ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

The Dixmier Trace and Asymptotics of Zeta Functions

Alan L. Carey
Adam Rennie
Aleksandr Sedaev
Fyodor Sukochev

Vienna, Preprint ESI 1872 (2006) November 28, 2006

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

THE DIXMIER TRACE AND ASYMPTOTICS OF ZETA FUNCTIONS

by

Alan L. Carey[1,3]
Mathematical Sciences Institute
Australian National University
Canberra ACT 0200
AUSTRALIA

Adam Rennie[2]
Institute for Mathematical Sciences
University of Copenhagen
Universitetsparken 5, DK-2100, Copenhagen, DENMARK

Aleksandr Sedaev[1,4]
Department of Mathematics,
Voronezh State University of Architecture and Construction,
20-letiya Oktyabrya 84, Voronezh 394006, RUSSIA

Fyodor Sukochev[1]
School of Informatics and Engineering
Flinders University
Bedford Park 5042
AUSTRALIA

AMS classification nos. Primary: 19K56, 46L80: secondary: 58B30, 46L87.
Keywords: Spectral triple, Dixmier trace, zeta function.

Supported by the Clay Mathematics Institute[3] and by grants from ARC[1] (Australia), SNF (Denmark)[2] and RFBR (5-01-00629)(Russia)[4].

Abstract

We obtain general theorems which enable the calculation of the Dixmier trace in terms of the asymptotics of the zeta function and of the trace of the heat semigroup. We prove our results in a general semi-finite von Neumann algebra. We find for $p > 1$ that the asymptotics of the zeta function determines an ideal strictly larger than $\mathcal{L}^{p,\infty}$ on which the Dixmier trace may be defined. We also establish stronger versions of other results on Dixmier traces and zeta functions.

1. Introduction

1.1. **Background.** The key role of the Dixmier trace in noncommutative geometry was discovered by Connes around 1990, [13]. Since then, it has become a cornerstone of noncommutative geometry. Notably, the Dixmier trace is used to define dimension, integration and has been used in physical applications, along with heat kernel type expansions, to define 'spectral actions' for noncommutative field theories, [9, 15]. The Dixmier trace (or more precisely Dixmier *traces*) are a family of non-normal traces on the bounded operators on a separable Hilbert space $\mathcal{H}$ measuring the logarithmic divergence of the trace of a compact operator. There is an ideal of compact operators denoted $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ consisting precisely of those operators with finite Dixmier trace. (This and the related ideals $\mathcal{L}^{(p,\infty)}(\mathcal{H})$, $p \geqslant 1$, are defined in detail in Section 2 *cf* also [13].) Following [13] connections between Dixmier traces, zeta functions and heat kernel asymptotics were systematically studied in [6]. Motivated by these results, and questions arising in connection with physical applications, we substantially extend the understanding of these matters in this article.

Briefly, for an important special case, we show that for a positive compact operator T, the existence of the limit $\lim_{r\to\infty} \frac{1}{r}\mathrm{Trace}(T^{p+\frac{1}{r}})$ implies that the operator T lies in an ideal $\mathcal{Z}_p$. The ideal $\mathcal{Z}_1$ is $\mathcal{L}^{(1,\infty)}(\mathcal{H})$, while for $p>1$ $\mathcal{Z}_p$ is strictly larger than $\mathcal{L}^{(p,\infty)}(\mathcal{H})$. (It is in fact precisely what is termed, in [32, Section 1.d], the p-convexification of $\mathcal{L}^{(1,\infty)}(\mathcal{H})$.) We then show that if $\lim_{r\to\infty} \frac{1}{r}\mathrm{Trace}(T^{p+\frac{1}{r}})$ exists it equals $p\mathrm{Trace}_\omega(T^p)$ for any state ω generating a Dixmier trace, Trace_ω. Thus we show that the asymptotics of the zeta function singles out the class of compact operators which have a finite Dixmier trace.

In fact the analogues of these statements are true for compact operators T in a semifinite von Neumann algebra $\mathcal{N}$ with faithful, normal, semifinite trace τ for which there are corresponding ideals $\mathcal{Z}_p(\mathcal{N})$ and $\mathcal{L}^{(p,\infty)}(\mathcal{N},\tau)$. Readers unfamiliar with ideal theory in such general algebras may restrict attention to the standard case of bounded operators on an infinite dimensional separable Hilbert space with its usual trace (denoted by 'Trace' here). Our reason for striving for generality stems from the emergence recently of applications of the semifinite von Neumann theory [1, 2, 18, 4, 5, 35].

Our results follow primarily from (strengthened versions of) deep facts from [6] and recent advances in the study of singular traces, some of which seem not to be well known. We also work in this paper with general Marcinkiewicz spaces and general 'Dixmier traces' as these spaces are already known to arise in the study of pseudodifferential operators [34].

Before giving a more precise account of our results, let us set out the motivations coming from noncommutative physics and geometry. In [28] it was shown that the Moyal 'plane' of dimension $2N$ defines a $(2N,\infty)$-summable spectral triple. In order to prove this, the authors used a variant of Cwikel's inequality, and to compute Dixmier traces, they employed the zeta function methods of [6]. Numerous other noncommutative spaces which are (p,∞)-summable have been studied, [7, 8, 9, 17, 19, 35, 36], some with physical applications or relevance.

Examining these examples shows that except for very special and/or simple examples, eg [7, 8, 35, 36], the determination of Dixmier summability of an operator relies on one of two methods: Weyl's theorem, or Cwikel type inequalities. In particular for operators arising from 'noncommutative action principles' (that is when we minimise functionals on noncommutative algebras), no (classical) geometric context need exist, and so Weyl's theorem is of no use.

The theorems presented here offer alternative techniques for proving Dixmier summability results, and computing Dixmier traces. This is likely to be relevant for (very) noncommutative examples and physically inspired examples. It is also likely that via zeta function regularisation of determinants, our techniques could provide criteria for one-loop renormalizability of noncommutative field theories.

1.2. **Summary of the main results.** We need some notation in order to present the results. We remark that in a semifinite von Neumann algebra $\mathcal{N}$ with faithful normal semifinite trace τ the $\tau-$compact operators are generated by projections P with $\tau(P) < \infty$. Suppose that T is a $\tau-$compact positive operator in $\mathcal{N}$. (If one has a semifinite spectral triple determined by an unbounded self adjoint operator D then one should think of T as $|D|^{-1}$ or $(1+D^2)^{-1/2}$.) For a given τ let τ_ω denote a Dixmier trace corresponding to an element $\omega \in \ell_\infty^*(\mathbb{N})$ or $\ell_\infty^*(\mathbb{R}_+)$. We remark that ω must satisfy some invariance properties which we will explain in detail in Section 3. By the zeta function of T we mean $\zeta(s) = \tau(T^s)$.

Consider the following hypothesis:
(*) Under the assumption that $\tau(T^s)$ exists for all $s > p$ suppose that $\lim_{r\to\infty} \frac{1}{r}\zeta(p+\frac{1}{r})$ exists. It is then natural to ask, in view of [6, 13], the following question:
A. If hypothesis (*) holds then does it follow that $T \in \mathcal{L}^{(p,\infty)}$?
We prove that the answer to Question A is yes if $p = 1$ and no if $p > 1$. This leads to a second question:
B. For $p > 1$ what constraint does hypothesis (*) place on the singular values of T?

We remark that in contrast to the situation with the classical Schatten ideals it is not true that if $T \in \mathcal{L}^{(1,\infty)}$ then $T^{1/p} \in \mathcal{L}^{(p,\infty)}$. In fact there is a strictly smaller ideal inside $\mathcal{L}^{(1,\infty)}$ characterized by this property. We prove correspondingly that there is an ideal $\mathcal{Z}_p$ strictly larger than $\mathcal{L}^{(p,\infty)}$ with the property that if $T^{1/p} \in \mathcal{Z}_p$ then $T \in \mathcal{L}^{(1,\infty)}$. We also prove that if hypothesis (*) holds then $T \in \mathcal{Z}_p$.

This leads to the further question:
C. If hypothesis (*) holds how does the limit relate to the Dixmier trace of T^p?
In fact we show that for a certain class of Dixmier traces τ_ω

$$\lim_{r\to\infty} \frac{1}{r}\zeta(p+\frac{1}{r}) = p \lim_{t\to\infty} \frac{1}{\log(1+t)} \int_0^t \mu_s(T)^p ds := p\tau_\omega(T^p).$$

Our methods then lead us to prove some stronger versions of several results in [6]. In that paper we were forced to consider a subset of the set of all Dixmier traces determined by requiring invariance under a certain transformation group. In the new approach of this article we can relax many of these invariance conditions.

Then, in view of [13, p 563] and the relationship of the zeta function to the heat kernel, it is natural to ask what hypothesis (*) implies concerning the small time asymptotics of the trace of the heat semigroup. (We note that hypothesis (*) implies that the heat semigroup $e^{-tT^{-2}}$, defined using the functional calculus, is trace class for all $t > 0$.) This matter is resolved in Theorem 5.1. Let $F(\lambda) = \lambda^{-1}\tau(e^{-\lambda^{-2}T^{-2}})$, then under hypothesis (*) for $p = 1$ this function is bounded on $(0,\infty)$ and Theorem 5.1 says that for certain $\omega \in L_\infty((0,\infty))^*$, $\omega(F)$ is a multiple of the Dixmier trace $\tau_\omega(T)$.

Conversely we know that if $\lambda^{-1}\tau(e^{-\lambda^{-2}T^{-2}})$ has an asymptotic expansion in λ as $\lambda \to \infty$ then the leading term in this expansion precisely determines the first singularity of $\tau(T^s)$ as $\mathrm{Re}(s)$ decreases. In this case, using the results described above, we find that $T \in \mathcal{Z}_1$ and the residue of the zeta function is equal to the Dixmier trace of T.

Finally, in Section 6, we revisit a question raised in [10]. Namely, for T in some general ideal $\mathcal{I}$ (in the $\tau-$compact operators), which admits a Dixmier trace τ_ω, what are the minimal conditions on an algebra $\mathcal{A}$ such that the functional $a \to \tau_\omega(aT)$ on $\mathcal{A}$ is actually a trace? This question is important in the manifold reconstruction theorem of [13]. We find that the methods of this paper enable us to substantially generalize [10] (who answer the question only for $\mathcal{I} = \mathcal{L}^{(p,\infty)}$). We find that, for the same minimal conditions as in [10], there is a very large class of Marcinkiewicz ideals $\mathcal{I}$ including $\mathcal{I} = \mathcal{Z}_p$ for which $a \to \tau_\omega(aT)$ is a trace.

We give in Section 2 a summary of the theory of singular traces and a careful discussion of ideals of compact operators needed in this paper. We follow this in Section 3 with some details on the construction of Dixmier traces. The main results are proved in Section 4, for the zeta function, and Section 5 for the heat operator. We finish with our generalization of [10].

Acknowledgement The fourth named author thanks Bruno Iochum for asking the question that led to Corollary 4.6 and the Université de Cergy-Pontoise for hosting his visit. We also thank Harald Grosse and Victor Gayral for advice and Evgenii Semenov for explaining to us some subtle facts about geometry of Marcinkiewicz spaces. The first named author thanks the Erwin Schrödinger Institute for its assistance with this research and the Clay Mathematics Institute for financial support.

2. Preliminaries: spaces and functionals

2.1. **Function spaces.** The theory of singular traces on operator ideals rests on some classical analysis which we now review for completeness.

