ESI The Erwin Schrödinger International Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Wien, Austria

The Spectrum of Resonances for One–Dimensional Schrödinger Operator with Compactly Supported Potential

Stanislav Stepin

Vienna, Preprint ESI 1903 (2007) April 11, 2007

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

THE SPECTRUM OF RESONANCES FOR ONE-DIMENSIONAL SCHRÖDINGER OPERATOR WITH COMPACTLY SUPPORTED POTENTIAL

Stanislav Stepin

Abstract. Asymptotic distribution of resonances or scattering poles is studied for one-dimensional Schrödinger operator with compactly supported potential having arbitrary tangency orders at the endpoints of the support. The main theorem of the present paper complements and strengthens the result of M.Zworski who treated the case of integer tangency orders by a different method.

MSC: primary 34L40, 34L25; secondary 34B07, 34E10

Keywords: spectrum of resonances, eigenvalue distribution, asymptotics of Laplace integrals

1. Introduction

In the present paper the following eigenvalue problem

$$-y''(x) + q(x)y(x) = k^2 y(x)\,, \tag{1}$$
$$y'(a) - iky(a) = 0, \quad y'(b) + iky(b) = 0 \tag{2}$$

is considered, where $k \in \mathbb{C}$ stands for a spectral parameter while the complex-valued function $q(x)$ vanishes at points a and b so that

$$q(x) = (x-a)^\alpha q_0(x)(b-x)^\beta, \;\; q_0(a)\cdot q_0(b) \neq 0, \;\; \alpha, \beta > 0.$$

If one assumes $q(x)$ to be zero for $x \leqslant a$ and $x \geqslant b$ then the boundary conditions (2) imply that an eigenfunction $y(x)$ corresponding to an eigenvalue k admits a continuation

$$y(x) \;=\; \begin{cases} c_1 e^{ikx}, & x \leqslant a, \\ c_2 e^{-ikx}, & x \geqslant b, \end{cases}$$

remaining a solution to equation (1) for all $x \in \mathbb{R}$. Thus, if k is an eigenvalue of the problem (1)-(2) and Im $k < 0$ then $y \in L_2(\mathbb{R})$ and hence $E = k^2$ is an eigenvalue of the corresponding Schrödinger operator with compactly supported potential $q(x)$. Provided that Im $k > 0$ the value $E = k^2$ turns out to be a resonance or scattering pole (i.e. a pole of the analytic continuation of the resolvent integral kernel) of the operator in question. In the case when

Im $k=0$ point $E=k^2$ is embedded into continuous spectrum and is called a spectral singularity of the Schrödinger operator mentioned above.

A number of papers by physicists and mathematicians deal with localization and asymptotic distribution of resonances in scattering theoretic context (see the monographs [1]-[4] and reviews [5],[6]). Below it is shown that at most finitely many points of the spectrum of the problem (1)-(2) are located in the lower half-plane $\mathbb{C}_- := \{\mathrm{Im} k \leqslant 0\}$, while the upper half-plane $\mathbb{C}_+ := \{\mathrm{Im} k \geqslant 0\}$ contains countably many eigenvalues for which an asymptotic distribution formula is established.

Theorem. *Suppose that $q_0 \in C^{\mathcal{N}}[a,b]$ and $\mathcal{N} \geqslant \max\{\alpha,\beta\}+1$. Then the spectrum of the problem (1)-(2) in the half-plane $\mathbb{C}_+$ consists of two series of eigenvalues*

$$\begin{aligned} k_n^{\pm} \;=\; \pm\frac{\pi n}{b-a} + \frac{i\gamma\ln n}{2(b-a)} \pm \frac{\pi\gamma}{4(b-a)} - \\ - \frac{i\ln\mathcal{C}}{2(b-a)} + \frac{i\gamma}{2(b-a)}\ln\left(\frac{2\pi}{b-a}\right) + \varepsilon_n^{\pm}, \quad (3) \end{aligned}$$

where $\gamma = \alpha+\beta+4$ and $\mathcal{C} = \Gamma(\alpha+1)\Gamma(\beta+1)q_0(a)q_0(b)(b-a)^{\alpha+\beta}$, while the remainder terms $\varepsilon_n^{\pm}$ as $n\to\infty$ admit the estimates

$$\varepsilon_n^{\pm} = \begin{cases} O(1/\ln n) \;\; if \;\; \alpha \;\; or \;\; \beta \;\; is \;\; integer \\ O(n^{m-1}(\ln n)^{-m}) \quad otherwise \end{cases}$$

with $m = \max\big\{\{\alpha\},\{\beta\}\big\}$ and, moreover, $\varepsilon_n^{\pm} = O\big(\ln n/n\big)$ provided that $\mathcal{N} \geqslant \max\{\alpha,\beta\}+2$.

The problem equivalent to (1)-(2) for integer α and β was studied in [7], where the eigenvalues $k_n^{\pm}$ have been calculated up to $o(1)$ remainder term. Theorem formulated above complements and strengthens the main result of [7], namely it enables one to consider complex-valued potentials $q(x)$ with zeroes of arbitrary (not necessarily integer) tangency orders at the endpoints of the support and for such potentials it gives asymptotic formulas for the resonances with the remainder term quantitatively estimated. In [8] the asymptotic distribution of the spectrum for the Regge problem is studied which differs from (1)-(2) by the change of the first boundary condition (2) to $y(a)=0$. The method used here to obtain the asymptotic formulas (3) is related to the approach developed in [8] and differs essentially from that of [7]. This paper presents the results obtained in colaboration with A.Tarasov a postgraduate student of Mathematics Department at Moscow State University.

