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OVERDETERMINED SYSTEMS, CONFORMAL DIFFERENTIAL GEOMETRY, AND THE BGG COMPLEX

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Abstract. This is an expanded version of a series of two lectures given at the IMA summer program "Symmetries and Overdetermined Systems of Partial Differential Equations". The main part of the article describes the Riemannian version of the prolongation procedure for certain overdetermined system obtained recently in joint work with T.P. Branson, M.G. Eastwood, and A.R. Gover. First a simple special case is discussed, then the (Riemannian) procedure is described in general.

The prolongation procedure was derived from a simplification of the construction of Bernstein–Gelfand–Gelfand (BGG) sequences of invariant differential operators for certain geometric structures. The version of this construction for conformal structures is described next. Finally, we discuss generalizations of both the prolongation procedure and the construction of invariant operators to other geometric structures.

Key words. Overdetermined system, prolongation, invariant differential operator, conformal geometry, parabolic geometry

AMS(MOS) subject classifications. 35N10, 53A30, 53A40, 53C15, 58J10, 58J70

1. Introduction. The plan for this article is a follows. I'll start by describing a simple example of the Riemannian version of the prolongation procedure of [4]. Next, I will explain how the direct observations used in this example can be replaced by tools from representation theory to make the procedure work in general. The whole procedure is based on an inclusion of the group $O(n)$ into $O(n+1,1)$. Interpreting this inclusion geometrically leads to a relation to conformal geometry, that I will discuss next. Via the conformal Cartan connection, the ideas used in the prolongation procedure lead to a construction of conformally invariant differential operators from a twisted de–Rham sequence. On manifolds which are locally conformally flat, this leads to resolutions of certain locally constant sheaves, which are equivalent to the (generalized) Bernstein–Gelfand–Gelfand (BGG) resolutions from representation theory. In the end, I will outline generalizations to other geometric structures.

It should be pointed out right at the beginning, that this presentation basically reverses the historical development. The BGG resolutions in representation theory were originally introduced in [3] and [15] in the 1970's. The constructions were purely algebraic and combinatorial, based

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on the classification of homomorphisms of Verma modules. It was known to the experts that there is a relation to invariant differential operators on homogeneous spaces, with conformally invariant operators on the sphere as a special case. However it took some time until the relevance of ideas and techniques from representation theory in conformal geometry was more widely appreciated. An important step in this direction was the work on the curved translation principle in [12]. In the sequel, there were some attempts to construct invariant differential operators via a geometric version of the generalized BGG resolutions for conformal and related structures, in particular in [2].

This problem was completely solved in the general setting of parabolic geometries in [9], and the construction was significantly simplified in [5]. In these constructions, the operators occur in patterns, and the first operators in each pattern form an overdetermined system. For each of these systems, existence of solutions is an interesting geometric condition. In [4] it was shown that weakening the requirement on invariance (for example forgetting conformal aspects and just thinking about Riemannian metrics) the construction of a BGG sequence can be simplified. Moreover, it can be used to rewrite the overdetermined system given by the first operator(s) in the sequence as a first order closed system, and this continues to work if one adds arbitrary lower order terms.

I am emphasizing these aspects because I hope that this will clarify two points which would otherwise remain rather mysterious. On the one hand, we will not start with some overdetermined system and try to rewrite this in closed form. Rather than that, our starting point is an auxiliary first order system of certain type which is rewritten equivalently in two different ways, once as a higher order system and once in closed form. Only in the end, it will follow from representation theory, which systems are covered by the procedure.

On the other hand, if one starts the procedure in a purely Riemannian setting, there are some choices which seem unmotivated. These choices are often dictated if one requires conformal invariance.

2. An example of the prolongation procedure.

2.1. The setup. The basics of Riemannian geometry are closely related to representation theory of the orthogonal group $O(n)$. Any representation of $O(n)$ gives rise to a natural vector bundle on n–dimensional Riemannian manifolds and any $O(n)$–equivariant map between two such representation induces a natural vector bundle map. This can be proved formally using associated bundles to the orthonormal frame bundle.

Informally, it suffices to know (at least for tensor bundles) that the standard representation corresponds to the tangent or cotangent bundle, and the correspondence is natural with respect to direct sums and tensor products. A linear map between two representations of $O(n)$ can be expressed in terms of a basis induced from an orthonormal basis in the standard representation. Starting from a local orthonormal frame of the (co)tangent bundle, one may locally use the same formula in induced frames on any Riemannian manifold. Equivariancy under the group $O(n)$ means that the result is independent of the initial choice of a local orthonormal frame. Hence one obtains a global, well defined bundle map.

The basic strategy for our prolongation procedure is to embed $O(n)$ into a larger Lie group G, and then look how representations of G behave when viewed as representations of the subgroup $O(n)$. In this way, representation theory is used as a way to organize symmetries. A well known inclusion of this type is $O(n) \hookrightarrow O(n+1)$, which is related to viewing the sphere S^n as a homogeneous Riemannian manifold. We use a similar, but slightly more involved inclusion.

Consider $\mathbb{V} := \mathbb{R}^{n+2}$ with coordinates numbered from 0 to $n+1$ and the inner product defined by

$$\langle (x_0, \dots, x_{n_1}), (y_0, \dots, y_{n+1}) \rangle := x_0 y_{n+1} + x_{n+1} y_0 + \sum_{i=1}^{n} x_i y_i .$$

For this choice of inner product, the basis vectors $e_1, \dots, e_n$ span a subspace $\mathbb{V}_1$ which is a standard Euclidean $\mathbb{R}^n$, while the two additional coordinates are what physicists call light cone coordinates, i.e. the define a signature $(1,1)$ inner product on $\mathbb{R}^2$. Hence the whole form has signature $(n+1,1)$ and we consider its orthogonal group $G = O(\mathbb{V}) \cong O(n+1,1)$. There is an evident inclusion $O(n) \hookrightarrow G$ given by letting $A \in O(n)$ act on $\mathbb{V}_1$ and leaving the orthocomplement of $\mathbb{V}_1$ fixed.

In terms of matrices, this inclusion maps $A \in O(n)$ to the block diagonal matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix}$ with blocks of sizes 1, n, and 1. The geometric meaning of this inclusion will be discussed later.

The representation of A as a block matrix shows that, as a representation of $O(n)$, $\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus V_2$, where $\mathbb{V}_0$ and $\mathbb{V}_2$ are trivial representations spanned by e_{n+1} and e_0, respectively. We will often denote elements of $\mathbb{V}$ by column vectors with three rows, with the bottom row corresponding to $\mathbb{V}_0$.

If we think of $\mathbb{V}$ as representing a bundle, then differential forms with values in that bundle correspond to the representations $\Lambda^k \mathbb{R}^n \otimes \mathbb{V}$ for $k = 0, \dots, n$. Of course, for each k, this representation decomposes as $\oplus_{i=0}^{2} (\Lambda^k \mathbb{R}^n \otimes \mathbb{V}_i)$, but for the middle component $\Lambda^k \mathbb{R}^n \otimes \mathbb{V}_1$, there is a finer decomposition. For example, if $k = 1$, then $\mathbb{R}^n \otimes \mathbb{R}^n$ decomposes as

$$\mathbb{R} \oplus S^2_0 \mathbb{R}^n \oplus \Lambda^2 \mathbb{R}^n$$

into trace–part, tracefree symmetric part and skew part. We actually need only $k = 0, 1, 2$, where we get the picture

$$\begin{tikzcd}
\mathbb{R} \arrow[dr, leftrightarrow] & \mathbb{R}^n \arrow[dr, leftrightarrow] & \Lambda^2\mathbb{R}^n \\
\mathbb{R}^n \arrow[dr, leftrightarrow] & \mathbb{R} \oplus S^2_0\mathbb{R}^n \oplus \Lambda^2\mathbb{R}^n \arrow[dr, leftrightarrow] & \mathbb{R}^n \oplus W_2 \oplus \Lambda^3\mathbb{R}^n \\
\mathbb{R} & \mathbb{R}^n & \Lambda^2\mathbb{R}^n
\end{tikzcd} \tag{2.1}$$

and we have indicated some components which are isomorphic as representations of $O(n)$. Observe that assigning homogeneity $k + i$ to elements of $\Lambda^k\mathbb{R}^n \otimes \mathbb{V}_i$, we have chosen to identify components of the same homogeneity.

We will make these identifications explicit in the language of bundles immediately, but let us first state how we will use them. For the left column, we will on the one hand define $\partial : \mathbb{V} \to \mathbb{R}^n \otimes \mathbb{V}$, which vanishes on $\mathbb{V}_0$ and is injective on $\mathbb{V}_1 \oplus \mathbb{V}_2$. On the other hand, we will define $\delta^* : \mathbb{R}^n \otimes \mathbb{V} \to \mathbb{V}$ by using inverse identifications. For the right hand column, we will only use the identifications from right to left to define $\delta^* : \Lambda^2\mathbb{R}^n \otimes \mathbb{R} \to \mathbb{R}^n \otimes \mathbb{V}$. Evidently, this map has values in the kernel of $\delta^* : \mathbb{R}^n \otimes \mathbb{V} \to \mathbb{V}$, so $\delta^* \circ \delta^* = 0$. By constructions, all these maps preserve homogeneity. We also observe that $\ker(\delta^*) = S^2_0\mathbb{R}^n \oplus \operatorname{im}(\delta^*) \subset \mathbb{R}^n \otimes \mathbb{V}$.

Now we can carry all this over to any Riemannian manifold of dimension n. Sections of the bundle V corresponding to $\mathbb{V}$ can be viewed as triples consisting of two functions and a one–form. Since the representation $\mathbb{R}^n$ corresponds to T^*M, the bundle corresponding to $\Lambda^k\mathbb{R}^n \otimes \mathbb{V}$ is $\Lambda^kT^*M \otimes V$. Sections of this bundle are triples consisting of two k–forms and one T^*M–valued k–form. If there is no danger of confusion with abstract indices, we will use subscripts $i = 0, 1, 2$ to denote the component of a section in $\Lambda^kT^*M \otimes V_i$. To specify our maps, we use abstract index notation and define $\partial : V \to T^*M \otimes V$, $\delta^* : T^*M \otimes V \to V$ and $\delta^* : \Lambda^2T^*M \otimes V \to T^*M \otimes V$ by

$$\partial \begin{pmatrix} h \\ \varphi_b \\ f \end{pmatrix} := \begin{pmatrix} 0 \\ hg_{ab} \\ -\varphi_a \end{pmatrix} \quad \delta^* \begin{pmatrix} h_b \\ \varphi_{bc} \\ f_b \end{pmatrix} := \begin{pmatrix} \frac{1}{n}\varphi_c^c \\ -f_b \\ 0 \end{pmatrix} \quad \delta^* \begin{pmatrix} h_{ab} \\ \varphi_{abc} \\ f_{ab} \end{pmatrix} := \begin{pmatrix} \frac{-1}{n-1}\varphi_{ac}{}^c \\ \frac{1}{2}f_{ab} \\ 0 \end{pmatrix}$$

The numerical factors are chosen in such a way that our example fits into the general framework developed in section 3.