Consider a Banach space $(E, \|\cdot\|_E)$ of real valued Lebesgue measurable functions on the interval $J = [0,\infty)$ or else on $J = \mathbb{N}$. Let x^* denote the non-increasing, right-continuous rearrangement of $|x|$ given by

$$x^*(t) = \inf\{s \geqslant 0 \mid \lambda(\{|x| > s\}) \leqslant t\},\ t > 0,$$

where λ denotes Lebesgue measure. Then E will be called rearrangement invariant (or r.i.) if

(i). E is an ideal lattice, that is if $y \in E$, and x is any measurable function on J with $0 \leqslant |x| \leqslant |y|$, then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$;
(ii). if $y \in E$ and if x is any measurable function on J with $x^* = y^*$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

In the case $J = \mathbb{N}$, it is convenient to identify x^* with the rearrangement of the sequence $|x| = \{|x_n|\}_{n=1}^\infty$ in descending order. (The theory is in the monographs [30], [31], [32].) A r.i. space E is said to be a fully symmetric Banach space if it has the additional property that if $y \in E$ and $L_1 + L_\infty(J) \ni x \prec\prec y$, then $x \in E$ and $\|x\|_E \leqslant \|y\|_E$. Here, $x \prec\prec y$ denotes submajorization in the sense of Hardy-Littlewood-Pólya:

$$\int_0^t x^*(s)ds \leqslant \int_0^t y^*(s)ds, \qquad \forall\, t > 0.$$

All these spaces E satisfy $L_1 \cap L_\infty(J) \subseteq E \subseteq (L_1 + L_\infty)(J)$, with continuous embeddings. In this paper, we consider only fully symmetric Banach spaces E, which satisfy in addition $E \subseteq L_\infty(J)$ (a non-commutative extension of the theory of such spaces placed in the setting of a semifinite von Neumann algebra $\mathcal{N}$ corresponds to ideals in $\mathcal{N}$ equipped with unitarily invariant norm [29, 21, 11, 39, 6]).

Recall (see [30]) that for an arbitrary rearrangement invariant function space $E = E(0,\infty)$ the fundamental function of E, $\varphi_E(\cdot)$, is given by

$$\varphi_E(t) = \|\chi_{[0,t)}\|_E, \ t > 0.$$

2.2. **Marcinkiewicz function and sequence spaces.** Our main examples of fully symmetric function and sequence spaces are given in the following discussion. Let Ω denote the set of concave functions $\psi : [0,\infty) \to [0,\infty)$ such that $\lim_{t\to 0^+} \psi(t) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. For $\psi \in \Omega$ define the weighted mean function

$$a(x,t) = \frac{1}{\psi(t)} \int_0^t x^*(s)\, ds \quad t > 0$$

and denote by $M(\psi)$ the (Marcinkiewicz) space of measurable functions x on $[0,\infty)$ such that

$$\|x\|_{M(\psi)} := \sup_{t>0} a(x,t) = \|a(x,\cdot)\|_\infty < \infty. \tag{1}$$

We assume in this paper that $\psi(t) = O(t)$ when $t \to 0$, which is equivalent to the continuous embedding $M(\psi) \subseteq L_\infty(J)$. The definition of the Marcinkiewicz sequence space $m(\psi)$ of functions on $\mathbb{N}$ is similar,

$$m(\psi) = \left\{ x = \{x_n\}_{n=1}^\infty \ : \ \|x\|_{m(\psi)} := \sup_{N\geqslant 1} \frac{1}{\psi(N)} \sum_{n=1}^N x_n^* < \infty \right\}.$$

Example (i). Introduce the following functions

$$\psi_1(t) = \begin{cases} t \cdot \log 2, & 0 \leqslant t \leqslant 1 \\ \log(1+t), & 1 \leqslant t < \infty \end{cases},$$

respectively (for $p > 1$),

$$\psi_p(t) = \begin{cases} t, & 0 \leqslant t \leqslant 1 \\ t^{1-\frac{1}{p}}, & 1 \leqslant t < \infty \end{cases}.$$

The spaces $\mathcal{L}^{(1,\infty)}$ and $\mathcal{L}^{(p,\infty)}$ are the Marcinkiewicz spaces $M(\psi_1)$ and $M(\psi_p)$ respectively. The norm given by formula (1) on the space $\mathcal{L}^{(p,\infty)}$ is denoted by $\|\cdot\|_{(p,\infty)}$, $1 \leqslant p < \infty$.

Example (ii). In [34], F. Nicola considers, in connection with a class of pseudo-differential operators, the Marcinkiwecz space $M(\psi)$, with $\psi(t) = \log^2(t+1)$, $t > 0$.

2.3. **Symmetric operator spaces and functionals.** We now go from function spaces to the setting of (noncommutative) spaces of operators. Let $\mathcal{N}$ be a semifinite von Neumann algebra on the separable Hilbert space $\mathcal{H}$, with a fixed faithful and normal semifinite trace τ. We recall from [26, 25] the notion of *generalized singular value function.* Given a self-adjoint operator A in $\mathcal{N}$, we denote by $E^A(\cdot)$ the spectral measure of A. Then $E^{|A|}(B) \in \mathcal{N}$ for all Borel sets $B \subseteq \mathbb{R}$, and there exists $s > 0$ such that $\tau(E^{|A|}(s,\infty)) < \infty$. For $t \geqslant 0$, we define

$$\mu_t(A) = \inf\{s \geqslant 0 : \tau(E^{|A|}(s,\infty)) \leqslant t\}.$$

The function $\mu(A) : [0,\infty) \to [0,\infty]$ is called the *generalized singular value function* (or decreasing rearrangement) of A; note that $\mu_{(\cdot)}(A) \in L_\infty(J)$.

If we consider $\mathcal{N} = L_\infty([0,\infty), m)$, where m denotes Lebesgue measure on $[0,\infty)$, as an abelian von Neumann algebra acting via multiplication on the Hilbert space $\mathcal{H} = L^2([0,\infty), m)$, with the trace given by integration with respect to m, it is easy to see that the generalized singular value function $\mu(f)$ is precisely the decreasing rearrangement f^*. If $\mathcal{N}$ is all bounded operators (respectively, $\ell_\infty(\mathbb{N})$) and τ is the standard trace (respectively, the counting measure on $\mathbb{N}$), then $A \in \mathcal{N}$ is compact if and only if $\lim_{t\to\infty} \mu_t(A) = 0$; moreover,

$$\mu_n(A) = \mu_t(A), \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots,$$

and the sequence $\{\mu_n(A)\}_{n=0}^\infty$ is just the sequence of eigenvalues of $|A|$ in non-increasing order and counted according to multiplicity.

Given a semifinite von Neumann algebra $(\mathcal{N}, \tau)$ and a fully symmetric Banach function space $(E, \|\cdot\|_E)$ on $([0,\infty), m)$, satisfying $E \subseteq L_\infty[0,\infty)$, we define the corresponding non-commutative space $E(\mathcal{N}, \tau)$ by setting

$$E(\mathcal{N}, \tau) = \{A \in \mathcal{N} : \mu(A) \in E\}.$$

The norm is $\|A\|_{E(\mathcal{N},\tau)} := \|\mu(A)\|_E$, and the space $(E(\mathcal{N}, \tau), \|\cdot\|_{E(\mathcal{N},\tau)})$ is called the (non-commutative) fully symmetric operator space associated with $(\mathcal{N}, \tau)$ corresponding to $(E, \|\cdot\|_E)$. We write $E(\mathcal{N}, \tau)_+$ for the positive operators in $E(\mathcal{N}, \tau)$. If $\mathcal{N} = \ell_\infty(\mathbb{N})$, then the space $E(\mathcal{N}, \tau)$ is simply the (fully) symmetric sequence space ℓ_E, which may be viewed as the linear span in E of the vectors $e_n = \chi_{[n-1,n)}$, $n \geqslant 1$ (*cf* [31]).

The spaces $M(\psi)(\mathcal{N}, \tau)$ associated to Marcinkiewicz function spaces are called operator Marcinkiewicz spaces and we mostly omit the symbol $(\mathcal{N}, \tau)$ as this should not cause any confusion. We use, for the usual Schatten ideals in $\mathcal{N}$, the notation $L_p(\mathcal{N}, \tau)$, $p \geqslant 1$.

Definition 2.1. A linear functional $\varphi \in E(\mathcal{N}, \tau)^*$ is called symmetric if φ is positive, (that is, $\varphi(A) \geqslant 0$ whenever $0 \leqslant A \in E(\mathcal{N}, \tau)$) and $\varphi(A) \leqslant \varphi(A')$ whenever $\mu(A) \prec\prec \mu(A')$. A symmetric $\varphi \in E(\mathcal{N}, \tau)^*$ is called singular if it vanishes on all finite trace projections from $\mathcal{N}$.

The important examples of singular symmetric functionals that arise in noncommutative geometry are the Dixmier traces which we describe in the next Section. For the discussion of these we will need the following fact.

Theorem 2.2 ([24]). *Let φ_0 be a symmetric functional on E. If $\varphi(A) := \varphi_0(\mu(A))$, for all $A \geqslant 0$, $A \in E(\mathcal{N}, \tau)$, then φ extends to a symmetric functional $0 \leqslant \varphi \in E(\mathcal{N}, \tau)^*$.*

3. Invariant states and Dixmier traces.

The construction of Dixmier traces τ_ω depends crucially on the choice of the "invariant mean" ω. Here we explain the invariance properties we need for these invariant means via the results summarized below (all of them are proved using fixed point theorems).

We define the shift operator $T \colon \ell_\infty \to \ell_\infty$, the Cesàro operator $H : \ell_\infty \to \ell_\infty$ and dilation operators $D_n : \ell_\infty \to \ell_\infty$ for $n \in \mathbb{N}$ by the formulas

$$T\,(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

$$H\left(x_1, x_2, x_3, \dots\right) = (x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \dots),$$
$$D_n\left(x_1, x_2, x_3, \dots\right) = (\underbrace{x_1, \dots, x_1}_{n}, \underbrace{x_2, \dots, x_2}_{n}, \dots),$$

for all $x = (x_1, x_2, x_3, \dots) \in \ell_\infty$.

Theorem 3.1. [23] *There exists a state $\tilde{\omega}$ on ℓ_∞ such that for all $n \geqslant 1$*

$$\tilde{\omega} \circ T = \tilde{\omega} \circ H = \tilde{\omega} \circ D_n = \tilde{\omega}.$$

Now we consider analogous results for L_∞. We let $\mathbb{R}_+^*$ denote the positive reals with multiplication as the group operation. We define the isomorphism $L : L_\infty(\mathbb{R}) \to L_\infty(\mathbb{R}_+^*)$ by $L(f) = f \circ \log$. Next we define the Cesaro means (transforms) on $L_\infty(\mathbb{R})$ and $L_\infty(\mathbb{R}_+^*)$, respectively by:

$$H(f)(u) = \frac{1}{u} \int_0^u f(v) dv \quad \text{for} \quad f \in L_\infty(\mathbb{R}),\ u \in \mathbb{R}$$

and,

$$M(g)(t) = \frac{1}{\log t} \int_1^t g(s) \frac{ds}{s} \quad \text{for} \quad g \in L_\infty(\mathbb{R}_+^*),\ t > 0.$$

A brief calculation yields for $g \in L_\infty(\mathbb{R}_+^*)$, $LHL^{-1}(g)(r) = M(g)(r)$, i.e L intertwines the two means.