The spectrum of the problem (1)-(2) is known to coincide with the zeroes of the function

$$\Delta(k) \;=\; \begin{vmatrix} y_1'(a,k) - ik\, y_1(a,k) & y_2'(a,k) - ik\, y_2(a,k) \\ y_1'(b,k) + ik\, y_1(b,k) & y_2'(b,k) + ik\, y_2(b,k) \end{vmatrix}$$

called the characteristic determinant corresponding to a fundamental set of solutions $\{y_1(x,k),\, y_2(x,k)\}$ to equation (1). In the course of the proof of the Theorem an appropriate fundamental set of solutions is to be chosen as follows $y_1(x,k) = e^{-ikx} v_1(x,k)$ and $y_2(x,k) = e^{ikx} v_2(x,k)$, where $v_1(x,k)$ and $v_2(x,k)$ satisfy the integral equations

$$v_1(x,k) \;=\; 1 + \frac{i}{2k} \int_a^x (1 - e^{2ik(x-t)}) q(t) v_1(t,k)\, dt\,, \qquad (4)$$

$$v_2(x,k) \;=\; 1 + \frac{i}{2k} \int_x^b (1 - e^{2ik(t-x)}) q(t) v_2(t,k)\, dt\,, \qquad (5)$$

so that one has

$$\begin{aligned} y_1'(a,k) - ik\, y_1(a,k) \;&=\; -2ik\, e^{-ika}, \\ y_2'(b,k) + ik\, y_2(b,k) \;&=\; 2ik\, e^{ikb}, \end{aligned}$$

and, besides,

$$\begin{aligned} y_1'(b,k) + ik\, y_1(b,k) \;&=\; e^{-ikb} v_1'(b,k), \\ y_2'(a,k) - ik\, y_2(a,k) \;&=\; e^{ika}\, v_2'(a,k). \end{aligned}$$

Thus the problem in question concerning the asymptotics of eigenvalues is equivalent to investigation of the roots distribution for the equation

$$e^{-2ik(b-a)} v_1'(b,k) v_2'(a,k) \;=\; 4k^2. \qquad (6)$$

In what follows the solutions $v_1(x,k)$ and $v_2(x,k)$ to equations (4) and (5) are constructed and their behavior as $\mathbb{C}_+ \ni k \to \infty$ is studied. Making use of the Laplace integrals evaluation technique we show that

$$v_1'(b,k) \;=\; q_0(b)(b-a)^{\alpha}\Gamma(\beta+1)\, z^{-\beta-1} \;+\; \Phi_1(z) \;+\; \Psi_1(z), \qquad (7)$$

$$-v_2'(a,k) \;=\; q_0(a)(b-a)^{\beta}\Gamma(\alpha+1)\, z^{-\alpha-1} \;+\; \Phi_2(z) \;+\; \Psi_2(z), \qquad (8)$$

where $z = -2ik$, $\Psi_{1,2}(z) = O(e^{-(b-a)\mathrm{Re}\, z})$, while $\Phi_1(z) = O(z^{-\beta-2})$ and $\Phi_2(z) = O(z^{-\alpha-2})$ provided $\mathcal{N} \geqslant \max\{\alpha, \beta\} + 2$. These formulas enable one to locate the roots of equation (6) which can be reduced (after the substitution of (7) and (8)) to the form

$$z - \lambda \ln z + \delta(z) \ = \ \omega_n, \quad n \in \mathbb{Z}, \tag{9}$$

where $\lambda := \dfrac{\gamma}{b-a}$, $\omega_n := \dfrac{2\pi i n}{b-a} - \dfrac{\ln \mathcal{C}}{b-a}$ and $\delta(z) = O(z^{-1})$. It is known that the transcendental equation

$$z \, - \, \ln z \ = \ \omega$$

possesses a unique solution

$$z(\omega) \ = \ \omega \, + \, \ln \omega \, + \, O(\ln \omega / \omega)$$

provided that ω is large enough. The same holds true for equation (9) appearing in the problem under consideration. Namely for $n \in \mathbb{Z}$ of sufficiently large absolute value it proves to have a unique solution

$$z_n \ = \ \omega_n \, + \, \lambda \ln \omega_n \, + \, O\left(\frac{\ln |n|}{|n|} \right).$$

The verification of this fact completes the proof of the assertion of the Theorem in the case $\mathcal{N} \geqslant \max\{\alpha, \beta\} + 2$. Further the following notation will be used $\|f\| := \sup\limits_{x \in (a,b)} |f(x)|$ and $\langle x \rangle := -[-x]$ for $x \in \mathbb{R}$.

2. Fundamental set of solutions to Schrödinger equation

Given $p \in C^1(a,b)$ such that $p' \in L_1(a,b)$ let us introduce a notation

$$D := \left\{ \, z = \sigma + i\tau \mid \sigma \geqslant 0, \ |z| \geqslant 4(b-a)\|p\| \, \right\}$$

and consider an integral equation

$$u(x,z) \ = \ 1 \, + \, \frac{1}{z} \int_a^x (1 - e^{-z(x-t)}) p(t) u(t,z) \, dt. \tag{10}$$

Proposition 1. *For $z \in D$ equation* (10) *has a solution*

$$u(x,z) \ = \ 1 \, + \, \frac{1}{z} \int_a^x p(t) \, dt \, + \, U(x,z), \tag{11}$$

where $\|U(\,\cdot\,,z)\| \ = \ O(z^{-2})$ *and, moreover,* $\|u'(\,\cdot\,,z)\| \ \leqslant \ 2(b-a)\|p\|$.

