frac{1-\delta}{U}\right)\right\} \tag{17}$$

with $\Lambda(x) := \frac{1}{100}x^2|\log x|^{-2}$ (for small $|n|$ one can use $\|f_n\|_2 \leq 1$).

Proof. First notice that $f \in L^\infty$, since $\exp(-t(-\frac{1}{2}\Delta + V))$ maps L^2 to L^∞ and the same is true for the magnetic Schrödinger operator by the diamagnetic inequality. Therefore

$$F^2(r) := \sum_{m\in\mathbf{Z}} |f_m(r)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(r,\theta)|^2 \mathrm{d}\theta \tag{18}$$

is also bounded.

Using that

$$\left(-\frac{1}{2}\Delta + U_n - E\right) f_n + \sum_{m\in\mathbf{Z}} V_{-m} f_{n+m} = 0 \tag{19}$$

one sees immediately that $\Delta f_n \in L^2_{loc}$, since

$$\| \sum_m V_{-m} f_{n+m} \|_{L^2_{loc}} \le \|V\|_{L^2_{loc}} \cdot \sup F(r) < \infty. \tag{20}$$

In particular, by the Sobolev imbedding, f_n is continuous on $[0,\infty)$.

Choose a nonnegative, radially symmetric C^∞-cutoff function φ on $\mathbf{R}^2$, supported in the ball of radius $R \ge 100$, $\varphi \equiv 1$ for $r \le R/2$, $|\varphi| \le 1$, $|\Delta\varphi| \le 1$, such that $W(r) + |V_0(r)|_- \le U\varepsilon$ for $r \ge R/2$ ($\varepsilon < 1/100$ is to be determined later). Then (by $\Delta f_n \in L^2_{loc} \subset L^1_{loc}$)

$$\int \varphi |\nabla f_n|^2 = \frac{1}{2}\int \Delta\varphi \cdot |f_n|^2 - \int \varphi f_n \overline{\Delta f_n}. \tag{21}$$

Choose $K := [2(\log\varepsilon)/(\log\delta)] + 1$ (the bracket here denotes the integer part). Let b_m be the infimum of U_m on the support of φ, clearly $b_m \to \infty$ as $|m| \to \infty$. For $j \in \mathbf{Z}$ let

$$F_j(r) := \sup\{|f_{(2K+1)j+k}(r)| \ : \ |k| \le K\} \tag{22}$$

$$A_j := \|F_j\|_2;$$

for $j \ne 0$ let

$$N_j := \sum_{k=-K}^{K} a_{(2K+1)j+k},$$

and

$$N_0 := 1 + \sup|\Delta\varphi| + \sum_{|m|\le K} a_m.$$

Using (21) along with (19) and the notations above, we have

$$0 \le \frac{1}{2}\int \varphi|\nabla f_n|^2 \le \frac{1}{4}\sup|\Delta\varphi| \int |f_n|^2 - (b_n - E)\int \varphi|f_n|^2 + \int |\varphi f_n| \sum_{m\in\mathbf{Z}} |V_{-m}||f_{n+m}|, \quad (23)$$

i.e. (for large $|n|$, such that $b_n > E$)

$$\int \varphi|f_n|^2 \le \frac{1}{b_n - E}\sum_{i\in\mathbf{Z}} N_i \|F_j\|_2 \left(\|F_{j+i-1}\|_2 + \|F_{i+j}\|_2 + \|F_{j+i+1}\|_2\right), \quad (24)$$

where $j = j(n)$ is determined by $n = (2K+1)j(n) + k$ with some $|k| \le K$.

It is well known that $\frac{1}{2}(\mathbf{p} - \mathbf{A})^2 \ge B = B_0/2$ if the magnetic field is bounded from below by $B_0 > 0$. Therefore, the same operator inequality is true on each sectors, namely $-\frac{1}{2}\Delta + U_n \ge B$. Thus,

$$(B - E)\|f_n\|^2 \le \int \overline{f_n}\left(-\frac{1}{2}\Delta + U_n - E\right) f_n \le \int |f_n| \sum_{m\in\mathbf{Z}} |V_{-m}||f_{n+m}| \le \quad (25)$$