Definition 3.2. Let T_b denote translation by $b \in \mathbb{R}$, D_a denote dilation by $\frac{1}{a} \in \mathbb{R}_+^*$ and let P^a denote exponentiation by $a \in \mathbb{R}_+^*$. That is,

$$T_b(f)(x) = f(x + b) \quad \text{for} \quad f \in L_\infty(\mathbb{R}),$$
$$D_a(f)(x) = f\left(a^{-1}x\right) \quad \text{for} \quad f \in L_\infty(\mathbb{R}),$$
$$P^a(f)(x) = f(x^a) \quad \text{for} \quad f \in L_\infty(\mathbb{R}_+^*).$$

Proposition 3.3 ([6])**.** *If a continuous functional $\tilde{\omega}$ on $L_\infty(\mathbb{R})$ is invariant under the Cesaro operator H, the shift operator T_a or the dilation operator D_a then $\tilde{\omega} \circ L^{-1}$ is a continuous functional on $L_\infty(\mathbb{R}_+^*)$ invariant under M, the dilation operator D_a or P^a respectively. Conversely, composition with L converts an M, D_a or P^a invariant continuous functional on $L_\infty(\mathbb{R}_+^*)$ into an H, T_a or D_a invariant continuous functional on $L_\infty(\mathbb{R})$.*

We denote by $C_0(\mathbb{R})$ (respectively, $C_0(\mathbb{R}_+^*)$) the continuous functions on $\mathbb{R}$ (respectively, $\mathbb{R}_+^*$) vanishing at infinity (respectively at infinity and at zero).

Theorem 3.4 ([6])**.** *There exists a state $\tilde{\omega}$ on $L_\infty(\mathbb{R})$ satisfying the following conditions:*
(1) $\tilde{\omega}(C_0(\mathbb{R})) \equiv 0$.
(2) If f is real-valued in $L_\infty(\mathbb{R})$ then

$$ess \liminf_{t \to \infty} f(t) \leqslant \tilde{\omega}(f) \leqslant ess \limsup_{t \to \infty} f(t).$$

(3) If the essential support of f is compact then $\tilde{\omega}(f) = 0$.
(4) For all $a > 0$ and $c \in \mathbb{R}$ $\tilde{\omega} = \tilde{\omega} \circ T_c = \tilde{\omega} \circ D_a = \tilde{\omega} \circ H$.

Combining Theorem 3.4 and Proposition 3.3, we obtain

Corollary 3.5. *There exists a state ω on $L_\infty(\mathbb{R}_+^*)$ satisfying the following conditions:*
(1) $\omega(C_0(\mathbb{R}_+^*)) \equiv 0$.
(2) If f is real-valued in $L_\infty(\mathbb{R}_+^*)$ then

$$ess \liminf_{t\to\infty} f(t) \leqslant \omega(f) \leqslant ess \limsup_{t\to\infty} f(t).$$

(3) If the essential support of f is compact then $\omega(f) = 0$.
(4) For all $a, c > 0$ $\omega = \omega \circ D_c = \omega \circ P^a = \omega \circ M$.

Remark 3.6. In the sequel we will consider pairs of functionals $\tilde{\omega}$ on $L_\infty(\mathbb{R})$, $\omega \in L_\infty(\mathbb{R}_+^*)$ related by $\tilde{\omega} \circ L^{-1} = \omega$.

If ω is a state on ℓ_∞ (respectively, on $L_\infty(\mathbb{R})$, $L_\infty(\mathbb{R}_+^*)$), then we denote its value on the element $\{x_i\}_{i=1}^\infty$ (respectively, $f \in L_\infty(\mathbb{R})$, $L_\infty(\mathbb{R}_+^*)$) by $\omega - \lim_{i\to\infty} x_i$ (respectively, $\omega - \lim_{t\to\infty} f(t)$). We saw in Theorems 3.1, 3.4 and Corollary 3.5 states on ℓ_∞, $L_\infty(\mathbb{R})$, and $L_\infty(\mathbb{R}_+^*)$ invariant under various (group) actions. Alain Connes in [13] suggested working with the set of states on $L_\infty(\mathbb{R}_+^*)$, which is larger then the set

$$\{\omega : \quad \omega \text{ is an } M\text{-invariant state on } L_\infty(\mathbb{R}_+^*)\}$$

namely

$$CD(\mathbb{R}_+^*) := \{\tilde{\omega} = \gamma \circ M \ : \ \gamma \text{ is an arbitrary singular state on } C_b[0,\infty)\}.$$

These states are automatically dilation invariant. In this paper, we find that for the zeta function asymptotics it suffices to consider states that are D_2 and P^α invariant for all $\alpha > 1$.

In Section 5 we need a smaller set of states, namely a subset of

$$\{\omega \in L_\infty(\mathbb{R}_+^*)^* : \quad \omega \text{ is an } M\text{-invariant and } P^a\text{-invariant state on } L_\infty(\mathbb{R}_+^*),\ a > 0\}.$$

This subset consists of states whose existence is guaranteed by Corollary 3.5. We refer to any state satisfying the conditions (1) to (4) of Corollary 3.5 as a *DPM state* (in [6] we used the vaguer term 'maximally invariant'). We now recall the construction of Dixmier traces for the compact operators.

Definition 3.7. Let ω be a D_2-invariant state on ℓ_∞. The associated Dixmier trace of $T \in \mathcal{L}_+^{(1,\infty)}(\mathcal{H})$ is the number

$$\tau_\omega(T) := \omega\text{-}\lim_{N\to\infty} \frac{1}{\log(1+N)} \sum_{n=1}^{N} \mu_n(T).$$

Notice that in this definition we have chosen ω to satisfy only the dilation invariance assumption even though Dixmier [20] originally imposed on ω the assumption of dilation and translation invariance.

Definition 3.7 extends to the Marcinkiewicz spaces $M(\psi)(\mathcal{N},\tau)$. Fix an arbitrary D_2-invariant state ω on $L_\infty(\mathbb{R}_+^*)$. Then the state ω is D_{2^n}-invariant, $n \in \mathbb{Z}$ and a simple argument shows that it also satisfies conditions (1)–(3) of Corollary 3.5. For the remainder of the paper, let $\psi \in \Omega$ satisfy

$$\lim_{t\to\infty} \frac{\psi(2t)}{\psi(t)} = 1. \tag{2}$$

This condition is sufficient for the existence of singular traces or singular symmetric functionals on the corresponding fully symmetric operator spaces [24]. Indeed, setting

$$\tau_\omega(x) := \omega\text{-}\lim_{t\to\infty} a(x,t), \quad 0 \leqslant x \in M(\psi)(\mathcal{N},\tau) \tag{3}$$

(see the details in [24, p. 51]), we obtain an additive homogeneous functional on $M(\psi)(\mathcal{N},\tau)_+$, which extends to a symmetric functional on $M(\psi)(\mathcal{N},\tau)$ by linearity. The proof of linearity of τ_ω in [24, p. 51] is based on the assumption that ω is $D_{\frac{1}{2}}$-invariant which is equivalent to D_2-invariance (see above).

4. The Dixmier trace on Marcinkiewicz operator spaces

4.1. **Preliminaries.** In this subsection we generalize and strengthen some results from [6].

Lemma 4.1. *For every $\psi \in \Omega$ satisfying* (2) *and every $1 > \alpha > 0$, there is $C = C(\alpha)$ such that $\psi(t) < Ct^\alpha$, $t > 0$.*

Proof. Let $0 < \alpha$ and let $Q > 0$ be so large that for $t > Q$

$$\frac{\psi(2t)}{\psi(t)} < 2^\alpha.$$

There is $C > 1$ so large that $\psi(t) \leqslant Ct^\alpha$ for all $t < Q$. Suppose there is a first $Q_0 \geqslant Q$ for which $\psi(Q_0) = CQ_0^\alpha$. Then

$$\frac{\psi(Q_0)}{\psi(Q_0/2)} \geqslant \frac{CQ_0^\alpha}{C(Q_0/2)^\alpha} = 2^\alpha,$$

which is a contradiction. Consequently, $\psi(t) < Ct^\alpha$ for all $t > 0$. $\square$

Recall that for any τ-measurable operator T, the distribution function of T is defined by

$$\lambda_t(T) := \tau(\chi_{(t,\infty)}(|T|)), \quad t > 0,$$

where $\chi_{(t,\infty)}(|T|)$ is the spectral projection of $|T|$ corresponding to the interval (t,∞) (see [26]). By Proposition 2.2 of [26],

$$\mu_s(T) = \inf\{t \geqslant 0 \ : \ \lambda_t(T) \leqslant s\}.$$

We infer that for any τ-measurable operator T, the distribution function $\lambda_{(\cdot)}(T)$ coincides with the (classical) distribution function of $\mu_{(\cdot)}(T)$. From this formula and the fact that λ is right-continuous, we can easily see that for $t > 0$, $s > 0$

$$s \geqslant \lambda_t \Longleftrightarrow \mu_s \leqslant t.$$

Or equivalently,

$$s < \lambda_t \Longleftrightarrow \mu_s > t.$$

Using Remark 3.3 of [26] this implies that:

$$\int_0^{\lambda_t} \mu_s(T)ds = \int_{[0,\lambda_t)} \mu_s(T)ds = \tau(|T|\chi_{(t,\infty)}(|T|)), \quad t > 0. \tag{4}$$

Lemma 4.2. *For $T \in M(\psi)$ $T \geqslant 0$ and any $\beta > 1$ there is a $C = C(\beta)$ such that $\lambda_{1/t}(T) < Ct^\beta$ for every $t > 0$.*

Proof. Let $\alpha = 1 - 1/\beta$ and $\lambda_{1/t}(T) = a$. Hence $\mu_{(a-0)}(T) \geqslant 1/t$. Then by Lemma 4.1 there is $C_1 > 0$ such that

$$\|T\|_\psi = \sup_{0<h<\infty} \frac{\int_0^h \mu_s(T)ds}{\psi(t)} \geqslant \frac{\int_0^a \mu_{(a-0)}(T)ds}{\psi(a)} = \frac{a\mu_{(a-0)}(T)}{\psi(a)} \geqslant \frac{a(1/t)}{C_1 a^\alpha} = a^{1-\alpha}/(C_1 t).$$

Consequently

$$\lambda_{1/t}(T) = a < (C_1 \|T\|_\psi t)^{1/(1-\alpha)} = Ct^\beta.$$

□

Remark. Since $\beta > 1$ could be arbitrary, it is obvious that the constant C could be replaced by 1 if t is sufficiently large.

In the sequel we will suppose that ψ possesses the following property

$$A(\beta) = sup_{t>0} \frac{\psi(t^\beta)}{\psi(t)} \to 1, \text{ if } \beta \downarrow 1. \tag{5}$$

Observe that if $\psi(t) = \log(1+t)^\gamma$, $\gamma > 0$, then condition (5) is satisfied.

Proposition 4.3. (cf. [6, Proposition 2.4]) *For $T \in \mathcal{M}(\psi)$ positive let ω be D_2 and P^α-invariant, $\alpha > 1$ state on $L^\infty(\mathbb{R}_+^*)$. Then*

$$\tau_\omega(T) = \omega - \lim_{t\to\infty} \frac{1}{\psi(t))} \int_0^t \mu_s(T)ds = \omega - \lim_{t\to\infty} \frac{1}{\psi(t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T))$$

and if one of the $\omega-$limits is a true limit then so is the other.

