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HIGHER ORDER INFINITESIMAL BENDING OF A CLASS OF TOROIDS

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Abstract

In this paper we consider higher order infinitesimal bending of a surface in E_3. Sufficient condition for a toroid generated by polygonal meridian to be non-rigid of higher order is given. This work is an extension of the results [1]-[7]. Examples of non-rigid surfaces, with visual presentation of infinitesimal bending are given.

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Key words: Higher order infinitesimal bending, toroid

1 Introduction

We shall consider here infinitesimal bending of toroid rotational surfaces generated by polygonal meridian.

It is well-known [8], [9] that a sphere and a torus are rigid surfaces. Among surfaces that are topologically equivalent to the torus Belov [1] pointed at a class of non-rigid toroids. Belov's non-rigid surfaces are generated by quadrangular meridian of a special form. Toroid surfaces non containing plane part, generated by triangular meridian are rigid [2]. Infinitesimal bending of a toroid generated by a quadrangular meridian, given by coordinates of apices, is studied at [3]. More general, sufficient condition for non-rigidity of a toroid obtained by revolving a polygonal meridian is given at [6]. A procedure for finding a field of bending and rotation for such a surface is also outlined.

The second order infinitesimal bending for toroids is studied at [5]. Finally, these considerations lead us to give sufficient condition for toroid rotational surfaces with polygonal meridian to be non-rigid of higher order.

Higher order flexibility was studied at [11], [12] and [13].

2 Infinitesimal bending of higher order of rotational surfaces

Let S be a piecewise regular surface, given by equation

$$S : \bar{r} = \bar{r}(u, v), \tag{2.1}$$

included in a family of surfaces

$$S_t : \bar{r}_t = \bar{r}(u, v) + \sum_{j=1}^{m} t^j \overset{(j)}{\bar{z}}(u, v), \tag{2.2}$$

where $t \in R$ is a small parameter and $\overset{(j)}{\bar{z}}(u, v)$ are continuous vector functions defined at the points of S.

Definition 2.1. *Surfaces $S_t(2.2)$, $t \in R$ are infinitesimal bending of the m-th order of the surface S if*

$$ds_t^2 - ds^2 = o(t^m). \tag{2.3}$$

This condition is equivalent to the system of equations ([8], [9])

$$d\bar{r}d\bar{z} = 0,\ 2d\bar{r}d\overset{(j)}{\bar{z}} + \sum_{l=1}^{j-1} d\overset{(l)}{\bar{z}}d\overset{(j-l)}{\bar{z}} = 0,\ \textit{for}\ j = 2, ..., m. \tag{2.3'}$$

We shall use Cohn-Vossen's method [8] for the investigation of the infinitesimal bending of rotational surfaces. In the plane of the meridian which rotates around the u-axis let's introduce Descartes' orthogonal coordinate system $uO\rho$ and let $\rho = \rho(u)$ be the equation of the meridian. If $\bar{e}$ is unit vector of the axis of rotation, $\bar{a}(v)$ unit vector of the ρ-axis, where v is the angle between the plane of initial position of the meridian and $\bar{a}(v)$ then $\bar{a}'(v) \perp \bar{a}(v)$ and $\bar{a}'(v) \perp \bar{e}$ (see [8], page 90, or [9] page 253).

The equation of a surface of rotation, in the coordinate system with the base $\bar{e}, \bar{a}, \bar{a}'$ is

$$S : \bar{r}(u, v) = u\bar{e} + \rho(u)\bar{a}(v), \tag{2.4}$$

where $\rho = \rho(u)$ is the equation of the meridian. The fields of infinitesimal bending $\overset{(j)}{\bar{z}}(u, v)$, $j = 1, ..., m$ can be represented at the system with orths $\bar{e}$, $\bar{a}(v)$, $\bar{a}'(v)$ in the form

$$\overset{(j)}{\bar{z}}(u, v) = \overset{(j)}{\alpha}(u, v)\bar{e} + \overset{(j)}{\beta}(u, v)\bar{a} + \overset{(j)}{\gamma}(u, v)\bar{a}'. \tag{2.5}$$

The coefficients $\overset{(j)}{\alpha}(u, v)$, $\overset{(j)}{\beta}(u, v)$, $\overset{(j)}{\gamma}(u, v)$, $j = 1, 2, ..., m$, are periodical functions with respect to v with a period 2π, and according to [8] can be presented for $j = 1$ in the form

$$\begin{aligned} \overset{(1)}{\alpha}(u,v) &= \overset{(1)}{\alpha}_k(u,v) = \overset{(1)}{\varphi}_k(u)e^{ikv} + \overset{(1)}{\varphi}_{-k}(u)e^{-ikv}, \\ \overset{(1)}{\beta}(u,v) &= \overset{(1)}{\beta}_k(u,v) = \overset{(1)}{\psi}_k(u)e^{ikv} + \overset{(1)}{\psi}_{-k}(u)e^{-ikv}, \\ \overset{(1)}{\gamma}(u,v) &= \overset{(1)}{\gamma}_k(u,v) = \overset{(1)}{\chi}_k(u)e^{ikv} + \overset{(1)}{\chi}_{-k}(u)e^{-ikv}, \end{aligned} \tag{2.6}$$