$$\le \sum_{|m|\le K} \int \varphi|f_n||V_{-m}||f_{n+m}| + \sum_{|m|\le K}\int (1-\varphi)|f_n||V_{-m}||f_{n+m}| +$$

$$+ \sum_{|m|>K}\int |f_n||V_{-m}||f_{n+m}|.$$

In these inequalities it is enough to consider only $|V_0|_-$ instead of $|V_0|$, since $\int |f_n|^2|V_0|_+$ is positive. Using (24), Schwarz inequalities and the properties of V, in particular

$$\sup\left(\sum_{|m|\le K}(1-\varphi)|V_{-m}|\right) \le \sqrt{2K}\sup[(1-\varphi)(W + |V_0|_-)] \le \sqrt{2K}U\varepsilon, \quad (26)$$

we get, for large $|n|$ ($b_n > E$),

$$(B - E)\|f_n\|^2 \le \frac{3UK}{\sqrt{b_n - E}}(A_{j-1} + A_j + A_{j+1})\left(\sum_i N_i A_j(A_{j+i-1} + A_{j+i} + A_{j+i+1})\right)^{1/2} + \quad (27)$$

$$+\sqrt{2K}U\varepsilon A_j(A_{j+1}+A_j+A_{j-1})+\sum_{i\neq 0}N_iA_j(A_{j+i-1}+A_{j+i}+A_{j+i+1}).$$

Let $|n|\geq n_0$ be large enough, so that $(b_n-E)^{-1}\leq\varepsilon$ (n_0, naturally, depends on the magnetic field, on E and on ε, and let $n_0\geq 4R\varepsilon^{-1/2}$). The inequality (27) is true for any $n\in[(2K+1)j-K,\ (2K+1)j+K]$, therefore, for $|j|\geq J_0:=\left[\frac{n_0}{2K+1}\right]+6$ (so that $(2K+1)J_0-K\geq n_0$),

$$(B-E)A_j^2\leq(3UK\varepsilon^{1/2}+\sqrt{2K}U\varepsilon)(A_{j-1}+A_j+A_{j+1})\left(\sum_i N_iA_j(A_{j+i-1}+A_{j+i}+A_{j+i+1})\right)^{1/2}+$$

$$+\sum_{i\neq 0}N_iA_j(A_{j+i-1}+A_{j+i}+A_{j+i+1}) \tag{28}$$

(we have also used that $A_j\leq\sqrt{N_0}A_j\leq(\sum_i N_iA_j(A_{j+i-1}+A_{j+i}+A_{j+i+1}))^{1/2}$). By some easy estimates, we obtain

$$\left(B-E-\frac{3}{2}(1+15UK\sqrt{\varepsilon})\sum_{i\neq 0}N_i\right)A_j^2\leq$$

$$\leq 25UK\sqrt{N_0\varepsilon}(A_{j-1}^2+A_j^2+A_{j+1}^2)+\frac{1+5UK\sqrt{\varepsilon}}{2}\sum_{i\neq 0}N_i(A_{j+i-1}^2+A_{j+i}^2+A_{j+i+1}^2). \tag{29}$$

If

$$\varepsilon\leq\varepsilon_0:=\min\left(\frac{1}{100},\ \delta,\ \frac{1-\delta}{2D}\right), \tag{30}$$

i.e.

$$\sum_{i\neq 0}N_i\leq 2\sum_{m>K}D\delta^m\leq\frac{2D\delta^K}{1-\delta}\leq\varepsilon, \tag{31}$$

then it follows

$$(B-E-3\varepsilon(1+15UK\sqrt{\varepsilon}))(1+5UK\sqrt{\varepsilon})^{-1}A_j^2\leq\sum_{i\in\mathbf{Z}}L_i(A_{j+i-1}^2+A_{j+i}^2+A_{j+i+1}^2) \tag{32}$$

with $L_i:=N_i$ for $i\neq 0$ and $L_0:=25UK\sqrt{\varepsilon}(2+2D)^{1/2}(1-\delta)^{-1/2}$ (use also that $N_0\leq 2D(1-\delta)^{-1}+2$). Equivalently, for $|j|\geq J_0$,

