

**Group Quantization of Parametrized Systems.
I. Time Levels**

P. Hájíček

Vienna, Preprint ESI 185 (1994)

December 27, 1994

Supported by Federal Ministry of Science and Research, Austria
Available via WWW.ESI.AC.AT

Group Quantization of Parametrized Systems. I. Time Levels

P. Hájíček

Institute for Theoretical Physics

University of Bern

Sidlerstrasse 5, CH-3012 Bern, Switzerland

December 1994

Abstract

Dirac's ideas on forms of relativistic dynamics are generalized to all finite-dimensional first-class parametrized constrained systems. The Poincaré group and the algebra of its generators is replaced by a canonical group of symmetries and by an algebra of elementary observables. The quantum theory is determined by two choices: that of the group (or algebra) and that of its representation. A choice of gauge and time is represented by a transversal surface in the phase space. Projection of symmetry transformations and of observables to transversal surfaces is defined and shown to preserve both the group and the algebraic structures. Thus, the choice of gauge and time has no influence on the structure of the quantum mechanics. Any observable moves a given transversal surface via Poisson bracket around the phase space. This defines a one-dimensional family of time levels (in the phase space) and time slices (in classical solutions). A family of observables is called a complete family of partial Hamiltonians, if each point of each classical solution is contained in at least one time slice. The corresponding more-dimensional family of time slices form a subset of the full many-finger time-slice system. Any two of the corresponding time levels in the phase space are related by a symmetry. This fact is exploited to define "the same measurement at different times", to develop a complete quantitative theory of "changes in time" on this basis, and, finally, to construct the Schrödinger and Heisenberg picture of quantum dynamics.

1 Introduction and summary

Systems with constraints are frequently met in the contemporary theoretical physics. One can distinguish between two quite different cases of these systems: the so-called *gauge systems* whose classical solutions (curves in the phase space) are transversal to the orbits of the gauge group (which is generated via Poisson brackets by the constraints), and the so-called *parametrized systems* each solution curve of which lies within an orbit. We will concentrate on some typical problems associated with the parametrized systems.

The most challenging parametrized system is Einstein's general theory of relativity. Attempts to quantize this theory meet a number of difficulties of a technical as well as conceptual character (for reviews, see [1], [2] and [3]). Many conceptual problems are more or less directly associated with the notion of time – which is an issue common to all parametrized systems. On one hand, the choice of time is to a certain degree analogous to the choice of gauge in a gauge theory; on the other hand, the time plays a very special role in quantum mechanics and it surely has some measurable aspects. This dichotomy is the source of several problems.

In particular, two typical problems can arise: the so-called “global time problem” and the “multiple choice problem” (for details and examples, see [1] and [2]). The essence of the former, which is reminiscent of the Gribov problem, is that given constraints need not admit a global choice of time. In the case that there is a global time, we meet the latter problem – there will be no uniqueness in the choice of time and different choices together with the quantization method at hand will often lead to unitarily inequivalent quantum theories.

The multiple choice problem may be related to another, much more general problem: a given classical theory does not determine “the corresponding quantum theory” uniquely. There are (at least) factor ordering ambiguities. However, the “spirit of symmetry” of the classical theory suggests that we are to look for such a factor ordering that the resulting quantum theory becomes as invariant as the original classical theory. A pioneer effort in this direction has been undertaken in ref. [4].

The present paper will describe another attempt of this kind; it will be based on an old paper [5] by Dirac. In fact, [5] was aimed at a special relativistic *mechanics* of elementary particles that avoids the infinities of the quantum field theory. Of course, the sweeping success of the quantum field theory as we know it today pushed the Dirac paper into oblivion (but see [6]). However, it turns out that some ideas of ref. [5] have a direct relevance to quantization of parametrized systems. They lead to clean separation between those structures of the theory that are independent of possible time choices (and which are considered more fundamental) and many possible time structures which can be constructed after the fundamental structure is given.

Dirac's fundamental structure was some sufficiently complete algebra of, or group generated by, observables. Observables were defined as those phase space functions that have weakly vanishing Poisson brackets with the constraints. For the special relativistic system considered by Dirac, these observables were naturally associated with the Poincaré group. Today, analogous algebras and groups are used for much broader class of parametrized systems: in their work on algebraic quantization, Ashtekar and his collaborators (see [7] and [8]) employ also this kind of algebras; the so-called group quantization ([9], [10] and [11]) can be based on similar groups [12]. In fact, both methods are related [8], and they could be combined.

Thus, Dirac's theory [5], if obviously generalized, will include some aspects of algebraic and group quantization; we will see that it includes both quantizations in a certain sense. However, it is much more to it! Indeed, the next important Dirac's idea is the so-called "time surface" in the phase space. Exactly as a gauge surface of any gauge system, the time surface has to be transversal to the orbits generated by the constraints. The observables and the group action are projected onto a given time surface. The resulting functions on, or transformations of, the time surface have the same algebraic, or group, structure independently of how the time surface is chosen. In such a manner the desired time-choice independence is achieved.

The most interesting idea by Dirac concerns the time evolution. Any element of the algebra of observables which, via Poisson brackets, moves the time surface non-trivially in the phase space is eligible as a "Hamiltonian". Thus one obtains a family of "time levels" in the phase space such that each two time levels can be mapped onto each other by a symmetry. Using this symmetry, we can easily give meaning to "the same measurement at different times". The only role of the Hamiltonian is to generate (to define) these structures; it need not be the total energy of the system, it need not be positive, or define a ground state. Dirac, as well as subsequent work like e.g. [6] showed that the idea works for well-known special relativistic systems. This generalization of time evolution can be felt as as liberating by those who try to quantize gravity.

There is a double freedom in the construction of time evolution à la Dirac: The choice of a time surface and that of an associated Hamiltonian. Formally, quantum mechanics constructed for different choices of Hamiltonian will not be unitarily equivalent. This can be shown even for the free relativistic particle [13]. This is a "remnant" of the multiple choice problem in Dirac's theory. However, it seems that this feature can be controlled and does not lead to any serious problem. The theories are "compatible", if not unitarily equivalent [13].

In the present paper, we will generalize Dirac's theory to all finite-dimensional first-class parametrized systems. The restriction to finite number of degrees of freedom is chosen just for the sake of simplicity and because it enables a relatively

rigorous exposition. The plan of the paper is as follows.

In Sec. 2, we describe the geometrical properties of the constraint surface of the first-class systems following essentially ref. [14]. Then, we give a definition of the parametrized system that is in agreement with this geometrical point of view and describe in some detail the meaning of curves within orbits at the constraint surface, as well as the relation of orbits to certain maximal classical solutions.

Sec. 3 defines the basic notions of observables and symmetries of a parametrized system. In fact, the name “observables” for the variables that are constant along orbits is misleading in the case of parametrized systems [15]. There are observable features of these systems that cannot be described by these observables; in particular, Dirac’s time structures are clearly such features. Thus, we will adhere to the proposal of [15] using the word “perennial” instead of “observable”. By the way, extended use of perennials in quantum gravity has been criticized in [15], because no single perennial is known for general relativity, and because the general relativity has no perennials of a certain kind [16]. However, an abstract existence of general perennials for any parametrized system is at least very plausible (see, e.g. [17] and an argument in Sec. 4 of the present paper). Such an existence will be sufficient for our purposes: we are giving an (abstract) theory rather than a (practical) method. Thus, a perennial is a function that is constant along orbits and a symmetry is a symplectic diffeomorphism that preserves the constraint surface (see also [12]). We study relations between perennials and symmetries, show some general properties and define the so-called “algebras of elementary perennials” and “first-class canonical groups” for parametrized systems (differing by some detail from [8] or [12]).

In Sec. 4, the so-called “transversal surfaces” are defined (we replace Dirac’s name “time surface” by it). Perennials and symmetries are projected to transversal surfaces and the projection is studied. We show that the Poisson algebra structure as well as the group structure is preserved by the projection. The projection also preserves the relation between one-dimensional subgroups of symmetries and the corresponding perennial generators.

In Sec. 5, we describe an obvious generalization of Dirac’s notion of Hamiltonian to any parametrized system. The crucial role is played by curves within the orbits and their relation to classical solutions. Using this relation we can define Hamiltonians or partial Hamiltonians associated with a given transversal surface. Any partial Hamiltonian generates a one-dimensional group of symmetries that moves the corresponding transversal surface in the phase space. In this way, we obtain time levels in the phase space and time slices in the classical solutions of the system. The time slices generated by partial Hamiltonian cover only some open part of some classical solution. A complete family of partial Hamiltonians generate a more-dimensional family of time slices that cover all classical solutions completely. A subfamily of

the full system of many-finger time slices results in this way. Any two time levels in the phase space that have been generated from a given transversal surfaces by a complete family of partial Hamiltonians are related to each other by a symmetry transformation. This transformation can be made to map perennials into perennials and thus define related measurements at different time levels. This is used to develop a quantitative and complete theory of changes in time. The corresponding classical dynamical equations are derived. The way to the Schrödinger and Heisenberg picture of quantum dynamics is straightforward. The families of perennials generated by partial Hamiltonians define some particular class of “evolving constants of motion” ([18]).

The Appendix A illustrates the theory by working out explicitly some of our new notions and theorems for a toy model: the system of free massive particles in Minkowski space. Finally, proofs of some theorems are collected in the Appendix B.

2 First-class parametrized systems

In this section we give basic definitions, explain the notation and recall some well-known facts about finite-dimensional first-class parametrized systems. Most examples of such systems that can be met in the literature are usually presented in the following form.

An N manifold \mathcal{C} is specified as the *configuration space*, and its cotangent bundle $T^*\mathcal{C} = \Gamma$ as the *phase space* of the system. Some coordinates $\{q^\mu\}$ are chosen on \mathcal{C} , and the corresponding coordinates $\{q^\mu, p_\mu\}$ on Γ . Then, the action S of the system is written as follows:

$$S = \int d\tau (p_\mu \dot{q}^\mu - N_\alpha C_\alpha), \quad (1)$$

where the dot means the derivative with respect to τ , N_α are *Lagrange multipliers* (additional independent variables) and $C_\alpha, \alpha = 1, \dots, \nu'$, are the *constraint functions* (shortly “constraints”).

Varying S with respect to N_α 's, we obtain the following system of eqs.:

$$C_\alpha = 0 \quad \forall \alpha, \quad (2)$$

the so-called *constraints*. The system (2) defines a subset $\tilde{\Gamma}$ of Γ which is called *constraint surface*.

The system is called *first-class*, if the constraints satisfy the following conditions

$$\{C_\alpha, C_\beta\} = f_{\alpha\beta\gamma} C_\gamma \quad \forall \alpha, \beta, \quad (3)$$

where $f_{\alpha\beta\gamma}(q, p)$ are some regular functions on Γ and the symbol $\{.,.\}$ denotes the Poisson bracket. It follows that

$$\{C_\alpha, C_\beta\} |_{\tilde{\Gamma}} = 0 \quad \forall \alpha, \beta; \quad (4)$$

one can show (see, e.g. [19]) that the condition (4) implies eqs. (3).

