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All electro–vacuum Majumdar–Papapetrou space–times with nonsingular black holes

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Abstract

We show that all Majumdar–Papapetrou electrovacuum space–times with a non–empty black hole region and with a non–singular domain of outer communications are the standard Majumdar–Papapetrou space–times.

1 Introduction

Consider an electrovacuum space–time with a non–empty black hole region $\mathcal{B}$ and with an asymptotically flat spacelike surface Σ such that $\partial\Sigma$ is a compact manifold lying in the black hole region. Suppose further that $|Q| = M$, where Q is the total electric charge as seen from the asymptotically flat region of Σ and M is the ADM mass of Σ. According to [9, 8, 19], (under perhaps some supplementary conditions on $\partial\Sigma$) one expects that

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1. On Σ there is a globally defined Killing vector field X which is timelike in the asymptotically flat region.
2. For any $p \in \Sigma$ such that X is timelike there exists a neighbourhood $\mathcal{O}_p$ thereof and a coordinate system $x^\mu \in \Omega_p \subset \mathbf{R}^4$ such that the gravitational and electromagnetic fields take the Israel–Wilson–Perjes [13, 17] form.
3. The ADM four–momentum of Σ is timelike.

This leads naturally to the question of classifying space–times with the above properties. To our knowledge no conclusive study of this problem has been done so far (*cf.*, however, [10] for some remarks related to this issue). In this letter we wish to settle this question under the supplementary assumption that the domain of outer communications is static, *i.e.*, that the twist of the Killing vector field vanishes. In that case in the local coordinates discussed above the metric g and the electromagnetic potential A can be written in the Majumdar–Papapetrou (MP) form [15, 16]

$$g = -u^{-2}dt^2 + u^2(dx^2 + dy^2 + dz^2)\,, \tag{1.1}$$

$$A = u^{-1}dt\,, \tag{1.2}$$

with some nowhere vanishing, say positive, function u. Einstein–Maxwell equations read then

$$\frac{\partial u}{\partial t} = 0\,, \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0\,. \tag{1.3}$$

A space–time will be called a standard MP space–time if the coordinates x^μ of (1.1)–(1.2) cover the range $\mathbf{R} \times (\mathbf{R}^3 \setminus \{\vec{a}_i\})$ for a finite set of points $\vec{a}_i \in \mathbf{R}^3$, $i = 1, \ldots, I$, and if the function u has the form

$$u = 1 + \sum_{i=1}^{I} \frac{m_i}{|\vec{x} - \vec{a}_i|}\,, \tag{1.4}$$

for some positive constants m_i. It has been shown by Hartle and Hawking [10] that every standard MP space–time can be analytically extended to an electro–vacuum space–time with a non–empty black hole region, and with a domain of outer communication which is non–singular in the sense described below.[1] We shall prove the following:

Theorem 1.1 *Consider an electro–vacuum space–time (M, g) with a non–empty black hole region $\mathcal{B}$. Suppose that there exists in M an asymptotically flat spacelike hypersurface Σ with compact interior, with boundary $\partial\Sigma \subset \mathcal{B}$ and with*

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[1] The case in which $I = \infty$ has been considered in [2, Appendix B], where it was pointed out that the scalar $F_{\mu\nu}F^{\mu\nu}$ is unbounded in such space–times if the $\vec{a}_i$'s have accumulation points. It follows from our analysis below that the case where $I = \infty$ and the $\vec{a}_i$'s do not have accumulation points cannot lead to regular asymptotically flat space–times in the sense of Theorem 1.1.

timelike (non–vanishing) ADM four–momentum. Assume moreover that on the closure of the domain of outer communication $\ll \mathcal{J} \gg$ there exists a Killing vector field X with complete orbits diffeomorphic to $\mathbf{R}$, X being timelike in an asymptotic region of Σ. If (M, g) is locally a MP space–time in the sense of point 2 above, then there exists a subset of $\ll \mathcal{J} \gg$ which is isometrically diffeomorphic to a standard MP space–time.

It is clear that all the hypotheses above are necessary in the sense that they are satisfied by the standard MP space–times. In section 3 we present another version of Theorem 1.1, and we discuss various ways of modifying the hypotheses above.