We can differentiate sections of V using the component–wise Levi–Civita connection, which we denote by ∇. Note that this raises homogeneity by one. The core of the method is to mix this differential term with an algebraic one. We consider the operation $\Gamma(V) \to \Omega^1(M, V)$ defined by $\Sigma \mapsto \nabla\Sigma + \partial\Sigma$. Since ∂ is tensorial and linear, this defines a linear connection $\tilde{\nabla}$ on the vector bundle V.

We are ready to define the class of systems that we will look at. Choose a bundle map (not necessarily linear) $A : V_0 \oplus V_1 \to S^2_0T^*M$, and view it as $A : V \to T^*M \otimes V$. Notice that this implies that A increases homogeneities.

Then consider the system

$$\tilde{\nabla}\Sigma + A(\Sigma) = \delta^*\psi \qquad \text{for some } \psi \in \Omega^2(M, V). \tag{2.2}$$

We will show that on the one hand, this is equivalent to a second order system on the V_0–component Σ_0 of Σ and on the other hand, it is equivalent to a first order system on Σ in closed form.

2.2. The first splitting operator. Since A by definition has values in $\ker(\delta^*)$ and $\delta^* \circ \delta^* = 0$, the system (2.2) implies $\delta^*(\tilde{\nabla}\Sigma) = 0$. Hence we first have to analyze the operator $\delta^* \circ \tilde{\nabla} : \Gamma(V) \to \Gamma(V)$. Using abstract indices and denoting the Levi–Civita connection by ∇_a we obtain

$$\Sigma = \begin{pmatrix} h \\ \varphi_b \\ f \end{pmatrix} \overset{\tilde{\nabla}_a}{\mapsto} \begin{pmatrix} \nabla_a h \\ \nabla_a \varphi_b + h g_{ab} \\ \nabla_a f - \varphi_a \end{pmatrix} \overset{\delta^*}{\mapsto} \begin{pmatrix} \frac{1}{n}\nabla^b \varphi_b + h \\ -\nabla_a f + \varphi_a \\ 0 \end{pmatrix}$$

From this we can read off the set of all solutions of $\delta^*\tilde{\nabla}\Sigma = 0$. We can arbitrarily choose f. Vanishing of the middle row then forces $\varphi_a = \nabla_a f$, and inserting this, vanishing of the top row is equivalent to $h = -\frac{1}{n}\nabla^b\nabla_b f = -\frac{1}{n}\Delta f$, where Δ denotes the Laplacian. Hence we get

Proposition 2.2. *For any $f \in C^\infty(M, \mathbb{R})$, there is a unique $\Sigma \in \Gamma(V)$ such that $\Sigma_0 = f$ and $\delta^*(\nabla\Sigma + \delta\Sigma) = 0$. Mapping f to this unique Σ defines a second order linear differential operator $L : \Gamma(V_0) \to \Gamma(V)$, which is explicitly given by*

$$L(f) = \begin{pmatrix} -\frac{1}{n}\Delta f \\ \nabla_a f \\ f \end{pmatrix} = \sum_{i=0}^{2} (-1)^i (\delta^*\nabla)^i \begin{pmatrix} 0 \\ 0 \\ f \end{pmatrix}.$$

The natural interpretation of this result is that V_0 is viewed as a quotient bundle of V, so we have the tensorial projection $\Sigma \mapsto \Sigma_0$. The operator L constructed provides a differential splitting of this tensorial projection, which is characterized by the simple property that its values are in the kernel of $\delta^*\tilde{\nabla}$. Therefore, L and its generalizations are called *splitting operators*.

2.3. Rewriting as a higher order system. We have just seen that the system $\tilde{\nabla}\Sigma + A(\Sigma) = \delta^*\psi$ from 2.1 implies that $\Sigma = L(f)$, where $f = \Sigma_0$. Now by Proposition 2.2, the components of $L(f)$ in V_0 and V_1 are f and ∇f, respectively. Hence $f \mapsto A(L(f))$ is a first order differential operator $\Gamma(V_0) \to \Gamma(S_0^2 T^*M)$. Conversely, any first order operator $D_1 : \Gamma(V_0) \to \Gamma(S_0^2 T^*M) \subset \Omega^1(M, V)$ can be written as $D_1(f) = A(L(f))$ for some $A : V \to T^*M \otimes V$ as in 2.1.

Next, for $f \in \Gamma(V_0)$ we compute

$$\tilde{\nabla}L(f) = \tilde{\nabla}_a \begin{pmatrix} -\frac{1}{n}\Delta f \\ \nabla_b f \\ f \end{pmatrix} = \begin{pmatrix} -\frac{1}{n}\nabla_a \Delta f \\ \nabla_a\nabla_b f - \frac{1}{n} g_{ab}\Delta f \\ 0 \end{pmatrix}.$$

The middle component of this expression is the tracefree part $\nabla_{(a}\nabla_{b)_0}f$ of $\nabla^2 f$.

PROPOSITION 2.3. *For any operator $D_1 : C^\infty(M,\mathbb{R}) \to \Gamma(S^2_0 T^*M)$ of first order, there is a bundle map $A : V \to T^*M \otimes V$ such that $f \mapsto L(f)$ and $\Sigma \mapsto \Sigma_0$ induce inverse bijections between the sets of solutions of*

$$\nabla_{(a}\nabla_{b)_0}f + D_1(f) = 0 \tag{2.3}$$

and of the basic system (2.2).

Proof. We can choose $A : V_0 \oplus V_1 \to S^2_0 T^*M \subset T^*M \otimes V$ in such a way that $D_1(f) = A(L(f))$ for all $f \in \Gamma(V_0)$. From above we see that $\tilde{\nabla} L(f) + A(L(f))$ has vanishing bottom component and middle component equal to $\nabla_{(a}\nabla_{b)_0}f + D_1(f)$. From (2.1) we see that sections of $\operatorname{im}(\delta^*) \subset T^*M \otimes V$ are characterized by the facts that the bottom component vanishes, while the middle one is skew symmetric. Hence $L(f)$ solves (2.2) if and only if f solves (2.3). Conversely, we know from 2.2 any solution Σ of (2.2) satisfies $\Sigma = L(\Sigma_0)$, and the result follows. □

Notice that in this result we do not require D_1 to be linear. In technical terms, an operator can be written in the form $f \mapsto \nabla_{(a}\nabla_{b)_0}f + D_1(f)$ for a first order operator D_1, if and only if it is of second order, quasi–linear and its principal symbol is the projection $S^2T^*M \to S^2_0 T^*M$ onto the tracefree part.

2.4. Rewriting in closed form. Suppose that Σ is a solution of (2.2), i.e. $\tilde{\nabla}\Sigma + A(\Sigma) = \delta^*\psi$ for some ψ. Then the discussion in 2.3 shows that the two bottom components of $\tilde{\nabla}\Sigma + A(\Sigma)$ actually have to vanish. Denoting the components of Σ as before, there must be a one–form τ_a such that

$$\begin{pmatrix} \nabla_a h + \tau_a \\ \nabla_a \varphi_b + h g_{ab} + A_{ab}(f,\varphi) \\ \nabla_a f - \varphi_a \end{pmatrix} = 0. \tag{2.4}$$

Apart from the occurrence of τ_a, this is a first order system in closed form, so it remains to compute this one–form.

To do this, we use the *covariant exterior derivative* $d^{\tilde{\nabla}} : \Omega^1(M,V) \to \Omega^2(M,V)$ associated to $\tilde{\nabla}$. This is obtained by coupling the exterior derivative to the connection $\tilde{\nabla}$, so in particular on one–forms we obtain

$$d^{\tilde{\nabla}}\omega(\xi,\eta) = \tilde{\nabla}_\xi(\omega(\eta)) - \tilde{\nabla}_\eta(\omega(\xi)) - \omega([\xi,\eta]).$$

Explicitly, on $\Omega^1(M,V)$ the operator $d^{\tilde{\nabla}}$ is given by

$$\begin{pmatrix} h_b \\ \varphi_{bc} \\ f_b \end{pmatrix} \mapsto 2 \begin{pmatrix} \nabla_{[a}h_{b]} \\ \nabla_{[a}\varphi_{b]c} - h_{[a}g_{b]c} \\ \nabla_{[a}f_{b]} + \varphi_{[ab]} \end{pmatrix}, \tag{2.5}$$

where square brackets indicate indicate an alternation of abstract indices.