2.1 Geometry of constraint surface

We will need a geometrical definition of first-class systems, as it is given e.g. in [14]

In this approach, the basic space is the phase space Γ . Γ need not be a cotangent bundle; one just assumes Γ to be a symplectic manifold. That is, Γ is a $2N$ -dimensional manifold equipped with a non-degenerate closed two-form Ω . For the system defined by the action (1), $\Omega = dp_\mu \wedge dq^\mu$. In local coordinates z^A on Γ , $A = 1, \dots, 2N$, Ω can be represented by an anti-symmetric tensor field Ω_{AB} with the properties

1. $\Omega_{AB}(x) = -\Omega_{BA}(x)$, $\forall x \in \Gamma, A, B$
2. $\Omega_{AB}(x)$ has an inverse matrix, $\Omega^{AB}(x)$, at each $x \in \Gamma$,
3. $\Omega_{AB,C} + \Omega_{BC,A} + \Omega_{CA,B} = 0$ on Γ ,
4. Poisson bracket of two functions f and g on Γ is defined as follows

$$\{f, g, \} = \Omega^{AB} f_{,A} g_{,B}.$$

The basic geometrical object of this approach is the constraint surface $\tilde{\Gamma}$. If $\tilde{\Gamma}$ is defined by eqs. (2), then, for many interesting systems, $\tilde{\Gamma}$ will not be a surface: the rank of the matrix

$$\begin{pmatrix} C_{1,1} & \dots & C_{1,2N} \\ \vdots & & \vdots \\ C_{\nu',1} & \dots & C_{\nu',2N} \end{pmatrix} \quad (5)$$

will not be constant along $\tilde{\Gamma}$, in some open dense subset of $\tilde{\Gamma}$, it will be ν , say, and in the rest of points, the so-called critical points, it will be lower than ν . In the critical points, $\tilde{\Gamma}$ will have cusps and singularities.

However, if one ‘‘cuts out’’ the critical points, then a smooth surface of dimension $2N - \nu$ results. Such a surface will, of course, not be closed in Γ . We will define the constraint surface $\tilde{\Gamma}$ in this way, and we will assume that a reasonable quantum theory can still be constructed (some discussion of this problem is given in [20]).

Finally, we have to express the conditions (3) or (4) by some geometrical property of $\tilde{\Gamma}$. This can be done as follows (see [14]).

Let $p \in \tilde{\Gamma}$. We define $T_p \tilde{\Gamma}$ as the set of all vectors from $T_p \Gamma$ that are tangential to $\tilde{\Gamma}$. $T_p \tilde{\Gamma}$ is a $(2N - \nu)$ -dimensional subspace of $T_p \Gamma$. Then, the linear space N_p at each $p \in \tilde{\Gamma}$ can be defined as follows:

$$N_p = \{n \in T_p^* \Gamma \mid \langle n, X \rangle = 0, \quad X \in T_p \tilde{\Gamma}\}.$$

N_p is a ν -dimensional subspace of $T^*\Gamma$, and it is spanned by the gradients (5) of the constraint functions (for the system defined by the action (1)). Finally, define the linear space X_p at all points of $\tilde{\Gamma}$ by

$$X_p = \{X \in T_p\Gamma \mid X^A = \Omega^{AB}n_B, n_B \in N_p\}.$$

As Ω is non-degenerate, X_p is a ν -dimensional subspace of $T_p\Gamma$. X_p is called *longitudinal subspace* of $T_p\tilde{\Gamma}$, and its elements are *longitudinal vectors*.

With these definitions, we have the following theorem:

Theorem 1: $\tilde{\Gamma}$ is a constraint surface of a first-class system, iff

$$X_p \subset T_p\tilde{\Gamma}. \quad (6)$$

Then, X_p is an integrable distribution in $\tilde{\Gamma}$.

A proof of the Theorem 1 is given in [14]. The condition (6) can be used as a geometrical definition of the first-class systems. The longitudinal vectors have the following property. Let $\tilde{\Omega}$ be the pull-back of Ω to $\tilde{\Gamma}$. Then,

$$X_p = \{X \in T_p\tilde{\Gamma} \mid \Omega(X, Y) = 0, \forall Y \in T_p\tilde{\Gamma}\}. \quad (7)$$

Thus, X_p is the subspace of degeneracy of $\tilde{\Omega}$. Indeed,

$$\Omega_{AB}X^AY^B = \Omega_{AB}\Omega^{AC}n_C Y^B = n_C Y^C = 0.$$

As the distribution X_p is integrable, there will be an integral manifold to X_p through any point p of $\tilde{\Gamma}$. Let us call the maximal integral manifolds of X_p in $\tilde{\Gamma}$ *orbits*. Orbits are ν -dimensional submanifolds of $\tilde{\Gamma}$ such that each point $p \in \tilde{\Gamma}$ lies at exactly one orbit, γ_p . The quotient, $\tilde{\Gamma}/\gamma$, is the physical phase space; it has dimension $2N - 2\nu$.

2.2 Classical solutions

Here, we describe the relation between classical solutions and orbits.

In a pure gauge theory, the constraints generate, via Poisson brackets, the gauge transformations. Any pair of points of a given orbit can be joined by a transformation which is generated by the constraints. Hence, an orbit is a collection of all gauge equivalent initial data of the gauge theory. For parametrized systems, the situation is different. Indeed, varying the action (1) with respect to q^μ and p_μ , we obtain the following eqs.:

$$\dot{q}^\mu = N_\alpha \frac{\partial C_\alpha}{\partial p_\mu} = N_\alpha \{q^\mu, C_\alpha\}, \quad (8a)$$

$$\dot{p}_\mu = N_\alpha \frac{\partial C_\alpha}{\partial q^\mu} = N_\alpha \{p_\mu, C_\alpha\}. \quad (8b)$$

Hence, the tangential vector $(\dot{q}^\mu, \dot{p}_\mu)$ to any classical solution $(q^\mu(\tau), p_\mu(\tau))$ lies in $X_{(q,p)}$ (as it must lie at $\tilde{\Gamma}$). Thus, classical solutions lie inside orbits.

In general, a classical solution consists of a *base manifold* \mathcal{M} of a fixed dimension, and a system Q of fields on \mathcal{M} . Two classical solutions are considered as equal, if their base manifolds are diffeomorphic and if the corresponding fields can be brought into coincidence by means of this diffeo plus a gauge transformation.

Given a parametrized system, then there is also an information on how a classical solution $(\mathcal{M}_\sigma, Q_\sigma)$ is to be constructed for a curve $\sigma : \mathbb{R} \rightarrow \gamma$ in an orbit γ . This information is different from system to system, but we will assume that it has the following general properties:

Each orbit γ defines a unique maximal solution $(\mathcal{M}_\gamma, Q_\gamma)$ such that

1. there is a curve $\sigma : \mathbb{R} \rightarrow \gamma$ satisfying $(\mathcal{M}_\sigma, Q_\sigma) = (\mathcal{M}_\gamma, Q_\gamma)$;
2. if σ' is an arbitrary curve in γ , then the corresponding classical solution (\mathcal{M}_c, Q_c) is a part of $(\mathcal{M}_\gamma, Q_\gamma) : \mathcal{M}_c \subset \mathcal{M}_\gamma, Q_c = Q_\gamma|_{\mathcal{M}_c}$;
3. the classical solutions $(\mathcal{M}_\gamma, Q_\gamma)$ and $(\mathcal{M}_{\gamma'}, Q_{\gamma'})$ defined by two different orbits γ and γ' are different.

An arbitrary curve σ' in γ need not correspond to any non-trivial piece $(\mathcal{M}_\gamma, Q_\gamma)$: for example, a constant curve $\sigma(\tau) = p \in \gamma, \forall \tau \in \mathbb{R}$ represents just initial data for $(\mathcal{M}_\gamma, Q_\gamma)$. We will say that a curve σ' in γ represents a non-trivial solution, if $\mathcal{M}_{\sigma'} \subset \mathcal{M}_\gamma$ contains an open subset of \mathcal{M}_γ .

In general relativity (which is not an example of our parametrized systems as it is infinite-dimensional, but the situation is analogous), the base manifold of the classical solutions is a fixed four-dimensional manifold \mathcal{M} , and the system of fields is the four-metric $g_{\mu\nu}(x)$ on \mathcal{M} . $g_{\mu\nu}(x)$ is constructed from the curve ${}^3g_{kl}(x, t), \pi^{kl}(x, t)$ in an orbit γ by calculating first the corresponding laps and shift multipliers N and N^k from equations analogous to (8), and then constructing $g_{\mu\nu}(x)$ from ${}^3g_{kl}(x), N(x)$ and $N^k(x)$ by the well-known formula (see, e.g. [21], p. 507). The maximal solution $(\mathcal{M}_\gamma, Q_\gamma)$ that is associated with an orbit γ will be a maximal dynamical evolution of initial data; thus “maximality” is one in the sense of [22] rather than inextendibility (see, e.g. [23]). Then, two maximal solutions corresponding to two different orbits can overlap. Examples are obtained by studying spacetime solutions which are not globally hyperbolic.

In Appendix A, the system of ν free massive particles in Minkowski space is considered. The relation between orbits and classical solutions is easily constructed and the properties 1)- 3) are proved.

3 Perennials and symmetries

3.1 Algebras of elementary perennials

In ref. [7] and [8], the so-called *algebraic quantization* method is described. The emphasis is on the practical methodical aspects; this gives the “quantization” a form of a procedure split in quite a number of intermediate steps. Our aim is different: we are going to study some general properties of the quantum theory constructed in this way and to perform some additional constructions. For example, we are interested in relations between the Ashtekar and Dirac approaches, or the Ashtekar and group approaches (see the next subsection), and in constructing time evolution and Hamiltonians. For this purpose it is advantageous to suppress a number of the procedural steps so that only some relevant properties of the end result of the procedure remain.

The basic notion of our approach to algebraic quantization will be that of *perennial*. What is a perennial Γ

In ref. [24], Dirac introduced the notion of the *first-class quantity* for constraint systems. In the current literature, the name *observable* is generally used for these quantities. However, in the case of parametrized systems, there seem to be observable aspects which are not described by first-class quantities; the first-class quantities form a rather narrower class of integrals of motions. These observations lead in [15] to introduce the name *perennials* for the first-class quantities of parametrized systems. This enables one to distinguish between perennials and observables; we will find such a distinction very useful, and we will adhere to Kuchar’s nomenclature.

For a system (1), a perennial o is a (complex) C^∞ -function on the phase space such that

$$\{o, C_\alpha\} |_{\tilde{\Gamma}} = 0, \quad \forall \alpha.$$

It follows that o is constant along each orbit and, vice versa any C^∞ -function that is constant along orbit is a perennial. We can use this property as a more geometrical definition of perennials.

Let o_1 and o_2 be two perennials. Then, their complex linear combination $\lambda_1 o_1 + \lambda_2 o_2$, their product $o_1 o_2$, their Poisson bracket $\{o_1, o_2\}$, and their complex conjugation \bar{o}_1 and \bar{o}_2 are again perennials. Any set of perennials that is closed with respect to these four operations is called *Poisson algebra (with involution)*.