One would like to strengthen the conclusion of Theorem 1.1 to conclude that $\ll \mathcal{J} \gg$ must be isometrically diffeomorphic to a standard MP space–time. To do that one would need to prove that there are no other extensions of a standard MP space–time than those constructed by Hartle and Hawking in [10]. This seems to be a difficult problem, the resolution of which lies outside the scope of this paper.

The reader will find the details of the proof of Theorem 1.1 in Section 2. Here we wish to give a rough outline of the ideas involved. First, one shows that the local MP coordinate systems can be patched together to a coordinate system which covers a set of the form $\mathbf{R} \times (\mathbf{R}^3 \setminus \mathcal{S})$, where $\mathcal{S}$ is a closed compact subset of $\mathbf{R}^3$ on which u blows up. This is done by passing to the universal cover and constructing such global coordinates there. By analyzing the properties of the resulting space–time one concludes that the initial set had to be simply connected to start with. Next one needs to analyze the blow up set $\mathcal{S}$. (Recall that the blow up set of a harmonic function can have a rather complicated structure, *e.g.* fractal objects can occur.) The claim that $\mathcal{S}$ must be a discrete set of points is our main technical result here, proved in Proposition 2.1 below.

2 Definitions and proof

Before passing to the proof of Theorem 1.1 we wish to give a few definitions and to make some preliminary remarks. Let Σ be a spacelike surface in an electro–vacuum space–time (M, g). A set $\Sigma_{\mathrm{ext}} \subset \Sigma$ will be said to be an *asymptotically flat three–end* if Σ_{ext} is diffeomorphic to $\mathbf{R}^3 \setminus B(R)$, where $B(R)$ is a closed ball of radius R in $\mathbf{R}^3$. Moreover we shall ask that in the coordinates induced on Σ_{ext} by this identification we have

$$|g_{ij} - \delta_{ij}| + |r\partial_k g_{ij}| + |rK_{ij}| + |A_\mu| + |rF_{\mu\nu}| \leq Cr^{-\epsilon}\,, \tag{2.1}$$

$$\forall X^i \in \mathbf{R}^3 \qquad C^{-1} \sum (X^i)^2 \leq g_{ij} X^i X^j \leq C \sum (X^i)^2\,, \tag{2.2}$$

for some constant C and some $\epsilon > 0$. Here K_{ij} is the extrinsic curvature tensor of Σ_{ext}. Finally we require that the Killing vector be timelike on Σ_{ext}. A spacelike hypersurface Σ will be said to have *compact interior* if there exists a manifold $\Sigma_{\rm int}$, the closure of which is a compact manifold with boundary, such that $\Sigma = \Sigma_{\rm int} \cup_{i=1}^{I} \Sigma_{{\rm ext},i}$, for some finite number of asymptotically flat ends $\Sigma_{{\rm ext},i}$. Moreover for each i the boundary $\partial\Sigma_{{\rm ext},i}$ and some connected component of $\partial\Sigma_{\rm int}$ are assumed to be identified by a diffeomorphism.

Let us mention that if $\ll \mathscr{I} \gg$ is globally hyperbolic, then Proposition 2.1 of [5] shows that there is no loss of generality in assuming that there is only one asymptotic end. We shall however not make the assumption of global hyperbolicity of $\ll \mathscr{I} \gg$.

Let us from now on choose one of the asymptotically flat ends, and to minimize notation let us use the symbol $\Sigma_{\rm ext}$ for the end in question. Consider an electro–vacuum space–time with an asymptotically flat end $\Sigma_{\rm ext}$ with timelike ADM four–momentum and with a Killing vector X which is timelike on $\Sigma_{\rm ext}$. It follows from [3] that there exists $\epsilon > 0$ such that $X^\mu X_\mu < -\epsilon$ for all $r \geq R_1$ for some R_1. (We use the signature $(-,+,+,+)$.) If the orbits of X through $\Sigma_{\rm ext}$ are complete then by [2, 14, 18, 6] there exists a conformal completion of M satisfying the usual completeness requirements [7]. We can then define a black hole region in the standard way [11] as $\mathcal{B} = M \setminus J^-(\mathscr{I}^+)$, a white hole region as $\mathcal{W} = M \setminus J^+(\mathscr{I}^-)$, and the domain of outer communications as $\ll \mathscr{I} \gg = M \setminus (\mathcal{B} \cup \mathcal{W})$. These definitions coincide then with those used in [4].