Proof. For $z \in D$ equation (10) possesses a unique solution which can be obtained by the standard successive approximations procedure so that

$$\|u(\,\cdot\,,z)-1\| \;\leqslant\; \frac{4(b-a)\|p\|}{|z|} \;\leqslant\; 1.$$

Further, one can rewrite equation (10) in the form

$$u(x,z) \;=\; 1 + \frac{1}{z}\int_a^x p(t)\,dt \;+\; U(x,z),$$

where

$$U(x,z) \;=\; \frac{1}{z^2}\left(e^{-z(x-a)}p(a) - p(x) + \int_a^x e^{-z(x-t)}p'(t)\,dt\right) + \\ + \frac{1}{z}\int_a^x (1-e^{-z(x-t)})p(t)\big(u(t,z)-1\big)\,dt.$$

Taking the above estimate into account for $z \in D$ one has

$$\|U(\,\cdot\,,z)\| \;\leqslant\; \frac{1}{|z|^2}\left(2\|p\|\big(1+4(b-a)^2\|p\|\big) \;+\; \int_a^b |p'(t)|\,dt\right)$$

and, besides, the inequality

$$|u'(x,z)| \;=\; \left|\int_a^x e^{-z(x-t)}p(t)u(t,z)\,dt\right| \;\leqslant\; 2(b-a)\|p\|$$

holds that completes the proof.

Let $v_1(x,k)$ and $v_2(x,k)$ be the solutions of integral equations (4) and (5) the first of which is of the form (10), while the second one can be rewritten in such a form:

$$v_2(a+b-y,k) \;=\; 1 \;+\; \frac{i}{2k}\int_a^y (1-e^{2ik(y-s)})\,q(a+b-s)\,v_2(a+b-s,k)\,ds.$$

It can be verified directly that

$$y_1(x,k) \;:=\; e^{-ikx}v_1(x,k), \quad y_2(x,k) \;:=\; e^{ikx}v_2(x,k)$$

satisfy equation (1) and, moreover, the Wronskian of these two solutions for $k \in \mathbb{C}_+$ is given by the expression

$$W\{y_1,y_2\} \;=\; 2ik\,v_1(b,k) \;-\; v_1'(b,k)$$

which is non-zero by virtue of Proposition 1 provided that $|k|$ is sufficiently large and, therefore, solutions $y_1(x,k)$ and $y_2(x,k)$ are linearly independent in this case. For $k \in \mathbb{C}_-$ one can choose $\widetilde{y}_1(x,k) = y_1(x,-k)$ and $\widetilde{y}_2(x,k) = y_2(x,-k)$ to be a fundamental set of solutions to equation (1).

The spectrum of the problem (1)-(2) is known to be discrete and coincide with the zeroes of the characteristic determinant

$$\Delta[\varphi_1,\varphi_2](k) \;=\; \begin{vmatrix} \varphi_1'(a,k) - ik\,\varphi_1(a,k) & \varphi_2'(a,k) - ik\,\varphi_2(a,k) \\ \varphi_1'(b,k) + ik\,\varphi_1(b,k) & \varphi_2'(b,k) + ik\,\varphi_2(b,k) \end{vmatrix}$$

where $\{\varphi_1(x,k),\, \varphi_2(x,k)\}$ is a certain fundamental set of solutions to (1). From the above considerations it follows that the eigenvalues of the problem (1)-(2) in $\mathbb{C}_+$ are just the zeroes of the analytic function

$$\Delta(k) \;:=\; \Delta[y_1,y_2](k) \;=\; \begin{vmatrix} -2ik\,e^{-ika} & e^{ika}v_2'(a,k) \\ e^{-ikb}v_1'(b,k) & 2ik\,e^{ikb} \end{vmatrix}$$

while the spectrum in $\mathbb{C}_-$ of the problem in question consists of the zeroes of the determinant

$$\widetilde{\Delta}(k) \;:=\; \Delta[\widetilde{y}_1,\widetilde{y}_2](k) \;= \\ =\; \begin{vmatrix} 0 & e^{-ika}[v_2'(a,-k) - 2ik\,v_2(a,-k)] \\ e^{ikb}[v_1'(b,-k) + 2ik\,v_1(b,-k)] & 0 \end{vmatrix}.$$

In virtue of Proposition 1 one has the inequality

$$\left| v_1(b,-k) + \frac{v_1'(b,-k)}{2ik} \right| \;\geqslant\; |v_1(b,-k)| - \frac{|v_1'(b,-k)|}{2|k|} \;\geqslant\; 1 - \frac{3(b-a)\|q\|}{|k|}$$

valid for for $k \in \mathbb{C}_-$ such that $|k| \geqslant 2(b-a)\|q\|$ and similarly

$$\left| v_2(a,-k) - \frac{v_2'(a,-k)}{2ik} \right| \;\geqslant\; 1 - \frac{3(b-a)\|q\|}{|k|}\,.$$

Consequently the determinant $\widetilde{\Delta}(k)$ does not vanish if $k \in \mathbb{C}_-$ and $|k| > 3(b-a)\|q\|$ that actually proves the following

Proposition 2. *The lower half-plane $\mathbb{C}_-$ contains just finite number of eigenvalues of the problem* (1)-(2) *and, moreover, all of them are located in the disc* $\big\{|k| \leqslant 3(b-a)\|q\|\big\}$.

