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COMPACT QUOTIENTS OF NEGATIVELY CURVED RIEMANNIAN MANIFOLDS WITH LARGE ISOMETRY GROUP

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Abstract. A definition of locally cohomogeneity one Riemannian manifold is given and it is proved that a compact, negatively curved Riemannian manifold of local cohomogeneity one is locally symmetric; this result extends a well known theorem, due to Heintze, stating the local symmetry of a negatively curved, compact, locally homogeneous Riemannian manifold. As a corollary, we get that a non symmetric chomogeneity one Riemannian manifold admits no compact quotients.

1. Introduction.

Let (M,g) be a Riemannian manifold, $G \subseteq Iso_g(M)$ a Lie group of isometries and Γ a discrete subgroup of isometries. In general, the quotient Riemannian manifold $N = M \setminus \Gamma$ is not to be expected to be acted on neither by the same group G nor by some other Lie group G', locally isomorphic to G.

Nonetheless, the projection map $\pi\colon M \to N$ being a local diffeomorphism, any Killing field X of the Lie algebra $\mathfrak{G}$, of infinitesimal transformations of G, induces a local vector field $\pi_* X$ around any point $p \in N$. The germs of local Killing fields on N, obtained in that way, determines a subsheaf $\mathfrak{K}_{\mathfrak{G}}$ of the sheaf $\mathfrak{K}$ of the local Killing vector fields, whose stalk at a point p is a union of Lie algebras $\mathfrak{g}_\alpha$, one per each element of $\pi^{-1}(p)$ and each one isomorphic to $\mathfrak{G}$.

On the other hand, when M is a homogenous Riemannian manifold, N is automatically locally homogeneous, and, hence, if one tries to characterize a class of quotients of homogeneous spaces, it is natural to confine the analysis to the class of the locally homogeneous manifolds.

To our knowledge up to now, there is no definition of a class of Riemannian manifolds, suitable to study the quotients of cohomogeneity q Riemannian manifolds. We recall that a manifold M is said to be of cohomogeneity q under the action of a Lie group G of transformations, if the principal orbits of the action of G are of codimension q (for general references on the subject, see [AA], [AA1], [Br]).

So, we find convenient to introduce the following definition in order to describe what we intend as *local cohomogeneity* q :

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Definition 1.1. Let (M,g) be a Riemannian manifold and let $\mathfrak{L} \subset \mathfrak{K}$ be a subsheaf of the sheaf $\mathfrak{K}$ of all germs of local Killing fields. (M,g) is of *local cohomogeneity* q *for the local action of* $\mathfrak{L}$ if the following two properties are satisfied by $\mathfrak{L}$:

i) for any $p \in M$, the stalk $\mathfrak{L}_p$ is union of $\{\mathfrak{g}_\alpha\}_{\alpha \in I}$, I being countable;

ii) there exists an open dense set $\Omega \subset M$ such that, for any point $p \in \Omega$ and for any Lie algebra $\mathfrak{g}_\alpha \in \mathcal{F}_p$, the valuation map

$$v_p \colon \mathfrak{g}_\alpha \to T_pM$$

$$v_p([X]) = X|_p$$

has (n-q)-dimensional image.

It is to underline the fact that the degree q of local cohomogeneity is strictly related to the sheaf $\mathfrak{L}$ considered; in other words, being of local cohomogeneity q with $q > 0$ does not exclude that the manifold M can be locally homogeneous. Indeed, if the stalk $\mathfrak{L}_p$ contains two Lie algebras $\mathfrak{g}_1$ e $\mathfrak{g}_2$ such that

$$v_p(\mathfrak{g}_1) + v_p(\mathfrak{g}_2) = T_pM$$

then we may infer that p admits a neighborhood on which the local isometries act transitively.

For what concerns the theory of locally homogeneous Riemannian manifolds with negative curvature, the following well known result is due to E. Heintze:

Theorem 1.2. *[H] If (M,g) is a compact, negatively curved, locally homogeneous Riemannian manifold, then it is locally symmetric.*

This result furnishes also a complete description of all compact manifold of negative curvature, which can be obtained as quotients of homogeneous manifolds.

Our aim is to analyze the compact manifolds of negative curvature, when they are obtained as quotients of manifolds which admit a large group of isometries, even if not homogeneous. Motivated by Heintze's result, we obtained the following theorem on negatively curved Riemannian manifolds of local cohomogeneity one:

Main Theorem. *Let (M,g) be a compact, negatively curved Riemannian manifold, which is of local cohomogeneity one for the action of a subsheaf $\mathfrak{L} \subset \mathfrak{K}$. Then (M,g) is locally symmetric.*

¿From the previous remarks, it follows that any compact quotient of a manifold of cohomogeneity one is a manifold of local cohomogeneity one. From this fact and the Main Theorem, the proof of the following Corollary 1.3 is immediate.

