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A HOLONOMY CHARACTERISATION OF FEFFERMAN SPACES

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Abstract. We prove that Fefferman spaces, associated to non–degenerate CR structures of hypersurface type, are characterised, up to local conformal isometry, by the existence of a parallel orthogonal complex structure on the standard tractor bundle. This condition can be equivalently expressed in terms of conformal holonomy. Extracting from this picture the essential consequences at the level of tensor bundles yields an improved, conformally invariant analogue of Sparling's characterisation of Fefferman spaces.

1. Introduction

Fefferman spaces provide a geometric relationship between CR geometry and conformal geometry; given a CR manifold M, of hypersurface type, one obtains a canonical conformal structure on the total space $\tilde{M}$ of a certain circle bundle over M. The first version of this construction in [13] applied to the boundaries of strictly pseudoconvex domains, and used Fefferman's ambient metric construction. Soon after that, a version for abstract non–degenerate CR structures of hypersurface type was given in [3]. This used the canonical Cartan connection for CR structures of Chern and Moser [12].

Recently there has been renewed interest in the Cartan connection approach to CR structures and, more generally, parabolic geometries. Several powerful tools, like tractor calculus and BGG sequences for these geometries have been developed. These tools provide a new and effective approach the construction of Fefferman spaces and to treating the complicated relationship between the natural objects on the Fefferman space with those on the underlying CR manifold. This approach has been taken up in the article [9], where several new results on Fefferman spaces their conformal geometry are obtained. Central in these developments is the result that on the (conformal) standard tractor bundle of a Fefferman space $\tilde{M}$, one obtains a parallel orthogonal complex structure. This complex structure makes the standard tractor bundle into a Hermitian vector bundle and the standard tractor connection into a Hermitian connection. Key to the power of this approach is that these data are rather simply related to the CR standard tractor bundle of M.

In this article we complete the picture by showing that Fefferman spaces are in fact characterised by this orthogonal parallel complex structure, up

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to local conformal isometry. This has a very nice interpretation in terms of conformal holonomy, an area of significant recent interest, see e.g. [1] and references therein. By definition, conformal holonomy is the holonomy group of the standard tractor connection. In the dimensions and signatures which are relevant for us, this is a subgroup of $SO(2p'+2, 2q'+2)$. Existence of a parallel orthogonal complex structure is clearly equivalent to the conformal holonomy being contained in the subgroup $U(p'+1, q'+1)$. Hence we obtain a characterisation of Fefferman spaces among general conformal structures analogous to the characterisation of Kähler manifolds among pseudo-Riemannian manifolds. By construction, the conformal holonomy of a Fefferman space is in fact automatically contained in the smaller subgroup $SU(p'+1, q'+1)$. As a corollary, we therefore obtain that if the conformal holonomy is contained in $U(p'+1, q'+1)$ then the holonomy Lie algebra already has to be contained in $\mathfrak{su}(p'+1, q'+1)$. In particular, $\mathfrak{u}(p'+1, q'+1)$ cannot be a conformal holonomy Lie algebra although it is a Berger algebra. In fact this follows easily and directly from a simple calculation using the tractor connection, see Proposition 2.1. This was proved in a more involved way in [16].

There is a characterisation of Fefferman spaces available in the literature, which is usually referred to as Sparling's characterisation. While Sparling's original work remained unpublished, the characterisation has been proved in a different way by C.R. Graham in [15]. The central ingredient for this characterisation is an isotropic Killing field with certain additional properties. In our picture, this arises as follows: A parallel orthogonal complex structure $\mathbb{J}$ on the standard tractor bundle can be interpreted as a parallel section of the adjoint tractor bundle. Any section of the adjoint tractor bundle has an underlying vector field, and for a parallel section this is automatically a conformal Killing field which inserts trivially into the Cartan curvature. Expanding the condition $\mathbb{J} \circ \mathbb{J} = -\operatorname{id}$ yields the remaining properties required in Sparling's characterisation (and other identities that are differential consequences of these).

In the last section of the article we that show that the tensorial consequences of $\mathbb{J} \circ \mathbb{J} = -\operatorname{id}$ lead to a characterisation of Fefferman spaces that is in the same spirit as Sparling's, but is conformally invariant. Using this, with the extra choice of a certain conformal scale, recovers Sparling's characterisation precisely. This provides additional insight into the structure of conformal manifolds which admit isotropic conformal Killing fields.

We assume that the reader is familiar with basic conformal geometry and conformal tractor calculus and we will follow the conventions of [14]. Other sources for background on conformal tractor calculus are [2] and [8], for generalisations to parabolic geometries see [7]. All facts about the construction of Fefferman spaces we need can be found in [9].

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2. A characterisation of Fefferman spaces

2.1. **Parallel adjoint tractors and conformal Killing fields.** The main technical input for the characterisation is provided by general results in conformal geometry. Hence we start in the general setting of a smooth manifold M of dimension $n \geq 3$ endowed with a conformal structure $[g]$ of signature (p, q). Then the *standard tractor bundle* (see [2, 8]) is a vector bundle $\mathcal{T} \to M$ of rank $n + 2$, endowed with a bundle metric h of signature $(p + 1, q + 1)$, a line subbundle $\mathcal{T}^1 \subset \mathcal{T}$ with isotropic fibres, and a linear connection $\nabla^{\mathcal{T}}$ which is compatible with h and satisfies a certain non–degeneracy condition with respect to the subbundle $\mathcal{T}^1$. These data are canonically associated with the conformal structure and are equivalent to its canonical Cartan connection.

The *adjoint tractor bundle* $\mathcal{A} \to M$ is then defined as the bundle $\mathfrak{so}(\mathcal{T})$ of skew symmetric endomorphisms of $\mathcal{T}$. The adjoint tractor connection $\nabla^{\mathcal{A}}$ is the linear connection on $\mathcal{A}$ induced by $\nabla^{\mathcal{T}}$. There is a canonical projection $\Pi : \mathcal{A} \to TM$ from the adjoint tractor bundle to the tangent bundle, so in particular, any section of $\mathcal{A}$ has an underlying vector field.

We will sometimes work with abstract indices, denoting tractor indices by $A, B, \dots$ and tensor indices by $a, b, \dots$. Tractor indices are raised and lowered using the tractor metric $h = h_{AB}$ and its inverse, while tensor indices are raised and lowered using the *conformal* metric $\mathbf{g}_{ab} \in \Gamma(\mathcal{E}_{ab}[2])$ and its inverse. Thus raising (lowering) a tensor index decreases (increases) the conformal weight by 2. If we use connections in abstract index computations, then we will always assume that a metric from the conformal class has been chosen. The symbol ∇_a then denotes a coupled Levi–Civita–tractor connection, i.e. it acts by the standard tractor connection on all tractor indices and by the Levi–Civita connection on all tensor indices.

The curvature Ω of the standard tractor connection $\nabla^{\mathcal{T}}$ can be naturally interpreted as an element of $\Omega^2(M, \mathcal{A})$. Its abstract index representation therefore has the form $\Omega_{ab}{}^A{}_B$. Lowering the tractor index A, the result is skew symmetric in both pairs of indices. The form Ω also describes the curvature of the canonical Cartan connection. It can be easily expressed explicitly (see the proof below) in terms of the *Weyl–curvature* $C = C_{ab}{}^c{}_d$ and the *Cotton–York tensor* A_{abc}.

Proposition. *Let $s = s^A{}_B$ be a parallel section of the adjoint tractor bundle $\mathcal{A}$ and let $\mathbf{k} = \mathbf{k}^a$ denote the underlying vector field $\Pi(s)$. Then we have:*
(1) $\mathbf{k}$ is a conformal Killing field which in addition satisfies $\mathbf{k}^a \Omega_{ab}{}^A{}_B = 0$;
(2) $\Omega_{ab}{}^A{}_B s^B{}_A = 0$.

Proof. (1) is proved in [14, Proposition 2.2] and in a much more general setting in [6, Corollary 3.5].