Proof. We first note that

$$\int_0^t \mu_s(T)ds \leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)ds + 1, \quad t > 0.$$

Indeed, the inequality above holds trivially if $t \leqslant \lambda_{\frac{1}{t}}(T)$. If $t > \lambda_{\frac{1}{t}}(T)$, then

$$\int_0^t \mu_s(T)ds = \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)ds + \int_{\lambda_{\frac{1}{t}}(T)}^t \mu_s(T)ds.$$

Now $s > \lambda_{\frac{1}{t}}(T)$ implies that $\mu_s(T) \leqslant \frac{1}{t}$ so we have

$$\int_0^t \mu_s(T)ds \leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)ds + \frac{1}{t}(t - \lambda_{\frac{1}{t}}(T)) \leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)ds + 1.$$

Using this observation and lemma and remark above we see that for $\alpha > 1$ eventually

$$\int_0^t \mu_s(T)ds \leqslant \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)ds + 1 \leqslant \int_0^{t^\alpha} \mu_s(T)ds + 1$$

and so eventually

$$\frac{1}{\psi(t)} \int_0^t \mu_s(T)ds \leqslant \frac{1}{\psi(t)}(\int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)ds + 1) \leqslant \frac{1}{\psi(t)}(\int_0^{t^\alpha} \mu_s(T)ds + 1)$$
$$\leqslant \frac{\psi(t^\alpha)}{\psi(t)\psi(t^\alpha)}(\int_0^{t^\alpha} \mu_s(T)ds + 1).$$

Taking the ω-limit we get

$$\tau_\omega(T) \leqslant \omega - \lim_{t\to\infty} \frac{1}{\psi(t)} \int_0^{\lambda_{\frac{1}{t}}(T)} \mu_s(T)ds \leqslant \omega - \lim_{t\to\infty} \frac{1}{\psi(t)} \int_0^{t^\alpha} \mu_s(T)ds$$
$$\leqslant \omega - \lim_{t\to\infty} \frac{A(\alpha)}{\psi(t^\alpha)} \int_0^{t^\alpha} \mu_s(T)ds = A(\alpha)\tau_\omega(T)$$

where the last line uses P^α, $\alpha > 1$, invariance. Due to equality (4) and since the previous inequalities hold for all $\alpha > 1$ and by assumption (5) we have $A(\alpha) \to 1$ we get the conclusion of the proposition for ω-limits.

To see the last assertion of the Proposition suppose that $\lim_{t\to\infty} \frac{1}{\psi(t)} \int_0^t \mu_s(T)ds = B$ then by the above argument for any $\epsilon > 0$ and sufficiently large $t > 0$ we get

$$B - \epsilon \leqslant \frac{1}{\psi(t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T)) \leqslant A(\alpha)(B + \epsilon)$$

for all $\alpha > 1$ and since $A(\alpha) \to 1$, $\lim_{t\to\infty} \frac{1}{\psi(t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T)) = B$. On the other hand if the limit $\lim_{t\to\infty} \frac{1}{\psi(t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T))$ exists and equals B say then

$$\limsup_{t\to\infty} \frac{1}{\psi(t)} \int_0^t \mu_s(T)ds \leqslant B \leqslant A(\alpha) \liminf_{t\to\infty} \frac{1}{\psi(t)} \int_0^t \mu_s(T)ds$$

for all $\alpha > 1$ and so $\lim_{t\to\infty} \frac{1}{\psi(t)} \int_0^t \mu_s(T)ds = B$ as well. $\square$

Corollary 4.4. *Under the conditions of the preceding Proposition the expression*

$$\omega - \lim_{t\to\infty} \frac{1}{\psi(t)} \tau(T\chi_{(\frac{1}{t},\infty)}(T))$$

can be replaced by

$$\omega - \lim_{t\to\infty} \frac{1}{\psi(t)} \tau(T\chi_{(\frac{1}{t},1)}(T)).$$

If the real limit exists then the prefix ω may be removed.

The proof is immediate since $\psi(\infty) = \infty$ and the difference of these limits is

$$\lim_{t\to\infty} \frac{1}{\psi(t)} \tau(T\chi_{(1,\infty)}(T)) = \lim_{t\to\infty} \frac{1}{\psi(t)} \int_0^{\lambda_1(T)} \mu_s(T)ds = 0.$$

4.2. **An alternative description of $\mathcal{L}^{(1,\infty)}$.** The zeta function of a positive compact operator T is given by $\zeta(s) = \tau(T^s)$ for real positive s on the assumption that there exists some s_0 for which the trace is finite. Note that it is then true that $\tau(T^s) < \infty$ for all $s > s_0$. In this subsection we will always assume $\tau(T^s) < \infty$ for all $s > 1$ and we are interested in the asymptotic behavior of $\zeta(s)$ as $s \to 1$.

Let us define the space

$$\mathcal{Z}_1 = \{T \in \mathcal{N} : \; \|T\|_{\mathcal{Z}_1} = \limsup_{p\downarrow 1} (p-1)\tau(|T|^p) < \infty\}.$$

Since we also have the other equivalent definition

$$\|T\|_{\mathcal{Z}_1} = \limsup_{p\downarrow 1} (p-1)(\int_0^\infty \mu_t(|T|)^p dt)^{1/p} = \limsup_{p\downarrow 1} (p-1)\|T\|_{L_p}$$

(recall that we use the notation L_p for the Schatten ideals in $(\mathcal{N},\tau)$) the ordinary properties of the semi-norm for $\|\cdot\|_{\mathcal{Z}_1}$ are immediate.

Theorem 4.5. (i) *Let* $T \geqslant 0$*,* $T\in\mathcal{N}$ *and* $\limsup_{s\to 0} s\tau(T^{1+s}) = C < \infty$*, then*

$$\limsup_{u\to\infty}\frac{1}{\ln u}\int_0^u \mu_t(T)dt \leqslant Ce.$$

(ii) *The spaces* $\mathcal{Z}_1$ *and* $\mathcal{L}^{1,\infty}$ *coincide. Moreover, if* $\mathcal{N}$ *is a type I factor with the standard trace, or else* $\mathcal{N}$ *is semifinite and the trace is non-atomic then denoting by* $\mathcal{L}_0^{1,\infty}$ *the closure of* $L_1(\mathcal{N},\tau)$ *in* $\mathcal{L}^{1,\infty}$*, we have for any* $T\in\mathcal{C}_1$

$$\mathrm{dist}_{\mathcal{L}^{1,\infty}}(T,\mathcal{L}_0^{1,\infty}) = \limsup_{u\to\infty}\frac{1}{\ln u}\int_0^u \mu_t(T)dt \leqslant e\|T\|_{\mathcal{Z}_1}$$

and $\|T\|_{\mathcal{Z}_1} \leqslant \|T\|_{1,\infty}$*.*

Proof. (i) By assumption for every $\epsilon > 0$ there is an $s_0 > 0$ such that for all $s\in[0,s_0]$

$$s\int_0^\infty \mu_t(T)^{1+s}dt \leqslant C+\epsilon. \tag{6}$$

Then, for $u\geqslant 1$ according to Hölder's inequality and (6) we have

$$\int_0^u \mu_t(T)dt \leqslant \left(\int_0^u \mu_t(T)^{1+s}dt\right)^{\frac{1}{1+s}}\left(\int_0^u 1^{\frac{1+s}{s}}dt\right)^{\frac{s}{1+s}} \leqslant$$

$$\left(\frac{s}{s}\int_0^\infty \mu_t(T)^{1+s}dt\right)^{\frac{1}{1+s}} u^{\frac{s}{1+s}} \leqslant ((C+\epsilon)/s)^{\frac{1}{1+s}}u^{\frac{s}{1+s}} \leqslant (C+\epsilon)\frac{1}{s}u^s.$$

Set $u_0 = e^{1/s_0}$ and for $u > u_0$ set $s = 1/\ln u (< s_0)$. Then $u = e^{\ln u}$ and by the previous inequality

$$\int_0^u \mu_t(T)dt \leqslant (C+\epsilon)\frac{1}{s}u^s = (C+\epsilon)\frac{e^{\ln u\frac{1}{\ln u}}}{\frac{1}{\ln u}} = (C+\epsilon)e\ln u.$$

That is we have the inequality

$$\frac{1}{\ln u}\int_0^u \mu_t(T)dt \leqslant (C+\epsilon)e \text{ for } u > u_0.$$

Since

$$\|T\|_{\mathcal{L}^{1,\infty}} = \sup_{1\leqslant u\leqslant\infty}\frac{1}{\ln(1+u)}\int_0^u \mu_t(T)dt$$

we conclude that $T\in\mathcal{L}^{1,\infty}$. Moreover, since $\epsilon>0$ is arbitrary

$$\limsup_{u\to\infty}\frac{1}{\ln u}\int_0^u \mu_t(T)dt \leqslant eC.$$

Hence (i) and the embedding $\mathcal{Z}_1\subset\mathcal{L}^{1,\infty}$ are established.

The equality $\mathrm{dist}_{\mathcal{L}^{1,\infty}}(T,\mathcal{L}_0^{1,\infty}) = \limsup_{u\to\infty}\frac{1}{\ln u}\int_0^u \mu_t(T)dt$ is well-known in the special case when the algebra $\mathcal{N}$ is commutative (see e.g. [22, Proposition 2.1] and references therein). The general case follows from this special case, due to the combination of the following facts. Firstly, the inequality $\mu(x)-\mu(y) \prec\prec \mu(x-y)$ (see [21]) together with the fact that $\mathcal{L}^{1,\infty}$ is fully symmetric yields the inequality $\mathrm{dist}_{\mathcal{L}^{1,\infty}}(T,\mathcal{L}_0^{1,\infty}) \geqslant \mathrm{dist}_{\mathcal{L}^{1,\infty}}(\mu(T),\mathcal{L}_0^{1,\infty}(0,\infty))$ or $\mathrm{dist}_{\mathcal{L}^{1,\infty}}(T,\mathcal{L}_0^{1,\infty}) \geqslant \mathrm{dist}_{\mathcal{L}^{1,\infty}}(\mu(T),\mathcal{L}_0^{1,\infty}(\mathbf{N}))$, depending whether $\mathcal{N}$ is of type II or

I. Secondly, fix an arbitrary $T \in \mathcal{L}^{1,\infty}(\mathcal{N})$. Due to [11], there exists a rearrangement-preserving (and thus, isometric) embedding φ_T of $\mathcal{L}^{1,\infty}(0,\infty)$ (respectively, $\mathcal{L}^{1,\infty}(\mathbf{N})$ in the type I setting) into $\mathcal{L}^{1,\infty}(\mathcal{N})$ such that $\varphi_T(\mu(T)) = T$. This observation shows that $\mathrm{dist}_{\mathcal{L}^{1,\infty}}(T, \mathcal{L}_0^{1,\infty}) \leqslant \mathrm{dist}_{\mathcal{L}^{1,\infty}}(\mu(T), \mathcal{L}_0^{1,\infty}(0,\infty))$.

The argument above also proves the equality and the first inequality in (ii).