Corollary 1.3. *If* (M,g) *is a cohomogeneity one Riemannian manifold of negative curvature, which admits a compact quotient, then it is locally symmetic.*

Remark 1.4. We remark here that every Riemannian homogeneous space of negative curvature is a cohomogeneity one Riemannian manifold for the action of some group of isometries (see [PS]); since, by Borel's theorem (see [Bo]), every Riemannian symmetric space of negative curvature admits compact quotients, we may infer that any of such spaces admits compact quotients of local cohomogeneity one.

Remark 1.5. Another interesting class of manifolds, which can be considered as an analogue of the locally homogeneous spaces in case of cohomogeneity q, is the class of $\mathfrak{g}$-manifolds, introduced by D. Alekseevsky and P. Michor ([AM]).

2. Proof of the Main Theorem.

The proof will be devided into several Lemmata. In the following, $(\tilde{M},\tilde{g})$ will be the universal covering manifold of (M,g), which trns out to be complete, since (M,g) is compact.

Lemma 2.1. *For every* $p \in \Omega$ *and every* $\mathfrak{g}_\alpha \subset \mathfrak{L}_p$*, there is a Lie group* G_α *whose Lie algebra is* $\mathfrak{g}_\alpha$ *and which acts on* $\tilde{M}$ *by isometries and with principal orbits of codimension one.*

Proof. Every $[X] \in \mathfrak{g}_\alpha$ gives rise to a local Killing vector field X in a neighborhood of p; such vector field X, when lifted to $\tilde{M}$, can be uniquely extended to a global Killing vector field (see e.g. [KN]), so that the Lie algebra $\mathfrak{g}_\alpha$ induces a finite dimensional Lie algebra of Killing vector fields on $(\tilde{M},\tilde{g})$, isomorphic to $\mathfrak{g}_\alpha$; since $(\tilde{M},\tilde{g})$ is complete, every Killing vector field is complete and, by Palais Theorem ([P]), there exists a Lie group G_α of isometries of $(\tilde{M},\tilde{g})$, whose Lie algebra is isomorphic to $\mathfrak{g}_\alpha$; condition (ii) in the definition of local cohomogeneity one implies that the principal orbits of G_α have codimension one. $\square$

We now proceed to prove the Main Theorem, showing that (M,g) is locally homogeneous and hence symmetric, by Heintze's result. By contradiction, we will suppose from now on, that (M,g) is not locally homogeneous with respect to any group of isometries. We pick a group G out of the G_α, $\alpha \in I$ as constructed in Lemma 1; we also denote by Δ the group of covering transformations of $\tilde{M}$ over M. Note that Δ does not necessarily centralize G.

Lemma 2.2. *There exists at least one singular orbit* $S \subset \tilde{M}$ *for the action of* G

Proof. Suppose that every orbit of G is principal and pick $\delta \in \Delta$. If we denote by $G(p)$ the orbit of the point p for the action of G, we claim that $\delta(G(p)) = G(\delta(p))$ for every $p \in M$; indeed, we have that $T_p(\delta(G(p))) = T_p(G(\delta(p)))$ for every $p \in \tilde{M}$, since otherwise $(\tilde{M},\tilde{g})$ would be locally homogeneous and, hence, by completeness, homogeneous. So, the two regular distributions $p \mapsto T_p(\delta(G(p)))$ and $p \mapsto T_p(G(\delta(p)))$ coincide and hence also their leaves coincide. This proves the claim.

As a consequence, the foliation $\mathfrak{F}$ of $\tilde{M}$ given by the G-orbits goes down to M, because it is Δ-invariant. But, the one-dimensional distribution $T\mathfrak{F}^{-}$ is totally geodesic and this can not exist on a compact negatively curved Riemannian manifold, as a result by Walczak ([W]) states. $\square$

Lemma 2.3. *For every $\delta \in \Delta$, we have $\delta(S) = S$.*

Proof. By a result of a previous paper ([PS]), $(\tilde{M}, \tilde{g})$ admits at most a singular orbit for the action of G; for the sake of completeness we reproduce the proof of this fact in the Appendix. By Lemma 2.2 we have indeed that there exists *exactly one* singular orbit S.

Suppose, by contradiction, there is a point $p \in S$ and an element $\delta \in \Delta$ such that $\delta(p)$ is a regular point. It is clear that we may suppose that $T_{\delta(p)}\delta(S) \subset T_{\delta(p)}G(\delta(p))$, otherwise $(\tilde{M}, \tilde{g})$ would be homogeneous.