The explicit formula for the tractor curvature from [14, formula (6)] reads as

$$\Omega_{abAB} = Z_A{}^c Z_B{}^d C_{abcd} - 2X_{[A} Z_{B]}{}^c A_{cab},$$

with the convention that the last two indices of the Cotton tensor are the two–form indices. Hence from (1) we get $\mathbf{k}^a C_{abcd} = 0$ and $\mathbf{k}^a A_{cab} = 0$. From Proposition 2.2 and Lemma 2.1 of [14] we get $s_{AB} = \frac{1}{n} D_A K_B$, where $K_B = Z_B{}^a \mathbf{k}_a - \frac{1}{n} X_B \nabla_a \mathbf{k}^a$ for any choice of metric in the conformal class.

Since K_B has conformal weight one, we obtain

$$s_{AB} = Y_A K_B + Z_A{}^a \nabla_a K_B - \tfrac{1}{n} X_A (\nabla^a \nabla_a + \mathsf{P}^a{}_a) K_B.$$

Contracting both tractor indices with Ω_{abAB}, the last summand does not contribute, since $\Omega_{abAB} X^A = 0$, and a short computation gives

$$\Omega_{abAB} s^{AB} = -2\mathbf{k}^c A_{cab} + C_{abcd} \nabla^c \mathbf{k}^d.$$

For the first term, we observe that the total alternation of the Cotton–York tensor vanishes. (This is a consequence of symmetry of the Schouten tensor P_{ab}.) Hence

$$\mathbf{k}^c A_{cab} = \mathbf{k}^c(-A_{bca} - A_{abc}),$$

and we have observed above that the right hand side vanishes. Finally, the symmetries of the Weyl tensor together with $\mathbf{k}^b C_{abcd} = 0$ give

$$C_{abcd} \nabla^c \mathbf{k}^d = -\mathbf{k}^d \nabla^c C_{abcd} = -(n-3)\mathbf{k}^d A_{dab} = 0,$$

where we have used the Bianchi identity in the last step. This completes the proof of (2). □

2.2. **Complex structures on standard tractors.** Let us now assume that $\dim(M)$ is even and $[g]$ is a conformal class of signature $(2p'+1, 2q'+1)$, so the tractor metric has signature $(2p'+2, 2q'+2)$. We will further assume that we have given an orthogonal complex structure $\mathbb{J}$ on the standard tractor bundle $\mathcal{T}$, which is possible for such a signature. Observe that since $\mathbb{J}^2 = \mathbb{J} \circ \mathbb{J} = -\operatorname{id}$, orthogonality of $\mathbb{J}$ is equivalent to skew–symmetry. Hence $\mathbb{J}$ can be considered as a section of the adjoint tractor bundle $\mathcal{A}$. Note also that, using $\mathbb{J}$, we can extend the tractor metric h to a Hermitian bundle metric $\mathcal{H}$.

By definition, $\mathbb{J}$ is parallel for $\nabla^{\mathcal{A}}$ if and only if covariant derivatives by $\nabla^{\mathcal{T}}$ are complex linear. In this case $\nabla^{\mathcal{T}}$ is a Hermitian connection for $\mathcal{H}$ and hence its holonomy is contained in $U(p'+1, q'+1) \subset SO(2p'+2, 2q'+2)$. Conversely, this condition on the holonomy is evidently equivalent to existence of a parallel orthogonal complex structure $\mathbb{J}$.

Using $\mathbb{J}$, we can view the standard tractor bundle $\mathcal{T}$ as a complex vector bundle. In particular, we can form the complex line bundle $\mathcal{V} := \Lambda^{p'+q'+2}_{\mathbb{C}} \mathcal{T}$, i.e. the highest complex exterior power of $\mathcal{T}$. The standard tractor connection $\nabla^{\mathcal{T}}$ induces a linear connection $\nabla^{\mathcal{V}}$ on $\mathcal{V}$, and the curvature of this linear connection is given by the trace of the tractor curvature, viewed as a complex linear map. More explicitly, this is the complex valued two–form $\Omega_{ab}{}^A{}_A - i\Omega_{ab}{}^A{}_B \mathbb{J}^B{}_A$. The first summand vanishes by skew symmetry while the second vanishes by part (2) of Proposition 2.1. Therefore, locally the bundle $\mathcal{V}$ admits smooth sections which are parallel for the linear connection induced by $\nabla^{\mathcal{T}}$.

We want to interpret the existence of $\mathbb{J}$ via a reduction of structure group of the canonical Cartan bundle $\mathcal{G}$ associated to the conformal structure. The Cartan bundle can be easily obtained from the standard tractor bundle $\mathcal{T}$ by a frame bundle construction, see [8, 2.2]. To prepare the grounds for using complex structures later, we use the underlying real vector space of $\mathbb{V} := \mathbb{C}^{p'+q'+2}$ as the modelling vector space for $\mathcal{T}$. Fix a Hermitian inner product $\langle\ ,\ \rangle$ of signature $(p'+1, q'+1)$ on $\mathbb{V}$ as well as a real isotropic line $L_{\mathbb{R}} \subset \mathbb{V}$, and a non-zero element $\nu_0 \in \Lambda_{\mathbb{C}}^{p'+q'+2}\mathbb{V}$. Let $\langle\ ,\ \rangle_{\mathbb{R}}$ be the real part of $\langle\ ,\ \rangle$. Let $\mathcal{T}^1 \subset \mathcal{T}$ be the distinguished real line subbundle in $\mathcal{T}$. Then the fibre $\mathcal{G}_x$ of the canonical Cartan bundle can be identified with the set of all orthogonal linear isomorphisms $u : \mathbb{V}_{\mathbb{R}} \to \mathcal{T}_x$ such that $u(L_{\mathbb{R}}) = \mathcal{T}_x^1$. Fixing an element $u \in \mathcal{G}_x$, we obtain a bijection from the stabiliser $P \subset SO(\mathbb{V}) =: G$ of the line $L_{\mathbb{R}}$ onto $\mathcal{G}_x$. Together with the fact that $\mathcal{T}$ can be locally trivialised as a filtered metric vector bundle, this implies that the disjoint union $\mathcal{G} = \sqcup_{x\in M}\mathcal{G}_x$ can be made into a smooth principal P–bundle over M. By construction, $\mathcal{T}$ is the associated bundle $\mathcal{G} \times_P \mathbb{V}$.

Let $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ be the Lie algebra of G and let $\mathfrak{p} \subset \mathfrak{g}$ be the subalgebra corresponding to P. The canonical conformal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ can be recovered from the standard tractor connection as follows. Since $\mathcal{T} = \mathcal{G} \times_P \mathbb{V}$, the space $\Gamma(\mathcal{T})$ of smooth sections can be naturally identified with the space $C^\infty(\mathcal{G}, \mathbb{V})^P$ of smooth functions $f : \mathcal{G} \to \mathbb{V}$ such that $f(u\cdot g) = g^{-1}(f(u))$ for all $u \in \mathcal{G}$, and all $g \in P \subset SO(\mathbb{V})$. Any vector field $\xi \in \mathfrak{X}(M)$ can be lifted (locally) to a vector field $\overline{\xi}$ on $\mathcal{G}$. For a section $s \in \Gamma(\mathcal{T})$ corresponding to $f : \mathcal{G} \to \mathbb{V}$, the function inducing $\nabla^{\mathcal{T}}_\xi s$ is then given by $\overline{\xi}\cdot f + \omega(\overline{\xi}) \circ f$, for any such lift ξ. It is shown in [7, 2.7] that this uniquely defines a Cartan connection ω on $\mathcal{G}$ and normality of ω is equivalent to normality of $\nabla^{\mathcal{T}}$.