where $k \geq 2$, $k \in Z$, and $\overset{(1)}{\varphi}_{-k}(u)$ is conjugate value of $\overset{(1)}{\varphi}_k(u)$ and so on. The functions $\overset{(1)}{\varphi}_k(u)$, $\overset{(1)}{\psi}_k(u)$, $\overset{(1)}{\chi}_k(u)$ satisfy the following systems of differential equations

$$\overset{(1)}{\varphi}{}'_k + \rho' \overset{(1)}{\psi}{}'_k = 0, \quad ik\overset{(1)}{\chi}_k + \overset{(1)}{\psi}_k = 0, \quad ik\overset{(1)}{\varphi}_k + \rho'(k\overset{(1)}{\psi}_k - \overset{(1)}{\chi}_k) + \rho\overset{(1)}{\chi}{}'_k = 0, \tag{2.7}$$

wherefrom one obtains a differential equation of the second order with respect to $\overset{(1)}{\psi}_k$

$$\rho\overset{(1)}{\psi}{}''_k + (k^2-1)\rho''\overset{(1)}{\psi}_k = 0. \tag{2.8}$$

We firstly solve the equation (2.8) and then determine $\overset{(1)}{\varphi}_k(u)$, $\overset{(1)}{\chi}_k(u)$ from (2.7). Thus we obtain

$$\begin{aligned} \overset{(1)}{\bar{z}}(u,v) &= \overset{(1)}{\bar{z}}_k(u,v) \\ &= e^{ikv}[\overset{(1)}{\varphi}_k(u)\bar{e} + \overset{(1)}{\psi}_k(u)\bar{a}(v) + \overset{(1)}{\chi}_k(u)\bar{a}'(v)] \\ &+ e^{-ikv}[\overset{(1)}{\varphi}_{-k}(u)\bar{e} + \overset{(1)}{\psi}_{-k}(u)\bar{a}(v) + \overset{(1)}{\chi}_{-k}(u)\bar{a}'(v)]. \end{aligned} \tag{2.9}$$

For $j > 1$ ([8, 10, 11]) the functions $\overset{(j)}{\alpha}_k(u,v)$, $\overset{(j)}{\beta}_k(u,v)$, $\overset{(j)}{\gamma}_k(u,v)$, $j = 2, ..., m$, have:
1) for even j, only even powers $e^{\pm ikv}$ and e^0,
2) for odd j, only odd powers $e^{\pm ikv}$, i.e.

$$\begin{aligned} \overset{(j)}{\alpha}_k(u,v) &= \sum_{(p)}[\overset{(j)}{\varphi}_{pk}(u)e^{ipkv} + \overset{(j)}{\varphi}_{-pk}(u)e^{-ipkv}], \\ \overset{(j)}{\beta}_k(u,v) &= \sum_{(p)}[\overset{(j)}{\psi}_{pk}(u)e^{ipkv} + \overset{(j)}{\psi}_{-pk}(u)e^{-ipkv}], \\ \overset{(j)}{\gamma}_k(u,v) &= \sum_{(p)}[\overset{(j)}{\chi}_{pk}(u)e^{ipkv} + \overset{(j)}{\chi}_{-pk}(u)e^{-ipkv}], \end{aligned} \tag{2.10}$$

where

1) p=0,2,4,...,j for even j

2) p=1,3,5,...,j for odd j

The fundamental field of infinitesimal bending of an order j is

$$\begin{aligned}\overset{(j)}{\bar{z}}_k(u,v) = &\big[\sum_{(p)}(\overset{(j)}{\varphi}_{pk}(u)e^{ipkv} + \overset{(j)}{\varphi}_{-pk}(u)e^{-ipkv})\big]\bar{e}\\ &+ \big[\sum_{(p)}(\overset{(j)}{\psi}_{pk}(u)e^{ipkv} + \overset{(j)}{\psi}_{-pk}(u)e^{-ipkv})\big]\bar{a}(v)\\ &+ \big[\sum_{(p)}(\overset{(j)}{\chi}_{pk}(u)e^{ipkv} + \overset{(j)}{\chi}_{-pk}(u)e^{-ipkv})\big]\bar{a}'(v).\end{aligned}$$

The functions $\overset{(j)}{\varphi}_{pk}(u)$, $\overset{(j)}{\psi}_{pk}(u)$, $\overset{(j)}{\chi}_{pk}(u)$, satisfy following system of differential equations

$$\begin{aligned}&\overset{(j)}{\varphi}{}'_{pk}(u) + \rho'(u)\overset{(j)}{\psi}{}'_{pk}(u) = \overset{(j)}{A}_{pk}(u),\\ &ipk\overset{(j)}{\chi}_{pk}(u) + \overset{(j)}{\psi}_{pk}(u) = \overset{(j)}{B}_{pk}(u),\\ &ipk\overset{(j)}{\varphi}_{pk}(u) + \rho'(u)(ipk\overset{(j)}{\psi}_{pk}(u) - \overset{(j)}{\chi}_{pk}(u)) + \rho(u)\overset{(j)}{\chi}{}'_{pk}(u) = \overset{(j)}{C}_{pk}(u).\end{aligned} \tag{2.11}$$