$$\alpha A_j^2\leq\sum_{i\in\mathbf{Z}}A_{i+j}^2S_i \tag{33}$$

with $\alpha := (B - E - 3\varepsilon(1 + 15UK\sqrt{\varepsilon}))(1 + 5UK\sqrt{\varepsilon})^{-1}$ and $S_i := L_{i-1} + L_i + L_{i+1}$. Assume that $\varepsilon \le \Lambda(\frac{1-\delta}{U})$ (with the function Λ defined in Proposition 3.1), then $UK\sqrt{\varepsilon} \le (const)$. If, in addition, $\varepsilon \le (const)(B-E)$, then $\alpha \ge (const)(B-E)$. ¿From the estimate (31), we have

$$S_i = S_{-i} \le 3\varepsilon^{i-1} \quad \text{for} \quad i > 1 \tag{34}$$

and

$$S_0, S_1 \le 3L_0.$$

Using $\sum_i A_i^2 \le 1$, one easily obtains from (33) that

$$A_j^2 \le \left(\frac{10L_0}{\alpha}\right)^{|j|-J_0} \tag{35}$$

for $|j| \ge J_0$, provided that ε is small enough such that $20L_0 < \alpha$.

Finally, for $|n| \ge 80K$ we have $|j(n)| \ge |n|/(4K)$, and, if in addition, $|n| \ge 10n_0$ (when $|j(n)| \ge 2J_0$), then

$$\|f_n\|_2^2 \le A_{j(n)}^2 \le \left(\frac{10L_0}{\alpha}\right)^{\frac{|n|}{8K}}. \tag{36}$$

Therefore, for small enough ε, (36) proves Proposition 3.1 (with constants depending on all data).

To establish the effective bound (16) for the constant field case, one calculates that for $\varepsilon \le 1/B$ and for any $|n| \ge n_0$ with

$$n_0 := \max(2BR^2,\ 4R\varepsilon^{-\frac{1}{2}}), \tag{37}$$

we have $b_n - E \ge \varepsilon^{-1}$. Then some straightforward calculation leads to (16) with $N_0 := 10n_0 = 40R\varepsilon^{-1/2}$ and ε given by (17). □

4 L^∞-estimates

In Section 3 we have shown the exponential decay of the L^2-norm of f_n. Armed with this, we are looking for the same decay of the L^∞-norms. By a Schwarz inequality in the path space, we can see from Proposition 2.3 that

$$e^{-2tE}|f_n(r)|^2 = \left|\left(e^{-tH}f\right)_n(r)\right|^2 \le e^{2t(U+2a)}\mathbf{E}_\varrho\mathbf{E}_N|f_{n+N_t}(\varrho_r(t))|^2, \tag{38}$$

($a := \sum_{m\ge1} a_m \le D/(1-\delta)$).

Let $\tilde f(r,\theta) := \sum_m |f_m(r)|^2e^{im\theta}$, then clearly $\tilde f\in L^2$, by $\int|\tilde f|^2 = 2\pi\sum_n\int|f_n(r)|^4 r\mathrm{d}r \le 2\pi C^2(t)e^{2tE}\|f\|_2^4$ with any $t>0$, using $\|f_n\|_\infty \le \|f\|_\infty \le C(t)e^{tE}\|f\|_2$, since $e^{-tH}: L^2\to L^\infty$ is bounded (with norm $C(t)\le (const)t^{-1/2}e^{t(U+2a)}$). Notice that

$$\mathbf{E}_\varrho\mathbf{E}_N|f_{n+N_t}(\varrho_r(t))|^2e^{2ta} = \left(e^{-t(-\frac12\Delta+\tilde V)}\tilde f\right)_n(r) \tag{39}$$

with $\tilde V(r,\theta) := \sum_{m\ne0}a_me^{im\theta}$. The heat kernel of $\tilde H := -\frac12\Delta+\tilde V$ maps L^2 into L^∞ as well (with norm $C'(t)\le(const)t^{-1/2}e^{2at}$), but this only bounds $\mathbf{E}_\varrho\mathbf{E}_N|f_{n+N_t}(\varrho_r(t))|^2$ by $\|f\|_2^2$ and not by $\|f_n\|_2^2$.