Now suppose that $\mathbb{J}$ is an orthogonal complex structure on $\mathcal{T}$. From above we then know that $\mathcal{V} = \Lambda_{\mathbb{C}}^{p'+q'+2}\mathcal{T}$ admits local non–vanishing parallel sections. Restricting to an open subset if necessary, we assume that ν is a global nonzero parallel section of $\mathcal{V}$. Then we define $\mathcal{G}_x^{SU} \subset \mathcal{G}_x$ to be the set of those maps u, which are complex linear with respect to $\mathbb{J}_x$, and which also have the property that the induced isomorphism on the $(p'+q'+2)$nd complex exterior power maps ν_0 to $\nu(x)$. The subset of complex linear maps in $SO(\mathbb{V})$ is exactly the unitary group $U(\mathbb{V})$, such a map lies in $SU(\mathbb{V})$, if in addition the induced map on $\Lambda_{\mathbb{C}}^{p'+q'+2}\mathbb{V}$ preserves ν_0. Thus, $\mathcal{G}^{SU} := \sqcup\mathcal{G}_x^{SU} \subset \mathcal{G}$ is a principal fibre bundle over M with structure group $P^{SU} := SU(\mathbb{V})\cap P$. This structure group is exactly the stabiliser of the real line $L_{\mathbb{R}}$ in the group $SU(\mathbb{V})$.

Note that by elementary linear algebra $SU(\mathbb{V})$ acts transitively on the space of real null lines, so the inclusion $SU(\mathbb{V}) \hookrightarrow G = SO(\mathbb{V})$ induces a diffeomorphism $SU(\mathbb{V})/P^{SU} \cong G/P$. Looking at derivatives at the base points, we see that the inclusion $\mathfrak{su}(\mathbb{V}) \hookrightarrow \mathfrak{g} = \mathfrak{so}(\mathbb{V})$ induces a linear isomorphism $\mathfrak{su}(\mathbb{V})/\mathfrak{p}^{SU} \to \mathfrak{g}/\mathfrak{p}$.

Proposition. *The canonical conformal Cartan connection ω restricts to a Cartan connection $\omega^{SU} \in \Omega^1(\mathcal{G}^{SU}, \mathfrak{su}(\mathbb{V}))$.*

Proof. The relationship between $s \in \Gamma(\mathcal{T})$ and the corresponding equivariant function $f \in C^\infty(\mathcal{G}, \mathbb{V})^P$ is characterised by the fact that for each $u \in \mathcal{G}_x$ we get $s(x) = u(f(u)) \in \mathcal{T}_x$. Elements of $\mathcal{G}^{SU}$ are by definition complex linear as isomorphisms $\mathbb{V} \to \mathcal{T}_x$. Consequently, if $\tilde{f} : \mathcal{G} \to \mathbb{V}$ is the equivariant function corresponding to $\mathbb{J}s$, then $\tilde{f}(u) = if(u)$ for all $u \in \mathcal{G}^{SU}$. Now assume that $\xi \in \mathfrak{X}(M)$ is a vector field, and consider a local lift $\overline{\xi}$ on $\mathcal{G}^{SU} \subset \mathcal{G}$, which is tangent to $\mathcal{G}^{SU}$. Then the integral curves of $\tilde{\xi}$ are contained in $\mathcal{G}^{SU}$, so along these integral curves we have $\tilde{f} = if$. Thus we have $\overline{\xi}\cdot\tilde{f} = i(\overline{\xi}\cdot f)$, and thus $\nabla^{\mathcal{T}}_{\xi}\mathbb{J}s$ corresponds to $i(\overline{\xi}\cdot f) + \omega(\overline{\xi}) \circ if$. On the other hand, $\nabla^{\mathcal{T}}_{\xi}\mathbb{J}s = \mathbb{J}\nabla^{\mathcal{T}}_{\xi}s$, which, along $\mathcal{G}^{SU}$ corresponds to the function $i(\overline{\xi}\cdot f+\omega(\overline{\xi})\circ f)$. But this exactly shows that $\omega(\overline{\xi})$ is complex linear, provided that $\overline{\xi}$ is tangent to $\mathcal{G}^{SU}$.

According to the observations above, we can view $\mathcal{T}$ as the associated bundle $\mathcal{G}^{SU} \times_{PSU} \mathbb{V}$. Thus the exterior power $\mathcal{V}$ is the associated bundle $\mathcal{G}^{SU} \times_{PSU} \Lambda^{p'+q'+2}\mathbb{V}$. Now elements of $SU(\mathbb{V})$ by definition act trivially on this exterior power, so $\mathcal{V}$ is actually the trivial bundle $M \times \mathbb{C}$. By construction, the section ν of $\mathcal{V}$ corresponds to the constant function ν_0. The relation between $\nabla^{\mathcal{V}}$ and ω is similar to the situation of $\nabla^{\mathcal{T}}$, as discussed above. Using this, we can conclude that for $\overline{\xi}$ tangent to $\mathcal{G}^{SU}$, we have $\omega(\overline{\xi}) \in \mathfrak{su}(\mathbb{V})$.

Hence we can restrict ω to $\omega^{SU} \in \Omega^1(\mathcal{G}^{SU}, \mathfrak{su}(\mathbb{V}))$. On each tangent space, the map ω^{SU} is evidently injective and hence bijective for dimensional reasons. Denoting by $r^g : \mathcal{G} \to \mathcal{G}$ the principal right action of $g \in P$, equivariancy of ω reads as $(r^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$. For $g \in P^{SU} \subset P$ we get $r^g(\mathcal{G}^{SU}) \subset \mathcal{G}^{SU}$ and equivariancy of ω immediately implies that $(r^g)^*\omega^{SU} = \mathrm{Ad}(g^{-1}) \circ \omega^{SU}$ for such g. Finally, take an element A in the Lie algebra $\mathfrak{p}^{SU}$ of P^{SU}. Viewing A as an element of $\mathfrak{p}$, we get the fundamental vector field $\zeta_A \in \mathfrak{X}(\mathcal{G})$ generated by A. By definition, in points of $\mathcal{G}^{SU} \subset \mathcal{G}$, the field ζ_A is tangent to $\mathcal{G}^{SU}$. Hence we can restrict ζ_A to a vector field on $\mathcal{G}^{SU}$ and by definition, this restriction coincides with the fundamental vector field generated by A on this smaller principal bundle. Thus $\omega(\zeta_A) = A$ implies that ω^{SU} reproduces the generators of fundamental vector fields. This completes the verification that ω^{SU} defines a Cartan connection on $\mathcal{G}^{SU}$. □

2.3. **Passing to a local leaf space.** By part (1) of Proposition 2.1, the vector field $\mathbf{k}^a$, underlying the parallel adjoint tractor $\mathbb{J}$, is a conformal Killing field. The argument of [9, Theorem 3.1] shows that $\mathbf{k}^a$ is nowhere vanishing and by [9, 4.4] it is actually a Killing field for appropriate metrics in the conformal class. Since $\mathbf{k}$ is nowhere vanishing, it defines a rank one foliation of M, and we may consider a local leaf space for this foliation. This means that for each $x \in M$ we find an open neighbourhood W of x and a smooth surjective submersion $\psi : W \to N$ onto some manifold N, such that $\ker(T_x\psi) = \mathbb{R}\mathbf{k}(x)$ for each $x \in W$.

A particular outcome from Proposition 2.2 is that the tangent bundle TM can be naturally viewed as the associated bundle $\mathcal{G}^{SU} \times_{PSU} \mathfrak{su}(\mathbb{V})/\mathfrak{p}^{SU}$, where $\mathfrak{p}^{SU}$ is the Lie algebra of P^{SU}. In this picture, there is a helpful interpretation of the distribution spanned by $\mathbf{k}$. It is represented by the real span of the class of any element $A \in \mathfrak{su}(\mathbb{V})$ such that $A - \mathbb{J} \in \mathfrak{p} \subset \mathfrak{so}(\mathbb{V})$. But the fact that $A - t\mathbb{J} \in \mathfrak{p}$ for some $t \in \mathbb{R}$ is equivalent to the fact that A stabilises the complex line $L \subset \mathbb{V}$, which is generated by $L_{\mathbb{R}}$.