This system is equivalent to a differential equation of the second order with respect to $\overset{(j)}{\psi}_{pk}$

$$\rho\overset{(j)}{\psi}_{pk}{}'' + (p^2k^2-1)\rho''\overset{(j)}{\psi}_{pk} = \overset{(j)}{R}_{pk}(u), \tag{2.12}$$

where

$$\overset{(j)}{R}_{pk}(u) = -\rho''\overset{(j)}{B}_{pk}(u) + \rho\overset{(j)}{B}{}''_{pk}(u) - p^2k^2\overset{(j)}{A}_{pk}(u) - ipk\overset{(j)}{C}{}'_{pk}(u). \tag{2.13}$$

The right sides of the system (2.11) can be written more detailed in the known way according to [10]

$$\begin{aligned}\overset{(j)}{A}_{pk}(u) = \overset{(j)}{A}_{(j-2h_j)k}(u) = &-\frac{1}{2}\sum_{s=1}^{j-1}\sum_{r_s}(\overset{(s)}{\varphi}{}'_{r_sk}\overset{(j-s)}{\varphi}{}'_{(j-2h_j-r_s)k}\\ &+ \overset{(s)}{\psi}{}'_{r_sk}\overset{(j-s)}{\psi}{}'_{(j-2h_j-r_s)k} + \overset{(s)}{\chi}{}'_{r_sk}\overset{(j-s)}{\chi}{}'_{(j-2h_j-r_s)k}),\end{aligned} \tag{2.14}$$

(2.15)

$$\overset{(j)}{B}_{pk}(u) = \overset{(j)}{B}_{(j-2h_j)k}(u)$$
$$= -\frac{1}{2\rho}\sum_{s=1}^{j-1}\sum_{(r_s)}\{-k^2 r_s(j-2h_j-r_s)(\overset{(s)}{\varphi}_{r_s k}\overset{(j-s)}{\varphi}_{(j-2h_j-r_s)k}$$
$$+ (ir_{sk}\overset{(s)}{\psi}_{r_s k} - \overset{(s)}{\chi}_{r_s k})[i(j-2h_j-r_s)k\overset{(j-s)}{\psi}{}'_{(j-2h_j-r_s)k} - \overset{(j-s)}{\chi}_{(j-2h_j-r_s)k}]$$
$$+ (ir_{sk}\overset{(s)}{\chi}_{r_s k} + \overset{(s)}{\psi}_{r_s k})[i(j-2h_j-r_s)k\overset{(j-s)}{\chi}_{(j-2h_j-r_s)k} + \overset{(j-s)}{\psi}_{(j-2h_j-r_s)k}]\},$$

(2.16)

$$\overset{(j)}{C}_{pk}(u) = \overset{(j)}{C}_{(j-2h_j)k}(u)$$
$$= -\sum_{s=1}^{j-1}\sum_{(r_s)}\{ik(j-2h_j-r_s)(\overset{(s)}{\varphi}{}'_{r_s k}\overset{(j-s)}{\varphi}_{(j-2h_j-r_s)k}$$
$$+ \overset{(s)}{\psi}{}'_{r_s k}[i(j-2h_j-r_s)k\overset{(j-s)}{\psi}_{(j-2h_j-r_s)k} - \overset{(j-s)}{\chi}_{(j-2h_j-r_s)k}]$$
$$+ \overset{(s)}{\chi}{}'_{r_s k}[i(j-2h_j-r_s)k\overset{(j-s)}{\chi}_{(j-2h_j-r_s)k} + \overset{(j-s)}{\psi}_{(j-2h_j-r_s)k}]\},$$

where
1) $j = 2, ..., m$
2) $p_j = \frac{j}{2}$ for even j and $p_j = \frac{j-1}{2}$ for odd j
3) $h_j = 0, ..., p_j, j = 2, ..., m$
4) the summation index r_s in (2.14-16) takes values in the set
$\{\pm(s-2h_s), h_s = 0, 1, ..., p_s\}$ so that the numbers $j-2h_j-r_s$ belong to the set $\{\pm(j-s-2h_{(j-s)}), h_{(j-s)} = 0, 1, ..., p_{(j-s)}\}$;
5) To every number j, $2 \le j \le m$, there correspond $p_j + 1$ systems of equations (2.11).

S. E. Cohn-Vossen [8] proved that at the breaking points of the meridian the next theorem is valid (at [8] this fact is not formulated like a theorem but we will here do this).