One can observe that any constraint function C_α satisfies the definition of perennial. However, from the physical point of view, these perennials are trivial being equal to zero on $\tilde{\Gamma}$. Still, the Poisson algebra of all perennials \mathcal{P} will contain all constraint functions, and even the whole ideal \mathcal{J}_c which is generated by them. Thus, the quotient space

$$\tilde{\mathcal{P}} = \mathcal{P}/\mathcal{J}_c$$

will again form a well-defined Poisson algebra (with involution, if $\tilde{\mathcal{J}}_c = \mathcal{J}_c$). Any two perennials in the same class of $\tilde{\mathcal{P}}$ are equal on $\tilde{\Gamma}$, so they are not distinguishable from the physical point of view.

Consider a subset $\tilde{\mathcal{S}}$ of $\tilde{\mathcal{P}}$ with the following properties.

- (i) $\tilde{\mathcal{S}}$ is a Lie algebra with involution, that is, $\tilde{\mathcal{S}}$ is closed with respect to complex linear combination, Poisson brackets, and complex conjugation.
- (ii) $\tilde{\mathcal{S}}$ has so many elements that they almost separate the orbits. That is, if γ_1 and γ_2 are two orbits, such that there is a perennial $o \in \mathcal{P}$ satisfying

$$o(\gamma_1) \neq o(\gamma_2),$$

then there is a class of perennials $\{o'\} \in \tilde{\mathcal{S}}$ such that

$$o'(\gamma_1) \neq o'(\gamma_2).$$

(For many parametrized systems, there will be pairs of different orbits which cannot be separated by any continuous function of orbits; that is, $\tilde{\Gamma}/\gamma$ is non-Hausdorff. For examples see e.g. [25]), $\tilde{\mathcal{S}}$ will be called *algebra of elementary perennials*.

In general, $\tilde{\mathcal{S}}$ is not uniquely determined. The construction of the quantum mechanics will, however, depend on the choice of $\tilde{\mathcal{S}}$. Thus, the choice of $\tilde{\mathcal{S}}$ reflects the non-uniqueness of construction of quantum mechanics from a classical theory. This non-uniqueness has, however, still another aspect. $\tilde{\mathcal{S}}$ can contain Lie subalgebras with involution which still almost separate orbits. These can be called *subalgebras of elementary perennials*.

A given Lie algebra $\tilde{\mathcal{S}}$ can contain elements that are not independent algebraically. For example, there can be f, g, h in $\tilde{\mathcal{S}}$ satisfying a *relation* in $\tilde{\mathcal{P}}$

$$h = fg. \tag{9}$$

For topologically nontrivial $\tilde{\Gamma}/\gamma$, there always will be such relations.

A *quantization* can now be defined as any faithful representation of the Lie algebra $\tilde{\mathcal{S}}$ in a Hilbert space \mathcal{H} satisfying the following requirements. Let us denote by \hat{f} the linear operator on \mathcal{H} that represents the element $f \in \tilde{\mathcal{S}}$. Then

$$(a) \quad (\lambda \hat{f} + \kappa \hat{g}) = \lambda \hat{f} + \kappa \hat{g},$$

$$(b) \quad \{\hat{f}, \hat{g}\} = -i[\hat{f}, \hat{g}],$$

$$(c) \quad \hat{f} = \hat{f}^\dagger,$$

(d) if a constant function “1” with value 1 is in $\tilde{\mathcal{S}}$, then

$$\hat{1} = id,$$

(e) all relations are satisfied, if the products are replaced by symmetrized products; for example, if (9) holds, then

$$\hat{h} = \frac{1}{2}\hat{f}\hat{g} + \frac{1}{2}\hat{g}\hat{f}, \quad (10)$$

(f) all subalgebras of elementary perennials in $\tilde{\mathcal{S}}$ are *irreducibly* represented in \mathcal{H} .

Again, there will be more such representations (unitarily inequivalent). Thus, the choice of the representation is another freedom in the construction of quantum mechanics.

Observe that the requirement of irreducibility concerns all subalgebras of elementary perennials in $\tilde{\mathcal{S}}$. This implies, of course, that $\tilde{\mathcal{S}}$ also is represented irreducibly. This requirement is analogous to the usual requirement for non-constrained systems: any physical representation of some Lie algebra of phase space functions must induce an irreducible representation of the Heisenberg algebra (see, e.g. the discussion of the Van Hove theorem in [26], p.435).

3.2 Canonical groups

A different method of quantization, which is however, quite closely related to the algebraic one, is the *group quantization method*. In refs. [9], [10] and [11], the method is explained for non-constrained systems, or only few first procedural steps are given for the constrained ones. In ref. [12], a sort of short-cut of the group method is given for constrained systems that is somewhat analogous to our previous subsection. In the present subsection we will describe a minor modification of Rovelli’s approach [12] that will be suitable for our purposes.

The basic notion of group quantization is that of a *symmetry* of a parametrized system.

Definition: Let $\varphi : \Gamma \rightarrow \Gamma$ satisfy the conditions:

1. φ is a diffeo with a domain $D(\varphi)$ dense in Γ ,
2. φ is a symplectic map, $\varphi^*\Omega = \Omega$,
3. φ preserves the constraint surface, $\varphi\tilde{\Gamma} \subset \tilde{\Gamma}$;

then φ is called a *symmetry* of the parametrized system.

Hence, the symmetries of a parametrized system just have to preserve the constraint surface – those of a gauge system have to preserve the Hamiltonian in addition.

Observe also that we admit symmetries that can become singular somewhere in Γ (and even on $\tilde{\Gamma}$).

The following theorem will often be used

Theorem 2:

1. Let φ be a symmetry. Then φ maps orbits onto orbits.
2. If the vector field ξ on Γ is an infinitesimal symmetry, then for any $p \in \tilde{\Gamma} \cap \text{Dom}\xi$ it holds:

a) $\xi(p) \in T_p\tilde{\Gamma}$

b) there is a neighbourhood U of p in Γ and a function $F : U \rightarrow \mathbb{C}$ s.t.
 $\xi^A = \Omega^{AB}\partial_B F$ in U ;

F satisfies the equation

$$\{F, C_\alpha\} |_{\tilde{\Gamma}} = 0 \quad \forall \alpha.$$

The proof of Theorem 2 is given in Appendix B. Theorem 2 also shows the way of how algebras of perennials could be related to groups of symmetries: the generators of the group are locally Hamiltonian vector fields, and if they are globally Hamiltonian, they define perennials.

The notion analogous to algebras of elementary perennials is that of *first-class canonical group*:

Definition: Let \mathcal{G} be a group of transformations of Γ satisfying the following requirements:

1. all elements of \mathcal{G} are symmetries,
2. all infinitesimal generators of \mathcal{G} are globally Hamiltonian vector fields, and the algebra of the corresponding perennials is identical with the Lie algebra of the group \mathcal{G}
3. there is $\gamma \subset \tilde{\Gamma}$ such that $\text{clos}(\mathcal{G}\gamma) = \tilde{\Gamma}$ (\mathcal{G} acts almost transitively on $\tilde{\Gamma}/\gamma$), where $\text{clos}(\mathcal{M})$ denotes the topological closure of the set \mathcal{M}
4. $\mathcal{N} = \{\varphi \in \mathcal{G} \mid \varphi\gamma = \gamma, \forall \gamma\}$ is a closed subgroup of \mathcal{G} .

Then, the Lie group $\tilde{\mathcal{G}} = \mathcal{G}/\mathcal{N}$ is called a first-class canonical group of the parametrized system (it is easy to show that \mathcal{N} is a normal subgroup of \mathcal{G}).

By the adjective “first-class”, we distinguish this group from the canonical group of ref. [10]; the canonical group whose elements need not be symmetries and which has to act transitively in Γ can still be useful for constrained systems just for the methodical reasons.

Such a first-class canonical group need not exist for an arbitrary parametrized system: because of condition 3, $\tilde{\Gamma}/\gamma$ must contain a dense subset which is a homogeneous space (of the form $\tilde{\mathcal{G}}/\tilde{\mathcal{G}}_0$, where $\tilde{\mathcal{G}}_0$ is a closed subgroup of $\tilde{\mathcal{G}}$). This problem must be further studied; here, we will suppose that a first-class canonical group exists for our system.

Then, there will be many such groups, in general. The choice of the $\tilde{\mathcal{G}}$ is the freedom that we have in construction of the quantum theory. In particular, a chosen group $\tilde{\mathcal{G}}$ can have subgroups that satisfy all conditions for a first-class canonical group. We call such subgroups *first-class canonical subgroups*.

The group quantization is finished by a choice of a unitary representation of the group $\tilde{\mathcal{G}}$ in a Hilbert space \mathcal{H} satisfying the condition that all first-class canonical subgroups of $\tilde{\mathcal{G}}$ are represented irreducibly on \mathcal{H} . As a rule, there will be a number of such representations, and this gives the second freedom in the quantization.

Let us close this section by a comparison of the group and algebraic quantizations as described above (see also [8]). Clearly, the Lie algebra of $\tilde{\mathcal{G}}$ defines a Lie algebra of classes of equivalent perennials because of condition 2. This Lie algebra satisfies property (i) of an algebra of elementary perennials of subsection 2.1, but not necessarily the orbit separation property (ii). Indeed, the separation property of the algebras and the transitivity (condition 3) of the groups do not have any direct relation in general. For example, if one quantizes systems with a discrete configuration space [27], then there are no algebras (canonical commutation rules), but one can quantize by the group method [28]; the group is not a Lie group, but a discrete one. Another example is given in ref. [29]: the canonical group $\tilde{\mathcal{G}}$ has to contain discrete elements similar to parity (these have to be represented by operators that are simultaneously unitary and self-adjoint, so their square is *id*) in order that it acts almost transitively. The Lie algebra of the component of identity of $\tilde{\mathcal{G}}$, however, defines a sufficient number of perennials so that the separation condition is satisfied.

Another difference is that a given algebra of elementary perennials need not define a group (if the Hamiltonian vector fields are not complete in Γ) and, if it defines one, then the group can contain more information than the algebra. The former problem may be a spurious one: there are some suggestions that all “quantizable” variables should generate a group (see, e.g. [11] and the discussion about the Van Hove theorem in [26]). The latter difference is due to the fact that “different” Lie

groups can have the “same” Lie algebras (example: $SO(2,1)$ and $Sl(2, \mathbb{R})$); then, not all representations of the Lie algebra are obtained from faithful representations of the group. This is, in fact, desirable, because it limits the second freedom in the quantization.

The third difference is due to the ability of the algebras to incorporate relations like (9). These cannot be built into the group structure! Thus, the group can have representations which do not respect eq. (10). Then, algebraic quantization is more advantageous in such a case.

It seems to follow that an obvious combination of both quantizations will be superior to any of them in general. In ref. [29], we show an example of such a quantization.

4 Transversal surface

The quantum mechanics constructed in the previous section have two strange features. 1) There is “no Hamiltonian” (better, the Hamiltonian is equal zero), and all quantum observables are integrals of motion. We are faced by the problem to reconstruct changes and time dependence, which surely are observable features of our world, within these quantum mechanics. This is the problem of “frozen dynamics” (see, e.g. [1], [2]). 2) There is “no gauge” (better, no time function and gauge choice was necessary). This could surely be considered as “an advantage”, but it is still of great interest to understand the relation of our quantum mechanics with those which are constructed via a choice of gauge. In fact, as a choice of gauge for a parametrized system has to do with time foliation ([1], [2]), both problems are related. In our theory, a gauge will be represented by a transversal surface.