Let us denote by $\mathfrak{q} \subset \mathfrak{su}(\mathbb{V})$ and $Q \subset SU(\mathbb{V})$ the stabilisers of L. Then Q is a parabolic subgroup of $SU(\mathbb{V})$ with Lie algebra $\mathfrak{q}$. By construction, Q acts on the line L and the stabiliser of the real line $L_{\mathbb{R}} \subset L$ is the subgroup

$P^{SU} \subset Q$. Since the action of Q on L is evidently transitive, we see that $Q/P^{SU} \cong \mathbb{R}P^1$, so in particular this quotient is connected.

Theorem. *Let $(M, [g])$ be a conformal manifold endowed with a parallel orthogonal complex structure $\mathbb{J}$ on the standard tractor bundle $\mathcal{T}$ and a non vanishing parallel section ν of $\mathcal{V}$. Let $(p : \mathcal{G}^{SU} \to M, \omega^{SU})$ be the Cartan geometry of type $(SU(\mathbb{V}), P^{SU})$ obtained in Proposition 2.2.*

Then, for sufficiently small local leaf spaces $\psi : W \to N$, we can find a Cartan connection $\underline{\omega} \in \Omega^1(N \times Q, \mathfrak{su}(\mathbb{V}))$ on the trivial principal bundle $N \times Q$ and a P^{SU}–equivariant diffeomorphism Φ from a P^{SU}–invariant open subset of $N \times Q$ onto a P^{SU}–invariant subset of $\mathcal{G}^{SU}$ which pulls back ω^{SU} to $\underline{\omega}$.

Proof. We follow the proofs of Proposition 2.6 and Theorem 2.7 of [5], and refer to the notation introduced just above the Theorem. By part (1) of Proposition 2.1, $\mathbf{k}$ hooks trivially into the Cartan curvature, so the mapping $A \mapsto (\omega^{SU})^{-1}(A) \in \mathfrak{X}(\mathcal{G}^{SU})$ is a Lie algebra homomorphism. This can be viewed as an action of $\mathfrak{q}$ on $\mathcal{G}^{SU}$, and such an action integrates to a local action of Q on $\mathcal{G}^{SU}$. Choosing the leaf space N so small that there is a local section s of $\mathcal{G}^{SU} \to W \to N$ one finds an open neighbourhood V of P^{SU} in Q which is invariant under right multiplication by elements of P^{SU}. Using s, one constructs a P^{SU}–equivariant diffeomorphism Φ from $N \times V$ onto a P^{SU}–invariant subset of $\mathcal{G}^{SU}$, such that $\psi \circ p \circ \Phi = \mathrm{pr}_1 : N \times Q \to N$ and for $A \in \mathfrak{q}$, with corresponding fundamental vector field $\zeta_A \in \mathfrak{X}(N \times Q)$, we have $T\Phi \circ \zeta_A = (\omega^{SU})^{-1}(A) \circ \Phi$.

The fullback $\Phi^*\omega^{SU} \in \Omega^1(N \times V, \mathfrak{su}(\mathbb{V}))$ restricts to a linear isomorphism on each tangent space. The restriction of this pullback to $N \times \{e\}$ can be extended equivariantly to a Cartan connection $\underline{\omega} \in \Omega^1(N \times Q, \mathfrak{su}(\mathbb{V}))$. Using that $\mathbf{k}$ inserts trivially into the Cartan curvature, one shows that $\underline{\omega}$ coincides with $\Phi^*\omega^{SU}$ locally around $N \times \{e\}$ and hence on $N \times V$ by equivariancy. □

Corollary. *A local leaf space N as in the Theorem inherits an almost CR structure of hypersurface type. Explicitly, the complex subbundle $H \subset TN$ of corank one is the image of the orthocomplement $\mathbf{k}^\perp \subset TM$.*

Proof. The Cartan geometry $(N \times Q, \underline{\omega})$ gives rise to an identification of TN with the associated bundle $(N \times Q) \times_Q (\mathfrak{su}(\mathbb{V})/\mathfrak{q}) \cong N \times (\mathfrak{su}(\mathbb{V})/\mathfrak{q})$. By definition, $\mathfrak{q}$ is the stabiliser of the complex line $L \subset \mathbb{V}$. Now we define $\mathfrak{h} \subset \mathfrak{su}(\mathbb{V})/\mathfrak{q}$ as the set of those $A + \mathfrak{q}$ for which $A(L) \subset L^\perp$, where $L^\perp$ is the complex orthocomplement of L. This is a well defined subspace in $\mathfrak{su}(\mathbb{V})/\mathfrak{q}$. Fixing a nonzero element $v \in L_\mathbb{R}$ the map $A + \mathfrak{q} \mapsto A(v) + L$ induces a linear isomorphism $\mathfrak{su}(\mathbb{V})/\mathfrak{q} \to v^{\perp_\mathbb{R}}/L$, where we use the real orthocomplement of v. This restricts to an isomorphism $\mathfrak{h} \cong L^\perp/L$, so $\mathfrak{h} \subset \mathfrak{su}(\mathbb{V})/\mathfrak{q}$ has real codimension one. Moreover, $L^\perp/L$ is a complex vector space, and the induced complex structure on $\mathfrak{h}$ is independent of the choice of v. Since both the subspace $\mathfrak{h}$ and its complex structure are invariant under the natural action of Q, they induce a complex subbundle $H \subset TN$ of real codimension one.

To describe H explicitly, consider a point $u \in \mathcal{G}^{SU} \subset \mathcal{G}$ and a tangent vector $\xi \in T_u\mathcal{G}^{SU}$. By definition $T\psi \cdot Tp \cdot \xi$ lies in the CR subbundle if and only if $A := \omega(\xi) \in \mathfrak{su}(\mathbb{V})$ maps L to $L^\perp$. Let $v \in L_\mathbb{R}$ be a nonzero element.

Since A is orthogonal, we have $\langle A(v), v\rangle_{\mathbb{R}} = 0$, and using that A is complex linear we see that $A(L) \subset L^\perp$ if and only if $\langle A(v), iv\rangle_{\mathbb{R}} = 0$. But, by construction, on the subbundle $\mathcal{G}^{SU}$, the element $i\,\mathrm{id} + \mathfrak{p} \in \mathfrak{so}(\mathbb{V})/\mathfrak{p}$ corresponds to the vector field $\mathbf{k}$. Since the conformal structure is induced by $\langle\ ,\ \rangle_{\mathbb{R}}$, the explicit description of H follows. □

2.4. **The curvature of the reduced Cartan connection.** The next step is to show that the almost CR structure from Corollary 2.3 is non–degenerate of signature (p', q') and integrable. We will prove more than that, namely that the Cartan geometry $(N \times Q, \underline{\omega})$ is torsion free and normal, which also implies that it is the canonical Cartan geometry associated to this CR structure. To study these issues, we have to understand the curvature of $\underline{\omega}$.

For a sufficiently small local leaf space $\psi : W \to N$, Theorem 2.3 provides us with a Cartan geometry of type $(SU(\mathbb{V}), Q)$ on N. Regular Cartan geometries of that type induce an underlying partially integrable almost CR structure, and conversely, a partially integrable almost CR structure gives rise to a unique regular normal Cartan connection, see [9, 2.3] and [10, 4.15].

Recall that the curvature of the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is the two form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by $K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)]$ for $\xi, \eta \in \mathfrak{X}(\mathcal{G})$. The defining properties of a Cartan connection immediately imply that this form is horizontal and and P–equivariant. Using the trivialisation of $T\mathcal{G}$ provided by ω, one obtains the curvature function $\kappa : \mathcal{G} \to L\big(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}\big)$, which is characterised by

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) := K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)).$$

This is well defined since K is horizontal and equivariancy of K implies that κ is equivariant for the natural P–action on $L\big(\Lambda^2(\mathfrak{g}/\mathfrak{p}), \mathfrak{g}\big)$. On the other hand, composing the inverse of the isomorphism $\mathfrak{su}(\mathbb{V})/\mathfrak{p}^{SU} \to \mathfrak{g}/\mathfrak{p}$ induced by inclusion, with a natural projection, we obtain a natural surjection $\mathfrak{g}/\mathfrak{p} \to \mathfrak{su}(\mathbb{V})/\mathfrak{q}$.