Theorem 2.1. (Cohn-Vossen [8])*If the functions $\varphi_k(u)$ and $\chi_k(u)$ are continuous at the breaking points $u = \sigma$ of the meridian $\rho = \rho(u)$, then at these points the functions $\psi_k(u)$ satisfy the equation*

()2.17
$$[\rho(u)\psi_k'(u) + (k^2-1)\rho'(u)\psi_k(u)]\,|_{\sigma-0}^{\sigma+0} = 0,$$

i.e

(2.17′)
$$\rho(\sigma)[\psi_k'(\sigma+0) - \psi_k'(\sigma-0)] + (k^2-1)[\rho'(\sigma+0) - \rho'(\sigma-0)]\psi_k(\sigma) = 0.$$

We can prove the next theorem for the infinitesimal bending of the higher order:

Theorem 2.2. *Suppose that at the breaking points $u = \sigma$ of the meridian $\rho =$*

$\rho(u)$, $(\rho'(\sigma+0)\neq\rho'(\sigma-0))$ the functions $\overset{(j)}{\varphi}_{pk}(u)$ and $\overset{(j)}{\chi}_{pk}(u)$ are continuous $(j=1,...,m, p=2,...,m,$ for even $m,$ $p=1,3,...,m,$ for odd $m)$. Then at these points the functions $\overset{(j)}{\psi}_{pk}(u)$, $j=1,...,m$ satisfy the equation

$$[\rho(u)\overset{(j)}{\psi}{}'_{pk}(u)+(p^2k^2-1)\rho'(u)\overset{(j)}{\psi}_{pk}(u)]\,|_{\sigma-0}^{\sigma+0}=\overset{(j)}{Q}_{pk}(\sigma), \tag{2.18}$$

where

$$\begin{aligned}&\overset{(j)}{Q}_{pk}(\sigma)\\ &=\rho'(\sigma-0)\overset{(j)}{B}_{pk}(\sigma-0)-\rho(\sigma)\overset{(j)}{B}_{pk}{}'(\sigma-0)+ipk\overset{(j)}{C}_{pk}(\sigma-0)\\ &\quad-\rho'(\sigma+0)\overset{(j)}{B}_{pk}(\sigma+0)+\rho(\sigma)\overset{(j)}{B}_{pk}{}'(\sigma+0)-ipk\overset{(j)}{C}_{pk}(\sigma+0).\end{aligned} \tag{2.19}$$

Proof. From (2.12.2):

$$ipk\overset{(j)}{\chi}_{pk}(u)=\overset{(j)}{B}_{pk}(u)-\overset{(j)}{\psi}_{pk}(u)$$

$$ipk\overset{(j)}{\chi}_{pk}{}'(u)=\overset{(j)}{B}_{pk}{}'(u)-\overset{(j)}{\psi}_{pk}{}'(u),$$

and from (2.12.3)

$$\begin{aligned}ipk\overset{(j)}{\varphi}_{pk}(u)=&-\rho'(u)[ipk\overset{(j)}{\psi}_{pk}(u)-\frac{1}{ipk}(\overset{(j)}{B}_{pk}(u)-\overset{(j)}{\psi}_{pk}(u))]\\ &-\frac{\rho(u)}{ipk}[\overset{(j)}{B}_{pk}{}'(u)-\overset{(j)}{\psi}_{pk}{}'(u)]+\overset{(j)}{C}_{pk}(u).\end{aligned}$$

Multiplying the left and the right side with ipk we get

$$\begin{aligned}-p^2k^2\overset{(j)}{\varphi}_{pk}(u)=&\rho(u)\overset{(j)}{\psi}_{pk}{}'(u)+(p^2k^2-1)\rho'(u)\overset{(j)}{\psi}_{pk}(u)\\ &+\rho'(u)\overset{(j)}{B}_{pk}(u)-\rho(u)\overset{(j)}{B}_{pk}{}'(u)+ipk\overset{(j)}{C}_{pk}(u).\end{aligned}$$

At the breaking points of the meridian we have

$$\overset{(j)}{\varphi}_{pk}(\sigma-0)=\overset{(j)}{\varphi}_{pk}(\sigma+0), \tag{2.20}$$

wherefrom

$$\begin{aligned}&\rho(\sigma+0)\overset{(j)}{\psi}{}'_{pk}(\sigma+0)+(p^2k^2-1)\rho'(\sigma+0)\overset{(j)}{\psi}_{pk}(\sigma+0)\\ &=\rho(\sigma-0)\overset{(j)}{\psi}{}'_{pk}(\sigma-0)+(p^2k^2-1)\rho'(\sigma-0)\overset{(j)}{\psi}_{pk}(\sigma-0)+\overset{(j)}{Q}_{pk},\end{aligned}$$

i.e. we obtained (2.18), where $\overset{(j)}{Q}_{pk}$ is given by (2.19).

3 Infinitesimal bending of higher order of rotational toroids generated by a polygonal meridian

We shall consider infinitesimal bending of a toroid generated by a polygonal meridian with the apexes $A_l(u_l,\ \rho_{_l})$, $l = 1, ..., n$, $A_{n+1} \equiv A_1$ at the coordinate system $uO\rho$. The equations of the sides are

$$A_l A_{l+1} : \rho_{_{(l)}} = \rho_{_l} + \frac{\rho_{_{l+1}} - \rho_{_l}}{u_{l+1} - u_l}(u - u_l) = k_l u + n_l \tag{3.1}$$