Definition: Let Γ_1 be a $(2N - 2\nu)$ -dimensional surface in $\tilde{\Gamma} \subset \Gamma$ satisfying the following requirements.

1. Let $p \in \Gamma_1$ and γ_p be the orbit through p ; then

$$T_p\Gamma_1 \oplus T_p\gamma_p = T_p\tilde{\Gamma}; \quad (11)$$

in particular, no non-zero vector is simultaneously tangential to both Γ_1 and γ_p .

2. Let γ be an arbitrary orbit, then γ intersects Γ_1 in at most one point.

Then, Γ_1 is called *transversal surface*.

Nice tool to work with Γ_1 is provided by two maps, i_1 and π_1 , which are defined as follows. i_1 is the injection of Γ_1 into Γ ; $i_1 : \Gamma_1 \rightarrow \Gamma$. π_1 is the projection from $\tilde{\Gamma}$ to Γ_1 given by

$$\pi_1 p = \Gamma_1 \cap \gamma_p,$$

where γ_p is the orbit through $p \in \tilde{\Gamma}$. As $\Gamma_1 \cap \gamma_p$ may be empty, we have a nontrivial domain $\text{Dom}\pi_1 \subset \tilde{\Gamma}$. This subset of $\tilde{\Gamma}$ will play an important role, so we introduce a special name for it: *domain of Γ_1 in $\tilde{\Gamma}$, $D(\Gamma_1)$* .

We will call Γ_1 a *global transversal surface*, if $D(\Gamma_1) = \tilde{\Gamma}$, and *maximal transversal surface*, if there is no transversal surface Γ_2 such that $D(\Gamma_2)$ contains $D(\Gamma_1)$ as a proper subset.

There are parametrized systems that do not admit any global transversal surface. This has to do with the so-called *global time problem* ([1], [2]), but is not identical to it. The global time problem has been studied for parametrized systems with a single constraint C ([30], [25]); it arises, if there is no “time function” T . T is a function on the phase space Γ that grows along each orbit,

$$\{T, C\} > 0.$$

Hence, if there is such a time function, then $T = \text{const.}$ will be a global transversal surface. However, the opposite is not true: there may be a global transversal surface, but no time function (if the orbits are closed). In the case of more constraints, some of them being pure gauges, the non-existence of global transversal surface can be due just to a Gribov problem for the gauge constraints, so that there is no problem with the time (parametrized SU(2) gauge theory). In the formalism which we are going to develop (following closely Dirac’s ideas [5]), the global time and Gribov problem will turn up to be completely analogous and no advantage seems to result from treating them separately.

The basic property of transversal surface is the following:

Theorem 3: Let Γ_1 be an arbitrary transversal surface, i_1 and π_1 the corresponding injection and projection, $\Omega_1 = i_1^*\Omega$ be the pull back of Ω to Γ_1 and $\tilde{\Omega}$ that of Ω to $\tilde{\Gamma}$. Then,

- a) (Γ_1, Ω_1) is a symplectic manifold,
- b)

$$\tilde{\Omega} = \pi_1^*\Omega_1 \quad \text{on } D(\Gamma_1). \quad (12)$$

The proof of Theorem 3 is given in Appendix B.

The symplectic manifold (Γ_1, Ω_1) can be considered as a reduced phase space, if Γ_1 is a global transversal surface: the gauges are chosen and the constraints are solved. For any transversal surface Γ_1 , the Poisson bracket on Γ_1 that is defined by Ω_1 will be denoted by $\{\cdot, \cdot\}_1$. $\tilde{\Omega}$ is a pre-symplectic form on $\tilde{\Gamma}$; it is degenerate such that

$$X_p = \{X \in T_p\tilde{\Gamma} \mid \tilde{\Omega}(X, Y) = 0, \forall Y \in T_p\tilde{\Gamma}\}.$$

Indeed, the distribution X_p of vectors tangential to orbits is mapped to zero vector by π_{1*} .

One would like, as next step, to quantize the system using a fixed transversal surface. This will be a quantization based on a “choice of gauge”. Then, one will meet the so-called *multiple choice problem*: the quantum theories which will result from different choices Γ_1 and Γ_2 of transversal surface, will not be unitarily equivalent (see [1] and [2]). However, this inequivalence seems to be quite dependent from the method used. In this section, we describe a method which is “gauge independent” (solving, in a sense, the multiple choice problem).

Let us consider first the algebraic method. For it we need a Lie algebra of elementary variables on (Γ_1, Ω_1) . The crucial observation is that a given algebra of elementary perennials as defined in the previous section determines a unique Lie algebra of elementary variables with the same algebraical structure on (Γ_1, Ω_1) . This can be seen as follows.

First, we define a “projection” of perennials to (Γ_1, Ω_1) . Let o be any perennial, and let o_1 be a function on Γ_1 given by

$$o_1 = o |_{\Gamma_1} (= i_1^* o = o \circ i_1).$$

In fact, i_1^* will map any function on Γ to a function on Γ_1 , but, for *perennials*, i_1^* preserves *all* algebraic operations:

Theorem 4: Let \mathcal{P} be the Poisson algebra of perennials on Γ , \mathcal{J}_1 its ideal of perennials vanishing at $D(\Gamma_1)$, and

$$\mathcal{P}_1 = \{o_1 \in C^\infty(\Gamma_1) \mid \exists o \in \mathcal{P} \text{ such that } o_1 = i_1^* o\}.$$

Then,

1. \mathcal{P}_1 is a Poisson algebra on (Γ_1, Ω_1) (closed with respect to linear combination and product of functions and $\{\cdot, \cdot\}_1$);
2. i_1^* is a Poisson algebra homomorphism with kernel \mathcal{J}_1 .

The proof of Theorem 4 is given in Appendix B. In particular, if $\tilde{\mathcal{S}}$ is any algebra of elementary perennials, then $i_1^* \tilde{\mathcal{S}}$ is a well-defined Lie algebra of elementary variables on (Γ_1, Ω_1) ; the relations (like (9)) as well as the separation property will be preserved, only some new subalgebras of elementary variables might emerge if Γ_1 is not global.

The algebraic structure itself is, however, not sensitive to the global properties of Γ_1 : the full algebra of elementary perennials can be projected to an arbitrary small piece of transversal surface.

The above projection procedure can be inverted, at least in some cases. Suppose e.g. that Γ_1 is a global transversal surface and o_1 is an arbitrary C^∞ -function on Γ_1 . Define $\tilde{o} : \tilde{\Gamma} \rightarrow \mathbb{C}$ by

$$\tilde{o} = o_1 \circ \pi_1 = \pi_1^* o_1.$$

\tilde{o} is a function on $\tilde{\Gamma}$ that is constant along orbits. Suppose that \tilde{o} can be extended to a C^∞ -function to a neighbourhood of $\tilde{\Gamma}$ in Γ ; if there is one such extension, of course, then there will be many, and they will form an equivalence class of perennials. In general, there will be a set of functions on Γ_1 for which this construction can be performed, and it seems likely that this set will be quite large (some non-trivial conditions might occur just in a neighbourhood of critical points).

This inversion would also lead to an abstract proof of existence of perennials and it is, therefore, worth of further study.

Projection of symmetries is a more complicated business than that of perennials. Suppose φ is a symmetry and Γ_1 a transversal surface. We define

$$\varphi_1 = \pi_1 \circ \varphi \circ i_1.$$

This is well-defined for any $p \in \Gamma_1$ only if $\varphi(p) \subset D(\Gamma_1)$ and this is equivalent to the condition that φ preserves $D(\Gamma_1)$. Of course, any symmetry can be projected to a global transversal surface.

Suppose that φ and ψ are two symmetries which preserve $D(\Gamma_1)$. Then, their composition $\psi \circ \varphi$ also does, and it is easy to prove that

$$(\psi \circ \varphi)_1 = \psi_1 \circ \varphi_1.$$

Indeed, if ψ is a symmetry, then

$$\pi_1(\psi(i_1(\pi_1(p)))) = \pi_1(\psi(p))$$

as $i_1(\pi_1(p))$ and p lie at the same orbit and ψ maps orbits onto orbits. Hence

$$\begin{aligned} \psi_1 \circ \varphi_1 &= (\pi_1 \circ \psi \circ i_1) \circ (\pi_1 \circ \varphi \circ i_1) = \\ &= (\pi_1 \circ \psi \circ i_1 \circ \pi_1) \circ \varphi \circ i_1 = \pi_1 \circ (\psi \circ \varphi) \circ i_1. \end{aligned}$$

Thus, the composition of maps is preserved by the projection only if the maps are symmetries.

Suppose further that φ is a symmetry preserving $D(\Gamma_1)$. Then, φ_1 is a symplectic map on (Γ_1, Ω_1) . Indeed,

$$\varphi_1^* \Omega_1 = i_1^*(\varphi^*(\pi_1^* \Omega_1)) = i_1^*(\varphi^* \tilde{\Omega}) = i_1^* \tilde{\Omega}_1 = \Omega_1;$$

we have used Theorem 3b.

Let \mathcal{G} be a group of symmetries and \mathcal{N} its normal subgroup that leaves orbits invariant. Let Γ_1 be a transversal surface. Define

$$\mathcal{G}_1 = \{\varphi \in \mathcal{G} \mid \varphi D(\Gamma_1) \subset D(\Gamma_1)\},$$

the subgroup of \mathcal{G} leaving $D(\Gamma_1)$ invariant. Clearly, \mathcal{N} is a normal subgroup of \mathcal{G}_1 . Denote by $a_1(\varphi)$ the projection of $\varphi \in \mathcal{G}$ to Γ_1 . Then, the following theorem is an immediate consequence of the above considerations:

Theorem 5: \mathcal{G}_1 acts via a_1 as a group of symplectic diffeomorphisms on (Γ_1, Ω_1) and

$$\mathcal{N} = \{\varphi \in \mathcal{G}_1 \mid a_1(\varphi) = id\}.$$

Suppose that $\tilde{\mathcal{G}} = \mathcal{G}/\mathcal{N}$ is a first-class canonical group of our system and Γ_1 is a global transversal surface. Then, $\mathcal{G}_1 = \mathcal{G}$, and all elements of any class from \mathcal{G}/\mathcal{N} have the same action on Γ_1 via a_1 . Hence, $a_1\tilde{\mathcal{G}}$ is a well-defined group of symplectic maps on (Γ_1, Ω_1) that is isomorphic to $\tilde{\mathcal{G}}$ (the action a_1 of $\tilde{\mathcal{G}}$ on Γ_1 is effective). Moreover, $a_1\tilde{\mathcal{G}}$ acts almost transitively on Γ_1 . Thus, $a_1\tilde{\mathcal{G}}$ is a canonical group on the phase space (Γ_1, Ω_1) (that is, canonical group in the sense of ref. [9], [10], [11]).

In this way, any given group quantization of our parametrized system as defined in the previous section ‘induces’ a group quantization on the reduced phase space. The result of this quantization is independent of the choice of the transversal surface Γ_1 (it is a fixed representation of $\tilde{\mathcal{G}}$). What can be done in a case when there is no global transversal surface? A particular example is studied in ref. [29].