The following idea helps. Fix $n\ge 2N_0$ (the case $n<-2N_0$ is similar), let

$$f^* := \sum_{m=n-k}^{n+k}|f_m|^2e^{im\theta} \tag{40}$$

with $k := [n/2]$. Then $\|f^*\|_2^2 \le (2k+1)c_1^2e^{-2c_2(n-k)}\cdot 2\pi C^2(t)e^{2tE}\|f\|_2^2$ using Proposition 3.1. Thus,

$$\begin{aligned}\left(e^{-t\tilde H}\tilde f\right)_n(r) &\le \left|\left(e^{-t\tilde H}f^*\right)_n(r)\right| + e^{2at}\mathbf{E}_\varrho\mathbf{E}_N\left(|f_{n+N_t}(\varrho_r(t))|^2\chi(|N_t|>k)\right)\le\\ &\le C'(t)\|f^*\|_2 + C^2(t)e^{2t(E+a)}\|f\|_2^2\mathbf{P}_N(|N_t|>k)\le \\ &\le C'(t)C(t)e^{tE}c_1\sqrt{2\pi(n+1)}e^{-c_2n/2} + C^2(t)e^{2t(E+a)}C(t,\eta,\delta,D)\delta^{n(1-\eta)/2}\end{aligned} \tag{41}$$

for any $1 > \eta > 0$, using $\|f\|_2 = 1$ and Theorem B.1 with the notation $C(t, \eta, \delta, D) := 2(1-\delta^{1-\eta})^{-1} \exp[Dt(\delta^\eta(1-\delta^\eta)^{-1} - \delta(1-\delta)^{-1})]$.

Therefore, combining (38), (39) and (41), we get (with the choice $t = 1$) for $|n| \geq 2N_0$

$$\|f_n\|_\infty \leq c_3 e^{-c_4|n|}, \tag{42}$$

where c_3 and c_4 are positive constants, effectively computable from c_1, c_2, E, U, D and a. In case of the constant magnetic field for $|n| \geq 2N_0 = 80R/\sqrt{\varepsilon}$ and small enough ε (see (16) and (17)) one obtains effective constants c_3 and c_4 which depend on the magnetic field only via ε. In particular, c_4 can be chosen $c_2/4$ (see (16)) if $\eta = 1/2$, and c_3 is either proportional with $\sqrt{|n|}$ or it can be included in the exponent, with $c_4 = c_2/8$, if $|n| \geq (const) n_0 \log n_0$.

5 Gaussian decay of the angular modes

To establish the Gaussian decay of f_n, we use a simple large deviation argument in the estimate obtained from the Feynman-Kac formula:

$$e^{-tE} f_n(r) = \left(e^{-tH} f\right)_n(r) \leq e^{(U+2a)t} \mathbf{E}_\varrho \mathbf{E}_N |f_{n+N_t}(\varrho_r(t))| \exp\left[-\int_0^t U_{n+N_\tau}(\varrho_r(\tau)) \mathrm{d}\tau\right]. \tag{43}$$

We will show that if

$$r \geq r^* := (const)\sqrt{\frac{N_0}{\min(1, B)}}, \tag{44}$$

then

$$|f_n(r)| \leq c_5 e^{-c_6|n|} e^{-c_7(r - r_{|n|})^2} \tag{45}$$

with constants depending on all data. We denote by r_m the unique nonnegative solution of $m = ra(r)$ $(m \geq 1)$. By $a(r) \geq B_0 r/2 = Br$, clearly $r_m \leq \sqrt{m/B}$. Fix n and r.

Case 1. $|n| > (r - r_{|n|})^2/4$.

Define the following event:

$$\mathcal{E} := \left\{ \sup_{\tau \in [0,t]} |N_\tau| \leq \frac{|n|}{2} \right\}. \tag{46}$$

For $t = 1$, on the event $\mathcal{E}$ we have $|f_{n+N_1}(\varrho_r(1))| \leq c_3 e^{-c_4|n|/2}$. We can apply (42), since (44) implies $|n| \geq 4N_0$ in Case 1. On the complementary event, $\mathcal{E}^c$, estimate $|f_{n+N_1}|$ trivially (by $(const)e^{U+E+2a}$ as in Section 4), and use the exponential estimate for $\mathbf{P}(\mathcal{E}^c)$ from Theorem B.1.