3. Asymptotics of Laplace type integrals

In what follows a notation $a_\sigma := a+\sigma^{-1}$ is used, where $\sigma > 1/(b-a)$ so that $a_\sigma \in (a,b)$. Asymptotic behavior of the integral

$$I(f,z) := \int_{a_\sigma}^{b} e^{-z(b-x)} f(x)\, dx$$

as $\sigma = \operatorname{Re} z \to \infty$ depending on the properties of function $f(x)$ will be studied here. Beforehand we introduce an appropriate

Definition. *Function $f \in C^N(a,b)$ belongs to the class $(\eta|N|\theta)$ if for arbitrary $\kappa \leqslant N$ and $\delta \in (a,b)$ there exists $c_\kappa^\delta > 0$ such that*

$$|f^{(\kappa)}(x)| \leqslant c_\kappa^\delta \begin{cases} [1+(x-a)^{\eta-\kappa}], & x \in (a,\delta] \\ (b-x)^{\theta-\kappa}, & x \in [\delta, b) \end{cases}$$

and indicate the properties of functions $f \in (\eta|N|\theta)$ we shall make use of in the sequel: $f' \in (\eta-1|N-1|\theta-1)$ and, besides,

$$f^{(-1)}(x) := -\int_x^b f(t)\, dt \in (\eta|N+1|\theta+1)$$

in the case when $\theta > -1$. Moreover, given $f \in (\eta|N|\theta)$ and $g \in (\eta'|N'|\theta')$ it can be verified that

$$f(x)g(x) \in \big(\min\{\eta,\eta',\eta+\eta'\}\,\big|\,\min\{N,N'\}\,\big|\,\theta+\theta'\big).$$

Lemma 1. *For arbitrary $f \in (\eta|N|\theta)$, $\theta > -1$, an asymptotic estimate*

$$\int_{a_\sigma}^{b} \left|e^{-z(b-x)} f(x)\right| dx = O(\sigma^{-\theta-1}),$$

is valid as $\sigma = \operatorname{Re} z \to \infty$. Provided that $N = \langle\theta\rangle$ one has

$$I(f,z) = I_1(f,z) + I_2(f,z),$$

where $I_1(f,z) = O(z^{-N}\sigma^{-\{\theta-0\}})$ and $I_2(f,z) = O([1+\sigma^{-\eta}]e^{-\sigma(b-a)})$.

Proof. For fixed $\delta \in (a,b)$ and $\sigma \geqslant 1/(\delta-a)$ the following inequality

$$|I(f,z)| \leqslant c_0^\delta \int_{a_\sigma}^{\delta} e^{-\sigma(b-x)}\left[1+(x-a)^\eta\right] dx + c_0^\delta \int_\delta^b e^{-\sigma(b-x)}(b-x)^\theta\, dx$$

holds, where

$$\int_{\delta}^{b} e^{-\sigma(b-x)}(b-x)^{\theta}\, dx \;\leqslant\; \Gamma(\theta+1)\,\sigma^{-\theta-1}$$

and, moreover,

$$\int_{a_\sigma}^{\delta} e^{-\sigma(b-x)}[1+(x-a)^{\eta}]\, dx \;\leqslant\; e^{-\sigma(b-\delta)}\int_{a_\sigma}^{\delta}\big[1+(x-a)^{\eta}\big]\, dx \;=\; O(\sigma^{-\theta-1}).$$

Applying integration by parts to $I(f,z)$ and taking into account that $f^{(\kappa)}(b)=0$ for $\kappa=0,\dots N-1$, one has

$$\int_{a_\sigma}^{b} e^{-z(b-x)} f(x)\, dx \;=\; \frac{(-1)^N}{z^N}\int_{a_\sigma}^{b} e^{-z(b-x)} f^{(N)}(x)\, dx \;+$$

$$+\;\sum_{\kappa=1}^{N}\frac{(-1)^{\kappa}}{z^{\kappa}} f^{(\kappa-1)}\,(a_\sigma)\, e^{-z(b-a_\sigma)} \;=\; I_1(f,z)+I_2(f,z),$$

where $I_1(f,z)=O(z^{-N}\sigma^{-\{\theta-0\}})$ as $\sigma\to\infty$ by virtue of the first assertion of the Lemma. The inequality

$$|I_2(f,z)| \;\leqslant\; \sum_{\kappa=1}^{N}|z|^{-\kappa}|f^{(\kappa-1)}(a_\sigma)|e^{-\sigma(b-a_\sigma)} \;\leqslant\; \big[1+\sigma^{-\eta}\big]\, e^{-\sigma(b-a)+1}\sum_{\kappa=1}^{N} c^{\delta}_{\kappa-1}$$

valid for sufficiently large $\sigma>0$ completes the proof.

Proposition 3. *Given $\theta>-1$ and $N\geqslant\langle\theta\rangle+1$ let $g\in(\eta|N|\theta)$ satisfy the condition $g(x)(b-x)^{-\theta}=:g_0(x)\in C^N(a,b]$. Then an asymptotic representation*

$$I(g,z) \;=\; g_0(b)\,\Gamma(\theta{+}1)\,z^{-\theta-1} \;+\; \tilde I_1(g,z) \;+\; \tilde I_2(g,z) \tag{12}$$

holds as $\sigma=\operatorname{Re}z\to\infty$, where

$$\widetilde{I}_1(g,z) \;=\; O(z^{-\langle\theta\rangle-1}\sigma^{-\{\theta-0\}}) \;\; \text{and} \;\; \widetilde{I}_2(g,z) \;=\; O([1+\sigma^{-\eta}]e^{-\sigma(b-a)}).$$

Moreover $\widetilde{I}_1(g,z)\;=\;O(z^{-\theta-2})$ provided that $N\geqslant\langle\theta\rangle+2$.