We have seen that there are certain relations between group and algebraic quantization. This is based on a relation between infinitesimal generators of the group and perennials. This relation survives the projection to a transversal surface, as the following theorem shows.

Theorem 6: Let Γ_1 be a transversal surface and let φ_t be a one-dimensional group of symmetries preserving $D(\Gamma_1)$ so that the projection φ_{1t} of φ_t to Γ_1 is well-defined. Let X be the generator of φ_t on Γ and X_1 that of φ_{1t} on Γ_1 . Let $p \in \Gamma_1$, U be a neighbourhood of p in Γ and $U \cap \Gamma_1 = U_1$ that of p in Γ_1 . Choose a local coordinate $\{z^A\}$ in U and $\{z^a\}$ in U_1 . Let Ω_{AB} be the components of Ω with respect to $\{z^A\}$ and Ω_{1ab} that of Ω_1 with respect to $\{z^a\}$. Let f be a function in U satisfying

$$\partial_A f = \Omega_{AB} X^B.$$

Let, finally, f_1 be the projection of f of U_1 :

$$f_1 = i_1^* f.$$

Then,

$$\partial_a f_1 = \Omega_{1ab} X_1^b.$$

The proof of Theorem 6 is given in Appendix B. In particular, if X is globally Hamiltonian, then $U = \Gamma$ and f is a perennial whose projection f_1 to Γ_1 generates φ_{1t} .

The last theorem of this section concerns the relation between structures at different transversal surfaces.

Theorem 7: Let Γ_1 and Γ_2 be two transversal surfaces such that $D(\Gamma_1) \cap D(\Gamma_2) \neq \emptyset$. Let Ω_1 and Ω_2 be the symplectic forms induced by Ω on Γ_1 and Γ_2 , let o_1 and o_2 be the projections of a perennial o to Γ_1 and Γ_2 , and let φ_1 and φ_2 be those of a symmetry φ that satisfies $\varphi D(\Gamma_1) \subset D(\Gamma_1), \varphi D(\Gamma_2) \subset D(\Gamma_2)$. Let, finally, $\sigma : \Gamma_1 \rightarrow \Gamma_2$ be defined by

$$\begin{aligned}\sigma(p) &= \pi_2(p) \quad \forall p \in \Gamma_1, \\ \text{Dom}\sigma &= \pi_1(D(\Gamma_1) \cap D(\Gamma_2)).\end{aligned}$$

Then:

- a) $\sigma^* \Omega_2 = \Omega_1$,
- b) $\sigma^* o_2 = o_1$,
- c) $\varphi_2 = \sigma \circ \varphi \circ \sigma^{-1}$.

The proof of Theorem 7 is given in Appendix B.

5 Time evolutions and Hamiltonians

In ref. [5], the notion of a Hamiltonian of a system of interacting relativistic point particles is generalized: A Hamiltonian is any perennial that moves (via Poisson bracket) a given transversal surface around in the phase space. Accordingly, a) a system will have many Hamiltonians and b) each Hamiltonian is associated with a transversal surface (the same perennial can be a Hamiltonian for a transversal surface Γ_1 , but no Hamiltonian for another surface Γ_2). In this way, conditions on Hamiltonians are weakened on one hand; although this weakening was developed by Dirac for other purposes, it seems to be very useful for the theory of general parametrized systems with their problem of time. On the other hand, the conditions are stronger in the sense, that such a Hamiltonian always generates a *symmetry* group of the system.

One of the building blocks used by Dirac to construct a time evolution is a particular transversal surface. (His “three forms of relativistic dynamics” are associated with the three most symmetrical “transversal” surfaces in Minkowski spacetime: the spacelike plane and hyperboloid and the null plane.) It seems that a time evolution of an isolated parametrized system cannot be constructed completely just by means of the system’s own perennials. This can quite convincingly be demonstrated by studying the dynamics of a particle in Minkowski spacetime (see, e.g. ref. [13]). That is one of the reasons for distinguishing between “perennials” and “observables”. However, it is hoped that the “non-perennial” observable aspects can be

obtained by means of perennials of some suitably extended system; for example, one could try to couple gravity to some matter clock, etc., see, e.g. ref. [31].

In this section we generalize and further develop Dirac's ideas on Hamiltonians and time evolution. The crucial notions are those of "time level" and of "the same measurement at different times". Indeed, to observe a "change" in a system between the "time levels" t_1 and t_2 , one has to perform two "equal measurements", one at t_1 and one at t_2 , and compare their results. This idea can be made precise as follows.

Choose a transversal surface Γ_0 with domain $D(\Gamma_0)$ in $\tilde{\Gamma}$ and a one-dimensional group of symmetries $h(t)$ preserving $D(\Gamma_0)$. Let the generator of $h(t)$ be the perennial h . Define

$$\Gamma_t = h(t)\Gamma_0.$$

Γ_t is a transversal surface with the same domain as Γ_0 , because $h(t)$ is a symmetry presenting $D(\Gamma_0)$. The one-dimensional family $\{\Gamma_t\}$ sweeps a $(2N - 2\nu + 1)$ -dimensional surface $\tilde{\Gamma}_0$ within the $(2N - \nu)$ -dimensional constraint surface $\tilde{\Gamma}$. We will call the surfaces Γ_t in $\tilde{\Gamma}$ *time levels*. Observe that the time levels cover only a small portion of the constraint hypersurface $\tilde{\Gamma}$ if there is more than one constraint ($\nu > 1$).

Suppose next that we perform a measurement at the time level Γ_0 . Let this be a measurement of the value $o(p)$ of a perennial o at some state $p \in \Gamma_0$ of the system at $t = 0$. What is the corresponding, or "the same", measurement at the time level Γ_t ? We define: it is the measurement of the value of the *time-shifted* perennial o_t , defined by

$$o_t = h^*(-t)o = o \circ h(-t),$$

at Γ_t . At least formally, this definition is reasonable because of the following theorem.

Theorem 8: Let o and o' be two perennials and φ a symmetry. Then, φ^*o and φ^*o' are again perennials, and the following relations hold

$$\alpha\varphi^*o + \beta\varphi^*o' = \varphi^*(\alpha o + \beta o'),$$

$$\varphi^*o \cdot \varphi^*o' = \varphi^*(o \cdot o'),$$

$$\{\varphi^*o, \varphi^*o'\} = \varphi^*\{o, o'\}.$$

The proof is very simple and we leave it to the reader. Thus, a symmetry preserves the perennial property and the Poisson algebra of perennials. Theorem 8 enables us to define a complete system of related ("the same") measurements at any two different time levels. This provides a formal reason why the time evolution should be defined by a symmetry group. The one-dimensional family $\{o_t\}$ of time-shifted perennials can be also considered as a kind of "evolving constant of motion" (cf. [18]).

The notion of time-shifted perennial enables us to study the motion of the system in a quantitative way. First, we introduce an *auxiliary rest frame*. This frame is formed by the orbits of the group $h(t)$ in $\tilde{\Gamma}_0$. That is, the points $p \in \Gamma_0$ and $h(t)p \in \Gamma_t$ are considered as “the same”. Indeed, any two “equal” measurements, of o and of $h^*(-t)o$, will give the same results at p and $h(t)p$:

$$o(p) = o \circ h(t)(h(-t)p). \quad (13)$$

To study the classical solutions – the possible motions of the system – relatively to the auxiliary rest frame, we have to represent the constraint orbits by curves in $\tilde{\Gamma}_0$. This is straightforward: Let γ be an orbit in $D(\Gamma_0)$ and set $\eta_\gamma(t)$ be the intersection point of Γ_t and γ :

$$\eta_\gamma(t) = \Gamma_t \cap \gamma;$$

$\eta_\gamma(t), t \in \mathbb{R}$ is desired representation as the point $\eta_\gamma(t) \in \gamma$ is well-defined for each $t \in \mathbb{R}$. Indeed, Γ_t is transversal, so Γ_t intersects γ at one point at most. γ intersects Γ_0 as $\gamma \in D(\Gamma_0)$. But $h(t)$ is a symmetry preserving $D(\Gamma_0)$, so $D(h(t)\Gamma_0) = D(\Gamma_0)$, and γ must also intersect Γ_t . From the definition of $\eta_\gamma(t)$, it follows that

$$\eta_\gamma(\mathbb{R}) = \tilde{\Gamma}_0 \cap \gamma.$$

Suppose, we measure the perennial o at the time level Γ_0 and the time-shifted perennial o_t at Γ_t if the system “moves” along $\eta_\gamma(t)$. The results are $o(\eta_\gamma(0))$ and $o_t(\eta_\gamma(t))$. The number $o_t(\eta_\gamma(t))$ can be expressed in two different ways. First, from the definition of o_t , it follows that

$$o_t(\eta_\gamma(t)) = o(\xi_\gamma(t)) = \tilde{o}(\xi_\gamma(t)),$$

where \tilde{o} is the projection of the perennial o to Γ_0 and

$$\xi_\gamma(t) = h(-t)\eta_\gamma(t). \quad (14)$$

$\xi_\gamma(t)$ is the projection of the curve $\eta_\gamma(t)$ down to Γ_0 by means of the auxiliary rest frame. This means that $\xi_\gamma(t)$ describes the motion $\eta_\gamma(t)$ relatively to the rest frame. Second, it follows from the constancy of o_t along orbits that

$$o_t(\eta_\gamma(t)) = o_t(\eta_\gamma(0))$$

as both points $\eta_\gamma(t)$ and $\eta_\gamma(0)$ lie at the same orbit γ . In this way, we have obtained the “(classical) Schrödinger” and the “(classical) Heisenberg picture of the dynamics”. Within the former, there is a time-independent observable $\tilde{o} = o|_{\Gamma_0}$ of the reduced system, and a time-dependent state $\xi_\gamma(t) \in \Gamma_0$ of the system. Within the

latter, there is a time-dependent observable $\tilde{o}_t = o_t |_{\Gamma_0}$ of the reduced system, and a time-independent state $\xi_\gamma(0) \in \Gamma_0$ of the system. The result of the measurement at time t is given by

$$\tilde{o}_0(\xi_\gamma(t)) = \tilde{o}_t(\xi_\gamma(0)).$$

The time-dependence of the measurement results can be calculated using the following theorem.

Theorem 9: Let (Γ_0, Ω_0) be the symplectic space with $\Omega_0 = i_0^* \Omega$, and let $H : \Gamma_0 \rightarrow \mathbb{R}$ be defined by

$$H = -h |_{\Gamma_0} .$$

Then:

A) The curve $\xi_\gamma(t)$ defined by (14) for each $\gamma \in D(\Gamma_0)$ is an integral curve of the Hamiltonian vector field of the function H in (Γ_0, Ω_0) .

B) The one-dimensional family $\{\tilde{o}_t\}$ of observables satisfies the equation

$$\frac{d\tilde{o}_t}{dt} = \{\tilde{o}_t, H\}_0. \quad (15)$$

The proof of the Theorem 9 is given in the Appendix B.