To complete the proof of (ii), let us take an arbitrary $T \in \mathcal{L}^{1,\infty}$ and note that by the definition of the norm in the Marcinkiewicz space $\mathcal{L}^{1,\infty}$ we have $x \prec\prec \|T\|_{1,\infty}/(1+t)$. Since the spaces $L_p(\mathcal{N},\tau)$, $1 \leqslant p \leqslant \infty$, are fully symmetric operator spaces we have

$$\|T\|_p \leqslant \|T\|_{1,\infty}\|1/(1+t)\|_p,\ p > 1.$$

Taking the p-th power we get

$$\int_0^\infty \mu_t(T)^p\,dt \leqslant \|T\|_{1,\infty}^p \int_0^\infty 1/(1+t)^p dt = \|T\|_{1,\infty}^p \frac{1}{p-1}.$$

If now $p \downarrow 1$ we conclude that

$$\|T\|_{\mathcal{Z}_1} = \limsup_{p\downarrow 1}(p-1)\int_0^\infty \mu_t(T)^p\,dt \leqslant \|T\|_{1,\infty}.$$

Hence, $\mathcal{L}^{1,\infty} \subset \mathcal{Z}_1$. Due to the first part of the proof we infer that the spaces $\mathcal{Z}_1$ and $\mathcal{L}^{1,\infty}$ are coincident. $\square$

Corollary 4.6. *Let $T \in \mathcal{N}$ be positive with $\tau(T^s) < \infty$ for all $s > 1$. If $\lim_{r\to\infty}\frac{1}{r}\tau(T^{1+\frac{1}{r}})$ exists then $T \in \mathcal{L}^{(1,\infty)}$.*

4.3. **The case $p > 1$.** Our approach above to the study of $\mathcal{Z}_1$ allows us to generalize immediately. Let us define a class of spaces $\mathcal{Z}_q,\ q \geqslant 1$ by:

$$\mathcal{Z}_q = \{T \in \mathcal{N}_+ :\ \|T\|_{\mathcal{Z}_q} = \limsup_{p\downarrow q}((p-q)\tau(T^p))^{1/p} < \infty\}.$$

Setting $r = 1 + \frac{p-q}{q} = \frac{p}{q}$, we have

$$\|T\|_{\mathcal{Z}_q} = \limsup_{p\downarrow q}((p-q)\tau(T^{q(1+(p-q)/q)}))^{1/p} = (q\limsup_{p\downarrow q}(p-q)/q\tau((T^q)^{(1+(p-q)/q)}))^{1/p}$$

$$= q^{1/q}(\limsup_{r\downarrow 1}((r-1)\tau((T^q)^r))^{1/(qr)} = (q\|T^q\|_{\mathcal{Z}_1})^{1/q}.$$

Now it is clear that $T \in \mathcal{Z}_q$ if and only if $T^q \in \mathcal{Z}_1$ and $\|T\|_{\mathcal{Z}_q} = (q\|T^q\|_{\mathcal{Z}_1})^{1/q}$.

We now state a few consequences of Theorem 4.5. The classical p-convexification procedure for an arbitrary Banach lattice X is described in [32, Section 1.d] and is sometimes termed power norm transformation. It is simply a direct generalization of the procedure of defining L_p-spaces from an L_1-space.

The proof of the first corollary below is immediate.

Corollary 4.7. (i) *There is a more convenient equivalent formula for the semi-norm $\|\cdot\|_{\mathcal{Z}_q}$ namely*

$$\|T\|_{\mathcal{Z}_q}^+ = \|T^q\|_{\mathcal{Z}_1}^{1/q},\ q \geqslant 1.$$

(ii) *The space* $\mathcal{Z}_q$ *coincides as a set with the* q*-convexification of the operator space* $\mathcal{L}^{1,\infty}$ *:*

$$\mathcal{L}_q^{1,\infty} = \{T \in \mathcal{N}_+ : \ \|T\|_{1,\infty}^q = \sup_{1<u<\infty} \left(\frac{\int_0^u \mu_t(T)^q dt}{\log(1+u)} \right)^{1/q} < \infty\}.$$

If $\mathcal{N}$ *is a type I factor with the standard trace, or else* $\mathcal{N}$ *is semifinite and the trace is non-atomic then the semi-norms* $\|\cdot\|_{\mathcal{Z}_q}$ *and* $\mathrm{dist}_{\mathcal{L}_q^{1,\infty}}(\cdot, \mathcal{L}_{q,0}^{1,\infty})$ *are equivalent. Here,* $\mathcal{L}_{q,0}^{1,\infty}$ *is the closure of* $L_1(\mathcal{N},\tau)$ *in* $\mathcal{L}_q^{1,\infty}$*.*

Corollary 4.8. *(i) An element* $T \in \mathcal{Z}_p,\ p \geqslant 1,$ *iff* $T^p \in \mathcal{L}^{1,\infty}$*. Moreover*

$$\frac{1}{r}\int_0^\infty \mu_t(T)^{p+1/r} dt = \frac{1}{r}\tau(T^{p+1/r}) = p\frac{1}{pr}\tau(T^{p(1+1/pr)}). \tag{7}$$

and for $r > 0$ *the expression in (7) belongs to* $L^\infty(\mathbb{R}_+^*)$*.*
(ii) If $T \in \mathcal{L}^{p,\infty}$ *then* $T \in \mathcal{Z}_p$*.*
(iii) If T *is a positive in* $\mathcal{N}$ *such that* $\lim_{r\to\infty} \frac{1}{r}\tau(T^{p+\frac{1}{r}})$ *exists, then* $T \in \mathcal{Z}_p$*.*

Proof. The first statement is immediate from earlier results. To prove (ii) we remind the reader that $T \in \mathcal{L}^{p,\infty}$ iff $\mu_t(T) \leqslant C\min(1, t^{-1/p})$ for some $C < \infty$. Then as $r \to \infty$

$$\frac{1}{r}\int_0^\infty \mu_t(T)^{p+1/r} dt \leqslant C\frac{1}{r}(1 + \int_1^\infty t^{-1-1/pr} dt) = C\frac{1}{r}(1 - prt^{-1/pr}|_1^\infty) = C\frac{(1+pr)}{r} < \infty.$$

For (iii), we note that if $\lim_{r\to\infty} \frac{1}{r}\tau(T^{p+\frac{1}{r}})$ exists, then $T^p \in \mathcal{Z}_1$ and by (i) $T \in \mathcal{Z}_p$ $\square$

In view of the preceding corollary we have the following implications

$$T \in \mathcal{L}^{p,\infty} \Longrightarrow T \in \mathcal{Z}_p,$$
$$T \in \mathcal{Z}_p \Longleftrightarrow T^p \in \mathcal{Z}_1 = \mathcal{L}^{1,\infty}.$$

Hence, everything which has been proved for $T \in \mathcal{Z}_1 = \mathcal{L}^{1,\infty}$ is automatically true for $S = T^p$ provided $T \in \mathcal{Z}_p$ or especially if $T \in \mathcal{L}^{p,\infty}$.

4.4. **The space** $\mathcal{Z}_p,\ p > 1$ **is strictly larger than** $\mathcal{L}^{p,\infty}$**.** We deduce the result in the title of this subsection by proving that the analogue of Theorem 4.5 does not hold when $p > 1$.

Proposition 4.9. *The assumption* $\sup_{r\geqslant 1} \frac{1}{r}\tau(T^{p+\frac{1}{r}}) < \infty$ *does not guarantee* $T \in \mathcal{L}^{(p,\infty)}$*.*

Proof. We use the notation $\mu_t(T) := x(t),\ t > 0$. The proof is based on the observation (see [30] and also detailed explanations in [39, Section 5]) that the ordinary norm

$$\|x\|_\psi = \sup_{t>0} \frac{\int_0^t x^*(s)ds}{\psi(t)}$$

in the Marcinkiewicz space $M(\psi)$ (here, $\psi \in \Omega$ as in Section 2) is equivalent to the quasi-norm

$$F_\psi(x) = \sup_{0<t<\infty} \frac{tx^*(t)}{\psi(t)}$$

provided that $\liminf_{t\to\infty} \frac{\psi(2t)}{\psi(t)} > 1$. For $\psi_p(t) = t^{1-1/p},\ p > 1,$ the norm $\|\cdot\|_{\psi_p}$ and quasi-norm $F_p(\cdot) = F_{\psi_p}(\cdot)$ are equivalent. In other words, the norm of any element T from the ideal $\mathcal{L}^{(p,\infty)}$ is equivalent to $F_p(x)$. This is not the case for $\psi_0(t) := \ln(1+t)$ (that is the functional $F_0(\cdot) = F_{\psi_0}(\cdot)$ and the norm in $\mathcal{L}^{(1,\infty)}$ are not equivalent) and it is easy to locate a function

$z(t) = z^*(t)$ such that $\|z\|_{\psi_0} < \infty$ but $F_0(z) = \sup_{t>0} z^*(t)t = \infty$. For example, we take $z(t) = n/2^{n^2}$ for $t \in (2^{(n-1)^2}, 2^{n^2}]$, $n = 1, 2, ...,$ and $z(t) = 1$ for $t \in [0, 1]$. It is easy to verify that there exists $0 < C < \infty$ such that

$$\int_0^t z^*(s)ds \leqslant C \ln(1+t)$$

(that is $z \prec\prec C/(1+t)$) and at the same time

$$z(t)t|_{t=2^{n^2}} = z(2^{n^2})2^{n^2} = n, \ n = 1, 2, ...$$

(that is $F_0(z) = \infty$).

Observe that since $z \prec\prec C/(1+t)$, we have for every $\nu > 0$

$$\int_0^\infty z(t)^{1+\nu}dt \leqslant C \int_0^\infty (1/(1+t))^{1+\nu}dt = C/\nu < \infty.$$

Now, let us fix $p > 1$ and set $x(t) = z^{1/p}(t)$ for $t > 0$. The estimate above gives

$$s\int_0^\infty x^{p+s}(t)dt = p(s/p\int_0^\infty z(t)^{1+s/p}dt) \leqslant Cp < \infty.$$

Nevertheless,

$$F_p(x) = \sup_{0<t<\infty} x(t)t^{1/p} = (F_0(z))^{1/p} = \infty.$$

That is the condition $\sup_{r\geqslant 1} \frac{1}{r}\tau(T^{p+\frac{1}{r}}) < \infty$ does not imply $T \in \mathcal{L}^{(p,\infty)}$. $\square$

We remark that while $\mathcal{Z}_p$, $p > 1$ is the p-convexification of the ideal $\mathcal{L}^{1,\infty}$; in turn, the ideal $\mathcal{L}^{p,\infty}$ is the p-convexification of some subideal in $\mathcal{L}^{1,\infty}$, which is termed the 'small ideal' in [6]. We will establish this latter fact in subsection 5.2.

4.5. **Limits of zeta functions.** Our earlier results enable us to considerably weaken the hypotheses in one of the main theorems of [6]. First we recall the following preliminary result proved in [6].

Proposition 4.10. (weak*-Karamata theorem) *Let $\tilde{\omega} \in L_\infty(\mathbb{R})^*$ be a dilation invariant state and let β be a real valued, increasing, right continuous function on $\mathbb{R}_+$ which is zero at zero and such that the integral $h(r) = \int_0^\infty e^{-\frac{t}{r}}d\beta(t)$ converges for all $r > 0$ and $C = \tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}h(r)$ exists. Then*

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}h(r) = \tilde{\omega} - \lim_{t\to\infty} \frac{\beta(t)}{t}.$$

The classical Karamata theorem has a similar statement with the $\tilde{\omega}$ limits replaced by ordinary limits.