Proof. Condition $g\in(\eta|N|\theta)$ implies that $g_0\in(\eta|N|0)$ and, therefore $\Phi(x):=g_0(x)-g_0(b)\in(\eta|N|1)$ since $g_0\in C^N(a,b]$. Let us now substitute expression

$$g(x) \;=\; g_0(b)(b-x)^{\theta} \;+\; r(x),$$

where $r(x):=\Phi(x)(b-x)^{\theta}\in(\eta|N|\theta{+}1)$, into the integral

$$I(g,z) \;=\; g_0(b)\int_{a_\sigma}^{b} e^{-z(b-x)}(b-x)^{\theta}\, dx \;+\; I(r,z)$$

and consider the summands on the r.h.s. separately. The integral from the first summand can be rewritten in the form

$$\int_{a_\sigma}^{b} e^{-z(b-x)}(b-x)^\theta\,dx \;=\; \int_0^{b-a_\sigma} e^{-zt}\,t^\theta\,dt \;=\; z^{-\theta-1}\Gamma(\theta+1)\;+\;R_1(z),$$

with the remainder term $R_1(z)$ which admits the estimate

$$|R_1(z)| \;\leqslant\; \int_{b-a_\sigma}^{\infty} e^{-\sigma t}\,t^\theta\,dt \;=\; e^{-\sigma(b-a_\sigma)}\int_{b-a_\sigma}^{\infty} e^{-\sigma(t-(b-a_\sigma))}\,t^\theta\,dt \;\leqslant$$
$$\leqslant\; e^{-\sigma(b-a)+1}\int_{b-a_\sigma}^{\infty} e^{-(t-(b-a_\sigma))}\,t^\theta\,dt \;\leqslant\; e^{1+(b-a)}\Gamma(\theta+1)\,e^{-\sigma(b-a)}$$

provided that $\sigma\geqslant 1$. Further, since $r\in(\eta|N|\theta+1)$ and $N\geqslant\langle\theta\rangle+1$, Lemma 1 can be applied to evaluate $I(r,z)$. Namely for σ large enough one has

$$I(r,z) \;=\; \int_{a_\sigma}^{b} e^{-z(b-t)}\,r(t)\,dt \;=\; I_1(r,z)\;+\;I_2(r,z),$$

where $I_1(r,z) \;=\; O\big(z^{-\langle\theta\rangle-1}\sigma^{-\{\theta-0\}}\big)$, $I_2(r,z) \;=\; O\left([1+\sigma^{-\eta}]e^{-\sigma(b-a)}\right)$. To complete the proof of the first assertion of the Proposition it suffices to note that functions

$$\widetilde{I}_1(g,z) \;:=\; I_1(r,z),\quad \widetilde{I}_2(g,z) \;:=\; g_0(b)R_1(z)+I_2(r,z)$$

satisfy the required estimates.

To prove the second assertion in the case when $N\geqslant\langle\theta\rangle+2$ the previous arguments have to be specified. Let us first introduce a modified function

$$\widetilde{\Phi}(x) \;:=\; g_0(x)-g_0(b)-g_0'(b)(x-b)$$

and note that $\widetilde{\Phi}\in(\eta|N|2)$. Now following the same lines as before one should substitute expression

$$g(x) \;=\; g_0(b)(b-x)^\theta \;-\; g_0'(b)(b-x)^{\theta+1} \;+\; \widetilde{r}(x),$$

where $\widetilde{r}(x):=\widetilde{\Phi}(x)(b-x)^\theta\in(\eta|N|\theta+2)$, into the integral

$$I(g,z) \;=\; g_0(b)\int_{a_\sigma}^{b} e^{-z(b-x)}(b-x)^\theta\,dx \;-$$
$$-\; g_0'(b)\int_{a_\sigma}^{b} e^{-z(b-x)}(b-x)^{\theta+1}\,dx \;+\; I(\widetilde{r},z)$$

and again consider the summands on the r.h.s. separately. The integral in the second summand here has the form

$$\int_{a_\sigma}^{b} e^{-z(b-x)}(b-x)^{\theta+1}\,dx \;=\; z^{-\theta-2}\Gamma(\theta+2) \;+\; R_2(z),$$

with the remainder term $R_2(z)$ admitting the estimate

$$|R_2(z)| \;\leqslant\; e^{1+(b-a)}\Gamma(\theta+2)\,e^{-\sigma(b-a)}\,, \quad \sigma \geqslant 1\,.$$

Since $\widetilde{r}\in(\eta|N|\theta+2)$ and $N\geqslant\langle\theta\rangle+2$ then by virtue of Lemma 1 one has

$$I(\widetilde{r},z) \;=\; \int_{a_\sigma}^{b} e^{-z(b-t)}\,\widetilde{r}(t)\,dt \;=\; I_1(\widetilde{r},z) \;+\; I_2(\widetilde{r},z),$$

where $I_1(\widetilde{r},z) \;=\; O\left(z^{-\langle\theta\rangle-2}\right)$, $I_2(\widetilde{r},z) \;=\; O\left([1+\sigma^{-\eta}]e^{-\sigma(b-a)}\right)$. Summing up we set

$$\begin{aligned}\tilde{I}_1(g,z) \;&=\; -g_0'(b)\Gamma(\theta{+}2)z^{-\theta-2} \;+\; I_1(\widetilde{r},z),\\ \tilde{I}_2(g,z) \;&=\; g_0(b)R_1(z)-g_0'(b)R_2(z) \;+\; I_2(\widetilde{r},z),\end{aligned}$$

and the proof of the second assertion is complete.