Thus, there are some reasons to call the function H a *Hamiltonian*. However, there is still a problem: the curve $\eta_\gamma(t)$ at γ need not represent any non-trivial motion. Indeed, $\eta_\gamma(t)$ represents some part of the maximal classical solution $(\mathcal{M}_\gamma, Q_\gamma)$ associated with γ . Let us denote this part by $(\mathcal{M}(\Gamma_0, h, \gamma), Q(\Gamma_0, h, \gamma))$. If $\mathcal{M}(\Gamma_0, h, \gamma)$ does not contain any open subset of \mathcal{M}_γ , then $\eta_\gamma(t)$ represents just a one-dimensional set of gauge-equivalent initial data. Then, there is no reason to call the corresponding function H a Hamiltonian. If $\eta_\gamma(t)$ is a non-trivial part of a classical solution, then we can distinguish the following two cases:

1. Let Γ_0 be a global transversal surface and let for each orbit $\gamma \in \tilde{\Gamma}$, $\mathcal{M}(\Gamma_0, h, \gamma) = \mathcal{M}_\gamma$. Then, we call H a *Hamiltonian associated with Γ_0* .

2. There is an orbit $\gamma \in D(\Gamma_0)$ such that $\mathcal{M}(\Gamma_0, h, \gamma)$ contains an open proper subset \mathcal{M}_γ . Then, H is called a *partial Hamiltonian associated with Γ_0* .

In general, there will be only partial Hamiltonians. In each of the two cases, however, $\eta_\gamma(t)$ represents a non-trivial piece of a maximal classical solution at least within an open subset of orbits in $\tilde{\Gamma}/\gamma$. That is, our construction determines a foliation and a gauge choice in some open subset of maximal classical solutions; these foliations will, however, cover only a part of each solution in general. One can hope to improve this unsatisfactory situation by a choice of a whole family of partial Hamiltonians such that, starting from one transversal surface Γ_0 , all points of all maximal classical solutions of $D(\Gamma_0)$ will be covered. Such a family of partial Hamiltonians associated with Γ_0 will be called *complete*.

Consider Γ_0 which is a global transversal surface and complete family of partial Hamiltonians associated with Γ_0 . Then, the family of the corresponding time levels

can be parametrized by $(t_1, \dots, t_k) \in \mathbb{R}^k$, where k is the number of partial Hamiltonians in the family. This corresponds to a k -parameter family of time slices inside of each solution. Thus, if we apply these ideas to general relativity, we obtain a k -dimensional subset of the full system of “many-finger time” slices in solutions. In this way, our method leads to a sort of compromise between a fixed choice of time foliation for each spacetime (i.e., the constant mean external curvature hypersurfaces, cf. ref. [32]) and a complete “many-finger time” formalism that would enable one to calculate the evolution between *any* two spacelike hypersurfaces in any spacetime (cf. ref. [1]).

As for the quantum mechanics, the construction of the corresponding Schrödinger and Heisenberg pictures of the dynamics is straightforward, because all perennials and symmetry groups needed for that aim possess already their operator representations.

Acknowledgement Useful discussions with R. Beig, B. Brügmann, D. Giulini, A. Higuchi, J. Tolar and L. Ziewer are thankfully acknowledged. The work was supported in part by Schweizerischer Nationalfonds. The author thanks for hospitality at the Schrödinger Institute where a part of the paper has been written, and for the support by the Grant.

Appendix A: System of ν free relativistic particles

The system of ν free relativistic massive particles is the simplest example of a parametrized system with several constraints. The configuration space C is \mathbb{R}^N , $N = 4\nu$ with coordinates x_α^a , $\alpha = 1, \dots, \nu$, $a = 0, 1, 2, 3$, which are defined by an inertial system in Minkowski spacetime. The phase space $\Gamma \cong \mathbb{R}^{2N}$, and the canonical coordinates can be chosen as x_α^a and $p_{\alpha a}$. The action in the form (1) can be written as

$$S = \int d\tau (p_{\alpha a} \dot{x}_\alpha^a - N_\alpha C_\alpha),$$

where the constraints C_α are defined by

$$C_\alpha = \frac{1}{2}(\eta^{ab} p_{\alpha a} p_{\alpha b} + m_\alpha^2),$$

where η^{ab} is the Minkowski metric and m_α is the mass of α -th particle. The constraint surface $\tilde{\Gamma}$ consists of 2^ν components, $\tilde{\Gamma}_+$ is that one at which all $p_{\alpha 0}$ are negative.

The orbits are maximal integral manifolds of the vector fields

$$\dot{x}_\alpha^a = \eta^{ab} p_{\alpha b}, \quad \dot{p}_\alpha^a = 0.$$

Thus the vector fields are constant along each orbit and so their integral can be written immediately

$$x_\alpha^a = X_\alpha^a - \zeta_\alpha \eta^{ab} P_{\alpha b}, \quad (A1a)$$

$$p_{\alpha b} = P_{\alpha b}, \quad (A1b)$$

where ζ_α are arbitrary real parameters and $X_\alpha^a, P_{\alpha a}$ is a set of 8ν constants satisfying the constraints

$$\eta^{ab} P_{\alpha a} P_{\alpha b} + m_\alpha^2 = 0, \quad \alpha = 1, \dots, \nu.$$

Thus, they are ν -dimensional planes, and ζ_α can be considered as coordinates on them.

A solution curve is determined by eqs. (8)

$$\dot{x}_\alpha^a = N_\alpha \eta^{ab} p_{\alpha b},$$

$$\dot{p}_{\alpha a} = 0.$$

If the Lagrange multipliers are given functions of the parameter τ , then we have

$$\zeta_\alpha(\tau) = - \int_0^\tau d\tau' N_\alpha(\tau').$$

Classical solutions can be considered as maps σ of ν real lines \mathbb{R} into Minkowski spacetime. Thus, the bare manifold \mathcal{M} is $\dot{\cup}_{\alpha=1}^\nu \mathbb{R}_\alpha$, where $\mathbb{R}_\alpha = \mathbb{R}, \forall \alpha$, and the fields are given by points of the Minkowski space V , as described by coordinates x^a . Hence, σ can be described by 4ν functions σ_α^a as follows

$$x^a(\tau_\alpha) = \sigma_\alpha^a(\tau_\alpha), \tau_\alpha \in \mathbb{R}_\alpha; \quad (A2)$$

x^α are scalar fields on \mathcal{M} .

Let $\sigma : \mathcal{M} \rightarrow V$ and $\sigma' : \mathcal{M} \rightarrow V$ be two solutions. We say that σ' is part of σ , if there is a map

$$\varphi : \mathcal{M} \rightarrow \mathcal{M}$$

such that $\varphi(\text{Dom}\sigma') \subset \text{Dom}\sigma$ and

$$\sigma' = \sigma \circ \varphi.$$

φ is called a *reparametrization*.

Any curve at the orbit (A1) can be given by

$$\zeta_\alpha = f_\alpha(\tau), \quad (A3)$$

where $f_\alpha(\tau)$ are piecewise smooth real functions. Then, the corresponding solution σ is given by

$$\sigma_\alpha^a(\tau_\alpha) = X_\alpha^a - f_\alpha(\tau_\alpha) \eta^{ab} P_{\alpha b}. \quad (A4)$$

This is the relation between a curve at an orbit and the corresponding classical solutions.

The maximal classical solution $(\mathcal{M}_\gamma, Q_\gamma)$ for the orbit γ given by eq. (A1) can be represented by the curve with

$$f_\alpha(\tau) = \tau,$$

so that the corresponding map σ_γ is

$$\sigma_{\gamma\alpha}^a(\tau_\alpha) = X_\alpha^a - \tau_\alpha \eta^{ab} P_{\alpha b}.$$

Indeed, consider any other curve at γ ; it is represented by eq. (A3) and the corresponding solution is given by (A4). Consider the map $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ defined by

$$\varphi(\tau_\alpha) = f_\alpha(\tau_\alpha).$$

As f_α are real functions, we have $f_\alpha(\tau_\alpha) \in \mathbb{R}_\alpha \quad \forall \alpha, \tau_\alpha \in \mathbb{R}_\alpha$, and

$$\sigma = \sigma_\gamma \circ \varphi.$$

Thus, σ is a part of σ_γ .

An example of transversal surface Γ_1 is given by

$$x_\alpha^0 = 0, \tag{A5a}$$

$$C_\alpha = 0 \quad \forall \alpha. \tag{A5b}$$

Γ_1 is a 6ν -dimensional surface in $\tilde{\Gamma}$. We must show that the two conditions on transversal surface are fulfilled.

1. Any tangential vector to the orbit γ given by eqs. (A1) has the form

$$\dot{x}_\alpha^a = N_\alpha \eta^{ab} P_{\alpha b},$$

$$\dot{p}_{\alpha a} = 0,$$

where

$$P_{\alpha 0} = \pm \sqrt{P_{\alpha 1}^2 + P_{\alpha 2}^2 + P_{\alpha 3}^2 + m_\alpha^2} \neq 0.$$

It will be tangential to Γ_1 , if $\dot{x}_\alpha^0 = 0$ for all α , but this implies $N_\alpha = 0, \quad \forall \alpha$, or $\dot{x}_\alpha^a = 0, \dot{p}_{2a} = 0$.

2. The point of intersection of the orbit (A1) with the surface (A5) is given by the solution of the equations

$$X_\alpha^0 - \zeta_\alpha \eta^{0b} P_{\alpha b} = 0.$$

This equation has always a unique solution

$$\zeta_\alpha = -\frac{X_\alpha^0}{P_{\alpha 0}}. \tag{A6}$$

Thus, Γ_1 is even globally transversal.

Let the coordinates on Γ_1 be $(y_\alpha^k, q_{\alpha k})$, $\alpha = 1, \dots, 4, k = 1, 2, 3$, such that the imbedding map i_1 is given by the equations

$$\begin{aligned} x_\alpha^0 &= 0, & x_\alpha^k &= y_\alpha^k, \\ p_{\alpha 0} &= \pm \sqrt{q_{\alpha 1}^2 + q_{\alpha 2}^2 + q_{\alpha 3}^2 + m_\alpha^2}, & p_{\alpha b} &= q_{\alpha b}. \end{aligned}$$

The pull back Ω_1 of Ω is given in these coordinates by

$$\Omega_1 = \sum_{\alpha} dq_{\alpha k} \wedge dy_\alpha^k.$$

The map π_1 is defined by eq. (A6): all points of the orbit (A1) are mapped by π_1 to the point

$$\begin{aligned} x_\alpha^0 &= 0, & x_\alpha^k &= X_\alpha^k + X_\alpha^0 \frac{P_{\alpha k}}{P_{\alpha 0}}, \\ p_{\alpha a} &= P_{\alpha a}. \end{aligned}$$

Finally, we give several examples of Hamiltonians and partial Hamiltonians.