Case 2. $|n| \leq (r - r_{|n|})^2/4$.

Let $\mathcal{F}$ and $\mathcal{G}$ be the following events:

$$\mathcal{F} := \left\{ \sup_{\tau \in [0,t]} |\varrho_r(\tau) - r| \leq \frac{|r - r_{|n|}|}{4} \right\}, \tag{47}$$

$$\mathcal{G} := \left\{ \sup_{\tau \in [0,t]} |N_\tau| \leq \frac{B(r - r_{|n|})^2}{8} \right\}. \tag{48}$$

Choose again $t = 1$, and on the event $\mathcal{G}^c$ use Theorem B.1. On the event $\mathcal{F}^c$, use a standard large deviation estimate for the Bessel process to obtain

$$\mathbf{P}_\varrho(\mathcal{F}^c) \leq (const) e^{-(const)(r - r_{|n|})^2}. \tag{49}$$

On $\mathcal{F} \cap \mathcal{G}$ we use the estimate

$$U_m(s) \geq \frac{B^2}{2}(s - r_{|m|})^2, \tag{50}$$

and the fact, that for $0 < n < m$ we have $r_m - r_n \leq \sqrt{(m-n)/B}$, to obtain

$$U_{n+N_\tau}(\varrho_r(\tau)) \geq \frac{B^2}{14}(r - r_{|n|})^2, \tag{51}$$

and by estimating $|f_{n+N_1}|$ again in a trivial way, (45) will follow from (43).

Finally, Theorem 2.1 is obtained by summing up the estimate (45) as it was indicated in (14) (one also has to use the fact that $r_{|n|} \leq \sqrt{|n|/B}$). For $r \leq r^*$, one easily obtains (6) from the boundedness of f by adjusting C_1.

In the case of a constant magnetic field, one can estimate the constant, C_2, in the exponent of the Gaussian decay. We just focus on the two Cases, I. and II., mentioned in Theorem 2.1.

In Case I. one uses exactly the proof above, just one keeps track of the constants given in (16) and (17). In particular, $\varepsilon \sim |(B-E)/\log(B-E)|^2$, $N_0 \sim R(\varepsilon)/\sqrt{\varepsilon}$, and c_2 and $c_4 \sim |\log(B-E)|^{-1}$.

For the proof of Case II. one obtains $\varepsilon \sim b^{-2}$, $N_0 \sim R(\varepsilon)/\sqrt{\varepsilon}$ and $c_2 \sim (\log b)^{-1}$ in Section 3. Use $t = 1/b$ in Section 4 instead of $t = 1$; $c_4 = c_2/2$ is at least proportional with $(\log b)^{-1}$ and (42) is valid for $|n| \geq (const)N_0$ (not forgetting that $C(t)$ and $C'(t)$ in (41) scale as $t^{-1/2}$). Finally, we have to use $t \sim (\log b)/b$ in the calculation of Section 5; the borderline between Case 1. and 2. is at $|n| \leq$ or $\geq b(r - r_{|n|})^2/4$. To obtain the best result, when we sum up for all n (as in (14)), we should use (45) with different c_6 and c_7 (obtained from Case 1. and 2., respectively) depending on n and r. This concludes the proof of Theorem 2.1. □

6 Example of a non-Gaussian decay

Proof of Theorem 2.2. First we note that the ground state, f, of H is positive in the $\mathcal{H}$-space representation, i.e. all functions f_n are positive. Using Theorem XIII. 44. from [RS], this follows from the fact that e^{-tH} is positivity preserving (on $\mathcal{H} = \bigoplus L^2(2\pi r dr)$), which is a direct consequence of Proposition 2.3 (note that for our potential, given by (10), $\log(a_m^{-1}V_m)$ is real). One also has to check that the ground state is below the bottom of the essential spectrum, $B := B_0/2$, but this is easy, e.g. by choosing a Gaussian trial function in the zero angular momentum sector.