Now almost by definition, $d^{\tilde\nabla}\tilde\nabla\Sigma$ is given by the action of the curvature of $\tilde\nabla$ on Σ. One easily computes directly that this coincides with the component–wise action of the Riemann curvature. In particular, this is only non–trivial on the middle component. On the other hand, since $A(\Sigma) = A_{ab}(f,\varphi)$ is symmetric, we see that $d^{\tilde\nabla}(A(\Sigma))$ is concentrated in the middle component, and it certainly can be written as $\Phi_{abc}(f,\nabla f,\varphi,\nabla\varphi)$ for an appropriate bundle map Φ. Together with the explicit formula, this shows that applying the covariant exterior derivative to (2.4) we obtain

$$\begin{pmatrix} 2\nabla_{[a}\tau_{b]} \\ -R_{ab}{}^{d}{}_{c}\varphi_d + \Phi_{abc}(f,\nabla f,\varphi,\nabla\varphi) - 2\tau_{[a}g_{b]c} \\ 0 \end{pmatrix} = 0.$$

Applying δ^*, we obtain an element with the bottom two rows equal to zero and top row given by

$$\tfrac{1}{n-1}\big(R_a{}^{cd}{}_c\varphi_d - \Phi_a{}^c{}_c(f,\nabla f,\varphi,\nabla\varphi)\big) + \tau_a,$$

which gives a formula for τ_a. Finally, we define a bundle map $C : V \to T^*M \otimes V$ by

$$C\begin{pmatrix} h \\ \varphi_b \\ f \end{pmatrix} := \begin{pmatrix} \frac{-1}{n-1}\big(R_a{}^{cd}{}_c\varphi_d - \Phi_a{}^c{}_c(f,\varphi,\varphi,-hg - A(f,\varphi))\big) \\ A_{ab}(f,\varphi) \\ 0 \end{pmatrix}$$

to obtain

THEOREM 2.4. *Let $D : C^\infty(M,\mathbb{R}) \to \Gamma(S^2_0T^*M)$ be a quasi–linear differential operator of second order whose principal symbol is the projection $S^2T^*M \to S^2_0T^*M$ onto the tracefree part. Then there is a bundle map $C : V \to T^*M \otimes V$ which has the property that $f \mapsto L(f)$ and $\Sigma \mapsto \Sigma_0$ induce inverse bijections between the sets of solutions of $D(f) = 0$ and of $\tilde\nabla\Sigma + C(\Sigma) = 0$. If D is linear, then C can be chosen to be a vector bundle map.*

Since for any bundle map C, a solution of $\tilde\nabla\Sigma + C(\Sigma) = 0$ is determined by its value in a single point, we conclude that any solution of $D(f) = 0$ is uniquely determined by the values of f, ∇f and Δf in one point. Moreover, if D is linear, then the dimension of the space of solutions is always $\leq n+2$. In this case, $\tilde\nabla + C$ defines a linear connection on the bundle V, and the maximal dimension can be only attained if this connection is flat.

Let us make the last step explicit for $D(f) = \nabla_{(a}\nabla_{b)_0}f + A_{ab}f$ with some fixed section $A_{ab} \in \Gamma(S^2_0T^*M)$. From formula (2.5) we conclude that

$$\Phi_{abc}(f,\nabla f,\varphi,\nabla\varphi) = 2f\nabla_{[a}A_{b]c} + 2A_{c[b}\nabla_{a]}f,$$

and inserting we obtain the closed system

$$\begin{cases} \nabla_a h - \frac{1}{n-1}\big(R_a{}^{cd}{}_c\varphi_d + f\nabla^c A_{ac} + \varphi^c A_{ac}\big) = 0 \\ \nabla_a\varphi_b + hg_{ab} + fA_{ab} = 0 \\ \nabla_a f - \varphi_a = 0 \end{cases}$$

which is equivalent to $\nabla_{(a}\nabla_{b)_0}f + A_{ab}f = 0$.

2.5. Remark. As a slight detour (which however is very useful for the purpose of motivation) let me explain why the equation $\nabla_{(a}\nabla_{b)_0}f + A_{ab}f = 0$ is of geometric interest. Let us suppose that f is a nonzero function. The we can use it to conformally rescale the metric g to $\hat{g} := \frac{1}{f^2}g$. Now one can compute how a conformal rescaling affects various quantities, for example the Levi–Civita connection. In particular, we can look at the conformal behavior of the Riemannian curvature tensor. Recall that the Riemann curvature can be decomposed into various components according to the decomposition of $S^2(\Lambda^2\mathbb{R}^n)$ as a representation of $O(n)$. The highest weight part is the *Weyl curvature*, which is independent of conformal rescalings.

Contracting the Riemann curvature via $\mathrm{Ric}_{ab} := R_{ca}{}^c{}_b$, one obtains the *Ricci curvature*, which is a symmetric two tensor. This can be further decomposed into the *scalar curvature* $R := \mathrm{Ric}^a{}_a$ and the tracefree part $\mathrm{Ric}^0_{ab} = \mathrm{Ric}_{ab} - \frac{1}{n}Rg_{ab}$. Recall that a Riemannian metric is called an *Einstein metric* if the Ricci curvature is proportional to the metric, i.e. if $\mathrm{Ric}^0_{ab} = 0$.

The behavior of the tracefree part of the Ricci curvature under a conformal change $\hat{g} := \frac{1}{f^2}g$ is easily determined explicitly, see [1]. In particular, $\hat{g}$ is Einstein if and only if

$$\nabla_{(a}\nabla_{b)_0}f + A_{ij}f = 0$$

for an appropriately chosen $A_{ij} \in \Gamma(S^2_0T^*M)$. Hence existence of a nowhere vanishing solution to this equation is equivalent to the possibility of rescaling g conformally to an Einstein metric.

From above we know that for a general non–trivial solution f of this system and each $x \in M$, at least one of $f(x)$, $\nabla f(x)$, and $\Delta f(x)$ must be nonzero. Hence $\{x : f(x) \neq 0\}$ is a dense open subset of M, and one obtains a conformal rescaling to an Einstein metric on this subset.

3. The general procedure. The procedure carried out in an example in section 2 can be vastly generalized by replacing the standard representation by an arbitrary irreducible representation of $G \cong O(n+1,1)$. (Things also work for spinor representations, if one uses $Spin(n+1,1)$ instead.) However, one has to replace direct observations by tools from representation theory, and we discuss in this section, how this is done.

3.1. The Lie algebra $\mathfrak{o}(n+1,1)$. We first have to look at the Lie algebra $\mathfrak{g} \cong \mathfrak{o}(n+1,1)$ of $G = O(\mathbb{V})$. For the choice of inner product used in 2.1 this has the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & Z & 0 \\ X & A & -Z^t \\ 0 & -X^t & -a \end{pmatrix} : A \in \mathfrak{o}(n), a \in \mathbb{R}, X \in \mathbb{R}^n, Z \in \mathbb{R}^{n*}, \right\}.$$

The central block formed by A represents the subgroup $O(n)$. The element $E := \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$ is called the *grading element*. Forming the commutator with

E is a diagonalizable map $\mathfrak{g} \to \mathfrak{g}$ with eigenvalues -1, 0, and 1, and we denote by $\mathfrak{g}_i$ the eigenspace for the eigenvalue i. Hence $\mathfrak{g}_{-1}$ corresponds to X, $\mathfrak{g}_1$ to Z and $\mathfrak{g}_0$ to A and a. Moreover, the Jacobi identity immediately implies that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ with the convention that $\mathfrak{g}_{i+j} = \{0\}$ unless $i + j \in \{-1, 0, 1\}$. Such a decomposition is called a $|1|$–*grading* of $\mathfrak{g}$. In particular, restricting the adjoint action to $\mathfrak{o}(n)$, one obtains actions on $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, which are the standard representation respectively its dual (and hence isomorphic to the standard representation).

Since the grading element E acts diagonalizably under the adjoint representation, it also acts diagonalizably on any finite dimensional irreducible representation $\mathbb{W}$ of $\mathfrak{g}$. If $w \in \mathbb{W}$ is an eigenvector for the eigenvalue j, and $Y \in \mathfrak{g}_i$, then $E \cdot Y \cdot w = Y \cdot E \cdot w + [E, Y] \cdot w$ shows that $Y \cdot w$ is an eigenvector with eigenvalue $i + j$. From irreducibility it follows easily that denoting by j_0 the lowest eigenvalue, the set of eigenvalues is $\{j_0, j_0 + 1, \dots, j_0 + N\}$ for some $N \geq 1$. Correspondingly, we obtain a decomposition $\mathbb{W} = \mathbb{W}_0 \oplus \cdots \oplus \mathbb{W}_N$ such that $\mathfrak{g}_i \cdot \mathbb{W}_j \subset \mathbb{W}_{i+j}$. In particular, each of the subspaces $\mathbb{W}_j$ is invariant under the action of $\mathfrak{g}_0$ and hence in particular under the action of $\mathfrak{o}(n)$. Notice that the decomposition $\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \mathbb{V}_2$ used in section 2 is obtained in this way.

One can find a Cartan subalgebra of (the complexification of) $\mathfrak{g}$ which is spanned by E and a Cartan subalgebra of (the complexification of) $\mathfrak{o}(n)$. The theorem of the highest weight then leads to a bijective correspondence between finite dimensional irreducible representations $\mathbb{W}$ of $\mathfrak{g}$ and pairs $(\mathbb{W}_0, r)$, where $\mathbb{W}_0$ is a finite dimensional irreducible representation of $\mathfrak{o}(n)$ and $r \geq 1$ is an integer. Basically, the highest weight of $\mathbb{W}_0$ is the restriction to the Cartan subalgebra of $\mathfrak{o}(n)$ of the highest weight of $\mathbb{W}$, while r is related to the value of the highest weight on E. As the notation suggests, we can arrange things in such a way that $\mathbb{W}_0$ is the lowest eigenspace for the action of E on $\mathbb{W}$. For example, the standard representation $\mathbb{V}$ in this notation corresponds to $(\mathbb{R}, 2)$. The explicit version of this correspondence is not too important here, it is described in terms of highest weights in [4] and in terms of Young diagrams in [10]. It turns out that, given $\mathbb{W}_0$ and r, the number N which describes the length of the grading can be easily computed.

3.2. Kostant's version of the Bott–Borel–Weil theorem. Suppose that $\mathbb{W}$ is a finite dimensional irreducible representation of $\mathfrak{g}$, decomposed as $\mathbb{W}_0 \oplus \cdots \oplus \mathbb{W}_N$ as above. Then we can view $\Lambda^k\mathbb{R}^n \otimes \mathbb{W}$ as $\Lambda^k\mathfrak{g}_1 \otimes \mathbb{W}$, which leads to two natural families of $O(n)$–equivariant maps. First we define $\partial^* : \Lambda^k\mathfrak{g}_1 \otimes \mathbb{W} \to \Lambda^{k-1}\mathfrak{g}_1 \otimes \mathbb{W}$ by

$$\partial^*(Z_1 \wedge \cdots \wedge Z_k \otimes w) := \textstyle\sum_{i=1}^k (-1)^i Z_1 \wedge \cdots \wedge \widehat{Z_i} \wedge \cdots \wedge Z_k \otimes Z_i \cdot w,$$

where the hat denotes omission. Note that if $w \in \mathbb{W}_j$, then $Z_i \cdot w \in W_{j+1}$, so this operation preserves homogeneity. On the other hand, we have $Z_i \cdot Z_j \cdot w - Z_j \cdot Z_i \cdot w = [Z_i, Z_j] \cdot w = 0$, since $\mathfrak{g}_1$ is a commutative subalgebra. This easily implies that $\partial^* \circ \partial^* = 0$.