$l = 1, 2, ..., n,\ A_{n+1} \equiv A_1,\ \rho_{n+1} \equiv \rho_1,\ u_{n+1} \equiv u_1$ where $\rho_{_{(l)}}$ is the value of ρ on the side $A_l A_{l+1}$

$$\rho_{_{(l)}}{}' = k_l,\ \rho_{_{(l)}}{}'' = 0,\ l = 1, 2, ..., n. \tag{3.2}$$

Let us find the field of infinitesimal bending $\overset{(j)}{\bar{z}}_k(u, v)$ on the side $A_l A_{l+1}$

$$\begin{aligned} \overset{(j)}{\bar{z}}_{k,l}(u, v) = {} & [\sum_{(p)}(\overset{(j)}{\varphi}_{pk,l}(u)e^{ipkv} + \overset{(j)}{\varphi}_{-pk,l}(u)e^{-ipkv})]\bar{e} \\ & + [\sum_{(p)}(\overset{(j)}{\psi}_{pk,l}(u)e^{ipkv} + \overset{(j)}{\psi}_{-pk,l}(u)e^{-ipkv})]\bar{a}(v) \\ & + [\sum_{(p)}(\overset{(j)}{\chi}_{pk,l}(u)e^{ipkv} + \overset{(j)}{\chi}_{-pk,l}(u)e^{-ipkv})]\bar{a}'(v), \end{aligned} \tag{3.3}$$

$l = 1, 2, ..., n,$
1) $p = 0, 2, 4, ..., j$ for even j,
2) $p = 1, 3, 5, ..., j$ for odd j.
Starting from (2.12), we get (according to (3.2)):

$$\rho_{_{(l)}} \overset{(j)}{\psi}{}''_{pk,l}(u) = \overset{(j)}{R}_{pk,l}(u) \tag{3.4}$$

where

$$\overset{(j)}{R}_{pk,l}(u) = \rho \overset{(j)}{B}{}''_{pk,l}(u) - p^2k^2 \overset{(j)}{A}_{pk,l}(u) - ipk\overset{(j)}{C}{}'_{pk,l}(u) \tag{3.5}$$

and $\overset{(j)}{A}_{pk,l}(u)$, $\overset{(j)}{B}_{pk,l}(u)$, $\overset{(j)}{C}_{pk,l}(u)$ are given by (2.14-16) on the side $A_l A_{l+1}$. Solving (3.4) we have

$$\overset{(j)}{\psi}_{pk,l}(u) = \int du \int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{_{(l)}}(u)} du + \overset{(j)}{M}_{pk,l} u + \overset{(j)}{N}_{pk,l} \tag{3.6}$$

where $\overset{(j)}{M}_{pk,l}$ and $\overset{(j)}{N}_{pk,l}$ are arbitrary constants. From (2.13.2) and (3.6) we have

$$\overset{(j)}{\chi}_{pk,l}(u) = \frac{1}{ipk}\Big[\overset{(j)}{B}_{pk,l}(u) - \int du \int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du - \overset{(j)}{M}_{pk,l}u - \overset{(j)}{N}_{pk,l}\Big]. \tag{3.7}$$

¿From (2.13.3) and (3.6),(3.7)

$$\begin{aligned}\overset{(j)}{\varphi}_{pk,l}(u) &= k_l(\frac{1}{p^2k^2} - 1)\int du \int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du \\ &- \frac{\rho_{(l)}(u)}{p^2k^2}\int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du + \frac{1}{ipk}\overset{(j)}{C}_{pk,l}(u) - \frac{k_l}{p^2k^2}\overset{(j)}{B}_{pk,l}(u) \\ &+ \frac{\rho_{(l)}(u)}{p^2k^2}\overset{(j)}{B}_{pk,l}{}'(u) + (k_lu + \frac{k_l}{p^2k^2}u - \frac{\rho_{(l)}(u)}{p^2k^2})\overset{(j)}{M}_{pk,l} \\ &+ k_l(1 + \frac{1}{p^2k^2})\overset{(j)}{N}_{pk,l}\end{aligned} \tag{3.8}$$

1) p=0,2,4,...,j for even j,
2) p=1,3,5,...,j for odd j.
For p=0 from (2.13.2) we have

$$\overset{(j)}{\psi}_{0,l}(u) = \overset{(j)}{B}_{0,l}(u), \tag{3.9}$$

and from (2.13.1):

$$\overset{(j)}{\varphi}_{0,l}(u) = \int \overset{(j)}{A}_{0,l}(u)du - k_l\overset{(j)}{B}_{0,l}(u) + \overset{(j)}{M}_{0,l}. \tag{3.10}$$

Using (2.13.3) and (3.9,10) we get:

$$\overset{(j)}{\chi}_{0,l}(u) = \rho_{(l)}(u)\Big[\int \frac{\overset{(j)}{C}_{0,l}(u)}{(\rho_{(l)}(u))^2}du + \overset{(j)}{N}_{0,l}\Big]. \tag{3.11}$$

The coefficients of the fundamental field $\overset{(j)}{\bar{z}}_{k,l}(u,v)$ are given by (3.6)-(3.11). The constants $\overset{(j)}{M}_{pk,l}$, $\overset{(j)}{N}_{pk,l}$ will be found using the fact that the field of infinitesimal bending is to be continuous on the whole surface including the circles, described by the apices of the meridian. At the apex $A_l(u_l,\ \rho_l)$ from

$$\overset{(j)}{\psi}_{pk,l}(u_{l+1}) = \overset{(j)}{\psi}_{pk,l+1}(u_{l+1}),\ l = 1,2,...,n,\ j = 1,2,...,m \tag{3.12}$$

and from (3.6) we have

$$\begin{aligned}&u_{l+1}\overset{(j)}{M}_{pk,l} - u_{l+1}\overset{(j)}{M}_{pk,l+1} + \overset{(j)}{N}_{pk,l} - \overset{(j)}{N}_{pk,l+1} \\ &= \Big[\int du \int \frac{\overset{(j)}{R}_{pk,l+1}(u)}{\rho_{(l+1)}(u)}du - \int du \int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du\Big]\Big|_{u=u_{l+1}}.\end{aligned} \tag{3.13}$$