Let

$$g(m) := \inf \left\{ f_m(r) \,:\, \left| r - \sqrt{\frac{m}{B}} \right| \le \frac{1}{\sqrt{B}} \right\} \tag{52}$$

for $m \ge 0$. Then Theorem 2.2 is a straightforward consequence of

Lemma 6.1 *For any $\alpha < 1$ there exist N and $C > 1$ such that for any $n \ge N$*

$$\alpha^n g(n^2) \le C g\left((n+1)^2 \right). \tag{53}$$

To prove Theorem 2.2 from this Lemma, simply note that for $n \ge 2N$

$$\inf_{|r-n/\sqrt{B}| \le 1/\sqrt{B}} \left(\sup_\theta |f(r,\theta)| \right) \ge \inf_{|r-n/\sqrt{B}| \le 1/\sqrt{B}} \left(\sum_{m \in \mathbf{Z}} f_m(r)^2 \right)^{1/2} \ge \tag{54}$$

$$\ge \inf_{|r-n/\sqrt{B}| \le 1/\sqrt{B}} f_{n^2}(r) = g_{n^2} \ge C^{N-n} \alpha^{n^2/2} g(N^2)$$

so

$$\frac{1}{n^2} \log \left(\sup_\theta \left| f\left(\frac{n}{\sqrt{B}}, \theta \right) \right| \right) \ge \frac{1}{n^2} \log \left(\inf_{|r-n/\sqrt{B}| \le 1/\sqrt{B}} \left(\sup_\theta |f(r,\theta)| \right) \right) \ge \tag{55}$$

$$\ge \frac{\log \alpha}{2} + \frac{\log g(N^2) - n \log C}{n^2},$$

and α is arbitrarily close to 1. $\square$

Proof of Lemma 6.1. By the Feynman-Kac formula and the positivity of f_n's we have

$$f_{(n+1)^2}(r) \ge e^{t(E+2a)} \mathbf{E}_\varrho \mathbf{E}_N \chi(\mathcal{E}) f_{(n+1)^2+N_t}(\varrho_r(t)) \times \tag{56}$$

$$\exp\left(- \int_0^t [U_{(n+1)^2+N_s}(\varrho_r(s)) + V_0(\varrho_r(s))] \mathrm{d}s + \sum_{m \ne 0} \int_0^t \log[a_m^{-1} V_{-m}(\varrho_r(s))] \mathrm{d}N_s^m \right)$$

for any event $\mathcal{E}$ ($a := \sum_{m \ge 1} a_m$ as before). Choose $t := 1/B$ and let $r \in [\frac{n}{\sqrt{B}}, \frac{n+2}{\sqrt{B}}]$.

We shall choose $\mathcal{E}$ in a such a way that we could control the various terms in (56) from below. Let

$$\mathcal{E}_1 = \left\{ N_t^m = 0 \quad \text{for all} \quad m, \quad \text{except} \quad N_{t/2}^{-2n-1} = N_t^{-2n-1} = 1 \right\}, \tag{57}$$

i.e. there is only one jump, this is of size $-(2n+1)$, before $t/2$, and no jumps between $t/2$ and t. The probability of this event is

$$\mathbf{P}(\mathcal{E}_1) = e^{-2ta} a_{2n+1} \cdot \frac{t}{2} \geq e^{-2ta} \alpha^{n/2} \cdot \frac{t}{2} \tag{58}$$

if n is large enough (depending on the decay of $|\log a_m|/m$).

Let

$$\mathcal{E}_2 := \left\{ \frac{n-2}{\sqrt{B}} \leq \varrho_r(s) \leq \frac{n+3}{\sqrt{B}} \quad \text{for} \quad s \leq \frac{t}{2}, \quad \text{and} \quad \left| \varrho_r(s) - \frac{n}{\sqrt{B}} \right| \leq \frac{1}{\sqrt{B}} \quad \text{for} \quad \frac{t}{2} \leq s \leq t \right\}. \tag{59}$$

Standard large deviation estimate for the Bessel process shows that $\mathbf{P}(\mathcal{E}_2) \geq (const) > 0$.