Lemma. *In the situation of Theorem 2.3, the restriction of the curvature function $\underline{\kappa}$ of $\underline{\omega}$ to the domain of Φ is related to the curvature function κ of the conformal Cartan connection ω by the commutative diagram*

$$\begin{tikzcd} \Lambda^2(\mathfrak{g}/\mathfrak{p}) \arrow[r, "\kappa\circ\Phi"] \arrow[d] & \mathfrak{g} \\ \Lambda^2\big(\mathfrak{su}(\mathbb{V})/\mathfrak{q}\big) \arrow[r, "\underline{\kappa}"] & \mathfrak{su}(\mathbb{V}), \arrow[u] \end{tikzcd}$$

where we view $\mathcal{G}^{SU}$ as a subspace of $\mathcal{G}$, and the vertical arrows are the natural surjection, respectively the natural inclusion. This completely determines $\underline{\kappa}$.

Proof. Since ω^{SU} is simply a restriction of ω, the definition of the curvature immediately implies that K^{SU} is the restriction of K to $\mathcal{G}^{SU}$. (This restriction automatically has values in the algebra $\mathfrak{su}(\mathbb{V}) \subset \mathfrak{g}$.) In the proof of Theorem 2.3 we have seen that on the domain of Φ, $\underline{\omega}$ coincides with $\Phi^*\omega^{SU}$. Again by definition of the curvature, this shows that $\underline{K} = \Phi^*(K^{SU})$. Moreover, for $X \in \mathfrak{su}(\mathbb{V})$ we get $T\Phi \circ \underline{\omega}^{-1}(X) = (\omega^{SU})^{-1}(X) \circ \Phi = \omega^{-1}(X) \circ \Phi$. Together with the above, we get for another element $Y \in \mathfrak{su}(\mathbb{V})$ and all points a in the domain of Φ the equation $\underline{K}(a)(\underline{\omega}^{-1}(X), \underline{\omega}^{-1}(Y)) = K(\Phi(a))(\omega^{-1}(X), \omega^{-1}(Y))$. But the natural surjection $\mathfrak{g}/\mathfrak{p} \to \mathfrak{su}(\mathbb{V})/\mathfrak{q}$ by definition maps $X + \mathfrak{p}$ to $X + \mathfrak{q}$ (and is even characterised by that), so we obtain the claimed relation between κ and $\underline{\kappa}$. Since the domain of Φ contains $N \times \{e\}$ and the curvature function $\underline{\kappa}$ is Q–equivariant, we see that this relation determines $\underline{\kappa}$. □

2.5. **The characterisation theorem.** To prove our main result, we have to verify one more property of the conformal Killing field underlying a parallel complex structure on the standard tractor bundle.

Lemma. *Let $(M, [g])$ be a conformal manifold endowed with a parallel complex structure $\mathbb{J}$ on the standard tractor bundle, let $\mathbf{k}^a$ be the conformal Killing field underlying $\mathbb{J}$ (see 2.1), and let $\Omega_{ab}{}^A{}_B$ be the Cartan curvature. Then for the Levi–Civita connection ∇ of a preferred scale in the sense of* [9, 4.4]*, we have $\Omega_{ab}{}^A{}_B \nabla^a \mathbf{k}^b = 0$.*

Proof. By the explicit formula for $\Omega_{ab}{}^A{}_B$ in the proof of Proposition 2.1 it suffices to show that $C_{abcd}\nabla^a \mathbf{k}^b = 0$ and $A_{cab}\nabla^a \mathbf{k}^b = 0$. In that proof, we have seen that $C_{abcd}\nabla^c \mathbf{k}^d = 0$, so the first equation follows by the symmetries of the Weyl curvature. Using the covariant derivative of this equation, we obtain

$$A_{cab}\nabla^a \mathbf{k}^b = \tfrac{1}{n-3}(\nabla^d C_{abdc})(\nabla^a \mathbf{k}^b) = \tfrac{-1}{n-3} C_{abdc}\nabla^d \nabla^a \mathbf{k}^b.$$

There is an explicit formula for $\mathbb{J}_{AB}$ in part (5) of Proposition 4.4 of [9] in terms of any preferred scale. Expanding $0 = Z^A{}_a Z^B{}_b \nabla_d \mathbb{J}_{AB}$ using this formula, we obtain

$$0 = \nabla_d \nabla_a \mathbf{k}_b + 2\mathsf{P}_{d[a}\mathbf{k}_{b]} + 2g_{d[a}\ell_{b]}.$$

Inserting this into the above equation we get zero by the trace-freeness of C_{abcd} and since $\mathbf{k}^a C_{abcd} = 0$, which was observed in Proposition 2.1. □

Having this technical result at hand, we can proceed to the main results for the characterisation.

Proposition. *The Cartan geometry $(N \times Q, \underline{\omega})$ obtained in Theorem 2.3 is torsion free (and hence regular) and normal. Thus it induces on the local leaf space N a CR structure of hypersurface type, which is non–degenerate of signature (p', q'), as well as an appropriate root $\mathcal{E}(1,0)$ of the canonical bundle, see* [9, 2.3]*.*

Proof. Lemma 2.4 describes the curvature $\underline{\kappa}$ of $\underline{\omega}$. For $u \in \mathrm{im}(\Phi) \subset \mathcal{G}^{SU}$, the map $\kappa(u) : \Lambda^2(\mathfrak{g}/\mathfrak{p}) \to \mathfrak{g}$ descends to $\Lambda^2(\mathfrak{su}(\mathbb{V})/\mathfrak{q})$ and has values in $\mathfrak{su}(\mathbb{V})$. First observe that since $\kappa(u)$ is normal in the conformal sense, it is torsion free, so we conclude that the values actually lie in $\mathfrak{su}(\mathbb{V}) \cap \mathfrak{p} = \mathfrak{p}^{SU} \subset \mathfrak{q}$. This shows that $\underline{\omega}$ is torsion free and hence regular as a Cartan connection on $N \times Q$. Regularity implies that the induced almost CR structure on N (as described in Corollary 2.3) is non–degenerate of signature (p', q') and partially integrable. Moreover, once we have proved that $\underline{\omega}$ is normal, torsion freeness implies that the structure is integrable (and hence CR) by [10, 4.16]. The bundle $\mathcal{E}(1,0)$ can be defined as $(N \times Q) \times_Q L$ which immediately implies that it has the required properties.

Hence the proof boils down to showing that, if we interpret $\kappa(u)$ as a map $\Lambda^2(\mathfrak{su}(\mathbb{V})/\mathfrak{q}) \to \mathfrak{su}(\mathbb{V})$, then it satisfies the normalisation condition $\partial^* \circ \kappa = 0$ for Cartan geometries of type $(SU(\mathbb{V}), Q)$. Having proved this, normality of $\underline{\omega}$ immediately follows from Q–equivariancy of ∂^*.

To compute $\partial^* \circ \kappa$ we have to choose various bases. First, there is an abelian subalgebra $\mathfrak{p}_+ \subset \mathfrak{p} \subset \mathfrak{g}$ which consists of all maps $Z \in \mathfrak{g}$ which vanish on $L_{\mathbb{R}}$ and map its real orthocomplement $L_{\mathbb{R}}^{\perp_{\mathbb{R}}}$ to $L_{\mathbb{R}}$. Skew symmetry then implies that all values of Z lie in $L_{\mathbb{R}}^{\perp_{\mathbb{R}}}$. The Killing form of $\mathfrak{g}$ induces a duality between $\mathfrak{p}_+$ and $\mathfrak{g}/\mathfrak{p}$.