In the mentioned points A_{l+1}, the equation (2.18) has the form

$$
\begin{aligned}
&(k_{l+1}u_{l+1}+n_{l+1})\Big[\Big(\int \frac{\overset{(j)}{R}_{pk,l+1}(u)}{\rho_{(l+1)}(u)}du\Big)\Big|_{u=u_{l+1}}+\overset{(j)}{M}_{pk,l+1}\Big]\\
&+(p^2k^2-1)k_{l+1}\Big[\Big(\int du\int \frac{\overset{(j)}{R}_{pk,l+1}(u)}{\rho_{(l+1)}(u)}du\Big)\Big|_{u=u_{l+1}}+\overset{(j)}{M}_{pk,l+1}u_{l+1}+\overset{(j)}{N}_{pk,l+1}\Big]\\
&-\rho_{(l)}(u_{l+1})\Big[\Big(\int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du\Big)\Big|_{u=u_{l+1}}+\overset{(j)}{M}_{pk,l}\Big]\\
&-(p^2k^2-1)k_{l}\Big[\Big(\int du\int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du\Big)\Big|_{u=u_{l+1}}+\overset{(j)}{M}_{pk,l}u_{l+1}+\overset{(j)}{N}_{pk,l}\Big]\\
&=\overset{(j)}{Q}_{pk,l}(u_{l+1})
\end{aligned}
$$

where $\overset{(j)}{Q}_{pk}(u)$ is given by (2.19) $l=1,2,...,n,\ j=1,2,...,m$. In that way, at the apices of the meridian we have the system of the equations

$$
\begin{aligned}
&-(n_l+p^2k^2k_lu_{l+1})\overset{(j)}{M}_{pk,l}+(n_{l+1}+p^2k^2k_{l+1}u_{l+1})\overset{(j)}{M}_{pk,l+1}\\
&-(p^2k^2-1)k_l\overset{(j)}{N}_{pk,l}+(p^2k^2-1)k_{l+1}\overset{(j)}{N}_{pk,l+1}=\overset{(j)}{V}_{pk}(u_{l+1})
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
&\overset{(j)}{V}_{pk}(u_{l+1})=k_l\overset{(j)}{B}_{pk,l}-\rho_{_l}\overset{(j)}{B}_{pk,l}{}'+ipk\overset{(j)}{C}_{pk,l}-k_{l+1}\overset{(j)}{B}_{pk,l+1}\\
&+\rho_{_l}\overset{(j)}{B}_{pk,l+1}{}'-ipk\overset{(j)}{C}_{pk,l+1}-\rho_{_l}\Big(\int \frac{\overset{(j)}{R}_{pk,l+1}(u)}{\rho_{(l+1)}(u)}du\Big)\Big|_{u=u_{l+1}}\\
&-(p^2k^2-1)k_{l+1}\Big(\int du\int \frac{\overset{(j)}{R}_{pk,l+1}(u)}{\rho_{(l+1)}(u)}du\Big)\Big|_{u=u_{l+1}}+\rho_{(l)}\Big(\int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du\Big)\Big|_{u=u_{l+1}}\\
&+(p^2k^2-1)k_l\Big(\int du\int \frac{\overset{(j)}{R}_{pk,l}(u)}{\rho_{(l)}(u)}du\Big)\Big|_{u=u_{l+1}}.
\end{aligned}
$$

¿From above exposed, the following theorem holds:

Theorem 3.1. *Sufficient condition for the toroid rotational surface generated by polygonal meridian to be non-rigid of the order m, i.e. to have the field of infinitesimal bending* $\overset{(m)}{\overline{z}}(u,v)$ *of the order m, as an extension of the fields* $\overset{(1)}{\overline{z}}(u,v)$, $\overset{(2)}{\overline{z}}(u,v)$,..., $\overset{(m-1)}{\overline{z}}(u,v)$ *is the systems of linear equations with respect to unknowns* $\overset{(j)}{M}_{pk,l}$, $\overset{(j)}{N}_{pk,l}$ (3.13) *and* (3.14) *to be compatible.*

4 Visualization of infinitesimal bending of toroid generated by a polygonal meridian

Using a computer enables us analyzing non rigidity conditions and determination a family of toroids which satisfy them. The class of nonrigid toroids "discovered" by Belov[1] is enlarged by some new examples of toroids satisfying non rigidity conditions. As a examples here is given toroid with non convex quadrangle meridian, one 5 apexes convex and one 9 apexes convex meridian. In examination toroids and determination theirs properties we have used symbolic program package Mathematica. Our program takes points of a meridian as input, then determines rigidity conditions and gives as output symbolical definitions of rotational surface together with field of infinitesimal bending of the first order. Then it can be pass as argument to some function for 3D representation. Considered bending on this way could have been graphically analyzed. Graphical representations of deformations is considered in articles [14]-[16]. It is useful to see rotational surfaces and influence of fields of infinitesimal bending on them. For the purpose of visual presentation animation of infinitesimal bending of surfaces we developed SurfBend. SurfBend, the programm devoted to visualize infinitesimal bending of toroids, developed in C++ using OpenGL, is partialy presented at the ESI Conference Rigidity and Flexibility, Viena, 2006. Obtained analytical expressions are graphically represented in the following examples and illustrate theoretical considerations.