In the following we will take $T \in \mathcal{L}^{(1,\infty)}$ positive, $||T|| \leqslant 1$ with spectral resolution $T = \int \lambda dE(\lambda)$. We would like to integrate with respect to $d\tau(E(\lambda))$; unfortunately, these scalars $\tau(E(\lambda))$ are, in general, all infinite. To remedy this situation, we instead must integrate with respect to the increasing (negative) real-valued function $N_T(\lambda) = \tau(E(\lambda) - 1)$ for $\lambda > 0$. Away from 0, the increments $\tau(\triangle E(\lambda))$ and $\triangle N_T(\lambda)$ are, of course, identical. The following theorem is a strengthened version of Theorem 3.1 of [6] made possible by Proposition 4.3.

Theorem 4.11. *For $T \in \mathcal{L}^{(1,\infty)}$ positive, $||T|| \leqslant 1$ let ω be a D_2-dilation and P^α-invariant, $\alpha > 1$ state on $L^\infty(\mathbb{R}_+^*)$. Let $\tilde{\omega} = \omega \circ L$ where L is given in Section 3, then we have:*

$$\tau_\omega(T) = \tilde{\omega} - \lim \frac{1}{r}\tau(T^{1+\frac{1}{r}}).$$

If $\lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}})$ exists then

$$\tau_\omega(T) = \lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}})$$

for an arbitrary dilation invariant functional $\omega \in L^\infty(\mathbb{R}_+^*)^*$.

Proof. The proof is just a minor rewriting of the corresponding argument in [6]. By Proposition 3.3, the state $\tilde{\omega}$ is dilation invariant and by Theorem 4.5(i) $h(r) = \frac{1}{r}\tau(T^{1+\frac{1}{r}}) \in L^\infty(\mathbb{R}_+)$. So, we can apply the weak*-Karamata theorem. First write $\tau(T^{1+\frac{1}{r}}) = \int_{0+}^1 \lambda^{1+\frac{1}{r}} dN_T(\lambda)$. Thus setting $\lambda = e^{-u}$

$$\tau(T^{1+\frac{1}{r}}) = \int_0^\infty e^{-\frac{u}{r}} d\beta(u)$$

where $\beta(u) = \int_u^0 e^{-v} dN_T(e^{-v}) = -\int_0^u e^{-v} dN_T(e^{-v})$. Since the change of variable $\lambda = e^{-u}$ is strictly decreasing, β is, in fact, nonnegative and increasing. By the weak*-Karamata theorem applied to $\tilde{\omega} \in L^\infty(\mathbb{R})^*$

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}}) = \tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u}.$$

Next with the substitution $\rho = e^{-v}$ we get:

$$\tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u} = \tilde{\omega} - \lim_{u\to\infty} \frac{1}{u} \int_{e^{-u}}^1 \rho dN_T(\rho). \tag{8}$$

Set $f(u) = \frac{\beta(u)}{u}$. We want to make the change of variable $u = \log t$ or in other words to consider $f \circ \log = Lf$. This is permissable by the discussion in Section 3 which tells us that if we start with a functional $\omega \in L^\infty(\mathbb{R}_+^*)^*$ as in the theorem we may replace it by the functional $\tilde{\omega} = \omega \circ L$ which is dilation invariant with

$$\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\tau(T^{1+\frac{1}{r}}) = \tilde{\omega} - \lim_{u\to\infty} \frac{\beta(u)}{u}$$

$$= \tilde{\omega} - \lim_{u\to\infty} f(u) = \omega - \lim_{t\to\infty} Lf(t) = \omega - \lim_{t\to\infty} \frac{1}{\log t} \int_{1/t}^1 \lambda dN_T(\lambda).$$

Now, by Proposition 4.3 and Corollary 4.4 applied to $\psi(t) = \log(1+t) \sim \log t$

$$\omega - \lim_{t\to\infty} \frac{1}{\log t} \int_{1/t}^1 \lambda dN_T(\lambda) = \omega - \lim_{t\to\infty} \frac{1}{\log t} \tau(\chi_{(\frac{1}{t},1]}(T)T) = \tau_\omega(T).$$

This completes the proof of the first part of the theorem.

The proof of the second part is similar. Using the classical Karamata theorem we obtain the following analogue of (8):

$$\lim_{r\to\infty} \frac{1}{r}\tau(T^{1+r}) = \lim \frac{\beta(u)}{u} = \lim_{u\to\infty} \frac{1}{u} \int_{e^{-u}}^1 \rho dN_T(\rho).$$

Making the substitution $u=\log t$ on the right hand side we have by Proposition 4.3

$$\lim_{u\to\infty}\frac{1}{u}\int_{e^{-u}}^{1}\rho dN_T(\rho)=\lim_{t\to\infty}\frac{1}{\log t}\int_{\frac{1}{t}}^{1}\lambda dN_T(\lambda)=\tau_\omega(T)=\lim_{t\to\infty}\frac{1}{\log(1+t)}\int_0^t\mu_s(T)ds.$$

□

We now deduce some corollaries of the discussion above. Retaining the notation as in the previous theorem we let ω be a D_2-dilation and P^α-invariant, $\alpha>1$ state on $L^\infty(\mathbb{R}_+^*)$. Let $\tilde\omega=\omega\circ L$. The assumption that $\frac{1}{r}\zeta(T^{1+\frac{1}{r}})$ is bounded in r means that, by Theorem 4.5, $T\in\mathcal{Z}_1=\mathcal{L}^{1,\infty}$. Then by Theorem 4.11

$$\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\zeta(T^{1+\frac{1}{r}})=\tau_\omega(T).$$

Consequently using (7) if either $T\in\mathcal{Z}_p$ or if $T\in\mathcal{L}^{p,\infty},\ p>1$ we have the formulae

$$\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\zeta(T^{p+\frac{1}{r}})=\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\tau(T^{p+1/r})=p\tilde\omega-\lim_{pr\to\infty}\frac{1}{pr}\tau(T^{p(1+1/pr)})=p\tau_\omega(T^p) \tag{9}$$

where the last step uses dilation invariance of $\tilde\omega$, which is guaranteed by our choice of ω. The equation (9) together with Theorem 4.11 tell us that if one of the limits in the previous equality is true then so are the others. In particular, if $\lim_{r\to\infty}\frac{1}{r}\zeta(T^{p+\frac{1}{r}})$ exists, then $T\in\mathcal{Z}_p$ and

$$\lim_{r\to\infty}\frac{1}{r}\zeta(T^{p+\frac{1}{r}})=p\lim_{t\to\infty}\frac{1}{\log(1+t)}\int_0^t\mu_s(T^p)ds.$$

5. The heat semigroup formula

5.1. **Asymptotics of the trace of the heat semigroup.** Throughout this section $T\geqslant 0$. For $q\in\mathbb{R}_+$ we define $e^{-T^{-q}}$ as the operator that is zero on $\ker T$ and on $\ker T^\perp$ is defined in the usual way by the functional calculus. We remark that if $T\geqslant 0,\ T\in\mathcal{Z}_p$ for some $p\geqslant 1$ then $e^{-tT^{-q}}$ is trace class for all $t>0$. This is because if $x\in E$, where $(E,\|\cdot\|_E)$ is any symmetric (or r.i.) space then

$$\|x\|_E\geqslant\|x^*(t)\chi_{[0,s]}(t)\|_E\geqslant x^*(s)\|\chi_{[0,s]}\|_E=x^*(s)\varphi(s),$$

where $\varphi(\cdot)$ is the fundamental function of E. Consequently, $x^*(s)\leqslant\|x\|_E/\varphi(s)$. For $E=\mathcal{Z}_p=\mathcal{L}_p^{1,\infty}$ (see Corollary 4.7(ii)) the fundamental function is $\varphi(s)=(s/log(1+s))^{1/p}$ Hence, for every $t>0$

$$\mu_s(e^{-tT^{-q}})=e^{-t/(\mu_s(T))^q}\leqslant e^{-tC(s/log(1+s))^{q/p}}\leqslant e^{-tCs^{q/p-\epsilon}}$$

for some $C>0$ all $0<p,q$ and $0<\epsilon<q/p$. Thus $\tau(e^{-tT^{-q}})<\infty$ for $q>0$ (since $\epsilon>0$ is arbitrary).

Theorem 5.1. (cf [6]) *If $T\geqslant 0$, $T\in\mathcal{Z}_p$, $1\leqslant p<\infty$ then, choosing ω to be DPM invariant and $\tilde\omega$ to be related with ω as in Remark 3.6, we have for $q>0$*

$$\omega-\lim_{\lambda\to\infty}\frac{1}{\lambda}\tau(e^{-T^{-q}\lambda^{-q/p}})=\frac{1}{q}\Gamma(p/q)\tilde\omega-\lim_{r\to\infty}\frac{1}{r}\zeta(p+\frac{1}{r})=\frac{p}{q}\Gamma(p/q)\tau_\omega(T^p).$$

Proof. We have, using the Laplace transform,

$$T^s=\frac{1}{\Gamma(s/q)}\int_0^\infty t^{s/q-1}e^{-tT^{-q}}dt.$$

Then

$$\Gamma(s/q)\zeta(s) = \Gamma(s/q)\tau(T^s) = \int_0^\infty t^{s/q-1}\tau(e^{-tT^{-q}})dt.$$

We split this integral into two parts, $\int_0^1$ and $\int_1^\infty$ and call the second integral $R(r)$ where $s = p + \frac{1}{r}$. Then

$$R(r) = \int_1^\infty t^{p/q+1/(qr)-1}\tau(e^{-tT^{-q}})dt.$$

The integrand decays exponentially in t as $t \to \infty$ because $T^{-q} \geqslant \|T^q\|^{-1}\mathbf{1}$ so that

$$\tau(e^{-tT^{-q}}) \leqslant \tau(e^{-T^{-q}}e^{-\frac{t-1}{\|T^q\|}}).$$

Then we can conclude that $R(r)$ is bounded independently of r and so $\lim_{r\to\infty}\frac{1}{r}R(r) = 0$. For the other integral $\int_0^1 t^{p/q+1/(qr)-1}\tau(e^{-tT^{-q}})dt$ we can make the substitution $t = e^{-\mu q/p}$. Then elementary calculus gives

$$\int_0^1 t^{p/q+1/(qr)-1}\tau(e^{-tT^{-q}})dt = -q/p\int_\infty^0 e^{-\mu(1+\frac{1}{pr})}\tau(e^{-e^{-\mu q/p}T^{-q}})d\mu = q/p\int_0^\infty e^{-\frac{\mu}{pr}}d\beta(\mu)$$

where $\beta(\mu) = \int_0^\mu e^{-v}\tau(e^{-e^{-vq/p}T^{-q}})dv$. Hence we can now write

$$\Gamma(p/q + \frac{1}{rq})\zeta(p+\frac{1}{r}) = q/p\int_0^\infty e^{-\frac{\mu}{pr}}d\beta(\mu) + R(r).$$