Let η be a normal geodesic to $G(\delta(p))$ starting from $\delta(p)$; since S is a singular orbit, the isotropy subgroup G_p acts transitively on the unit sphere of the normal space T_pS^{-} (see e.g. [AA1]) and the same holds for the isotropy subgroup

$$G'_p \overset{\text{def}}{=} \delta \circ G_p \circ \delta^{-1} \quad .$$

It follows that we can find a one-parameter group of isometries $H = \exp(tX)$, fixing $\delta(p)$, and such that $T_{\eta(\epsilon)}H(\eta(\epsilon)) \neq \{0\}$ for a sufficiently small $\epsilon \in \mathbb{R}^{+}$. We claim that there is a $t_o \in \mathbb{R}$ such that $T_{\exp(t_oX)(\eta(\epsilon))}H(\eta(\epsilon))$ is transverse to $T_{\exp(t_oX)(\eta(\epsilon))}G((\eta(\epsilon))$.

Indeed, if for every $t \in \mathbb{R}$ the curve $c(t) = \exp(tX)(\eta(\epsilon))$ satisfies $c'(t) \in T_{c(t)}G(c(t))$, we would have $c(t) \in T_{c(0)}G(c(0))$ for every $t \in \mathbb{R}$. We then consider for each $t \in \mathbb{R}$ the geodesic arc

$$\gamma_t : [0, \epsilon] \ni s \mapsto \exp(tX)\eta(s)$$

and note that

$$\operatorname{dist}(\exp(tX)\eta(\epsilon), \delta(p)) = \operatorname{dist}(\exp(tX)\eta(\epsilon), \exp(tX)(\delta(p))) = \operatorname{dist}(\eta(\epsilon), \delta(p)).$$

It then follows that, for any $t \in [0, \varepsilon]$ γ_t is a minimizing geodesic arc from the regular orbit $G(\delta(p))$ to the regular orbit $G(\eta(\epsilon))$ (actually ϵ can be chosen so small that $G(\eta(\epsilon))$ is regular); moreover, we have $\gamma_t(0) = \eta(0)$. But such a minimizing arc is unique, hence

$$\exp(tX)\eta(s) = \eta(s)$$

for all $s, t \in \mathbb{R}$; it then follows that $T_{\eta(\epsilon)}H(\eta(\epsilon)) = \{0\}$, contrary to our assumption. So, our claim is proved and we get local homogeneity of $(\tilde{M}, \tilde{g})$ (and hence global homogeneity), which was excluded at the beginning. $\square$

The proof of our main result concludes as follows. By Lemma 2.3, δ maps regular points into regular points and, using the same arguments as in the proof of Lemma 2.2, we obtain that δ maps regular orbits onto regular orbits. It follows that every $\delta \in \Delta$ maps G-orbits onto G-orbits, hence induces an isometry δ^* of the orbit space $\tilde{M}/G \cong [0, +\infty)$ fixing the origin (the origin corresponds to the singular orbit S). It follows that δ^* is the identity, that is δ maps each orbit onto itself. Since $\tilde{M}$ is not compact, also $M = \tilde{M}/\Delta$ is not compact: contradiction. $\square$

3. Appendix.

In this appendix we reproduce a result quoted in the proof of Lemma 2.3 and which we obtained in a previous paper ([PS]).

We consider a simply connected Riemannian manifold (M, g) of negative curvature and of cohomogeneity one under the action of a Lie group of isometries G.

Lemma. *If B is any singular orbit and H is the isotropy subgroup at some point $b \in B$, then H is maximal compact in G.*

Proof. Let us suppose that H is not maximal compact and that $H \subsetneq H'$, with H' compact subgroup of G. Then, by Cartan's theorem, there is a point p fixed by H'. We note that p is necessarily a singular point and does not belong to the orbit B, since otherwise H and H' would be conjugate, hence equal. We take a geodesic segment γ joining p and b; since γ is not contained in any singular orbit, it must contain at least one regular point z. Now, any $h \in H$ fixes b and p and hence it fixes the geodesic segment γ pointwise, by uniqueness; it follows that H is contained in the isotropy subgroup G_z. But z is regular and G_z must be conjugate to a proper subgroup of H: contradiction. $\square$

Using the previous Lemma, we obtain:

Proposition. *If M is simply connected, there is at most one singular orbit.*

Proof. Let us suppose there exist two singular orbits $B_i = G/H_i$ $(i = 1, 2)$, where H_1, H_2 are maximal compact in G, by Lemma 3.1. By a classical result in Lie group theory, the two subgroups H_1 and H_2 are conjugate to each other and it then follows that $B_1 = B_2$. Indeed, if not, there would exist two points $z_1 \in B_1$ and $z_2 \in B_2$ with the same isotropy subgroup, say H_1; but then the unique geodesic joining z_1 to z_2 would be left pointwise fixed by H_1, so that H_1 would be a subgroup of the regular isotropy subgroup, a contradiction. $\square$

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