Let us now pass to the proof of Theorem 1.1. Consider the set

$$\tilde{\Sigma} = \{p \in \Sigma : X(p) \ \text{ is timelike}\} . \tag{2.3}$$

If $\tilde{\Sigma}$ is simply connected, let $\check{\Sigma} = \tilde{\Sigma}$, otherwise let $\check{\Sigma}$ be the universal cover of $\tilde{\Sigma}$. Note that if $\check{\Sigma} \neq \tilde{\Sigma}$, then $\check{\Sigma}$ will have more than one asymptotically flat end. Choose one of those ends and, by a slight abuse of notation, call it $\Sigma_{\rm ext}$. Define finally $\hat{\Sigma}$ to be that connected component of $\check{\Sigma}$ which contains $\Sigma_{\rm ext}$. We define $\hat{M}$ to be $\mathbf{R} \times \hat{\Sigma}$ with a metric $\hat{g}$ defined uniquely by the requirements that

1. The vector $\partial/\partial t$ tangent to the $\mathbf{R}$ factor of $\hat{M}$ is a Killing vector,
2. on $\hat{\Sigma} \equiv \{0\} \times \hat{\Sigma}$ the metric and the extrinsic curvature coincide with those of the original space–time

(*cf. e.g.* [4, Appendix A, eqs. (A.15)–(A.17)] for an explicit construction).

On $\tilde{\Sigma}$ let us define the function u by

$$u^{-2} \equiv -g_{\mu\nu} X^\mu X^\nu . \tag{2.4}$$

Consider a sequence $p_i \in \tilde{\Sigma}$ such that $p_i \to p \in \partial\Sigma$. By definition of $\tilde{\Sigma}$ either $p \in \partial\Sigma$ or $u^{-2}(p_i) \to 0$. In the former case the arguments of [4, Prop. 3.3] show that $u^{-2}(p_i) \to 0$ as well. Let us by an abuse of notation denote by X the Killing vector on $\hat{M}$, and by u the corresponding quantity as in (2.4). It follows that

$$u^{-2}(p) \to_{p \to \partial\hat{\Sigma}} 0 . \tag{2.5}$$

On $\hat{M}$ we can define an auxiliary metric $h = h_{\mu\nu}dx^\mu dx^\nu$ by the equation

$$h_{\mu\nu} = u^{-2}(\hat{g}_{\mu\nu} + u^2 X_\mu X_\nu) - u^4 X_\mu X_\nu \, .$$

By hypothesis around every p in $\hat{\Sigma}$ there exists a coordinate system in which $\hat{g}$ takes the form (1.1). It follows that h is a flat Lorentzian metric on a neighbourhood of $\hat{\Sigma}$, with X being a covariantly constant vector field with respect to h. By isometry invariance this must hold throughout $\hat{M}$.

Choose a point $p \in \Sigma_{\rm ext}$ and let $\hat{e}^a$, $a = 0, \ldots, 3$ be a tetrad of vector fields at p such that $\hat{e}^0 = X(p)$. $\hat{e}^a$ should be chosen orthonormal with respect to the metric h. As $\hat{M}$ is simply connected and h is flat it follows that the set of equations

$$\hat{\nabla}_\nu e^{a\mu} = 0 \, , \qquad e^{a\mu}(p) = \hat{e}^{a\mu} \, , \tag{2.6}$$

admits a unique solution on $\hat{M}$. Here $\hat{\nabla}$ is the Levi–Civita connection of the metric h. It then follows from simple connectedness of $\hat{M}$ that the set of equations

$$x^a{}_{,\mu} = e^a_\mu \, , \qquad x^\mu(p) = 0 \, , \tag{2.7}$$

also admits a unique solution on $\hat{M}$. The x^a's provide a global coordinate system on $\hat{M}$ in which g takes the form (1.1). It should be clear that the coordinates x^a take values in $\mathbf{R} \times (\mathbf{R}^3 \setminus \mathcal{S})$ for some closed set $\mathcal{S} \subset \mathbf{R}^3$. When asymptotic flatness is taken into account in the above construction, it is not too difficult to show that $\mathcal{S}$ is *compact.*