Example 1:

$$h = \sum_{\alpha} p_{\alpha 0}.$$

h is clearly a perennial. The group $h(t)$ generated by h contains the transformations:

$$\begin{aligned} x_\alpha^0 &\rightarrow x_\alpha^0 + t, \\ x_\alpha^k &\rightarrow x_\alpha^k, \quad p_{\alpha a} \rightarrow p_{\alpha a}. \end{aligned}$$

Thus, Γ_t is given by

$$x_\alpha^0 = t, \quad C_\alpha = 0 \quad \forall \alpha$$

(we rename Γ_1 to Γ_0). The point $\eta_\gamma(t)$ of intersection of Γ_t with the orbit γ given by eq. (A1) is determined by the equation

$$X_\alpha^0 + \zeta_\alpha P_{\alpha 0} = t.$$

Hence, the curve $\eta_\gamma(t)$ is defined by

$$\zeta_\alpha(t) = \frac{t - X_\alpha^0}{P_{\alpha 0}}.$$

This curve represents the maximal solution σ_γ , as the reparametrization

$$t'_\alpha = \frac{t_\alpha - X_\alpha^0}{P_{\alpha 0}}$$

is invertible.

Hence, $H = -h|_{\Gamma_0} = \pm \sum_{\alpha} \sqrt{q_{\alpha 1}^2 + q_{\alpha 2}^2 + q_{\alpha 3}^2 + m_{\alpha}^2}$ is a Hamiltonian. The projection $\xi_{\gamma}(t)$ of $\eta_{\gamma}(t)$ to Γ_0 is

$$y_{\alpha}^k = X_{\alpha}^k - \frac{P_{\alpha k}}{P_{\alpha 0}}(t - X_{\alpha}^0),$$

$$q_{\alpha k} = P_{\alpha k};$$

this is just the motion of ν particles in the reduced phase space.

Example 2:

$$h = \sum_{\alpha} p_{\alpha 3}.$$

$h(t)$ is the translation in 3rd direction:

$$x_{\alpha}^a \rightarrow x_{\alpha}^a + \delta_3^a t, \quad p_{\alpha a} \rightarrow p_{\alpha a}.$$

The time surfaces Γ_t :

$$\Gamma_t = h(t)\Gamma_0 = \Gamma_0.$$

The point $\eta_{\gamma}(t)$ of intersection between the orbit γ given by eq. (A1) and Γ_t has the coordinates

$$\zeta_{\alpha}(t) = -\frac{X_{\alpha}^0}{P_{\alpha 0}}.$$

Thus, the curve $\eta_{\gamma}(t)$ consists of just one point. The coordinates of this point in Γ are

$$x_{\alpha}^a = X_{\alpha}^a + X_{\alpha}^0 \eta^{ab} \frac{P_{\alpha b}}{P_{\alpha 0}},$$

$$p_{\alpha a} = P_{\alpha a}.$$

This is just one initial data for the maximal solution σ_{γ} at the point

$$t_{\alpha} = -\frac{X_{\alpha}^0}{P_{\alpha 0}}$$

of \mathcal{M}_{γ} . Thus, $\mathcal{M}(\Gamma_0, h, \gamma)$ is just one point in \mathcal{M}_{γ} and does not contain an open subset of \mathcal{M}_{γ} : h is not suitable to define a Hamiltonian for Γ_0 .

Example 3:

$$h = p_{10}.$$

Γ_t is given by the equations

$$x_{\alpha}^a = \delta_{\alpha}^1 \delta_0^a t, \quad C_{\alpha} = 0, \quad \forall \alpha.$$

The intersection point $\eta_{\gamma}(t)$ has the coordinates

$$\zeta_1(t) = \frac{t - X^{01}}{P_{10}},$$

$$\zeta_\alpha(t) = -\frac{X_\alpha^0}{P_{\alpha 0}} \quad \alpha \neq 1.$$

The corresponding solution can be mapped in the maximal solution by the reparametrization φ that is given by

$$t'_1 = \frac{t - X_1^0}{P_{10}}, \quad t'_\alpha = \frac{X_\alpha}{P_{\alpha 0}}, \quad \alpha \neq 1.$$

The range of φ is an open proper subset of \mathcal{M}_γ . It follows that

$$H_1 = -h|_{\Gamma_0} = \pm \sqrt{q_{11}^2 + q_{12}^2 + q_{13}^2 + m_1^2}$$

is a partial Hamiltonian.

Clearly, the ν partial Hamiltonians

$$H_\alpha = \pm \sqrt{q_{\alpha 1}^2 + q_{\alpha 2}^2 q_{\alpha 3}^2 + m_\alpha^2}$$

are sufficient to move from any initial data to any other initial data of any maximal solution.

Appendix B: Proofs

B1. Proof of the Theorem 2

1) Let $p \in \tilde{\Gamma} \cap \text{Dom}\varphi$ and $q = \varphi p$. Then there is a neighbourhood U of p in $\tilde{\Gamma}$ such that φU is a neighbourhood of q in $\tilde{\Gamma}$, because $\varphi \tilde{\Gamma} \subset \tilde{\Gamma}$. It follows that

$$\varphi_* T_p \tilde{\Gamma} = T_q \tilde{\Gamma}$$

and

$$(\varphi^{-1})^* N_p = N_q.$$

As $X_p = \Omega^{-1} N_p$, $X_q = \Omega^{-1} N_q$ and φ preserves Ω , we obtain also that

$$\varphi_* X_p = X_q.$$

Thus, the distribution X_p is preserved by φ . This implies that the orbit through p is mapped by φ into the orbit through q . However, φ has an inverse; for an analogous reason, this inverse must map the orbit through q into the orbit through p . Hence, the map is onto.

2) The claim a) follows directly from the definition (φ preserves $\tilde{\Gamma}$).

b) ξ^A is an infinitesimal symplectic map,

$$\mathcal{L}_\xi \Omega = 0,$$

and the existence of U and f is a well-known property of such maps (see, e.g. [2]).

Finally,

$$\{f, C_\alpha\}_{\tilde{\Gamma}} = (\Omega^{AB} \partial_A f \partial_B C_\alpha)_{\tilde{\Gamma}} = -(\xi^B \partial_B C_\alpha)_{\tilde{\Gamma}} = 0$$

because of 2a), QED.

B2. Proof of Theorem 3

a) The form Ω is closed, so $i_1^*\Omega$ is also closed; we have, therefore, to show that $i_1^*\Omega$ is non-degenerate. The pulback $\tilde{\Omega}$ of Ω to $\tilde{\Gamma}$ is degenerate, it is well-known that

$$X_p = \{X \in T_p\tilde{\Gamma} \mid \Omega(X, Y) = 0, \forall Y \in T_p\tilde{\Gamma}\} \quad (B1)$$

for any $p \in \tilde{\Gamma}$ (see e.g. [2]). Let $p \in \Gamma_1 \subset \tilde{\Gamma}$, and suppose that $X \in T_p\Gamma_1$ such that

$$\Omega_1(X, Y) = 0, \quad \forall Y \in T_p\Gamma_1. \quad (B2)$$

We will show that $X = 0$. Indeed, let $W \in T_p\tilde{\Gamma}$. Using eq. (11), we find a unique vector $U \in T_p\Gamma_1$ and $V \in X_p$ such that

$$W = U + V;$$

then

$$\Omega(X, W) = \Omega(X, U) + \Omega(X, V).$$

However, the first term of the R.H.S. vanishes because of assumption (B2), the second because of (B1), so $\Omega(X, W) = 0$ for any $W \in T_p\tilde{\Gamma}$. From (B.1), it follows then that $X \in X_p$, so that $X \in X_p \cap T_p\Gamma_1$. The transversality (11) implies, however, that $X = 0$.

b) We show eq. (12) at a point $p \in \Gamma_1 \subset \tilde{\Gamma}$ first. Let $X, Y \in T_p\tilde{\Gamma}$. Using eq. (11), we can write

$$X = X_- + X_{\parallel}, \quad Y = Y_- + Y_{\parallel},$$

where

$$X_- \in T_p\Gamma_1, Y_- \in T_p\Gamma_1, X_{\parallel} \in X_p, Y_{\parallel} \in X_p$$

and

$$\pi_{1*}X = X_- \quad , \quad \pi_{1*}Y = Y_-.$$

Then, because of eq. (B1),

$$\tilde{\Omega}(X, Y) = \tilde{\Omega}(X_-, Y_-) = \tilde{\Omega}(\pi_{1*}X, \pi_{1*}Y) = \Omega_1(\pi_{1*}X, \pi_{1*}Y) = (\pi_1^*\Omega_1)(X, Y).$$

Thus, at $p \in \Gamma_1$, $\tilde{\Omega} = \pi_1^*\Omega_1$.

Let now $q \in D(\Gamma_1)$, $q \in \Gamma_1$. Then, there is $p \in \Gamma_1$ such that $\pi_1 q = p$, that is, q and p lie at the same orbits γ_q .

As γ_q is an integral manifold of the Hamiltonian vector fields of the constraints, there will be a symplectic diffeomorphism φ generated by these vector fields that preserves γ_q , and maps p on q . Hence, φ must satisfy the identity

$$\pi_1 = \pi_1 \circ \varphi,$$

or

$$\pi_{1*} = \pi_{1*} \circ \varphi_*.$$

Let $X, Y \in T_q \tilde{\Gamma}$; define

$$U = \varphi_*^{-1} X, V = \varphi_*^{-1} Y.$$

Then

$$\tilde{\Omega}(X, Y) = \tilde{\Omega}(\varphi_* U, \varphi_* V) = \varphi^* \tilde{\Omega}(U, V) = \tilde{\Omega}(U, V).$$

However, U and V are in $T_p \tilde{\Gamma}$, hence

$$\begin{aligned} \tilde{\Omega}(U, V) &= \Omega_1(\pi_{1*} U, \pi_{1*} V) = \Omega_1(\pi_{1*}(\varphi_* U), \pi_{1*}(\varphi_* V)) = \\ &= \Omega_1(\pi_{1*} X, \pi_{1*} Y) = (\pi_{1*}^* \Omega_1)(X, Y), \text{ QED.} \end{aligned}$$

B3. Proof of Theorem 4

First, we show the following lemma:

Lemma: Let o be a perennial and X its Hamiltonian vector field on (Γ, Ω) . Let o_1 be the projection of o to Γ_1 and X_1 the Hamiltonian vector field of o_1 on $(\Gamma_1, \Omega)_1$. Let $p \in \Gamma_1$; then,

$$X(p) = X_-(p) + X_{\parallel}(p), \tag{B3}$$

where $X_-(p) \in T_p \Gamma_1$, $X_{\parallel}(p) = X_p$ is the unique decomposition of $X(p)$. Then,

$$X_-(p) = X_1(p). \tag{B4}$$

Proof of the Lemma. Let U_p be a neighbourhood of p in Γ , x^A some coordinates in U_p and y^k some coordinates in $U_p \cap \Gamma_1$. Then, the imbedding i_1 is represented by the functions

$$x^A = x^A(y^k). \tag{B5}$$

The Hamiltonian vector fields X of o is then given by

$$X^A = \Omega^{AB} \partial_B o.$$

As o is a perennial, X must be tangential to $\tilde{\Gamma}$ at $\tilde{\Gamma}$ and the decomposition (B3) is well-defined (Γ_1 is transversal!). Then, there are coefficients Y^k s.t.

$$X^A_-(p) = \frac{\partial X^A}{\partial y^k} Y^k.$$

By means of the functions (B.5), the relation between o_1 and o is

$$o_1(y^k) = o(X^A(y^k)).$$

Thus

$$\partial_k o_1 = \frac{\partial x^A}{\partial y^k} \partial_A o.$$

Moreover,

$$\Omega_{1kl} = \Omega_{AB} \frac{\partial x^A}{\partial y^k} \frac{\partial x^B}{\partial y^l}.$$