4. Asymptotic behavior of solution to integral equation

Below it will be assumed without saying that function $p(x)$ vanishes at points a and b so that

$$p(x) \;=\; (x-a)^{\mu}p_0(x)(b-x)^{\nu},$$

where $\mu,\nu>0$ and, besides, $p_0\in C^{\mathcal{N}}[a,b]$, $\mathcal{N}\geqslant 1$. Let $u(x,z)$ be a solution to equation (10) specified in Proposition 1. In the present section the asymptotic behavior of the integral

$$\int_{a}^{b} e^{-z(b-x)}p(x)u(x,z)\,dx \;=\; u'(b,z) \tag{13}$$

will be studied for absolutely large values of parameter $z\in D$, namely we shall prove the following

Proposition 4. *Provided that* $p_0 \in C^{\mathcal{N}}[a,b], \mathcal{N} \geqslant \langle\nu\rangle + 1,$ *an asymptotic formula*

$$u'(b,z) \;=\; p_0(b)(b-a)^{\mu}\Gamma(\nu{+}1)\, z^{-\nu-1} \;+\; \Phi(z) \;+\; \Psi(z) \tag{14}$$

is valid, where $\Phi(z) = O\big(z^{-\langle\nu\rangle-1}\sigma^{-\{\nu-0\}}\big)$ *as* $\sigma \to \infty$ *and, moreover,* $\Phi(z) = O\big(z^{-\nu-2}\big)$ *if* $\mathcal{N} \geqslant \langle\nu\rangle + 2$, *while* $\Psi(z) = O\big(e^{-\sigma(b-a)}\big)$.

Integration by parts is known to be useful as far as the study of the asymptotic behavior of the Laplace type integrals is concerned; in this connection it should be mentioned that the form of the leading term in the asymptotic formula and the estimates of the remainder terms depend drastically on the smoothness of the integrand as well as on the multiplicities of its zeroes at the endpoints. Below the integration by parts formula is adapted to our setting, i.e. with regard for the specific nature of the integrand. Firstly let us introduce an integro-differential operation

$$l\colon f(x) \;\longmapsto\; -\,f'(x) \,+\, p(x)f^{(-1)}(x)\,;$$

due to the properties of functions of the class $(\eta|N|\theta)$ indicated in Section 3 one has

$$l\colon (\eta|N|\theta) \to (\eta-1|N-1|\theta-1)$$

provided that $\eta \leqslant \mu$, $N \leqslant \mathcal{N}$ and $\theta > -1$.

Lemma 2. *Suppose that function* $f \in C^1(a,b)$ *has an integrable singularity at point* $x = b$. *Then for an arbitrary segment* $[c,d] \subset (a,b)$ *an equality*

$$\begin{aligned}\int_c^d e^{-z(b-x)} f(x)u(x,z)\,dx \;&=\\ =\;\frac{1}{z}\bigg(f(x)u(x,z)e^{-z(b-x)} &- f^{(-1)}(x)\int_a^x e^{-z(b-t)}p(t)u(t,z)\,dt\bigg)\bigg|_c^d \;+\\ &+\;\frac{1}{z}\int_c^d e^{-z(b-x)}\, l(f)(x)\, u(x,z)\,dx\end{aligned} \tag{15}$$

holds. Formula (15) *remains valid in the case* $c = a$ *and/or* $d = b$ *provided that* $l(f)$ (*as well as* f') *is integrable in the left/right neighbourhoods of the endpoints* a *and/or* b.

Proof. Making use of integration by parts

$$\begin{aligned}\int_c^d f(x)u(x,z)\,de^{-z(b-x)} \;=\; f(x)u(x,z)e^{-z(b-x)}\Big|_c^d \,-\\ -\int_c^d e^{-z(b-x)}f'(x)u(x,z)\,dx - \int_c^d e^{-z(b-x)}f(x)u'(x,z)\,dx\end{aligned}$$

followed by a substitution of the integral representation for $u'(x,z)$ we obtain the required formula, since

$$\int_c^d e^{-z(b-x)} f(x)u'(x,z)\,dx \;=\; \int_c^d \left(\int_a^x e^{-z(b-x+x-t)}p(t)u(t,z)\,dt\right)df^{(-1)}(x) \;=$$
$$= \left(f^{(-1)}(x)\int_a^x e^{-z(b-t)}p(t)u(t,z)\,dt\right)\Big|_c^d - \int_c^d e^{-z(b-x)}\,p(x)f^{(-1)}(x)\,u(x,z)\,dx.$$