Now let $\mathcal{E} := \mathcal{E}_1 \cap \mathcal{E}_2$, then on the event $\mathcal{E}$ the following estimates hold for large n

$$U_{n^2+N_s}(\varrho_r(s)) \leq (const)B, \tag{60}$$

$$\sum_{m \neq 0} \int_0^t \log(a_m^{-1} V_{-m}(\varrho_r(s)) \mathrm{d}N_s \geq \log \left(\inf_{u \leq (n+3)/\sqrt{B}} h(u) \right). \tag{61}$$

Putting everything into (56) we have

$$f_{(n+1)^2}(r) \geq C(E, a, B, U) \alpha^n g(n^2) \tag{62}$$

with a constant $C(E, a, B, U)$ if

$$\frac{1}{2} \log \alpha \leq \frac{1}{n} \log \left(\inf_{u \geq (n+3)/\sqrt{B}} h(u) \right), \tag{63}$$

which is satisfied for large enough n, due to the condition on the decay of h. The inequality (62) is valid for any $r \in [n/\sqrt{B}, (n+2)/\sqrt{B}]$, therefore Lemma 6.1 follows. □

Appendices

A Proof of the generalized Feynman-Kac formula

The ideas and tools for proving Proposition 2.3 is found in [DJS], where Feynman-Kac representation is proved for a general Pauli type equation on the direct sum of finitely many function spaces. There, the diagonal operators were magnetic Laplacians plus potential, the offdiagonal operators were just multiplications by functions. Our setup is the same (we need only nonmagnetic Laplacians in the diagonals, since, in our decomposition, the effect of the magnetic field has already become an effective potential on the angular momentum sectors), with the only difference that we have infinitely many sectors. The formula (6.3) in [DJS] cannot be valid for this case, moreover the process N_t defined in (6.2) of [DJS] does not exist since the sum is divergent. Fortunately, a simple renormalization, by introducing decaying rates a_m, helps.

First we note that (13) is finite a.s. and the right hand side of (12) is well defined (the infinite sum in the last term can be $-\infty$). Assume first that $f \in C^2(\mathbf{R}^2)$, then $f_n(r) \in C^2([0,\infty))$.

By the semigroup property we only need to show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (\text{ RHS. of (12)}) = -(Hf)_n. \tag{64}$$

Define the stochastic process

$$\xi(t) = \exp\left[-\int_0^t \left(U_{n+N_s}(\varrho_r(s)) + V_0(\varrho_r(s)) - \sum_{m\neq 0} a_m\right)\mathrm{d}s + \right.$$
$$\left. + \sum_{m\neq 0}\int_0^t \log\left(a_m^{-1}V_{-m}(\varrho_r(s))\right)\mathrm{d}N_s^m\right]. \tag{65}$$

By the Ito-calculus (extended to Poisson processes) we can calculate the following stochastic differential (see [DJS])

$$\mathrm{d}[f_{n+N_t}(\varrho_r(t))\xi(t)] = \xi(t)\left[\nabla f_{n+N_t}(\varrho_r(t))\left(\mathrm{d}\varrho_r(t) - \frac{1}{\varrho_r(t)}\mathrm{d}t\right) + \frac{1}{2}\Delta f_{n+N_t}(\varrho_r(t))\mathrm{d}t + \right.$$

$$+f_{n+N_t}(\varrho_r(t))\left\{-\left(U_{n+N_t}(\varrho_r(t))+V_0(\varrho_r(t))-\sum_{m\neq 0}a_m\right)\mathrm{d}t+\sum_{m\neq 0}[a_m^{-1}V_{-m}(\varrho_r(t))-1]\mathrm{d}N_t^m\right\}+ \tag{66}$$
$$+\left(f_{n+N_t}(\varrho_r(t))-f_n(\varrho_r(t))\right)\mathrm{d}N_t\Big].$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(\text{ RHS. of (12)})=\lim_{s\to 0}\mathbf{E}_\varrho\mathbf{E}_N\frac{1}{s}\int_0^s\mathrm{d}[f_{n+N_t}(\varrho_r(t))\xi(t)]=$$
$$=\frac{1}{2}\Delta f_n(r)-f_n(r)\left[U_n(r)+V_0(r)-\sum_{m\neq 0}a_m\right]+ \tag{67}$$
$$+\sum_{m\neq 0}\Big[(a_m^{-1}V_{-m}(r)-1)f_{n+m}a_m+(f_{n+m}(r)-f_n(r))a_m\Big]=-(Hf)_n(r),$$

where we used that $\mathbf{E}_N\mathrm{d}N_t^M=a_m\mathrm{d}t$ and that $\mathrm{d}\varrho_r(t)-\mathrm{d}t/\varrho_r(t)$ is a martingale differential.