Next, there is an evident duality between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, which is compatible with Lie theoretic methods since it is induced by the Killing form of $\mathfrak{g}$. Using this, we can identify $\Lambda^k\mathfrak{g}_1 \otimes \mathbb{W}$ with the space of k–linear alternating maps $\mathfrak{g}_{-1}^k \to \mathbb{W}$. This gives rise to a natural map $\partial = \partial_k : \Lambda^k\mathfrak{g}_1 \otimes \mathbb{W} \to \Lambda^{k+1}\mathfrak{g}_1 \otimes \mathbb{W}$ defined by

$$\partial\alpha(X_0, \dots, X_k) := \textstyle\sum_{i=0}^k (-1)^i X_i \cdot \alpha(X_0, \dots, \widehat{X_i}, \dots, X_k).$$

In this picture, homogeneity boils down to the usual notion for multilinear maps, i.e. $\alpha : (\mathfrak{g}_{-1})^k \to \mathbb{W}$ is homogeneous of degree ℓ if it has values in $\mathbb{W}_{\ell-k}$. From this it follows immediately that ∂ preserves homogeneities, and $\partial \circ \partial = 0$ since $\mathfrak{g}_{-1}$ is commutative.

As a first step towards the proof of his version of the Bott–Borel–Weil–theorem (see [14]), B. Kostant proved the following result:

Lemma 3.2. *The maps ∂ and ∂^* are adjoint with respect to an inner product of Lie theoretic origin. For each degree k, one obtains an algebraic Hodge decomposition*

$$\Lambda^k\mathfrak{g}_1 \otimes \mathbb{W} = \operatorname{im}(\partial) \oplus (\ker(\partial) \cap \ker(\partial^*)) \oplus \operatorname{im}(\partial^*),$$

with the first two summands adding up to $\ker(\partial)$ *and the last two summands adding up to* $\ker(\partial^*)$.

In particular, the restrictions of the canonical projections to the subspace $\mathbb{H}_k := \ker(\partial) \cap \ker(\partial^*)$ *induce isomorphisms* $\mathbb{H}_k \cong \ker(\partial)/\operatorname{im}(\partial)$ *and* $\mathbb{H}_k \cong \ker(\partial^*)/\operatorname{im}(\partial^*)$.

Since ∂ and ∂^* are $\mathfrak{g}_0$–equivariant, all spaces in the lemma are naturally representations of $\mathfrak{g}_0$ and all statements include the $\mathfrak{g}_0$–module structure. Looking at the Hodge decomposition more closely, we see that for each k, the map ∂ induces an isomorphism

$$\Lambda^k\mathfrak{g}_1 \otimes \mathbb{W} \supset \operatorname{im}(\partial^*) \to \operatorname{im}(\partial) \subset \Lambda^{k+1}\mathfrak{g}_1 \otimes \mathbb{W},$$

while ∂^* induces an isomorphism in the opposite direction. In general, these two map are not inverse to each other, so we replace ∂^* by the map δ^* which vanishes on $\ker(\partial^*)$ and is inverse to ∂ on $\operatorname{im}(\partial)$. Of course, $\delta^* \circ \delta^* = 0$ and it computes the same cohomology as ∂^*.

Kostant's version of the BBW–theorem computes (in a more general setting to be discussed below) the representations $\mathbb{H}_k$ in an explicit and algorithmic way. We only need the cases $k = 0$ and $k = 1$ here, but to formulate the result for $k = 1$ we need a bit of background. Suppose that $\mathbb{E}$ and $\mathbb{F}$ are finite dimensional representations of a semisimple Lie algebra. Then the tensor product $\mathbb{E} \otimes \mathbb{F}$ contains a unique irreducible component whose highest weight is the sum of the highest weights of $\mathbb{E}$ and $\mathbb{F}$. This component is called the *Cartan product* of $\mathbb{E}$ and $\mathbb{F}$ and denoted by $\mathbb{E} \circledcirc \mathbb{F}$. Moreover, there is a nonzero equivariant map $\mathbb{E} \otimes \mathbb{F} \to \mathbb{E} \circledcirc \mathbb{F}$, which is unique up to multiples. This equivariant map is also referred to as the *Cartan product.*

The part of Kostant's version of the BBW–theorem that we need (proved in [4] in this form) reads as follows,

THEOREM 3.2. *Let $\mathbb{W} = \mathbb{W}_0 \oplus \dots \oplus \mathbb{W}_N$ be the irreducible representation of $\mathfrak{g}$ corresponding to the pair $(\mathbb{W}_0, r)$.*
(i) In degree zero, $\operatorname{im}(\partial^*) = \mathbb{W}_1 \oplus \dots \oplus \mathbb{W}_N$ and $\mathbb{H}_0 = \ker(\partial) = \mathbb{W}_0$.
(ii) The subspace $\mathbb{H}_1 \subset \mathfrak{g}_1 \otimes \mathbb{W}$ is isomorphic to $S^r_0 \mathfrak{g}_1 \circledcirc \mathbb{W}_0$. It is contained in $\mathfrak{g}_1 \otimes \mathbb{W}_{r-1}$ and it is the only irreducible component of $\Lambda^* \mathfrak{g}_1 \otimes \mathbb{W}$ of this isomorphism type.

3.3. Some more algebra. Using Theorem 3.2 we can now deduce the key algebraic ingredient for the procedure. For each $i \geq 1$ we have $\partial : \mathbb{W}_i \to \mathfrak{g}_1 \otimes \mathbb{W}_{i-1}$. Next, we consider $(\operatorname{id} \otimes \partial) \circ \partial : \mathbb{W}_i \to \otimes^2 \mathfrak{g}_1 \otimes \mathbb{W}_{i-2}$, and so on, to obtain $\mathfrak{g}_0$–equivariant maps

$$\varphi_i := (\operatorname{id} \otimes \dots \otimes \operatorname{id} \otimes \partial) \circ \ldots \circ (\operatorname{id} \otimes \partial) \otimes \partial : \mathbb{W}_i \to \otimes^i \mathfrak{g}_1 \otimes \mathbb{W}_0$$

for $i = 1, \ldots, N$, and we put $\varphi_0 = \operatorname{id}_{\mathbb{W}_0}$.

PROPOSITION 3.3. *Let $\mathbb{W} = \mathbb{W}_0 \oplus \dots \oplus \mathbb{W}_N$ correspond to $(\mathbb{W}_0, r)$ and let $\mathbb{K} \subset S^r \mathfrak{g}_1 \otimes \mathbb{W}_0$ be the kernel of the Cartan product. Then we have*
(i) For each i, the map $\varphi_i : \mathbb{W}_i \to \otimes^i \mathfrak{g}_1 \otimes \mathbb{W}_0$ is injective and hence an isomorphism onto its image. This image is given by

$$\operatorname{im}(\varphi_i) = \begin{cases} S^i \mathfrak{g}_1 \otimes \mathbb{W}_0 & i < r \\ (S^i \mathfrak{g}_1 \otimes \mathbb{W}_0) \cap (S^{i-r} \mathfrak{g}_1 \otimes \mathbb{K}) & i \geq r. \end{cases}$$

(ii) For each $i < r$, the restriction of the map $\delta^* \otimes \varphi_{i-1}^{-1}$ to $S^i \mathfrak{g}_1 \otimes \mathbb{W}_0 \subset \mathfrak{g}_1 \otimes S^{i-1} \mathfrak{g}_1 \otimes \mathbb{W}_0$ coincides with φ_i^{-1}.

Proof. (sketch) (i) Part (i) of Theorem 3.2 shows that $\partial : \mathbb{W}_i \to \mathfrak{g}_1 \otimes \mathbb{W}_{i-1}$ is injective for each $i \geq 1$, so injectivity of the φ_i follows. Moreover, for $i \neq r$, the image of this map coincides with the kernel of $\partial_1 : \mathfrak{g}_1 \otimes \mathbb{W}_{i-1} \to \Lambda^2 \mathfrak{g}_1 \otimes \mathbb{W}_{i-2}$, while for $i = r$ this kernel in addition contains a complementary subspace isomorphic to $S^k \mathfrak{g}_1 \circledcirc \mathbb{W}_0$. A moment of thought shows that ∂_1 can be written as $2\mathrm{Alt} \circ (\operatorname{id} \otimes \partial_0)$, where Alt denotes the alternation. This immediately implies that the φ_i all have values in $S^i \mathfrak{g}_1 \otimes \mathbb{W}_0$ as well as the claim about the image for $i < r$.

It further implies that $\operatorname{id} \otimes \varphi_{r-1}$ restricts to isomorphisms

$$\begin{tikzcd}
\mathfrak{g}_1 \otimes \mathbb{W}_{r-1} \arrow[r] & \mathfrak{g}_1 \otimes S^{r-1} \mathfrak{g}_1 \otimes \mathbb{W}_0 \\
\ker(\partial) \arrow[r] \arrow[u, hook] & S^r \mathfrak{g}_1 \otimes \mathbb{W}_0 \arrow[u, hook] \\
\operatorname{im}(\partial) \arrow[r] \arrow[u, hook] & \mathbb{K} \arrow[u, hook]
\end{tikzcd}$$

which proves the claim on the image for $i = r$. For $i > r$ the claim then follows easily as above.
(ii) This follows immediately from the fact that $\delta^*|_{\operatorname{im}(\partial)}$ inverts $\partial|_{\operatorname{im}(\delta^*)}$. ∎

3.4. Step one of the prolongation procedure. The developments in 3.1–3.3 carry over to an arbitrary Riemannian manifold (M, g) of dimension n. The representation $\mathbb{W}$ corresponds to a vector bundle $W = \oplus_{i=0}^{N} W_i$. Likewise, $\mathbb{H}_1$ corresponds to a direct summand $H_1 \subset T^*M \otimes W_{r-1}$ which is isomorphic to $S_0^r T^*M \circledcirc W_0$. The maps ∂, ∂^*, and δ^* induce vector bundle maps on the bundles $\Lambda^k T^*M \otimes W$ of W–valued differential forms, and for $i = 0, \dots, N$, the map φ_i induces a vector bundle map $W_i \to S^i T^*M \otimes W_0$. We will denote all these maps by the same symbols as their algebraic counterparts. Finally, the Cartan product gives rise to a vector bundle map $S^r T^*M \otimes W_0 \to H_1$, which is unique up to multiples.