Now we shall take advantage of formula (15) in order to evaluate the derivative $u'(b,z)$ of the solution to equation (10) and obtain the estimate to be used in the proof of the Theorem. Let us assume that $p_0 \in C^{\langle \widetilde{m}\rangle}[a,b]$, where $\widetilde{m} = \min\{\mu,\nu\}$. Then $l^n(p) \in C^{\langle \widetilde{m}\rangle - n}(a,b)$ for $n \leqslant \langle \widetilde{m}\rangle$ has integrable singularities at points a and b. Taking this into account and applying formula (15) to the integral (13) one has

$$\int_a^b e^{-z(b-x)}p(x)u(x,z)\;dx \;=\; \frac{1}{z^{\langle \widetilde{m}\rangle}}\int_a^b e^{-z(b-x)}\;l^{\langle \widetilde{m}\rangle}(p)(x)\,u(x,z)\;dx.$$

Because of the integrability of the function $l^{\langle \widetilde{m}\rangle}(p)$ on the segment $[a,b]$ and uniform boundedness of the norms $\|u(\,\cdot\,,z)\|$ (cf. the proof of Proposition 1) the integral on the r.h.s. of the latter equality is bounded uniformly in $z \in D$. This justifies the following

Lemma 3. *Provided that $p_0 \in C^{\langle \widetilde{m}\rangle}[a,b]$, where $\widetilde{m} = \min\{\mu,\nu\}$, the estimate*

$$|u'(b,z)| \;=\; \left|\int_a^b e^{-z(b-x)}p(x)u(x,z)\,dx\right| \;=\; O(|z|^{-\langle \widetilde{m}\rangle})$$

is valid as $|z| \to \infty$, $\operatorname{Re} z \geqslant 0$.

Subsequently in addition to the above estimate the refined asymptotic formula for $u'(b,z)$ as $\sigma = \operatorname{Re} z \to \infty$ will be required. To this end formula (15) should be applied to the integral (13) as many times as prescribes the tangency order at which $p(x)$ vanishes at the point b. Function $p(x)$ in general may have the zeroes at points a and b of different (and not necessary integer) tangency orders, hence the derivatives of $p(x)$ may have singularities at point a so that successive application of formula (15) becomes impossible. To avoid this difficulty it makes sense to divide the segment $[a,b]$ by the point $a_\sigma = a + \sigma^{-1}$ and thus decompose integral (13) into two summands to be treated separately. To start with let us introduce a notation

$$R(p,z) \;:=\; \int_a^{a_\sigma} e^{-z(b-x)}p(x)u(x,z)\,dx$$

and point out that (by virtue of the estimate $\|u(\,\cdot\,,z)\| \leqslant 2$ obtained in fact in the course of the proof of Proposition 1) the inequality

$$|R(p,z)| \;\leqslant\; \int_a^{a_\sigma} e^{-\sigma(b-x)}\,|p(x)u(x,z)|\,dx \;\leqslant\; 4\|p\|e^{-\sigma(b-a)} \tag{16}$$

holds provided that $z \in D$ and $\sigma \geqslant 1$.

Lemma 4. *Suppose that $f \in (\eta\,|\,\langle\theta\rangle\,|\,\theta)$ while $0 \leqslant \langle\theta\rangle \leqslant \mathcal{N}$ and $\eta \leqslant \mu$. Then*

$$J(f,z) \;:=\; \int_{a_\sigma}^{b} e^{-z(b-x)} f(x)u(x,z)\,dx \;=\; J_1(f,z) \,+\, J_2(f,z),$$

where $J_1(f,z) \,=\, O(z^{-\langle\theta\rangle})$ and $J_2(f,z) \,=\, O\big([1+\sigma^{-\eta}]e^{-\sigma(b-a)}\big)$ as $\sigma \to \infty$.

Proof. By Lemma 2 one can several ($= \langle\theta\rangle$ to be exact) times apply formula (15) to decompose and evaluate the integral

$$\begin{aligned} J(f,z) \;&=\; \frac{J\big(l^{\langle\theta\rangle}(f),z\big)}{z^{\langle\theta\rangle}} \;+\; \Bigg(\sum_{n=0}^{\langle\theta\rangle-1} \frac{l^n(f)(x)}{z^{n+1}} u(x,z)\,e^{-z(b-x)} \;- \\ &\quad - \sum_{n=0}^{\langle\theta\rangle-1} \frac{[l^n(f)]^{(-1)}(x)}{z^{n+1}} \int_a^x e^{-z(b-t)} p(t)u(t,z)\,dt \Bigg)\Bigg|_{a_\sigma}^{b} \;= \\ &= \frac{J\big(l^{\langle\theta\rangle}(f),z\big)}{z^{\langle\theta\rangle}} \;-\; \sum_{n=0}^{\langle\theta\rangle-1} \frac{l^n(f)(a_\sigma)}{z^{n+1}} u(a_\sigma,z)\,e^{-z(b-a_\sigma)} \;+ \\ &\qquad\qquad + \sum_{n=0}^{\langle\theta\rangle-1} \frac{[l^n(f)]^{(-1)}(a_\sigma)}{z^{n+1}}\, R(p,z). \end{aligned}$$

The estimate $\|u(\,\cdot\,,z)\| \leqslant 2$ valid for $z \in D$ implies the inequality

$$\big|J\big(l^{\langle\theta\rangle}(f),\,z\big)\big| \;\leqslant\; 2\int_{a_\sigma}^{b} \big|e^{-z(b-x)}\, l^{\langle\theta\rangle}(f)(x)\big|\;dx,$$

where $l^{\langle\theta\rangle}(f) \in (\eta - \langle\theta\rangle\,|\,0\,|\,\theta - \langle\theta\rangle)$ and hence by Lemma 1 the latter integral is bounded uniformly in $z \,=\, \sigma + i\tau$ provided that σ is sufficiently large. Consequently one has

$$J_1(f,z) \;:=\; J\big(l^{\langle\theta\rangle}(f),\,z\big)\; z^{-\langle\theta\rangle} \;=\; O\left(z^{-\langle\theta\rangle}\right), \quad \sigma \to \infty.$$