Then we have (remembering that the term $\frac{1}{r}R(r)$ has limit zero as $r \to \infty$)

$$\tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\Gamma(p/q + \frac{1}{pr})\zeta(p+\frac{1}{r}) = \Gamma(p/q)\tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\zeta(p+\frac{1}{r})$$

$$= \tilde{\omega} - \lim_{r\to\infty}\frac{q}{pr}\int_0^\infty e^{-\mu/pr}d\beta(\mu) = q\tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\int_0^\infty e^{-\mu/r}d\beta(\mu)$$

where the last step uses the assumed dilation invariance of $\tilde{\omega}$. So

$$\tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\Gamma(p/q + \frac{1}{pr})\zeta(p+\frac{1}{r}) = q\tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\int_0^\infty e^{-\frac{\mu}{r}}d\beta(\mu)$$

Now we are exactly in a position to use the weak*-Karamata theorem above to evaluate the RHS. Indeed, we now conclude

$$\tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\int_0^\infty e^{-\frac{\mu}{r}}d\beta(\mu) = \tilde{\omega} - \lim_{\mu\to\infty}\frac{\beta(\mu)}{\mu}.$$

We can summarise the preceding in the equation

$$\Gamma(p/q)\tilde{\omega} - \lim_{r\to\infty}\frac{1}{r}\zeta(p+\frac{1}{r}) = q\tilde{\omega} - \lim_{\mu\to\infty}\frac{\beta(\mu)}{\mu}. \tag{10}$$

Now make the change of variable $\lambda = e^v$ in the defining expression for $\beta(\mu)$ to obtain

$$\frac{\beta(\mu)}{\mu} = \frac{1}{\mu}\int_1^{e^\mu}\lambda^{-2}\tau(e^{-\lambda^{-q/p}T^{-q}})d\lambda$$

Make the substitution $\mu = \log t$ so the RHS becomes

$$\frac{1}{\log t}\int_1^t \lambda^{-2}\tau(e^{-T^{-q}\lambda^{-q/p}})d\lambda = g_1(t)$$

This is the Cesaro mean of

$$g_2(\lambda) = \frac{1}{\lambda}\tau(e^{-T^{-q}\lambda^{-q/p}}).$$

Thus as we chose $\omega \in L_\infty(\mathbb{R}_+^*)^*$ to be M invariant and $\tilde{\omega}$ to be related to ω as in Remark 3.6 we have

$$\tilde{\omega} - \lim_{\mu\to\infty} \frac{\beta(\mu)}{\mu} = \omega(g_1) = \omega(g_2).$$

Then using (10), we obtain

$$\Gamma(p/q)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta(p+\frac{1}{r}) = q\omega(g_2) = q\omega - \lim_{\lambda\to\infty} \frac{1}{\lambda}\tau(e^{-T^{-q}\lambda^{-q/p}})$$

Thus by (9) we obtain the statement of the theorem:

$$\Gamma(p/q)\tilde{\omega} - \lim_{r\to\infty} \frac{1}{r}\zeta(p+\frac{1}{r}) = q\omega - \lim_{\lambda\to\infty} \frac{1}{\lambda}\tau(e^{-T^{-q}\lambda^{-q/p}}) = p\Gamma(p/q)\tau_\omega(T^p).$$

□

5.2. **The $L^{p,\infty}$-case and the 'small' ideal.** As $T \in \mathcal{L}^{p,\infty}$ means that $\mu_t(T)t^{1/p} < C < \infty$ and $\mu_t(T^p) = \mu_t(T)^p$ we conclude that $\mu_t(T^p)t < C^p < \infty$. That is $T \in \mathcal{L}^{p,\infty} \Longrightarrow S = T^p \in \mathcal{I}$ where $\mathcal{I}$ is the so called 'small' subideal of $\mathcal{L}^{1,\infty}$ identified in [6]. Recall that $\mathcal{I}$ is specified by the condition on the singular values of $T \geqslant 0, T \in \mathcal{L}^{1,\infty}$: $\mu_s(T) \leqslant C/s$ for some constant $C > 0$. In subsection 4.1 [6] we proved the following result by a direct argument that avoids the use of the zeta function. If ω is M invariant and satisfies conditions (1),(2),(3) of Theorem 3.4 and $T \in \mathcal{I}$ then

$$\omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(e^{-\lambda^{-2}T^{-2}}) = \Gamma(3/2)\tau_\omega(T).$$

We may now apply this stronger result of [6] to operators $S \in \mathcal{I}$ where $S = T^p$ and $T \in \mathcal{L}^{p,\infty}$ to obtain the equality

$$\omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(e^{-\lambda^{-2}S^{-2}}) = \Gamma(3/2)\tau_\omega(S).$$

Hence we obtain the following result

(11) *If* $T \in \mathcal{L}^{p,\infty}$ *then* $\omega - \lim_{\lambda\to\infty} \lambda^{-1}\tau(e^{-\lambda^{-2}T^{-2p}}) = \Gamma(3/2)\tau_\omega(T^p)$.

Note that we have obtained this result under weaker conditions on ω than the more general Theorem 5.1 where $T \in \mathcal{Z}_p$. It would be interesting to understand an example in noncommutative geometry where $\mathcal{Z}_p$ arises naturally. We remark that in classical geometric examples such as differential operators on manifolds it is $\mathcal{L}^{p,\infty}$ $p \geqslant 1$ and the 'small ideal' $\mathcal{I}$ that arise naturally.

A further idea motivated by the geometric case is that one may argue the other way, from a knowledge of the asymptotics of the trace of the heat semigroup, to information on the zeta function. Thus let us assume that the trace of the heat operator $\tau(e^{-tT^{-2}})$ exists for all $t > 0$ and in addition has an asymptotic expansion in inverse powers of t as $t \to 0$. These assumptions hold for Dirac Laplacians for example in classical geometry and it is well known in this case that one can infer from the asymptotic expansion the nature of the first singularity of $\zeta(s)$ (as $\operatorname{Re} s$ decreases) from the leading term in inverse powers of t. We now explain this in some detail.

Thus assume that $\tau(e^{-tT^{-2}}) = Ct^{-p/2}$ + lower order powers of t^{-1} as $t \to 0$. We recall that as in Theorem 5.1 $\tau(e^{-tT^{-2}}) \to 0$ exponentially as $t \to \infty$. We introduce

$$\zeta_1(s) = \frac{1}{\Gamma(s/2)} \int_0^1 t^{s/2-1} \tau(e^{-tT^{-2}}) dt, \ s > p \tag{12}$$

and

$$\zeta_2(s) = \frac{1}{\Gamma(s/2)} \int_1^\infty t^{s/2-1} \tau(e^{-tT^{-2}}) dt, s > 0.$$

Then ζ_2 is analytic in a neighborhood of $s = p$ and we may write $\zeta(s) = \tau(T^s) := \zeta_1(s) + \zeta_2(s)$ for $\mathrm{Re}\, s > p$. Then the only contribution to the singularity at $s = p$ comes from ζ_1. Now

$$\frac{1}{\Gamma(s/2)} \int_0^1 t^{s/2-1} C t^{-p/2} dt = \frac{C}{\Gamma(s/2)(s/2 - p/2)}$$

and thus substitution in (12) gives

$$\zeta(s) = \tau(T^s) = \frac{C}{\Gamma(s/2)(s/2 - p/2)} + K(s)$$

where $K(s)$ is holomorphic for $s = p$. (We note that the lower order terms in the asymptotic expansion do contribute to the term $K(s)$ but these contributions are analytic near $s = p$.) Thus we may take the limit $\lim_{s\to p}(s-p)\zeta(s)$ and only the first term contributes as $\lim_{s\to p}(s-p)K(s) = 0$.

Proposition 5.2. If $\tau(e^{-tT^{-2}})$ has an asymptotic expansion in inverse powers of t with the leading term being $C/t^{p/2}$ for some constant C then $T \in \mathcal{Z}_p$ and

$$\lim_{s\to p}(s-p)\tau(T^s) = p\tau_\omega(T^p)$$

for any D_2 (and M) invariant ω.

6. Application to spectral triples

Throughout this Section the following assumptions hold. We let $\mathcal{D}$ be an unbounded self adjoint densely defined operator on $\mathcal{H}$ affiliated to $\mathcal{N}$ (this amounts to $(1+\mathcal{D}^2)^{-1} \in \mathcal{N}$). We suppose that $\mathcal{A}$ is a *-algebra in $\mathcal{N}$ consisting of operators a such that $[\mathcal{D}, a]$ is bounded and refer to the triple $(\mathcal{D}, \mathcal{A}, \mathcal{N})$ as a semifinite spectral triple.

Denote for brevity $\mathcal{M}^\psi := M(\psi)(\mathcal{N}, \tau)$ with ψ as in Section 4, satisfying (2). As in Corollary 4.7, we consider the following p-convexification of $\mathcal{M}^\psi$

$$\mathcal{M}^{\psi,p} := \{T \in \mathcal{N}_+ : \ \|T\|_{\psi,p} = \sup_{1<u<\infty} \frac{(\int_0^u \mu_t(T)^p dt)^{1/p}}{\psi^{1/p}(u)} < \infty\}, \quad p > 1.$$

We let τ_ω be a Dixmier trace on $\mathcal{M}^\psi$ corresponding to a suitable singular state ω. Suppose that $(1+\mathcal{D}^2)^{-p/2} \in \mathcal{M}^\psi$, or equivalently that $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{M}^{\psi,p}$. In applications of noncommutative geometry the functional φ_ω on $\mathcal{A}$ given by $\varphi_\omega(a) = \tau_\omega(a(1+D^2)^{-p/2})$ plays a key role. In particular it is of interest to know if this functional is a trace on $\mathcal{A}$. In [10] this question was answered in the affirmative for the case of $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{L}^{p,\infty}$. Their proof generalizes to our setting. In particular, it holds under the weaker assumption $(1+\mathcal{D}^2)^{-1/2} \in \mathcal{Z}_p$.

Theorem 6.1. *Under the immediately preceding hypotheses we have*

$$\varphi_\omega(ab) = \varphi_\omega(ba) \qquad a, b \in \mathcal{A}.$$

The proof is an extension of the approach in [10]. We need four preliminary facts. Some may be proved in a similar way to the corresponding results in [10].

Lemma 6.2. *Given a spectral triple $(\mathcal{D}, \mathcal{A}, \mathcal{N})$ we have*
(i) For $a, b \in \mathcal{N}$ the Hölder inequality

$$\tau_\omega(ab) \leqslant \tau_\omega(|a|^p)^{1/p} \tau(|b|^q)^{1/q}$$

for $p, q \geqslant 1$, $\frac{1}{p} + \frac{1}{q} = 1$, holds.
(ii) For any r with $0 < r < 1$ and $a \in \mathcal{A}$ the operator $[(1+\mathcal{D}^2)^{r/2}, a]$ is bounded and satisfies

$$||[(1+\mathcal{D}^2)^{r/2}, a]|| \leqslant C||[\mathcal{D}, a]||$$

where the constant $C > 0$ does not depend on a.
(iii) Let $T \in \mathcal{M}^\psi$ and $f(t) = \mu_t(T)$ so that f is a bounded decreasing function on $(0, \infty)$ from $M(\psi)$, then $f^\alpha \in L_1(\mathbb{R}_+)$ for every $\alpha > 1$.
(iv) The statement of the theorem (for $(1+\mathcal{D}^2)^{-p/2} \in \mathcal{M}^\psi$) is implied by

$$\tau_\omega(|[(1+\mathcal{D}^2)^{-p/2}, a]|) = 0 \text{ for all } a \in \mathcal{A}.$$

Proof. (i) We have by [11, Proposition 1.1] and by the Hölder inequality for function spaces

$$\int_0^t \mu_s(ab)ds \leqslant \int_0^t \mu_s(a)\mu_s(b)ds \leqslant (\int_0^t \mu_s(a)^p ds)^{1/p} (\int_0^t \mu_s(b)^q ds)^{1/q}$$

Dividing by $\psi(t)$ and applying the functional ω we get

$$\tau_\omega(ab) \leqslant \omega\left[\left(\frac{\int_0^t \mu_s(a)^p ds}{\psi(t)}\right)^{1/p} \left(\frac{\int_0^t \mu_s(b)^q ds}{\psi(t)}\right)^{1/q}\right]$$

$$\leqslant \omega\left(\frac{\int_0^t \mu_s(a)^p ds}{\psi(t)}\right)^{1/p} \omega\left(\frac{\int_0^t \mu_s(b)^q ds}{\psi(t)}\right)^{1/q} = \tau_\omega(|a|^p)^{1/p}\tau_\omega(|b|^q)^{1/q}$$

using Hölder inequality for states on abelian C^*-algebras. We omit the proof for $p = 1$, $q = \infty$.