Following [10], we note that

$$F_{\mu\nu}F^{\mu\nu} = -2 \left((\frac{\partial u^{-1}}{\partial x})^2 + (\frac{\partial u^{-1}}{\partial y})^2 + (\frac{\partial u^{-1}}{\partial y})^2 \right) \, . \tag{2.8}$$

The asymptotic conditions and the interior compactness condition show that there exists a constant C such that $F_{\mu\nu}F^{\mu\nu}$ is bounded on $\hat{M}$, which in turns implies that

$$|\text{grad } u^{-1}| \le C_1 \tag{2.9}$$

for some constant C_1. Here the norm of the gradient refers to the flat metric on $\mathbf{R}^3$. Clearly, u^{-1} is uniformly Lipschitz continuous on $\mathbf{R}^3 \setminus \mathcal{S}$.

We now claim that $\mathcal{S}$ must be a finite set of points. More precisely, we have the following:

Proposition 2.1 *Let $\mathcal{S} \subset \mathbf{R}^3$ be closed with $0 \notin \mathcal{S}$ and let u be harmonic on $\mathbf{R}^3 \setminus \mathcal{S}$. Suppose moreover that (2.9) holds. Then $\mathcal{S}$ is discrete; in fact, for any $R > 0$ we must have*

$$\#(\mathcal{S} \cap B(R)) < C_1 R u(0) + 1 \, . \tag{2.10}$$

Here C_1 is the constant of (2.9).

Proof: Let N be the smallest integer larger than or equal to $C_1Ru(0)+1$ and suppose that there exist points $x_1,\dots,x_N \in \partial\Sigma \cap B(R)$. Set

$$\rho = \inf_{i\neq j} |x_i - x_j| \,. \tag{2.11}$$

Choose any $\delta \in (0,1)$ and consider the function

$$v(x) = C_1^{-1}(1-\delta)\sum_{i=1}^{N} \frac{1}{|x-x_i|} \,.$$

Let $\mathcal{S}_\epsilon$ denote an ϵ–thickening of $\mathcal{S}$ and let $x \in \partial\mathcal{S}_\epsilon$. Let x_1 be the point closest to x, we then have $|x-x_1| \geq \epsilon$. Consider now a ball $B_{\rho/2,x}$ of radius $\rho/2$ centered at x, with ρ defined in (2.11). If $x_1 \in B_{\rho/2,x}$, then no other point x_i can also be in $B_{\rho/2,x}$, hence for $i \neq 1$ we must have $|x_i - x| \geq \rho/2$. If $x_1 \notin B_{\rho/2,x}$, we must also have $|x_i - x| \geq \rho/2$ for $i \neq 1$ as x_1 was the closest point. This gives the estimate

$$v\Big|_{\partial\mathcal{S}_\epsilon} < \frac{1-\delta}{C_1\epsilon} + 2\frac{N}{C_1\rho} \,.$$

By (2.5) the function u^{-1} vanishes on $\partial\mathcal{S}$, and the estimate (2.9) shows that

$$u^{-1}(x) \leq C_1 d(x,\partial\mathcal{S}) \,,$$

where $d(x,\partial\mathcal{S})$ denotes the distance from x to $\partial\mathcal{S}$. It follows that

$$u\Big|_{\partial\mathcal{S}_\epsilon} \geq \frac{1}{C_1\epsilon} \,.$$

We thus have, for all $\delta > 0$ and $\epsilon \leq \epsilon_0(\delta)$ for some $\epsilon_0(\delta)$,

$$(u-v)\Big|_{\partial\mathcal{S}_\epsilon} > 0 \,. \tag{2.12}$$

On the other hand for large r the function v tends to zero while u tends to 1 by asymptotic flatness, in fact $u > 1$ by the maximum principle. Hence we also have that $(u-v)(x)$ is positive for $r(x)$ large enough. Both u and v are harmonic on $\mathbf{R}^3 \setminus \mathcal{S}_\epsilon$ and thus, by the maximum principle, we must have $u - v > 0$ on $\mathbf{R}^3 \setminus \mathcal{S}_\epsilon$.