We have the component–wise Levi–Civita connection ∇ on W. We will denote a typical section of W by Σ. The subscript i will indicate the component in $\Lambda^k T^*M \otimes W_i$. Now we define a linear connection $\tilde{\nabla}$ on W by $\tilde{\nabla}\Sigma := \nabla\Sigma + \partial(\Sigma)$, i.e. $(\tilde{\nabla}\Sigma)_i = \nabla\Sigma_i + \partial(\Sigma_{i+1})$. Next, we choose a bundle map $A : W_0 \oplus \cdots \oplus W_{r-1} \to H_1$, view it as $A : W \to T^*M \otimes W$ and consider the system

$$\tilde{\nabla}\Sigma + A(\Sigma) = \delta^*\psi \quad \text{for some } \psi \in \Omega^2(M, W). \tag{3.1}$$

Since A has values in $\ker(\delta^*)$ and $\delta^* \circ \delta^* = 0$, any solution Σ of this system has the property that $\delta^*\tilde{\nabla}\Sigma = 0$.

To rewrite the system equivalently as a higher order system, we define a linear differential operator $L : \Gamma(W_0) \to \Gamma(W)$ by

$$L(f) := \textstyle\sum_{i=0}^{N} (-1)^i (\delta^* \circ \nabla)^i f.$$

Proposition 3.4. *(i) For $f \in \Gamma(W_0)$ we have $L(f)_0 = f$ and $\partial^*\tilde{\nabla}L(f) = 0$, and $L(f)$ is uniquely determined by these two properties.*
(ii) For $\ell = 0, \dots, N$ the component $L(f)_\ell$ depends only on the ℓ–jet of f. More precisely, denoting by $J^\ell W_0$ the ℓth jet prolongation of the bundle W_0, the operator L induces vector bundle maps $J^\ell W_0 \to W_0 \oplus \cdots \oplus W_\ell$, which are isomorphisms for all $\ell < r$.

Proof. (i) Putting $\Sigma = L(f)$ it is evident that $\Sigma_0 = f$ and $\Sigma_{i+1} = -\delta^*\nabla\Sigma_i$ for all $i \geq 0$. Therefore,

$$(\tilde{\nabla}\Sigma)_i = \nabla\Sigma_i + \partial(\Sigma_{i+1}) = \nabla\Sigma_i - \partial\delta^*\nabla\Sigma_i$$

for all i. Since $\delta^*\partial$ is the identity on $\operatorname{im}(\delta^*)$, we get $\delta^*\tilde{\nabla}\Sigma = 0$.

Conversely, expanding the equation $0 = \delta^*\tilde{\nabla}\Sigma$ in components we obtain

$$\Sigma_{i+1} = \delta^*\partial\Sigma_{i+1} = -\delta^*\nabla\Sigma_i,$$

which inductively implies $\Sigma = L(\Sigma_0)$.
(ii) By definition, $L(f)_\ell$ depends only on ℓ derivatives of f. Again by definition, $L(f)_1 = \delta^*\nabla f$, and if $r > 1$, this equals $\varphi_1^{-1}(\nabla f)$. Naturality of δ^* implies that

$$L(f)_2 = \delta^*\nabla\delta^*\nabla f = \delta^* \circ (\mathrm{id} \otimes \delta^*)(\nabla^2 f) = \delta^* \circ (\mathrm{id} \otimes \varphi_1^{-1})(\nabla^2 f).$$

Replacing ∇^2 by by its symmetrization changes the expression by a term of order zero, so we see that, if $r > 2$ and up to lower order terms, $L(f)_2$ is obtained by applying φ_2^{-1} to the symmetrization of $\nabla^2 f$. Using part (ii) of Proposition 3.3 and induction, we conclude that for $\ell < r$ and up to lower order terms $L(f)_\ell$ is obtained by applying φ_ℓ^{-1} to the symmetrized ℓth covariant derivative of ℓ, and the claim follows. □

Note that part (ii) immediately implies that for a bundle map A as defined above, $f \mapsto A(L(f))$ is a differential operator $\Gamma(W_0) \to \Gamma(H_1)$ of order at most $r-1$ and any such operator is obtained in this way.

3.5. The second step of the procedure. For a section $f \in \Gamma(W_0)$ we next define $D^{\mathbb{W}}(f) \in \Gamma(H_1)$ to be the component of $\tilde{\nabla} L(f)$ in $\Gamma(H_1) \subset \Omega^1(M, W)$. We know that $(\tilde{\nabla} L(f))_{r-1} = \nabla L(f)_{r-1} + \partial L(f)_r$, and the second summand does not contribute to the H_1–component. Moreover, from the proof of Proposition 3.4 we know that, up to lower order terms, $L(f)_{r-1}$ is obtained by applying φ_{r-1}^{-1} to the symmetrized $(r-1)$–fold covariant derivative of f. Hence up to lower order terms, $\nabla L(f)_{r-1}$ is obtained by applying $\mathrm{id} \otimes \varphi_{r-1}^{-1}$ to the symmetrized r–fold covariant derivative of f. Using the proof of Proposition 3.3 this easily implies that the principal symbol of $D^{\mathbb{W}}$ is (a nonzero multiple of) the Cartan product $S^r T^*M \otimes W_0 \to S_0^r T^*M \circledcirc W_0 = H_1$.

Proposition 3.5. *Let $D : \Gamma(W_0) \to \Gamma(H_1)$ be a quasi–linear differential operator of order r whose principal symbol is given by the Cartan product $S^r T^*M \otimes W_0 \to S_0^r T^*M \circledcirc W_0$. Then there is a bundle map $A : W \to T^*M \otimes W$ as in 3.4 such that $\Sigma \mapsto \Sigma_0$ and $f \mapsto L(f)$ induce inverse bijections between the sets of solutions of $D(f) = 0$ and of the basic system* (3.1).

Proof. This is completely parallel to the proof of Proposition 2.3: The conditions on D exaclty means that it can be written in the form $D(f) = D^{\mathbb{W}}(f) + A(L(f))$ for an appropriate choice of A as above. Then $\tilde{\nabla} L(f) + A(L(f))$ is a section of the subbundle $\ker(\delta^*)$ and the component in H_1 of this section equals $D(f)$. Of course, being a section of $\mathrm{im}(\delta^*)$ is equivalent to vanishing of the H_1–component.

Conversely, Proposition 3.4 shows that any solution Σ of (3.1) is of the form $\Sigma = L(\Sigma_0)$. □

3.6. The last step of the procedure. To rewrite the basic system (3.1) in first order closed form, we use the covariant exterior derivative $d^{\tilde{\nabla}}$. Suppose that $\alpha \in \Omega^1(M, W)$ has the property that its components α_i vanish for $i = 0, \dots, \ell$. Then one immediately verifies that $(d^{\tilde{\nabla}}\alpha)_i = 0$ for $i = 0, \dots, \ell - 1$ and $(d^{\tilde{\nabla}}\alpha)_\ell = \partial(\alpha_{\ell+1})$, so $(\delta^* d^{\tilde{\nabla}}\alpha)_i$ vanishes for $i \leq \ell$ and equals $\delta^*\partial\alpha_{\ell+1}$ for $i = \ell + 1$. If we in addition assume that α is a section of the subbundle $\mathrm{im}(\delta^*)$, then the same is true for $\alpha_{\ell+1}$ and hence $\delta^*\partial\alpha_{\ell+1} = \alpha_{\ell+1}$.

Suppose that Σ solves the basic system (3.1). Then applying $\delta^* d^{\tilde{\nabla}}$, we obtain

$$\delta^*(R\bullet\Sigma+d^{\tilde\nabla}(A(\Sigma)))=\delta^*d^{\tilde\nabla}\delta^*\psi,$$

where we have used that, as in 2.4, $d^{\tilde\nabla}\tilde\nabla\Sigma$ is given by the action of the Riemann curvature R. From above we see that we can compute the lowest nonzero homogeneous component of $\delta^*\psi$ from this equation. We can then move this to the other side in (3.1) to obtain an equivalent system whose right hand side starts one homogeneity higher. The lowest nonzero homogeneous component of the right hand side can then be computed in the same way, and iterating this we conclude that (3.1) can be equivalently written as

$$\tilde\nabla\Sigma+B(\Sigma)=0 \tag{3.2}$$

for a certain differential operator $B:\Gamma(W)\to\Omega^1(M,W)$.

While B is a higher order differential operator in general, it is crucial that the construction gives us a precise control on the order of the individual components of B. From the construction it follows that $B(\Sigma)_i\in\Omega^1(M,V_i)$ depends only on $\Sigma_0,\dots,\Sigma_i$, and the dependence is tensorial in Σ_i, first order in Σ_{i-1} and so on up to ith order in Σ_0.

In particular, the component of (3.2) in $\Omega^1(M,W_0)$ has the form $\nabla\Sigma_0=C_0(\Sigma_0,\Sigma_1)$. Next, the component in $\Omega^1(M,W_1)$ has the form $\nabla\Sigma_1=\tilde C_1(\Sigma_0,\Sigma_1,\Sigma_2,\nabla\Sigma_0)$, and we define

$$C_1(\Sigma_0,\Sigma_1,\Sigma_2):=\tilde C_1(\Sigma_0,\Sigma_1,\Sigma_2,-C_0(\Sigma_0,\Sigma_1)).$$

Hence the two lowest components of (3.2) are equivalent to

$$\begin{cases}\nabla\Sigma_1=C_1(\Sigma_0,\Sigma_1,\Sigma_2)\\ \nabla\Sigma_0=C_0(\Sigma_0,\Sigma_1)\end{cases}$$

Differentiating the lower row and inserting for $\nabla\Sigma_0$ and $\nabla\Sigma_1$ we get an expression for $\nabla^2\Sigma_0$ in terms of $\Sigma_0,\Sigma_1,\Sigma_2$. Continuing in this way, one proves

THEOREM 3.6. *Let $D:\Gamma(W_0)\to\Gamma(H_1)$ be a quasi–linear differential operator of order r with principal symbol the Cartan product $S^rT^*M\otimes W_0\to S^rT^*M_0\circledcirc W_0$. Then there is a bundle map $C:W\to T^*M\otimes W$ such that $\Sigma\mapsto\Sigma_0$ and $f\mapsto L(f)$ induce inverse bijections between the sets of solutions of $D(f)=0$ and of $\tilde\nabla\Sigma+C(\Sigma)=0$. If D is linear, then C can be chosen to be a vector bundle map.*

This in particular shows that any solution of $D(f)=0$ is determined by the value of $L(f)$ in one point, end hence by the N–jet of f in one point. For linear D, the dimension of the space of solutions is bounded by $\dim(\mathbb{W})$ and equality can be only attained if the linear connection $\tilde\nabla+C$ on W is flat. A crucial point here is of course that $\mathbb{W}$, and hence $\dim(\mathbb{W})$ and N

can be immediately computed from $\mathbb{W}_0$ and r, so all this information is available in advance, without going through the procedure. As we shall see later, both the bound on the order and the bound on the dimension are sharp.