(ii) If $\mathcal{N}$ is taken in its left regular representation, then the claim follows immediately from [38, Theorem 3.1]. The general case is done in [37, Theorem 2.4.3]. Note, that the assumption made in [10] that $\mathcal{D}$ has a bounded inverse is now redundant.

(iii) Using the inequalities preceding Lemma 4.3, we have for any $\beta > 1$ $f(t) \leqslant C' \frac{1}{t^{1/\beta}}$ for some $C' > 0$ and all sufficiently large t's. Since $\alpha > 1$ is given, we can choose β so that $\frac{\alpha}{\beta} = \gamma > 1$, and so $f^\alpha(t) \leqslant C'/t^\gamma$ which gives the required result.

(iv) Let $T = (1+\mathcal{D}^2)^{-p/2}$ and $a, b \in \mathcal{A}$. Then we know that for $T' \in \mathcal{M}^\psi$ $\tau_\omega(T'a) = \tau_\omega(aT')$ (see [13] or [6, Lemma 3.2(i)]) and hence

$$\varphi_\omega([a, b]) = \tau_\omega(Tab - aTb) = \tau_\omega([T, a]b).$$

Then

$$|\tau_\omega([T, a]b)| \leqslant \tau_\omega(|[T, a]|)||b||) = 0$$

with the last equality is implied by the hypothesis of the lemma. □

Choose r with $0 < r < 1$ such that $k = p/r \in \mathbb{N}$. Following [10], we see that the proof of the theorem rests on the identity (for $k \in \mathbb{N}$)

$$[a, (1+\mathcal{D}^2)^{-kr/2}] = \sum_{j=1}^{k} (1+\mathcal{D}^2)^{-jr/2}[(1+\mathcal{D}^2)^{r/2}, a](1+\mathcal{D}^2)^{(j-k-1)r/2}$$

where we are using part (ii) of the Lemma to give boundedness of $[(1+\mathcal{D}^2)^{r/2}, a]$. We now apply the previous identity to obtain:

$$\tau_\omega(|[a, (1+\mathcal{D}^2)^{-p/2}]|) = \tau_\omega(|[a, (1+\mathcal{D}^2)^{-kr/2}]|)$$

$$\leqslant \sum_{j=1}^{k} \tau_\omega[|(1+\mathcal{D}^2)^{-jr/2}[(1+\mathcal{D}^2)^{r/2}, a](1+\mathcal{D}^2)^{(j-k-1)r/2}|]$$

Hence choosing $p_j = \frac{2p}{r(2j-1)}$, $q_j = \frac{2p}{r(2k-2j+1)}$ and applying part (i) of the Lemma,

$$\tau_\omega(|[a, (1+\mathcal{D}^2)^{-p}]|) \leqslant ||[(1+\mathcal{D}^2)^{r/2}, a]|| \sum_{j=1}^{k} (\tau_\omega((1+\mathcal{D}^2)^{-p_j jr/2}))^{1/p_j} (\tau_\omega((1+\mathcal{D}^2)^{(j-k-1)q_j r/2}))^{1/q_j}$$

The exponents $p_j jr/2$ and $(j-k-1)q_j r/2$ are larger than p so using part (iii) of the Lemma, the Dixmier trace in the last two terms vanishes. Now use part (iv) of the Lemma to complete the proof of the Theorem.

References

[1] M.T. Benameur, A.L. Carey, J. Phillips, A. Rennie, F.A. Sukochev, K.P. Wojciechowski, *Analytic formulae for Spectral Flow in von Neumann algebras*, in Analysis, Geometry and topology of elliptic operators. eds B. Booss-Bavnbek, S.Klimek, M. Lesch, W. Zhang, World Scientific, Singapore, 2006

[2] M-T. Benameur, T. Fack, *Type II Noncommutative Geometry. I. Dixmier Trace in von Neumann Algebras*, Advances in Math, **199**, 29-87 (2006)

[3] A.L. Carey, J. Phillips, *Spectral flow in Θ-summable Fredholm modules, eta invariants and the JLO cocycle*, K Theory **31**, 135-194 (2004).

[4] A.L. Carey, J. Phillips, A. Rennie, F. Sukochev, *The local index formula in semifinite von Neumann algebras I. Spectral Flow*, Advances in Math. **202**, 451-516 (2006)

[5] A.L. Carey, J. Phillips, A. Rennie, F. Sukochev, *The local index formula in semifinite von Neumann algebras II: the even case*, Advances in Math. **202**, 517-554 (2006).

[6] A.L. Carey, J. Phillips and F. A. Sukochev, *Spectral Flow and Dixmier Traces*, Advances in Math. **173**, 68-113 (2003).

[7] P.S. Chakraborty, A. Pal, *Equivariant spectral triples on the quantum $SU(2)$ group*, K-Theory **¿¡DEFANGED.91347 28**, 107-126 (2003).

[8] P.S. Chakraborty, A. Pal, *On equivariant Dirac operators for $SU_q(2)$*, math/0501019

[9] A. Chamseddine, A. Connes, *The Spectral Action Principle*, Comm. Math. Phys. **186**, 731-750 (1997).

[10] F. Cipriani, E. Guido, S. Scarlatti, *A remark on trace properties of K-cycles*, J. Operator Theory **35**, 179-189 (1996).

[11] V.I. Chilin, F.A. Sukochev, *Weak convergence in non-commutative symmetric spaces,* J. Operator Theory **31**, 35-65 (1994).

[12] A. Connes, *Noncommutative Differential Geometry*, Publ. Math. Inst. Hautes Etudes Sci., **62**, 1-44 (1985).

[13] A. Connes, *Noncommutative Geometry*, Acad. Press, San Diego, 1994.

[14] A. Connes, *Geometry From the Spectral Point of View*, Lett. Math. Phys., **34**, 203-238 (1995).

[15] A. Connes, *The action functional in noncommutative geometry*, Commun. Math. Phys. **117**, 673-683 (1988).

[16] A. Connes, *Traces de Dixmier, modules de Fredholm et geometrie riemannienne*, in Conformal field theories and related topics (Annecy-le-Vieux, 1988). Nuclear Phys. In Proc. Suppl. **5b**, 65-70 (1988).

[17] A. Connes, J. Lott, *Particle models and noncommutative geometry*, Nuclear Physics **B18** supplement 29-47 (1991).

[18] A. Connes, M. Marcolli, *A walk in the noncommutative garden* math.QA/0601054

[19] L. Dabrowski, G. Landi, A. Sitarz, W. van Suijlekom, J.C. Varilly, *The Dirac operator on* $SU_q(2)$, Commun. Math. Phys., **259**, 729-759 (2005).

[20] J. Dixmier, *Existence de traces non normales*, (French) C. R. Acad. Sci. Paris **262**, 1107–1108 (1966).

[21] P.G. Dodds, T.K. Dodds and B. de Pagter, *Non-commutative Banach function spaces*, Math. Z. **201**, 583-597 (1989).

[22] P.G. Dodds, B. de Pagter, F.A. Sedaev, E.M. Semenov, F.A. Sukochev, *Singular symmetric functionals*, (Russian) Zap. Nauchn. Sem. S.-Peterburg., Otdel. Mat. Inst. Steklov. (POMI) **290** (2002), Issled. po Linein. Oper. i Teor. Funkts. 30, 42–71, 178; translation in J. Math. Sci. (N. Y.) **124**, 4867–4885 (2004). ¿¡DEFANGED.91348

[23] P.G. Dodds, B. de Pagter, F.A. Sedaev, E.M. Semenov, F.A. Sukochev, *Singular symmetric functionals and Banach limits with additional invariance properties*, (Russian) Izv. Ross. Akad. Nauk, Ser. Mat. **67**, 111–136 (2003).

[24] P. G. Dodds, B. de Pagter, E. M. Semenov, F. A. Sukochev *Symmetric functionals and singular traces*, Positivity **2**, 47-75 (1998).

[25] T. Fack, *Sur la notion de valeur caractéristique*, J. Operator Theory **7**, 307-333 (1982).

[26] T. Fack, H. Kosaki, *Generalised* s*-numbers of* τ*-measurable operators*, Pacific J. Math., **123**, 269-300 (1986).

[27] V. Gayral, *Heat kernel approach to UV/IR mixing on isospectral deformation manifolds* hep-th/0412233.

[28] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker, J. C. Várilly, *Moyal planes are spectral triples*, Commun. Math. Phys. **246**, 569–623 (2004).

[29] I. C. Gohberg, M. G. Krein, *Introduction to the Theory of Non-selfadjoint Operators*, Translations of Mathematical Monographs, **18**, AMS, 1969.

[30] S.G. Kreĭn, Yu.I. Petunīn, E.M Semenov, *Interpolyatsiya lineĭnykh operatorov.* (Russian). *Interpolation of linear operators.* Translated by J. Szűcs. Translations of Mathematical Monographs, 54. American Mathematical Society, Providence, R.I., 1982.

[31] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. I. Sequence spaces,* Results in Mathematics and Related Areas, 92. Springer-Verlag, Berlin-New York, 1977.

[32] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. II. Function spaces,* Results in Mathematics and Related Areas, 97. Springer-Verlag, Berlin-New York, 1979. ¿¡DEFANGED.91349

[33] S. Lord, A. Sedaev, F. Sukochev, *Dixmier traces as singular symmetric functionals and applications to measurable operators*, J. Funct. Analysis **224**, no. 1, 72–106 (2005).

[34] F. Nicola, *Trace functionals for a class of pseudo-differential operators in* $\mathbb{R}^n$, Math. Physics, Anal. and Geometry **6**, 89-105 (2003).

[35] D. Pask, A. Rennie, *The Noncommutative Geometry of Graph* C^**-Algebras I: The Index Theorem*, J. Funct. An. **233**, 92–134 (2006).

[36] D. Pask, A. Rennie, A. Sims, *The Noncommutative Geometry of* k*-Graph* C^**-Algebras*, archive math.oa/0512438.

[37] D. Potapov, *Lipschitz and Commutator estimates, a unified approach*, PhD Thesis, Flinders University of South Australia.

[38] D. Potapov, F.A. Sukochev *Lipschitz and commutator estimates in symmetric operator spaces*, J. Operator Theory (to appear).

[39] F. A. Sukochev, *Operator estimates for Fredholm modules*, Canad. J. Math. **52**, 849-896 (2000).