Similarly, we obtain a subalgebra $\mathfrak{q}_+ \subset \mathfrak{q} \subset \mathfrak{su}(\mathbb{V})$ by taking all maps $\underline{Z}$ which vanish on L and map $L^\perp$ to L. Again, the Killing form induces a duality between $\mathfrak{su}(\mathbb{V})/\mathfrak{q}$ and $\mathfrak{q}_+$. However, in the complex case, there is a finer decomposition. Namely, there is a subspace $\mathfrak{q}_2 \subset \mathfrak{q}_+$ of real dimension 1, which consists of those maps that vanish identically on $L^\perp$. (In the real case, the analogous condition already forces a map to vanish identically). The annihilator of $\mathfrak{q}_2$ is the (real) codimension one subspace $\mathfrak{h} \subset \mathfrak{su}(\mathbb{V})/\mathfrak{q}$ used in the proof of Corollary 2.3.

Since the Killing form on a simple Lie algebra is uniquely determined up to multiples by invariance, we may in both cases use the trace form on $\mathfrak{g}$ for obtaining the dualities on both algebras. This only replaces ∂^* by a non-zero multiple, so it has no effect on normality. Now let us chose a non-zero element $X_0 \in \mathfrak{su}(\mathbb{V})$ such that $X_0 + \mathfrak{q} \notin \mathfrak{h}$. Next, choose elements $X_1, \dots, X_{2n'}$ (where $n' = p' + q'$) such that $\{X_1 + \mathfrak{q}, \dots, X_{2n'} + \mathfrak{q}\}$ is a basis of $\mathfrak{h}$. Finally, put $X_{2n'+1} := i\cdot \mathrm{id} \in \mathfrak{g}$. Then $\{X_0 + \mathfrak{p}, \dots, X_{2n'+1} + \mathfrak{p}\}$ is a basis for $\mathfrak{g}/\mathfrak{p}$. By duality, there are unique elements $Z_0, \dots, Z_{2n'+1} \in \mathfrak{p}_+$ such that $\mathrm{tr}(X_i \circ Z_j) = \delta_{ij}$, so $\{Z_0, \dots, Z_{2n'+1}\}$ is the dual basis of $\mathfrak{p}_+$.

On the other hand, for $j = 0, \dots, 2n'$ let $\underline{Z}_j$ be the complex linear part of the linear map Z_j, i.e. $\underline{Z}_j(v) = \frac{1}{2}(Z_j(v) - iZ_j(iv))$. This means that $Z_j - \underline{Z}_j$ is conjugate linear, and since each X_i is complex linear, the map $X_i \circ (Z_j - \underline{Z}_j)$ is conjugate linear. But such a map has vanishing real trace, so we conclude that $\mathrm{tr}(X_i \circ \underline{Z}_j) = \delta_{ij}$. Using this for $X_{2n'+1} = i\cdot \mathrm{id}$, we in particular see that the $\underline{Z}_j \in \mathfrak{su}(\mathbb{V})$ for $j = 0, \dots, 2n'$.

Next, take a nonzero element $v \in L_{\mathbb{R}}$. By the definition of $\mathfrak{p}_+$, each $Z_j(L_{\mathbb{R}}^{\perp_{\mathbb{R}}}) \subset L_{\mathbb{R}}$, so $Z_j(iv) = av$ for some $a \in \mathbb{R}$. But then $i\cdot \mathrm{id} \circ Z_j$ maps iv to aiv, and looking at an appropriate basis extending $\{v, iv\}$ one concludes that $\mathrm{tr}(i\cdot \mathrm{id} \circ Z_j) = 2a$. Hence $Z_j|_L = 0$, and therefore $\underline{Z}_j|_L = 0$ for $j = 0, \cdots, 2n'$. For $w \in L^\perp$ all complex multiples of w lie in $L_{\mathbb{R}}^{\perp_{\mathbb{R}}}$, so $\underline{Z}_j(w) \in L$. Thus we conclude that $\underline{Z}_j \in \mathfrak{q}_+$ for all $j = 0, \dots, 2n'$ and hence $\{\underline{Z}_0, \dots, \underline{Z}_{2n'}\}$ is the basis of $\mathfrak{q}_+$ which is dual to the basis $\{X_0 + \mathfrak{q}, \dots, X_{2n'} + \mathfrak{q}\}$ of $\mathfrak{su}(\mathbb{V})/\mathfrak{q}$.

Now the formula for $\partial^*(\kappa(u)) : \mathfrak{su}(\mathbb{V})/\mathfrak{q} \to \mathfrak{su}(\mathbb{V})$ has two summands, see [17, section 5.1]. The first summand is

$$X + \mathfrak{q} \mapsto \sum_{j=0}^{2n'} [\kappa(u)(X + \mathfrak{q}, X_j + \mathfrak{q}), \underline{Z}_j],$$

where the bracket is the commutator of linear maps. Since $\kappa(u)(X+\mathfrak{q}, X_j+\mathfrak{q})$ is complex linear, this is the complex linear part of $\sum_{j=0}^{2n'} [\kappa(u)(X + \mathfrak{q}, X_j + \mathfrak{q}), Z_j]$. Now inside $\kappa(u)$, we can replace $X+\mathfrak{q}$ and $X_j+\mathfrak{q}$ by $X+\mathfrak{p}$ and $X_j+\mathfrak{p}$, respectively, which just corresponds to viewing $\kappa(u)$ as a map $\Lambda^2\mathfrak{g}/\mathfrak{p} \to \mathfrak{su}(\mathbb{V})$. Moreover, we may sum up to $2n'+1$ without changing the result since $\kappa(u)$ descends to $\mathfrak{su}(\mathbb{V})/\mathfrak{q}$. (This corresponds to the fact that $\mathbf{k}$ hooks trivially into κ.) Thus have expressed the first summand as the complex linear part of $\sum_{j=0}^{2n'+1}[\kappa(u)(X+\mathfrak{p}, X_j+\mathfrak{p}), Z_j]$, and the bracket is the same as in $\mathfrak{g}$. But by the choice of our bases, the condition that the conformal Cartan connection ω is normal exactly reads as $\sum_{j=0}^{2n'+1}[\kappa(u)(X+\mathfrak{p}, X_j+\mathfrak{p}), Z_j]=0$.

Thus we are left with the second summand, which is

$$(*) \qquad X+\mathfrak{q} \mapsto \tfrac{1}{2}\sum_{j=0}^{2n'} \kappa(u)([X, \underline{Z}_j]+\mathfrak{q}, X_j+\mathfrak{q}).$$

We have already observed that $\underline{Z}_j$ vanishes on L, so $[X, \underline{Z}_j](L) = \underline{Z}_j(X(L))$. If $X+\mathfrak{q} \in \mathfrak{h} \subset \mathfrak{su}(\mathbb{V})/\mathfrak{q}$, then $\underline{Z}_j(X(L)) \subset \underline{Z}_j(L^\perp) \subset L$, and hence $[X, \underline{Z}_j] \in \mathfrak{q}$. Therefore, it suffices to show that $(*)$ vanishes for $X = X_0$.

Take a nonzero element $v \in L_{\mathbb{R}}$. In the proof of Corollary 2.3 we have seen that $X+\mathfrak{q} \mapsto X(v)+L$ induces a linear isomorphism $\mathfrak{h} \to L^\perp/L$, which was used to define the complex structure on $\mathfrak{h}$. Fixing $w \in \mathbb{V}$ such that $\langle v, w\rangle = 1$, one easily verifies, in a basis, that the restriction of the traceform to $\mathfrak{h} \times \mathfrak{q}_+$ is a nonzero real multiple of $(X+\mathfrak{q}, \underline{Z}) \mapsto \langle \underline{Z}(X(v)), w\rangle_{\mathbb{R}}$.