To get a feeling for what is going on, let us consider some examples. If we look at operators on smooth functions, we have $\mathbb{W}_0 = \mathbb{R}$. The representation associated to $(\mathbb{R}, r)$ is $S_0^{r-1}\mathbb{V}$, the tracefree part of the $(r-1)$st symmetric power of the standard representation $\mathbb{V}$. A moment of thought shows that the eigenvalues of the grading element E on this representation range from $-r+1$ to $r-1$, so $N = 2(r-1)$. On the other hand, for $r \geq 3$ we have

$$\dim(S_0^{r-1}\mathbb{V}) = \dim(S^{r-1}\mathbb{V}) - \dim(S^{r-3}\mathbb{V}) = (n+2r-2)\frac{(n+2r-2)!}{n!(r-1)!},$$

and this is the maximal dimension of the space of solutions of any system with principal part $f \mapsto \nabla_{(a_1}\nabla_{a_2}\dots\nabla_{a_r)_0} f$ for $f \in C^\infty(M, \mathbb{R})$.

As an extreme example let us consider the conformal Killing equation on tracefree symmetric tensors. Here $W_0 = S_0^k TM$ for some k and $r = 1$. The principal part in this case is simply

$$f^{a_1 \dots a_k} \mapsto \nabla^{(a} f^{a_1 \dots a_k)_0}.$$

The relevant representation $\mathbb{W}$ in this case turns out to be $\circledcirc^k \mathfrak{g}$, i.e. the highest weight subspace in $S^k\mathfrak{g}$. In particular $N = 2k$ in this case, so even though we consider first order systems, many derivatives a needed to pin down a solution. The expression for $\dim(\mathbb{W})$ is already reasonably complicated in this case, namely (see [11])

$$\dim(\mathbb{W}) = \frac{(n+k-3)!(n+k-2)!(n+2k)!}{k!(k+1)!(n-2)!n!(n+2k-3)!}$$

The conformal Killing equation $\nabla^{(a} f^{a_1 \dots a_k)_0} = 0$ plays an important role in the description of symmetries of the Laplacian on a Riemannian manifold, see [11].

4. Conformally invariant differential operators. We now move to the method for constructing conformally invariant differential operators, which gave rise to the prolongation procedure discussed in the last two sections.

4.1. Conformal geometry. Let M be a smooth manifold of dimension $n \geq 3$. As already indicated in 2.4, two Riemannian metrics g and $\hat{g}$ on M are called *conformally equivalent* if and only if there is a positive smooth function φ on M such that $\hat{g} = \varphi^2 g$. A *conformal structure* on M is a conformal equivalence class $[g]$ of metrics, and then $(M, [g])$ is called a conformal manifold. A *conformal isometry* between conformal manifolds

$(M,[g])$ and $(\tilde{M},[\tilde{g}])$ is a local diffeomorphism which pulls back one (or equivalently any) metric from the class $[\tilde{g}]$ to a metric in $[g]$.

A Riemannian metric on M can be viewed as a reduction of structure group of the frame bundle to $O(n) \subset GL(n,\mathbb{R})$. In the same way, a conformal structure is a reduction of structure group to $CO(n) \subset GL(n,\mathbb{R})$, the subgroup generated by $O(n)$ and multiples of the identity.

We want to clarify how the inclusion $O(n) \hookrightarrow G \cong O(n+1,1)$ which was the basis for our prolongation procedure is related to conformal geometry. For the basis $\{e_0,\dots,e_{n+1}\}$ used in 2.1, this inclusion was simply given by $A \mapsto \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$. In 3.1 we met the decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of the Lie algebra $\mathfrak{g}$ of G. We observed that this decomposition is preserved by $O(n) \subset G$ and in that way $\mathfrak{g}_{\pm 1}$ is identified with the standard representation. But there is a larger subgroup with these properties. Namely, for elements of

$$G_0 := \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{smallmatrix}\right) : a \in \mathbb{R} \setminus 0, A \in O(n) \right\} \subset G,$$

the adjoint action preserves the grading, and maps $X \in \mathfrak{g}_{-1}$ to $a^{-1}AX$, so $G_0 \cong CO(\mathfrak{g}_{-1})$. Note that $G_0 \subset G$ corresponds to the Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$.

Now there is a more conceptual way to understand this. Consider the subalgebra $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathfrak{g}$ and let $P \subset G$ be the corresponding Lie subgroup. Then P is the subgroup of matrices which are block–upper–triangular with blocks of sizes 1, n, and 1. Equivalently, P is the stabilizer in G of the isotropic line spanned by the basis vector e_0. The group G acts transitively on the space of all isotropic lines in $\mathbb{V}$, so one may identify this space with the homogeneous space G/P.

Taking coordinates z_i with respect to an orthonormal basis of $\mathbb{V}$ for which the first $n+1$ vectors are positive and the last one is negative, a vector is isotropic if and only if $\sum_{i=0}^n z_i^2 = z_{n+1}^2$. Hence for a nonzero isotropic vector the last coordinate is nonzero and any isotropic line contains a unique vector whose last coordinate equals 1. But this shows that the space of isotropic lines in $\mathbb{V}$ is an n–sphere, so $G/P \cong S^n$.

Given a point $x \in G/P$, choosing a point v in the corresponding line gives rise to an identification $T_xS^n \cong v^\perp/\mathbb{R}v$ and that space carries a positive definite inner product induced by $\langle\ ,\ \rangle$. Passing from v to λv, this inner product gets scaled by λ^2, so we get a canonical conformal class of inner products on each tangent space, i.e. a conformal structure on S^n. This conformal structure contains the round metric of S^n.

The action ℓ_g of $g \in G$ on the space of null lines by construction preserves this conformal structure, so G acts by conformal isometries. It turns out, that this identifies $G/\{\pm \mathrm{id}\}$ with the group of all conformal isometries of S^n. For the base point $o = eP \in G/P$, the tangent space $T_o(G/P)$ is naturally identified with $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$. Let $P_+ \subset P$ be the subgroup of those $g \in P$ for which $T_o\ell_g = \mathrm{id}$. Then one easily shows that

$P/P_+ \cong G_0$ and the isomorphism $G_0 \cong CO(\mathfrak{g}_{-1})$ is induced by $g \mapsto T_o\ell_g$. Moreover, P_+ has Lie algebra $\mathfrak{g}_1$ and $\exp : \mathfrak{g}_1 \to P_+$ is a diffeomorphism.

4.2. Conformally invariant differential operators. Let $(M, [g])$ be a conformal manifold. Choosing a metric g from the conformal class, we get the Levi–Civita connection ∇ on each Riemannian natural bundle as well as the Riemann curvature tensor R. Using g, its inverse, and R, we can write down differential operators, and see how they change if g is replaced by a conformally equivalent metric $\hat{g}$. Operators obtained in that way, which do not change at all under conformal rescalings are called *conformally invariant*. In order to do this successfully one either has to allow density bundles or deal with conformal weights, but I will not go into these details here. The best known example of such an operator is the conformal Laplacian or Yamabe operator which is obtained by adding an appropriate amount of scalar curvature to the standard Laplacian.

The definition of conformally invariant operators immediately suggests a naive approach to their construction. First choose a principal part for the operator. Then see how this behaves under conformal rescalings and try to compensate the changes by adding lower order terms involving curvature quantities. This approach (together with a bit of representation theory) easily leads to a complete classification of conformally invariant first order operators, see [13]. Passing to higher orders, the direct methods get surprisingly quickly out of hand.

The basis for more invariant approaches is provided by a classical result of Elie Cartan, which interprets general conformal structures as analogs of the homogeneous space $S^n \cong G/P$ from 4.1. As we have noted above, a conformal structure $[g]$ on M can be interpreted as a reduction of structure group. This means that a conformal manifold $(M, [g])$ naturally carries a principal bundle with structure group $CO(n)$, the *conformal frame bundle*. Recall from 4.1 that the conformal group $G_0 = CO(\mathfrak{g}_{-1}) \cong CO(n)$ can be naturally viewed as a quotient of the group P. Cartan's result says that the conformal frame bundle can be canonically extended to a principal fiber bundle $\mathcal{G} \to M$ with structure group P, and $\mathcal{G}$ can be endowed with a canonical Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. The form ω has similar formal properties as the Maurer–Cartan form on G, i.e. it defines a trivialization of the tangent bundle $T\mathcal{G}$, which is P–equivariant and reproduces the generators of fundamental vector fields.

While the canonical Cartan connection is conformally invariant, it is not immediately clear how to use it to construct differential operators. The problem is that, unlike principal connections, Cartan connections do not induce linear connections on associated vector bundles.

4.3. The setup for the conformal BGG machinery. Let us see how the basic developments from 3.1–3.3 comply with our new point of view. First of all, for $g \in P$, the adjoint action does not preserve the grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, but it preserves the subalgebras $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, and

$\mathfrak{g}_1$. More generally, if $\mathbb{W} = \mathbb{W}_0 \oplus \cdots \oplus \mathbb{W}_N$ is an irreducible representation of $\mathfrak{g}$ decomposed according to eigenspaces of the grading element E, then each of the subspaces $\mathbb{W}_i \oplus \cdots \oplus \mathbb{W}_N$ is P–invariant. Since P naturally acts on $\mathfrak{g}_1$ and on $\mathbb{W}$, we get induced actions on $\Lambda^k\mathfrak{g}_1 \otimes \mathbb{W}$ for all k. The formula for $\partial^* : \Lambda^k\mathfrak{g}_1 \otimes \mathbb{W} \to \Lambda^{k-1}\mathfrak{g}_1 \otimes \mathbb{W}$ uses only the action of $\mathfrak{g}_1$ on $\mathbb{W}$, so ∂^* is P–equivariant.