4.1 Examples

1. The first example of non rigid toroidal surfaces is the surface with non convex quadrangle with apexes $A(-1,1)$, $B(0,\frac{5}{8})$, $C(1,1)$, $D(0,\frac{1}{4})$. The quadrangle rotates around u-axis of the coordinate system $uO\rho$. Revolving AB generates $S_{(1)}$ and by revolving BC, CD, DA, the parts $S_{(2)}S_{(3)}S_{(3)}$, the same for $\bar{z}_{(i)}$, are generated respectively. Let us denote that the deformed toroid S_ϵ.

$$S_\epsilon : \bar{r}(u,v) = \bar{r}(u,v) + \epsilon\bar{z}(u,v)$$

$$S_{(1)} : \bar{r}_{(1)}(u,v) = [-\frac{3}{8}u + \frac{5}{8}]cos(v)\bar{i} + [-\frac{3}{8}u + \frac{5}{8}]sin(v)\bar{j} + u\bar{k}, u \in [-1,0], v \in [0,2\pi],$$

$$S_{(2)} : \bar{r}_{(1)}(u,v) = [\frac{3}{8}u + \frac{5}{8}]cos(v)\bar{i} + [\frac{3}{8}u + \frac{5}{8}]sin(v)\bar{j} + u\bar{k}, u \in [0,1], v \in [0,2\pi],$$

$$S_{(3)} : \bar{r}_{(1)}(u,v) = [\frac{3}{4}u + \frac{1}{4}]cos(v)\bar{i} + [\frac{3}{4}u + \frac{1}{4}]sin(v)\bar{j} + u\bar{k}, u \in [1,0], v \in [0,2\pi],$$

$$S_{(4)} : \bar{r}_{(1)}(u,v) = [-\frac{3}{4}u + \frac{1}{4}]cos(v)\bar{i} + [-\frac{3}{4}u + \frac{1}{4}]sin(v)\bar{j} + u\bar{k}, u \in [0,-1], v \in [0,2\pi].$$

For the field $\bar{z}(u,v)$ of infinitesimal bending of the first order we have:

$$\begin{aligned}
\bar{z}_{(1)}(u,v) &= [2(u+\frac{5}{9})cos(2v)cos(v) + (u+\frac{5}{9})sin(2v)sin(v)]\bar{i} \\
&+ [2(u+\frac{5}{9})cos(2v)sin(v) - (u+\frac{5}{9})sin(2v)cos(v)]\bar{j} \\
&+ [\frac{3}{4}ucos(2v)]\bar{k}, u \in [-1,0], v \in [0,2\pi] \\
\bar{z}_{(2)}(u,v) &= [2(-u+\frac{5}{9})cos(2v)cos(v) + (-u+\frac{5}{9})sin(2v)sin(v)]\bar{i} \\
&+ [2(-u+\frac{5}{9})cos(2v)sin(v) - (-u+\frac{5}{9})sin(2v)cos(v)]\bar{j} \\
&+ [\frac{3}{4}ucos(2v)]\bar{k}, u \in [0,1], v \in [0,2\pi] \\
\bar{z}_{(3)}(u,v) &= [2(-\frac{1}{2}u+\frac{5}{90})cos(2v)cos(v) + (-\frac{1}{2}u+\frac{5}{90})sin(2v)sin(v)]\bar{i} \\
&+ [2(-\frac{1}{2}u+\frac{5}{90})cos(2v)sin(v) - (-\frac{1}{2}u+\frac{5}{90})sin(2v)cos(v)]\bar{j} \\
&+ [\frac{3}{4}ucos(2v)]\bar{k}, u \in [1,0], v \in [0,2\pi] \\
\bar{z}_{(4)}(u,v) &= [2(\frac{1}{2}u+\frac{5}{90})cos(2v)cos(v) + (\frac{1}{2}u+\frac{5}{90})sin(2v)sin(v)]\bar{i} \\
&+ [2(\frac{1}{2}u+\frac{5}{90})cos(2v)sin(v) - (\frac{1}{2}u+\frac{5}{90})sin(2v)cos(v)]\bar{j} \\
&+ [\frac{3}{4}ucos(2v)]\bar{k}, u \in [0,-1], v \in [0,2\pi]
\end{aligned}$$

Following figures show rotational surfaces and influence of infinitesimal bending. For the purpose of insight view rotational parameter v is taken to be less than 2π.