Now for our basis element X_0, by definition we have $X_0(v) \notin L^\perp$. Since $X_0(v) \neq 0$, it must be congruent to a nonzero, purely imaginary multiple of w modulo $L^\perp$. Further, for $j = 1, \dots, 2n'$ we have $\underline{Z}_j \in \mathfrak{q}_+$ so $\underline{Z}_j|_L = 0$ and since $\underline{Z}_j$ is skew Hermitian, this also implies $\underline{Z}_j(\mathbb{V}) \subset L^\perp$. Hence $[X_0, \underline{Z}_j](v) = -\underline{Z}_j(X_0(v)) \in L^\perp$. On the one hand, this shows that $[X_0, \underline{Z}_j]+\mathfrak{q} \in \mathfrak{h}$. On the other hand, together with the above we see that there is a nonzero real number a such that

$$\langle X(v), i[X_0, \underline{Z}_j](v)\rangle_{\mathbb{R}} = -\langle X(v), i\underline{Z}_j(X_0(v))\rangle_{\mathbb{R}} = a\operatorname{tr}(X \circ \underline{Z}_j).$$

Now $(X+\mathfrak{q}, Y+\mathfrak{q}) \mapsto \langle X(v), Y(v)\rangle_{\mathbb{R}}$ defines a non–degenerate inner product on $\mathfrak{h}$. Hence there are unique elements $Y_j+\mathfrak{q} \in \mathfrak{h}$ for $j = 1, \dots, 2n'$ such that $\langle X_k(v), Y_j(v)\rangle_{\mathbb{R}} = \delta_{kj}$. Applying the last displayed formula to $X = X_k$ we conclude that $[X_0, \underline{Z}_j]+q = -ia(Y_j+q)$. Therefore, normality is equivalent to

$$(**) \qquad \sum_{j=1}^{2n'} \kappa(u)(i(Y_j+\mathfrak{q}), X_j+\mathfrak{q}) = 0.$$

This expression admits an interpretation on M. Similar to the linear isomorphism $\mathfrak{h} \to L^\perp/L$, a nonzero element $v \in \mathbb{L}_{\mathbb{R}}$ also gives rise to a linear isomorphism $\mathfrak{g}/\mathfrak{p} \to L_{\mathbb{R}}^{\perp_{\mathbb{R}}}/L_{\mathbb{R}}$. Now $\langle\ ,\ \rangle_{\mathbb{R}}$ induces an inner product on $L_{\mathbb{R}}^{\perp_{\mathbb{R}}}/L_{\mathbb{R}}$ which can be carried over to $\mathfrak{g}/\mathfrak{p}$. Passing to the associated bundle $\mathcal{G} \times_P (\mathfrak{g}/\mathfrak{p})$ the resulting class of inner products induces the conformal structure on M. In particular, $L^\perp/L$ (together with the class of inner products induced by $\langle\ ,\ \rangle_{\mathbb{R}}$) corresponds to a subquotient of the tangent spaces of M. Choosing a preferred metric from the conformal class, sections 4.4 and 4.5 of [9] show that this subquotient can be naturally identified with $\tilde{H} = \mathbf{k}^\perp \cap \ell^\perp \subset TM$. Moreover, by [9, Proposition 4.4], the endomorphism $\nabla_a \mathbf{k}^b$ of TM vanishes on $\mathbf{k}$ and ℓ, and it represents the complex structure on $\tilde{H}$ obtained from the identification with $L^\perp/L$. But this exactly means that the left hand side of $(**)$ corresponds to $\Omega_{abAB}\nabla^a \mathbf{k}^b$, which vanishes by the Lemma above. □

Theorem. *Let $(M,[g])$ be a conformal structure of signature $(2p'+1, 2q'+1)$ such that there is a parallel orthogonal complex structure $\mathbb{J}$ on the standard tractor bundle. Then locally M is conformally isometric to the Fefferman space of a CR manifold of hypersurface type, which is non–degenerate of signature (p',q') endowed with an appropriate root $\mathcal{E}(1,0)$ of the canonical bundle in such a way the $\mathbb{J}$ corresponds to the canonical complex structure introduced in* [9].

Proof. Take an open subset $W \subset M$ and a local leaf space $\psi : W \to N$ as in Theorem 2.3. By the Proposition above we get a CR structure of the required type on N, and $(N \times Q, \underline{\omega})$ is the canonical normal Cartan geometry associated to this structure. By construction, $P^{SU} \subset Q$ is the stabiliser of the real isotropic line $L_{\mathbb{R}} \subset L \subset \mathbb{V}$. Taking $(N \times Q) \times_Q L$ as $\mathcal{E}(-1,0)$, the Fefferman space $\tilde{N}$ of N is simply $N \times (Q/P^{SU})$, endowed with the Cartan geometry $(N \times Q \to N \times (Q/P^{SU}), \underline{\omega})$. Theorem 2.3 exactly says that this Cartan geometry is locally isomorphic to $(p : \mathcal{G}^{SU} \to M, \omega^{SU})$, which by the construction in [9, Theorem 2.4 and 3.1] implies that M is locally isometric to $\tilde{N}$ in a way compatible with the complex structures on standard tractors. □

3. A strengthening of Sparling's characterisation

From some points of view, the local characterisation of Fefferman spaces in Theorem 2.5 is rather satisfactory. The characterisation uses only conformally invariant data, and it is very conceptual through its relation to conformal holonomy. On the other hand, at first sight it would seem to be much weaker than the characterisation due to Sparling as described in Theorem 3.1 of [15]. This characterisation says that if (M, g) is a pseudo-Riemannian manifold, of appropriate signature, which admits an isotropic Killing field $\mathbf{k}$ with the properties that it inserts trivially into both the Weyl curvature and the Cotton tensor, while $Ric(\mathbf{k}, \mathbf{k}) > 0$, then M is locally conformally isometric to a Fefferman space.

Now as shown in [14, Proposition 2.2] and in [6, Corollary 3.5], parallel sections of the adjoint tractor bundle $\mathcal{A}$ of any conformal manifold M are in bijective correspondence with conformal Killing fields which insert trivially into the Weyl curvature and the Cotton–York tensor. Given such a conformal Killing field $\mathbf{k}$, one can view the corresponding parallel adjoint tractor as an endomorphism $\mathbb{J}$ of the standard tractor bundle $\mathcal{T}$, and express the condition that $\mathbb{J} \circ \mathbb{J} = -id$ in terms of $\mathbf{k}$ and its covariant derivatives. Expressing $\mathbb{J} \circ \mathbb{J} = -id$ in this way, one is quickly led to see this includes the properties required in Sparling's characterisation (see subsection 4.4 of [9]). However pushing through the calculation naïvely at first suggests that many more conditions arise than are needed for the Sparling characterisation. In fact a direct but tedious calculation reveals that the remaining identities are all differential consequences of the conditions in Sparling's characterisation.

There is however an elegant and insightful way to do this using deeper methods from conformal geometry. As a bonus we obtain a variant of Sparling's characterisation which is stronger by dint of conformal invariance.

Theorem. *Let $(M, [g])$ be a conformal manifold of dimension $n \geq 3$ and let $s \in \Gamma(\mathcal{A})$ be a parallel section with underlying conformal Killing field $\mathbf{k}$, and suppose that $\mathbf{k}$ is isotropic. Then for any metric in the conformal class, with Levi–Civita connection ∇ and Schouten tensor P, the function*

$$\tfrac{1}{n^2}(\nabla_a \mathbf{k}^a)^2 - \mathbf{k}^a \mathsf{P}_{ab} \mathbf{k}^b - \tfrac{1}{n} \mathbf{k}^a \nabla_a \nabla_b \mathbf{k}^b$$

is equal to some constant $\lambda \in \mathbb{R}$. This constant is independent of the choice of metric from the conformal class and $s \circ s = \lambda \operatorname{id} \in \Gamma(\operatorname{End}(\mathcal{T}))$.