Consider now $v(0)$; we clearly have

$$v(0) \geq \frac{N(1-\delta)}{C_1R} \,.$$

Since $\delta > 0$ can be chosen arbitrarily small we conclude that

$$u(0) \geq \frac{N}{C_1R} \,,$$

that is,

$$u(0) \geq \frac{C_1 R u(0) + 1}{C_1 R} > u(0)\,,$$

which gives a contradiction, and (2.10) follows. $\square$

Returning to the proof of Theorem 1.1, as $\mathcal{S}$ is compact by construction we can choose R to be large enough so that $\mathcal{S} \cap B(R) = \mathcal{S}$. This shows that $\mathcal{S}$ must be a finite set of points, as claimed. It is now a standard result of potential theory that u has the form (1.4).

One of the consequences of what has been shown is that $\hat{M}$ has only one asymptotically flat region. Now if the set $\tilde{\Sigma}$ defined by (2.3) had been non–simply connected, then $\hat{M}$ would have had more than one such region. We conclude that $\tilde{\Sigma}$ is simply connected. This together with the assumed properties of Killing orbits of X on $\ll \mathcal{J} \gg \subset M$ allows us to identify $\hat{M}$ with a subset of M in the obvious way, and Theorem 1.1 follows.

3 Some alternative results

Let us start by pointing out that in Theorem 1.1 the hypothesis of existence of the spacelike surface Σ can be replaced by the requirement that there exists a Cauchy surface for $\ll \mathcal{J} \gg$ which is a complete Riemannian manifold with respect to the induced metric, and which has at least one asymptotically flat end. Note that such a Cauchy surface will not be asymptotically flat with compact interior, rather it will have some number of asymptotic ends in which the metric is not asymptotically flat.

It might be desirable for some purposes to have a formulation of the result at hand in which no mention of a black hole is made. There are several reasons for that. First note, that the discussion of [9, 8] can be carried through in a purely three–dimensional context, in which no global properties of the resulting developments need to be assumed. In this way one avoids the rather difficult question of existence of a development with a sufficiently regular conformal completion. Next, after having assumed that there exists a Killing vector field on $\ll \mathcal{J} \gg$ one would still need to establish completeness of the orbits thereof (the completeness of the Killing orbits does not *a fortiori* follow from the results of [1] under the hypotheses made here). All these issues can be avoided in a Cauchy data setting if one is willing to replace the condition that $\partial\Sigma$ be a subset of the black hole region by the requirement that X becomes null, or perhaps vanishes, on $\partial\Sigma$. More precisely, we have the following:

Theorem 3.1 *Consider an electro–vacuum space–time (M, g) and suppose that there exists in M an asymptotically flat spacelike hypersurface Σ with compact interior, with non–empty boundary $\partial\Sigma$ and with timelike (non–vanishing) ADM four–momentum. Assume moreover that there exists a Killing vector field X defined in a neighbourhood of Σ, X being timelike in an asymptotic region of*

Σ and null (perhaps vanishing) on $\partial\Sigma$. If (M, g) is locally a MP space–time in the sense of point 2 of Section 1, then there exists a neighbourhood of $\hat{\Sigma}$ which is isometrically diffeomorphic to a subset of a standard MP space–time. Here $\hat{\Sigma}$ is defined as that connected component of $\{p \in \Sigma : X^\mu$ is timelike$\}$ which intersects the asymptotically flat region.

The proof of Theorem 3.1 is a somewhat simpler version of the proof of Theorem 1.1.

4 Closing remarks

Recall now that while static electrovacuum black holes with non–degenerate horizons are well understood (*cf.* [12] and references therein), those which contain degenerate horizons have so far elluded any attempts for systematic classification. We hope that the results of [9, 8, 19] together with our paper provide a step in this direction. A complete classification could be achieved if one could prove that the existence of some component of the horizon which is degenerate implies $M = |Q|$. Unfortunately, it seems that even in the case of a connected degenerate horizon in a static electro–vacuum black hole space–time such an equality has not been established so far.

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