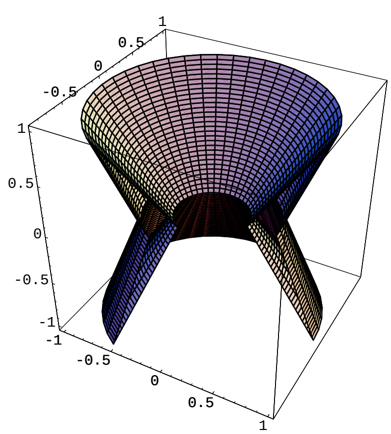

Fig.1.

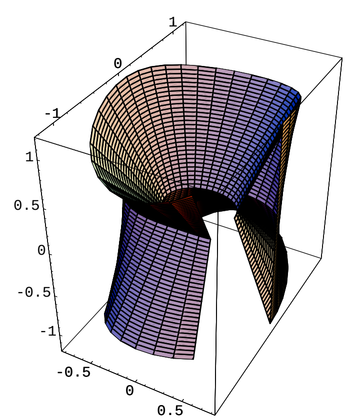

Fig.2.

On the Fig.1. are shown surfaces with all four parts without any deformations $\epsilon = 0.0$ and with $v \in [0, \frac{4}{3}\pi]$. On the Fig.2. are shown surfaces with all four parts with deformations $\epsilon = 0.15$ and with $v \in [0, \frac{3}{2}\pi]$.

2. The second example of non rigid toroidal surfaces is surface with convex pentagon with apexes $A(-1,2)$, $B(-2,3)$, $C(0,4)$, $D(2,3)$ $E(1,\frac{119-\sqrt{2641}}{45})$, . The polygon rotates around u-axis of the coordinate system $uO\rho$.

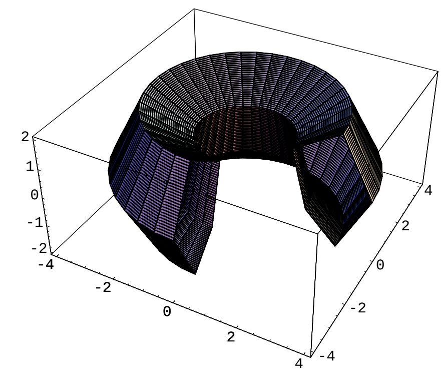

Fig.3.

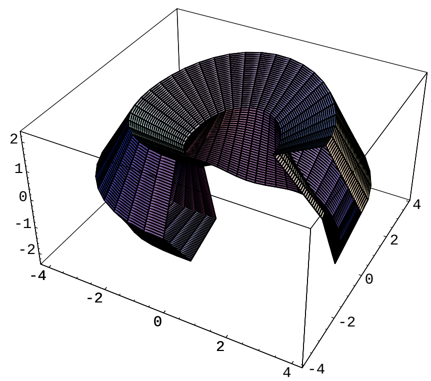

Fig.4.

On the Fig.3. are shown surfaces with all five parts without any deformations $\epsilon = 0.0$ and with $v \in [0, \frac{3}{2}\pi]$. On the Fig.4. are shown surfaces with all five parts with deformations $\epsilon = 0.25$ and with $v \in [0, \frac{3}{2}\pi]$.

3. The third example of non rigid toroidal surfaces is surface with convex poligon with 9 apexes $A(-1,1)$, $B(-2,2)$, $C(-4,3)$, $D(-3,4)$, $E(-2,6)$, $F(0,8)$, $G(3,12)$, $H(\frac{7}{2}, \frac{3839}{304}+\frac{(-173450877+836\sqrt{43583479149})^{1/3}}{304^{2/3}}-\frac{50005}{304(3(-173450877+836\sqrt{43583479149}))^{1/3}})$, $I(0,3)$. The polygon rotates around u-axis of the coordinate system $uO\rho$.

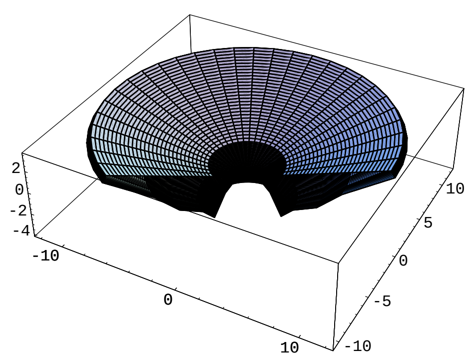

Fig.5.

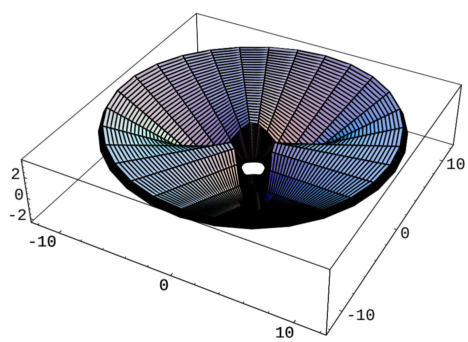

Fig.6.

On the Fig.5. are shown surfaces with all nine parts without deformations $\epsilon = 0.0$ and with $v \in [0, \frac{4}{3}\pi]$. On the Fig.6. are shown surfaces with parts GH,HI and IA with deformations $\epsilon = 0.02$ and with $v \in [0, 2\pi]$.

Remark. Let us remark that the main results at [1]-[7] can be considered as particular cases of Theorem 3.

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