In contrast to this, the only way to make P act on $\mathfrak{g}_{-1}$ is via the identification with $\mathfrak{g}/\mathfrak{p}$. However, the action of $\mathfrak{g}_{-1}$ on $\mathbb{W}$ has no natural interpretation in this identification, and $\partial : \Lambda^k\mathfrak{g}_1 \otimes \mathbb{W} \to \Lambda^{k+1}\mathfrak{g}_1 \otimes \mathbb{W}$ is *not* P–equivariant.

Anyway, given a conformal manifold $(M, [g])$ we can now do the following. Rather than viewing $\mathbb{W}$ just as sum of representations of $G_0 \cong CO(n)$, we can view it as a representation of P, and form the associated bundle $\mathcal{W} := \mathcal{G} \times_P \mathbb{W} \to M$. Bundles obtained in this way are called *tractor bundles*. I want to emphasize at this point that the bundle $\mathcal{W}$ is of completely different nature than the bundle W used in section 3. To see this, recall that elements of the subgroup $P_+ \subset P$ act on G/P by diffeomorphisms which fix the base point $o = eP$ to first order. Therefore, the action of such a diffeomorphism on the fiber over o of any tensor bundle is the identity. On the other hand, it is easy to see that on the fiber over o of any tractor bundle, this action is always non–trivial. Hence tractor bundles are unusual geometric objects.

Examples of tractor bundles have already been introduced as an alternative to Cartan's approach in the 1920's and 30's, in particular in the work of Tracy Thomas, see [17]. Their key feature is that the canonical Cartan connection ω induces a canonical linear connection, called the *normal tractor connection* on each tractor bundle. This is due to the fact that these bundles do not correspond to general representations of P, but only to representations which extend to the big group G. We will denote the normal tractor connection on $\mathcal{W}$ by $\nabla^{\mathcal{W}}$. These connections automatically combine algebraic and differential parts.

The duality between $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ induced by the Killing form, is more naturally viewed as a duality between $\mathfrak{g}_1$ and $\mathfrak{g}/\mathfrak{p}$. Via the Cartan connection ω, the associated bundle $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ is isomorphic to the tangent bundle TM. Thus, the bundle $\mathcal{G} \times_P (\Lambda^k\mathfrak{g}_1 \otimes \mathbb{W})$ is again the bundle $\Lambda^k T^*M \otimes \mathcal{W}$ of $\mathcal{W}$–valued forms. Now it turns out that in a well defined sense (which however is rather awkward to express), the lowest nonzero homogeneous component of $\nabla^{\mathcal{W}}$ is of degree zero, it is tensorial and induced by the Lie algebra differential ∂.

Equivariancy of ∂^* implies that it defines bundle maps

$$\partial^* : \Lambda^k T^*M \otimes \mathcal{W} \to \Lambda^{k-1} T^*M \otimes \mathcal{W}$$

for each k. In particular, $\operatorname{im}(\partial^*) \subset \ker(\partial^*) \subset \Lambda^k T^*M \otimes \mathcal{W}$ are natural subbundles, and we can form the subquotient $H_k := \ker(\partial^*)/\operatorname{im}(\partial^*)$. It turns out that these bundles are always naturally associated to the conformal frame bundle, so they are usual geometric objects like tensor bundles. The explicit form of the bundles H_k can be computed algorithmically using Kostant's version of the BBW–theorem.

4.4. The conformal BGG machinery. The normal tractor connection $\nabla^{\mathcal{W}}$ extends to the covariant exterior derivative, which we denote by $d^{\mathcal{W}} : \Omega^k(M, \mathcal{W}) \to \Omega^{k+1}(M, \mathcal{W})$. The lowest nonzero homogeneous component of $d^{\mathcal{W}}$ is of degree zero, tensorial, and induced by ∂.

Now for each k, the operator $\partial^* d^{\mathcal{W}}$ on $\Omega^k(M, V)$ is conformally invariant and its lowest nonzero homogeneous component is the tensorial map induced by $\partial^*\partial$. By Theorem 3.2, $\partial^*\partial$ acts invertibly on $\operatorname{im}(\partial^*)$. Hence we can find a (non–natural) bundle map β on $\operatorname{im}(\partial^*)$ such that $\beta\partial^* d^{\mathcal{W}}$ reproduces the lowest nonzero homogeneous component of sections of $\operatorname{im}(\partial^*)$. Therefore, the operator $\operatorname{id} - \beta\partial^* d^{\mathcal{W}}$ is (at most N–step) nilpotent on $\Gamma(\operatorname{im}(\partial^*))$, which easily implies that

$$\left(\textstyle\sum_{i=0}^{N} (\operatorname{id} - \beta\partial^* d^{\mathcal{W}})^i\right) \beta$$

defines a differential operator Q on $\Gamma(\operatorname{im}(\partial^*))$ which is inverse to $\partial^* d^{\mathcal{W}}$ and therefore conformally invariant.

Next, we have a canonical bundle map

$$\pi_H : \ker(\partial^*) \to \ker(\partial^*)/\operatorname{im}(\partial^*) = H_k,$$

and we denote by the same symbol the induced tensorial projection on sections. Given $f \in \Gamma(H_k)$ we can choose $\varphi \in \Omega^k(M, \mathcal{W})$ such that $\partial^*\varphi = 0$ and $\pi_H(\varphi) = f$, and consider $\varphi - Q\partial^* d^{\mathcal{W}}\varphi$. By construction, φ is uniquely determined up to adding sections of $\operatorname{im}(\partial^*)$. Since these are reproduced by $Q\partial^* d^{\mathcal{W}}$, the above element is independent of the choice of φ and hence defines $L(f) \in \Omega^k(M, \mathcal{W})$. Since Q has values in $\Gamma(\operatorname{im}(\partial^*))$ we see that $\pi_H(L(f)) = f$, and since $\partial^* d^{\mathcal{W}} Q$ is the identity on $\Gamma(\operatorname{im}(\partial^*))$ we get $\partial^* d^{\mathcal{W}} L(f) = 0$. If φ satisfies $\pi_H(\varphi) = f$ and $\partial^* d^{\mathcal{W}}\varphi = 0$, then

$$L(f) = \varphi - Q\partial^* d^{\mathcal{W}}\varphi = \varphi,$$

so $L(f)$ is uniquely determined by these two properties.

By construction, the operator $L : \Gamma(H_k) \to \Omega^k(M, \mathcal{W})$ in conformally invariant. Moreover, $d^{\mathcal{W}} L(f)$ is a section of $\ker(\partial^*)$, so we can finally define the BGG–operators $D^{\mathcal{W}} : \Gamma(H_k) \to \Gamma(H_{k+1})$ by $D^{\mathcal{W}}(f) := \pi_H(d^{\mathcal{W}} L(f))$. They are conformally invariant by construction.

To obtain additional information, we have to look at structures which are locally conformally flat or equivalently locally conformally isometric to the sphere S^n. It is a classical result that local conformal flatness is equivalent to vanishing of the curvature of the canonical Cartan connection.

Proposition 4.4. *On locally conformally flat manifolds, the BGG operators form a complex* $(\Gamma(H_*), D^{\mathcal{W}})$*, which is a fine resolution of the constant sheaf* $\mathbb{W}$*.*

Proof. The curvature of any tractor connection is induced by the Cartan curvature (see [7]), so on locally conformally flat structures, all tractor connections are flat. This implies that the covariant exterior derivative satisfies $d^{\mathcal{W}} \circ d^{\mathcal{W}} = 0$. Thus $(\Omega^*(M, \mathcal{W}), d^{\mathcal{W}})$ is a fine resolution of the constant sheaf $\mathbb{W}$.

For $f \in \Gamma(H_k)$ consider $d^{\mathcal{W}} L(f)$. By construction, this lies in the kernel of ∂^* and since $d^{\mathcal{W}} \circ d^{\mathcal{W}} = 0$, it also lies in the kernel of $\partial^* d^{\mathcal{W}}$. From above we know that this implies that

$$d^{\mathcal{W}} L(f) = L(\pi_H(d^{\mathcal{W}} L(f))) = L(D^{\mathcal{W}}(f)).$$

This shows that $L \circ D^{\mathcal{W}} \circ D^{\mathcal{W}} = d^{\mathcal{W}} \circ d^{\mathcal{W}} \circ L = 0$ and hence $D^{\mathcal{W}} \circ D^{\mathcal{W}} = 0$, so $(\Gamma(H_*), D^{\mathcal{W}})$ is a complex. The operators L define a chain map from this complex to $(\Omega^*(M, \mathcal{W}), d^{\mathcal{W}})$, and we claim that this chain map induces an isomorphism in cohomology.

First, take $\varphi \in \Omega^k(M, \mathcal{W})$ such that $d^{\mathcal{W}} \varphi = 0$. Then

$$\tilde{\varphi} := \varphi - d^{\mathcal{W}} Q \partial^* \varphi$$

is cohomologous to φ and satisfies $\partial^* \tilde{\varphi} = 0$. Moreover, $d^{\mathcal{W}} \tilde{\varphi} = d^{\mathcal{W}} \varphi = 0$, so $\partial^* d^{\mathcal{W}} \varphi = 0$. Hence $\tilde{\varphi} = L(\pi_H(\tilde{\varphi}))$ and $D^{\mathcal{W}}(\pi_H(\tilde{\varphi})) = 0$, so the induced map in cohomology is surjective.

Conversely, assume that $f \in \Gamma(H_k)$ satisfies $D^{\mathcal{W}}(f) = 0$ and that $L(f) = d^{\mathcal{W}} \varphi$ for some $\varphi \in \Omega^{k-1}(M, \mathcal{W})$. As before, replacing φ by $\varphi - d^{\mathcal{W}} Q \partial^* \varphi$ we may assume that $\partial^* \varphi = 0$. But together with $\partial^* L(f) = 0$ this implies $\varphi = L(\pi_H(\varphi))$ and thus $f = \pi_H(L(f)) = D^{\mathcal{W}}(\pi_H(\varphi))$. Hence the induced map in cohomology is injective, too. Since this holds both locally and globally, the proof is complete. □

Via a duality between invariant differential operators and homomorphisms of generalized Verma modules, this reproduces the original BGG resolutions as constructed in [15]. Via the classification of such homomorphisms, one also concludes that this construction produces a large subclass of all those conformally invariant operators which are non–trivial on locally conformally flat structures.