If $f\in L^2$, then choose a sequence $f^{(j)}\in C^2(\mathbf{R}^2)$, $f^{(j)}\to f$. Clearly $e^{-tH}f^{(j)}\to e^{-tH}f$ in L^2-sense, and by a Schwarz inequality in the path space one sees that the right hand side of (12), as an L^2-function of r, with f replaced by $f^{(j)}$, converges as $j\to\infty$. To establish the validity of (12) for each r, one needs continuity in r on the right hand side (then one can choose a continuous representative of $e^{-tH}f$ exactly by (12)). But for a continuous and bounded f, this just follows from the dominated convergence theorem in the path space.

Remark. Much weaker conditions are probably enough for the validity of Proposition 2.3. As in the case of the usual Feynman-Kac formula, the potential should likely be required to lie only in the Kato class for the L^2-formula. For the pointwise formula, one needs the smoothing property of the magnetic heat kernel which has been established in [BHL]. But since this is not the main point in this paper, we simply assume continuity and boundedness on V.

B Large deviation estimate for N_t

Let $\{N_t^m\}_{m\in\mathbf{Z}\setminus\{0\}}$ be a family of independent Poisson processes with rates $a_m := \mathbf{E}N_1^m$, $a_m = a_{-m}$, such that $\sum_{m=1}^{\infty} m a_m < \infty$. Consider the process

$$N_t = \sum_{m\in\mathbf{Z}\setminus\{0\}} m N_t^m; \tag{68}$$

this infinite sum converges in L^1-sense, therefore N_t is finite a.s. Let us denote $a := \sum_{m=1}^{\infty} a_m$.

Theorem B.1 *Assume that*

$$a_m \le D\delta^m \tag{69}$$

for some constants D and $\delta < 1$. Then, for any $0 < \eta < 1$, we have

$$\mathbf{P}\left(\sup_{s\in[0,t]} |N_s| \ge n\right) \le \frac{2}{1-\delta^{1-\eta}} \exp\left[tD\left(\frac{\delta^\eta}{1-\delta^\eta} - \frac{\delta}{1-\delta}\right)\right] \cdot \delta^{(1-\eta)n} \tag{70}$$

for any $n \in \mathbf{N}_+$.

Proof. Let $N_t^+ := \sum_{m\ge 1} m N_t^m$, $N_t^- := \sum_{m\le -1} m N_t^m$, then clearly

$$\mathbf{P}\left(\sup_{s\in[0,t]} |N_s| \ge n\right) \le 2\mathbf{P}\left(N_t^+ \ge n\right), \tag{71}$$

since N_t^+ and $-N_t^-$ have the same distribution and they are monotone processes. Clearly we overestimate the tail $\mathbf{P}(N_t^+ \ge n)$ if we increase the rates a_m to $D\delta^m$, so we can assume that $a_m = D\delta^m$.

By the Poisson law, we have

$$\mathbf{P}\left(N_t^+ = n\right) = e^{-ta}\left(\sum_{k_1+2k_2+\ldots+nk_n=n} \prod_{m=1}^{n} \frac{(ta_m)^{k_m}}{(k_m)!}\right), \tag{72}$$

where the summation goes over all nonnegative integer tuples $(k_1, k_2, \ldots, k_n)$ with $\sum_m m k_m = n$. Consider the Laplace transform

$$p_t(x) := \sum_{n=0}^{\infty} \mathbf{P}(N_t^+ = n) x^n, \tag{73}$$

then one easily gets

$$p_t(x) = e^{-ta} e^{t \sum_{m=1}^{\infty} a_m x^m}. \tag{74}$$

Now

$$\mathbf{P}(N_t^+ = n) \le x^{-n} p_t(x) \le x^{-n} e^{-ta} \exp\left(\frac{Dt\delta x}{1-\delta x}\right) \tag{75}$$

for any $x < \delta^{-1}$. Choose $x := \delta^{\eta-1}$ and sum up the estimate (75) for all integer $\ge n$ to get (70). □

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