Proof. As an endomorphism of $\mathcal{T}$, s is skew symmetric and hence $s \circ s \in \Gamma(S^2\mathcal{T})$. Now $S^2\mathcal{T}$ invariantly decomposes into the trace-free part $S^2_0\mathcal{T}$ and $\mathbb{R}\operatorname{id}$. Since s is a parallel section of $\operatorname{End}(\mathcal{T})$, $s \circ s$ is a parallel section of $S^2\mathcal{T}$, and hence it is the sum of a parallel section of $S^2_0\mathcal{T}$ and a constant multiple of the identity. Now the canonical filtration $\mathcal{T}^1 \subset \mathcal{T}^0 \subset \mathcal{T}$ with $\mathcal{T}^0 := (\mathcal{T}^1)^\perp$ induces a filtration of $S^2_0\mathcal{T}$. The largest filtration component is the image of $\mathcal{T}^0 \otimes \mathcal{T}$ in $S^2_0\mathcal{T}$ (under the obvious projection) and the quotient of $S^2_0\mathcal{T}$ by this filtration component is isomorphic to $S^2(\mathcal{T}/\mathcal{T}^0) \cong \mathcal{E}[2]$. If t^{AB} is a section of $S^2_0\mathcal{T}$ then the projection to this quotient is given by $X_A t^{AB} X_B$.

Now we can use the explicit formula for s_{AB} in terms of $K_B := Z_{Bb}k^b - \frac{1}{n} X_B \nabla_b \mathbf{k}^b$ from the proof of Proposition 2.1. This, in particular, implies $X^A s_{AB} = K_B$, as well as

$$K^B s_{BC} = -\tfrac{1}{n} K_C \nabla_b \mathbf{k}^b + \mathbf{k}^a \nabla_a K_C.$$

This shows that $X^A s_A{}^C s_C{}^B X_B = -\mathbf{k}^a \mathbf{k}_a = 0$. But a simple corollary of the machinery of BGG sequences developed in [11] and [4] is that any parallel section of an indecomposable tractor bundle can be recovered, by an invariant differential operator, from its projection to the quotient by the largest filtration component.

Hence $s \circ s = \lambda \operatorname{id}_{\mathcal{T}}$ for some constant $\lambda \in \mathbb{R}$, and we can compute this constant for example as $X^A s_A{}^C s_C{}^B Y_B = K^B s_{BC} Y^C$. The formula for λ then follows easily by expanding the above expression for $K^B s_{BC}$. □

Using this, we obtain a strengthening of Sparling's characterisation as well as surprising information about odd dimensional manifolds admitting isotropic normal conformal Killing fields.

Corollary. *Let M be a pseudo-Riemannian manifold of dimension $n \geq 3$ and let $\mathbf{k}$ be an isotropic conformal Killing field, which inserts trivially into the Weyl curvature and (the 2–form indices of) the Cotton–York tensor.*
(1) If n is even and the constant

$$\tfrac{1}{n^2}(\nabla_a \mathbf{k}^a)^2 - \mathbf{k}^a \mathsf{P}_{ab} \mathbf{k}^b - \tfrac{1}{n} \mathbf{k}^a \nabla_a \nabla_b \mathbf{k}^b$$

is negative, then M is locally conformally isometric to a Fefferman space $\tilde{N}$, of a CR manifold N, in such a way that $\mathbf{k}$ is mapped to a generator of the vertical subbundle of $\tilde{N} \to N$.
(2) If n is odd, then

$$\tfrac{1}{n^2}(\nabla_a \mathbf{k}^a)^2 - \mathbf{k}^a \mathsf{P}_{ab} \mathbf{k}^b - \tfrac{1}{n} \mathbf{k}^a \nabla_a \nabla_b \mathbf{k}^b = 0$$

and the adjoint tractor field s corresponding to $\mathbf{k}$ satisfies $s \circ s = 0$.

Proof. (1) Passing to a constant rescaling of $\mathbf{k}$, we obtain a conformal Killing field such that the associated parallel adjoint tractor defines an almost complex structure. Then the result follows from the theorem.
(2) Since n is odd, the tractor bundle $\mathcal{T}$ has odd rank, so the skew symmetric map $s : \mathcal{T} \to \mathcal{T}$ has to be degenerate. Then $s \circ s = \lambda\,\mathrm{id}$ is only possible for $\lambda = 0$, and the result follows. □

Remarks. (1) If in part (1) of the corollary we assume that $\mathbf{k}$ is a Killing field rather than just a conformal Killing field, then $\nabla_a \mathbf{k}^a = 0$, and the constant in question simply becomes $-\mathbf{k}^a \mathsf{P}_{ab} \mathbf{k}^b$. Since $\mathbf{k}$ is isotropic, this is a nonzero multiple of $Ric(\mathbf{k}, \mathbf{k})$, and we recover Sparling's characterisation. Note that in contrast to $Ric(\mathbf{k}, \mathbf{k}) > 0$, negativity of the constant in (1) does not evidently imply that $\mathbf{k}$ is nowhere vanishing. We get this as a consequence of the corollary.
(2) In section 4 of [15] it is shown how to deduce a global characterisation of Fefferman spaces from the local one.
(3) In the odd–dimensional case, the kernel of s is nontrivial in each point. If the dimension of these kernels is constant, then they form a smooth subbundle in the standard tractor bundle. Since s is parallel, this subbundle is invariant under the standard tractor connection. In particular, this implies that the conformal holonomy cannot act irreducibly.

References

[1] S. Armstrong, Definite signature conformal holonomy: a complete classification, preprint math.DG/0503388

[2] T.N. Bailey, M.G. Eastwood, A.R. Gover, Thomas's structure bundle for conformal, projective and related structures, Rocky Mountain J. **24** (1994), 1191–1217.

[3] D. Burns, K. Diederich, S. Shnider, Distinguished curves in pseudoconvex boundaries, Duke Math. J. **44** No. 2 (1977) 407–431.

[4] D.M.J. Calderbank, T. Diemer, Differential invariants and curved Bernstein–Gelfand–Gelfand sequences, J. Reine Angew. Math. **537** (2001), 67–103.

[5] A. Čap, Correspondence spaces and twistor spaces for parabolic geometries, J. Reine Angew. Math. **582** (2005) 143–172.

[6] A. Čap, Infinitesimal Automorphisms and Deformations of Parabolic Geometries, preprint math.DG/0508535

[7] A. Čap, A.R. Gover, Tractor calculi for parabolic geometries, Trans. Amer. Math. Soc. **354** (2002), 1511-1548.

[8] A. Čap, A.R. Gover, Standard tractors and the conformal ambient metric construction, Ann. Global Anal. Geom. **24**, 3 (2003) 231-259.

[9] A. Čap, A.R. Gover, CR tractors and the Fefferman space, preprint ESI 1874, electronically available at http://www.esi.ac.at

[10] A. Čap, H. Schichl, Parabolic geometries and canonical Cartan connections, Hokkaido Math. J. **29** No.3 (2000) 453–505.

[11] A. Čap, J. Slovák, V. Souček, Bernstein–Gelfand–Gelfand sequences, Ann. of Math. **154**, no. 1 (2001) 97-113.

[12] S.S. Chern, J. Moser, Real hypersurfaces in complex manifolds, Acta Math. **133** (1974), 219–271.

[13] C. Fefferman, Monge–Ampère equations, the Bergman kernel and geometry of pseudoconvex domains, Ann. of Math. **103** (1976) 395–416; Erratum **104** (1976) 393–394

[14] A.R. Gover, Laplacian operators and Q-curvature on conformally Einstein manifolds, preprint math.DG/0506037, to appear in Math. Ann.

[15] C.R. Graham, On Sparling's characterization of Fefferman metrics, Amer. J. Math. **109**, no. 5 (1987), 853–874.

[16] F. Leitner, A remark on conformal $SU(p,q)$-holonomy, preprint math.DG/0604393.

[17] K. Yamaguchi, Differential Systems Associated with Simple Graded Lie Algebras, Advanced Studies in Pure Math. **22** (1993), 413–494.

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