Local exactness of the BGG sequence implies that all the operators $D^{\mathcal{W}}$ are nonzero on locally conformally flat manifolds. Passing to general conformal structures does not change the principal symbol of the operator $D^{\mathcal{W}}$, so we always get non–trivial operators.

On the other hand, we can also conclude that the bounds obtained from Theorem 3.6 are sharp. From Theorem 3.2 we conclude that any choice of metric in the conformal class identifies $H_0 = \mathcal{W}/\operatorname{im}(\partial^*)$ with the bundle W_0 and H_1 with its counterpart from section 3, and we consider the operator $D^{\mathcal{W}} : \Gamma(H_0) \to \Gamma(H_1)$. By conformal invariance, the system

$D^{\mathcal{W}}(f) = 0$ must be among the systems covered by Theorem 3.6, and the above procedure identifies its solutions with parallel sections of $\mathcal{W}$. Since $\nabla^{\mathcal{W}}$ is flat in the locally conformally flat case, the space of parallel sections has dimension $\dim(\mathbb{W})$. Moreover, two solutions of the system coincide if and only if their images under L have the same value in one point.

5. Generalizations. In this last part, we briefly sketch how the developments of sections 3 and 4 can be carried over to larger classes of geometric structures.

5.1. The prolongation procedure for general $|1|$–graded Lie algebras. The algebraic developments in 3.1–3.3 generalize without problems to a semisimple Lie algebra $\mathfrak{g}$ endowed with a $|1|$–grading, i.e. a grading of the form $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Given such a grading it is easy to see that it is the eigenspace decomposition of $\mathrm{ad}(E)$ for a uniquely determined element $E \in \mathfrak{g}_0$. The Lie subalgebra $\mathfrak{g}_0$ is automatically the direct sum of a semisimple part $\mathfrak{g}_0'$ and a one–dimensional center spanned by E. This gives rise to E–eigenspace decompositions for irreducible representations. Again irreducible representations of $\mathfrak{g}$ may be parametrized by pairs consisting of an irreducible representation of $\mathfrak{g}_0'$ and an integer ≥ 1. Then all the developments of 3.1–3.3 work without changes.

Next choose a Lie group G with Lie algebra $\mathfrak{g}$ and let $G_0 \subset G$ be the subgroup consisting of those elements whose adjoint action preserves the grading of $\mathfrak{g}$. Then this action defines an infinitesimally effective homomorphism $G_0 \to GL(\mathfrak{g}_{-1})$. In particular, the semisimple part G_0' of G_0 is a (covering of a) subgroup of $GL(\mathfrak{g}_{-1})$, so this defines a type of geometric structure on manifolds of dimension $\dim(\mathfrak{g}_{-1})$. This structure is linked to representation theory of G_0' in the same way as Riemannian geometry is linked to representation theory of $O(n)$.

For manifolds endowed with a structure of this type, there is an analog of the prolongation procedure described in 3.4–3.6 with closely parallel proofs, see [4]. The only change is that instead of the Levi–Civita connection one uses any linear connection on TM which is compatible with the reduction of structure group. There are some minor changes if this connection has torsion. The systems that this procedure applies to are the following. One chooses an irreducible representation $\mathbb{W}_0$ of G_0' and an integer $r \geq 1$. Denoting by W_0 the bundle corresponding to $\mathbb{W}_0$, one then can handle systems whose principal symbol is (a multiple of) the projection from $S^rTM \otimes W_0$ to the subbundle corresponding to the irreducible component of maximal highest weight in $S^r\mathfrak{g}_1 \otimes \mathbb{W}_0$.

The simplest example of this situation is $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ endowed with the grading $\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}$ with blocks of sizes 1 and n. Then $\mathfrak{g}_{-1}$ has dimension n and $\mathfrak{g}_0 \cong \mathfrak{gl}(n, \mathbb{R})$. For the right choice of group, one obtains $G_0' = SL(n, \mathbb{R})$, so the structure is just a volume form on an n–manifold.

There is a complete description of $|1|$–gradings of semisimple Lie algebras in terms of structure theory and hence a complete list of the other geometries for which the procedure works. One of these is related to almost quaternionic structures, the others can be described in terms of identifications of the tangent bundle with a symmetric or skew symmetric square of an auxiliary bundle or with a tensor product of two auxiliary bundles.

5.2. Invariant differential operators for AHS–structures. For a group G with Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as in 5.1, one defines $P \subset G$ as the subgroup of those elements whose adjoint action preserves the subalgebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1 =: \mathfrak{p}$. It turns out that $\mathfrak{p}$ is the Lie algebra of P and $G_0 \subset P$ can also be naturally be viewed as a quotient of P.

On manifolds of dimension $\dim(\mathfrak{g}_{-1})$ we may consider reductions of structure group to the group G_0. The passage from G_0' as discussed in 5.1 to G_0 is like the passage from Riemannian to conformal structures. As in 4.2, one may look at extensions of the principal G_0–bundle defining the structure to a principal P–bundle $\mathcal{G}$ endowed with a normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. In the example $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{R})$ with $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$ from 5.1, the principal G_0–bundle is the full frame bundle, so it contains no information. One shows that such an extension is equivalent to the choice of a projective equivalence class of torsion free connections on TM. In all other cases (more precisely, one has to require that no simple summand has this form) Cartan's result on conformal structures can be generalized to show that such an extension is uniquely possible for each given G_0–structure, see e.g. [8].
The structures equivalent to such Cartan connections are called AHS–structures in the literature. Apart from conformal and projective structures, they also contain almost quaternionic and almost Grassmannian structures as well as some more exotic examples, see [8]. For all these structures, the procedure from section 4 can be carried out without changes to construct differential operators which are intrinsic to the geometry.

5.3. More general geometries. The construction of invariant differential operators from section 4 applies to a much larger class of geometric structures. Let $\mathfrak{g}$ be a semisimple Lie algebra endowed with a $|k|$–grading, i.e. a grading of the form $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ for some $k \geq 1$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ and such that the Lie subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. For any such grading, the subalgebra $\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k \subset \mathfrak{g}$ is a *parabolic* subalgebra in the sense of representation theory. Conversely, any parabolic subalgebra in a semisimple Lie algebra gives rise to a $|k|$–grading. Therefore, $|k|$–gradings are well understood and can be completely classified in terms of the structure theory of semisimple Lie algebras.

Given a Lie group G with Lie algebra $\mathfrak{g}$ one always finds a closed subgroup $P \subset G$ corresponding to the Lie algebra $\mathfrak{p}$. The homogeneous space G/P is a so–called *generalized flag variety.* Given a smooth manifold M of the same dimension as G/P, a *parabolic geometry* of type (G, P) on M is given by a principal P–bundle $p : \mathcal{G} \to M$ and a Cartan connection

$\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

In pioneering work culminating in [16], N. Tanaka has shown that assuming the conditions of regularity and normality on the curvature of the Cartan connection, such a parabolic geometry is equivalent to an underlying geometric structure. These underlying structures are very diverse, but during the last years a uniform description has been established, see the overview article [6]. Examples of these underlying structures include partially integrable almost CR structures of hypersurface type, path geometries, as well as generic distributions of rank two in dimension five, rank three in dimension six, and rank four in dimension seven. For all these geometries, the problem of constructing differential operators which are intrinsic to the structure is very difficult.

The BGG machinery developed in [9] and [5] offers a uniform approach for this construction, but compared to the procedure of section 4 some changes have to be made. One again has a grading element E which leads to an eigenspace decomposition $\mathbb{W} = \mathbb{W}_0 \oplus \cdots \oplus \mathbb{W}_N$ of any finite dimensional irreducible representation of $\mathfrak{g}$. As before, we have $\mathfrak{g}_i \cdot \mathbb{W}_j \subset \mathbb{W}_{i+j}$. Correspondingly, this decomposition is only invariant under a subgroup $G_0 \subset P$ with Lie algebra $\mathfrak{g}_0$, but each of the subspaces $\mathbb{W}_i \oplus \cdots \oplus \mathbb{W}_N$ is P–invariant. The theory of tractor bundles and tractor connections works in this more general setting without changes, see [7].

Via the Cartan connection ω, the tangent bundle TM can be identified with $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ and therefore $T^*M \cong \mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})^*$. Now the annihilator of $\mathfrak{p}$ under the Killing form of $\mathfrak{g}$ is the subalgebra $\mathfrak{p}_+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. For a tractor bundle $\mathcal{W} = \mathcal{G} \times_P \mathbb{W}$, the bundles of $\mathcal{W}$–valued forms are therefore associated to the representations $\Lambda^k \mathfrak{p}_+ \otimes \mathbb{W}$.

Since we are now working with the nilpotent Lie algebra $\mathfrak{p}_+$ rather than with an Abelian one, we have to adapt the definition of ∂^*. In order to obtain a differential, we have to add terms which involve the Lie bracket on $\mathfrak{p}_+$. The resulting map ∂^* is P–equivariant, and the quotients $\ker(\partial^*)/\operatorname{im}(\partial^*)$ can be computed as representations of $\mathfrak{g}_0$ using Kostant's theorem. As far as ∂ is concerned, we have to identify $\mathfrak{g}/\mathfrak{p}$ with the nilpotent subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$. Then we can add terms involving the Lie bracket on $\mathfrak{g}_-$ to obtain a map ∂ which is a differential. As the identification of $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{g}_-$, the map ∂ is not equivariant for the P–action but only for the action of a subgroup G_0 of P with Lie algebra $\mathfrak{g}_0$.

The P–equivariant map ∂^* again induces vector bundle homomorphisms on the bundles of $\mathcal{W}$–valued differential forms. We can extend the normal tractor connection to the covariant exterior derivative $d^{\mathcal{W}}$. As in 4.4, the lowest homogeneous component of $d^{\mathcal{W}}$ is tensorial and induced by ∂ which is all that is needed to get the procedure outlined in 4.4 going. Also the results for structures which are locally isomorphic to G/P discussed in 4.4 extend to general parabolic geometries.

The question of analogs of the prolongation procedure from section 3 for arbitrary parabolic geometries has not been completely answered yet. It is clear that some parts generalize without problems. For other parts, some modifications will be necessary. In particular, the presence of non–trivial filtrations of the tangent bundle makes it necessary to use the concept of weighted order rather than the usual concept of order of a differential operator and so on. Research in this direction is in progress.

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