Hence, at p ,

$$\begin{aligned} \partial_k o_1 &= \frac{\partial x^A}{\partial y^k} \partial_A o = \frac{\partial x^A}{\partial y^k} \Omega_{AB} X^B = \\ &= \frac{\partial x^A}{\partial y^k} \Omega_{AB} X_{\parallel}^B + \frac{\partial x^A}{\partial y^k} \Omega_{AB} X_{-}^B = \\ &= \frac{\partial x^A}{\partial y^k} \Omega_{AB} \frac{\partial x^B}{\partial y^l} Y^l = \Omega_{1kl} Y^l \end{aligned}$$

QED. Returning to the proof of Theorem 4, let us choose two arbitrary perennials o and o' . Then, clearly

$$i_1^*(o + o') = i_1^*o + i_1^*o',$$

and

$$i_1^*(o \cdot o') = (o \cdot o')|_{\Gamma_1} = o|_{\Gamma_1} \cdot o'|_{\Gamma_2} = (i_1^*o) \cdot (i_1^*o').$$

Finally,

$$\{o_1, o'_1\}_1|_p = \Omega_1(X_1, X'_1)|_p = \Omega(X_1, X'_1)|_p.$$

Here, we have used that $\Omega_1 = i_1^*\Omega$ and that X_1 and X'_1 are tangential to Γ_1 . Then, using the Lemma and the relation (B1) we obtain:

$$\begin{aligned} \Omega(X_1, X'_1)|_p &= \Omega(X_{-}, X'_{-})|_p = \Omega(X_{-} + X_{\parallel}, X'_{-} + X'_{\parallel})|_p = \\ &= \Omega(X, X')|_p = \{o, o'\}|_p. \end{aligned}$$

Hence: i_1^* preserves all operations of the Poisson algebra. The claim 1) of Theorem 4 follows immediately, as well as the claim that i_1^* is a Poisson algebra homomorphism.

Finally, any perennial o such that i_1^*o is zero everywhere on $D\Gamma_1$) means that o itself vanishes on $D(\Gamma_1)$, QED.

B4. Proof of Theorem 6

The group φ_{1t} is related to φ_t by

$$\varphi_{1t} = \pi_1 \circ \varphi_t \circ i_1.$$

From this, the relation between the generators X_1 and X follows:

$$X_1 = \pi_{1*}X.$$

Hence, $X_1 = X_-$ and

$$X = X_{\parallel} + X_1.$$

Then, the Lemma of subsection B3. implies immediately that X_1 is the Hamiltonian vector field of f in (Γ_1, Ω_1) , QED.

B5. Proof of Theorem 7

We can write σ as follows

$$\sigma = \pi_2 \circ i_1 \tag{B6}$$

and σ^{-1} as follows

$$\sigma^{-1} = \pi_1 \circ i_2. \tag{B7}$$

Then, using eq. (12),

$$\sigma^* \Omega_2 = i_1^*(\pi_2^*(\Omega_2)) = i_1^* \tilde{\Omega} = \Omega_1,$$

and, using the obvious relation

$$\pi_2^*(o_2) = o_2 \big|_{\tilde{\Gamma}},$$

we have

$$\sigma^* o_2 = i_1^*(\pi_2^*(o_2)) = i_1^*(o) = o_1.$$

To show the point c), we substitute for σ and σ^{-1} from (B6) and (B7):

$$\sigma \circ \varphi_1 \circ \sigma^{-1} = \pi_2 \circ i_1 \circ \varphi_1 \circ \pi_1 \circ i_2 = \pi_2 \circ \varphi_1 \circ \pi_1 \circ i_2,$$

as $i_1 \circ \varphi_1 = \varphi_1$. If we use the definition of φ_1 ,

$$\varphi_1 = \pi_1 \circ \varphi \circ i_1,$$

we obtain, as $i_1 \circ \pi_1 = \pi_1$, that

$$\sigma \circ \varphi_1 \circ \sigma^{-1} = \pi_2 \circ \pi_1 \circ \varphi \circ \pi_1 \circ i_2.$$

However,

$$\pi_1 \circ \varphi \circ \pi_1 = \pi_1 \circ \varphi,$$

as π_1 moves points along orbits and φ maps orbits onto orbits. Similarly,

$$\pi_2 \circ \pi_1 = \pi_2,$$

hence

$$\sigma \circ \varphi_1 \circ \sigma^{-1} = \pi_2 \circ \varphi \circ i_2 = \varphi_2,$$

QED.

B6. Proof of Theorem 9

A) Consider the curve $\xi_\gamma(t)$ in Γ_0 . Let γ_t the orbit that contains the point $\xi_\gamma(t)$. Then, as $h(t)\xi_\gamma(t) \in \gamma$, and $h(t)$ maps orbits onto orbits, we have

$$\gamma_t = h^{-1}(t)\gamma.$$

Next consider two points $p = \xi_\gamma(t)$ and $\xi_\gamma(t + \Delta t)$; as

$$\gamma_t = h^{-1}(t)\gamma.$$

we obtain that

$$\gamma_{t+\Delta t} = h^{-1}(\Delta t)\gamma_t.$$

Hence, the two points $\xi_\gamma(t + \Delta t)$ and $h^{-1}(\Delta t)p$ lie at the same orbit $\gamma_{t+\Delta t}$. Going to the limit $\Delta t \rightarrow 0$, we obtain the relation:

$$\lim_{\Delta t \rightarrow 0} \frac{h^{-1}(\Delta t)}{\Delta t} = \frac{d\xi_\gamma(t)}{dt} + X_{\parallel},$$

where X_{\parallel} is a vector tangential to γ_t . The vector on L.H.S is, however $-X(p)$, X being the Hamiltonian vector field of h , so the equation above is equivalent to

$$(-X(p))_{-} = \frac{d\xi_\gamma(t)}{dt}.$$

Then, the Lemma implies

$$\frac{d\xi_\gamma(t)}{dt} = -X_0(p)$$

where X_0 is the Hamiltonian vector field of the projection, $-H$, of h to (Γ_0, Ω_0) , and the A-part of Theorem 9 is true.

B) Eq. (13) implies that

$$o_t(h(t)p) = o(p), \quad p \in \Gamma_0.$$

Let γ be the orbit through p . Then, as o_t is a perennial, and as $\xi_\gamma(-t)$ lies at the same orbit as $h(t)p$, we obtain

$$\tilde{o}_t(\xi_\gamma(-t)) = o_t(h(t)p) = \tilde{o}(p).$$

The A-part of Theorem 9 implies then

$$\frac{d\tilde{o}_t}{dt} |_{\xi_\gamma(-t)} - \{\tilde{o}_t, H\} |_{\xi_\gamma(-t)} = 0.$$

This implies eq. (15) immediately, QED.

References

- [1] K.V. Kuchař, Time and Interpretation of Quantum Gravity, in: *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, eds. G. Kunstatter, D. Vincent, J. Williams. World Sci., Singapore, 1992.
- [2] C.J. Isham, Conceptual and Geometrical Problems in Quantum Gravity, in *Recent Aspects of Quantum Fields*, eds H. Mitter and H. Gausterer. Springer, Berlin, 1992.
- [3] C.J. Isham, Canonical Quantum Gravity and the Problem of Time, in *Recent Problems in Mathematical Physics*, NATO Advanced Study Insititute, Salamanca, June 17-27, 1992.
- [4] K.V. Kuchař, The Problem of Time in Canonical Quantization, in *Conceptual Problems of Quantum Gravity*, eds. A. Ashtekar and J. Stachel. Birkhäuser, Boston, 1991.
- [5] P.A.M. Dirac, *Rev. Mod. Phys.* **21** (1949) 392.
- [6] H. Leutwyler and J. Stern, *Annals of Phys.* **112** (1978) 75.
- [7] A. Ashtekar, "Lectures on Non-Perturbative Canonical Gravity". World Sci, Singapore, 1991.
- [8] A. Ashtekar and R. Tate, An Algebraic Extension of Dirac Quantization: Examples. Preprint gr-qc/9405073, CGPG-94/6-1
- [9] A. Kirilov, Elements of the Theory of Representations, *Grundlehren der Mathematischen Wissenschaften* 220, eds. S.S. Chern et al., Springer, Berlin, 1976.
- [10] C.J. Isham and A.C. Kakas, *Class. Quantum Grav.* **1** (1984) 621, 633.
- [11] C.J. Isham, Topological and Global Aspects of Quantum Theory, in *Relativity, Groups and Topology II.*, eds. B.S. DeWitt and R. Stora. Elsevier, 1984.
- [12] C. Rovelli, *Nuovo Cim.* **100B** (1987) 343.
- [13] P. Hájíček, Quantization of Systems with Constraints, in *Canonical Gravity: From Classical to Quantum*, eds. J. Ehlers and H. Friedrich. Springer, Berlin, 1994.
- [14] A. Ashtekar and M. Stillerman, *J. Math. Phys.* **27** (1986) 1319.

- [15] K.V. Kuchař, Canonical Quantum Gravity, in *Proceedings of the 13th International Conference on General Relativity and Gravitation*, ed. C. Kozameh. IOP Publishing, Bristol, 1993.
- [16] K.V. Kuchař, J. Math. Phys. **22** (1981) 2640.
- [17] D. Marolf, Quantum Observables and Recollapsing Dynamics. Preprint, gr-gc/9404053, 1994.
- [18] C. Rovelli, Phys. Rev. **D42** (1990) 2638, **D43** (1991) 442.
- [19] A. Wipf, Hamilton's Formalism for Systems with Constraints, in *Canonical Gravity: From Classical to Quantum*, eds. J. Ehlers and H. Friedrich, Springer, Berlin, 1994.
- [20] C. Emmrich and H. Römer, Commun. Math. Phys. **129** (1990) 69.
- [21] C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*. Freeman, San Francisco, 1970.
- [22] Y. Choquet-Bruhat and R. Geroch, Commun. Math. Phys. **14** (1969) 329.
- [23] S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-Time*. Cambridge University Press, Cambridge, 1973.
- [24] P.A.M. Dirac, *Lectures on Quantum Mechanics*. Yeshiva University Press, New York, 1964.
- [25] M. Schön and P. Hájíček, Class. Quantum Grav. **7** (1990) 861.
- [26] R. Abraham and J. Marsden, *Foundation of Mechanics*. Benjamin, London, 1985.
- [27] P. Šťovíček and J. Tolar, Reports Math. Phys. **20** (1984) 157.
- [28] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover 1951.
- [29] P. Hájíček, A. Higuchi, J. Tolar, subsequent paper.
- [30] P. Hájíček, J. Math. Phys. **30** (1989) 2488.
- [31] B.C. DeWitt, The Quantization of Geometry, in *Gravitation: an Introduction to Current Research*, ed. L. Witten. Wiley, New York, 1962.
- [32] R. Beig, The Classical Theory of Canonical General Relativity, in: *Canonical Gravity: From Classical to Quantum*, ed. J. Ehlers and H. Friedrichs. Springer, Berlin, 1994.