Further, since $l^n(f)$ and $[l^n(f)]^{(-1)}$ belong to the class $(\eta - n|\langle\theta\rangle - n|\theta - n)$ it follows that both $l^n(f)(a_\sigma)z^{-n-1}$ and $[l^n(f)]^{(-1)}(a_\sigma)z^{-n-1}$ are $O\big([1+\sigma^{-\eta}]\big)$

and, besides, $l^n(f)(b) = [l^n(f)]^{(-1)}(b) = 0$ if $n < \langle\theta\rangle$. Making use of this fact and taking the estimate (16) (as well as the uniform boundedness of the norm $\|u(\,\cdot\,,z)\|$) into account we get

$$J_2(f,z) \;:=\; -\sum_{n=0}^{\langle\theta\rangle-1} \frac{l^n(f)(a_\sigma)}{z^{n+1}}\, u(a_\sigma,z)\, e^{-z(b-a_\sigma)} \;+\; + \sum_{n=0}^{\langle\theta\rangle-1} \frac{[l^n(f)]^{(-1)}(a_\sigma)}{z^{n+1}}\, R(p,z) \;=\; O\big([1+\sigma^{-\eta}]e^{-\sigma(b-a)}\big), \quad \sigma\to\infty.$$

Proof of Proposition 4. Due to Lemma 4 and in view of the estimate (16) it suffices to verify an asymptotic formula

$$z^{\langle\nu\rangle}\, J_1(p,z) \;=\; J(l^{\langle\nu\rangle}(p),z) \;=\; = p_0(b)(b-a)^\mu\Gamma(\nu+1)\, z^{-\nu+\langle\nu\rangle-1} \;+\; \Upsilon(z) \;+\; O\left(\sigma^{\langle\nu\rangle}\, e^{-\sigma(b-a)}\right)$$

to be valid for $\sigma > 0$ large enough, where $\Upsilon(z) = O\big(z^{-1}\sigma^{-\{\nu-0\}}\big)$ if $\mathcal{N} \geqslant \langle\nu\rangle + 1$ and $\Upsilon(z) = O\big(z^{\langle\nu\rangle-\nu-2}\big)$ provided that $\mathcal{N} \geqslant \langle\nu\rangle + 2$.

Beforehand let us show (by induction arguments) that for $n \leqslant \langle\nu\rangle$ the following representation

$$l^n(p)(x) \;=\; (x-a)^{\mu-n}p_n(x)(b-x)^{\nu-n} \;+\; \widetilde{p}_n(x) \tag{17}$$

holds in which $p_n \in C^{\mathcal{N}-n}[a,b]$, $\widetilde{p}_n \in (\mu{-}n{+}1|\mathcal{N}{-}n{+}1|\nu{-}n{+}2)$ and, moreover,

$$p_n(b) \;=\; \frac{\Gamma(\nu+1)}{\Gamma(\nu-n+1)}(b-a)^n p_0(b).$$

The starting point of the induction ($n = 0$) is guaranteed by the assumption of the Proposition. Suppose now that for a certain $n < \langle\nu\rangle$ induction hypothesis is fulfilled. Then

$$l^{n+1}(p)(x) \;=\; (x-a)^{\mu-n-1}p_{n+1}(x)(b-x)^{\nu-n-1} \;+\; \widetilde{p}_{n+1}(x),$$

where

$$p_{n+1}(x) \;=\; [(\nu-n)(x-a)-(\mu-n)(b-x)]p_n(x) \;-\; (x-a)(b-x)p_n'(x)$$

and

$$\widetilde{p}_{n+1}(x) \;=\; -\widetilde{p}_n{}'(x) \;+\; p(x)\,[l^n(p)]^{(-1)}(x) \;\in\; (\mu-n|\mathcal{N}-n|\nu-n+1)\,.$$

Induction hypothesis (with regard for the explicit form of $p_{n+1}(x)$) implies that $p_{n+1} \in C^{\mathcal{N}-n-1}[a,b]$ and

$$p_{n+1}(b) \;=\; (\nu-n)(b-a)p_n(b) \;=\; \frac{\Gamma(\nu+1)}{\Gamma(\nu-n)}(b-a)^{n+1}p_0(b).$$

In particular it follows that function $l^{\langle\nu\rangle}(f)$ admits a representation of the form (17) with $n=\langle\nu\rangle$ and $\mathcal{N}\geqslant\langle\nu\rangle+1$, hence $p_{\langle\nu\rangle}\in C^1[a,b]$, $\widetilde{p}_{\langle\nu\rangle}\in(\mu-\langle\nu\rangle+1\,|\,2\,|\,\theta+2\,)$, where $\theta=\nu-\langle\nu\rangle$ and, besides,

$$p_{\langle\nu\rangle}(b) \;=\; \frac{\Gamma(\nu+1)}{\Gamma(\nu-\langle\nu\rangle+1)}\,(b-a)^{\langle\nu\rangle}p_0(b).$$

Let us now decompose the integral

$$\begin{aligned} J(l^{\langle\nu\rangle}(p),z) \;&=\; \int_{a_\sigma}^{b} e^{-z(b-x)}\, l^{\langle\nu\rangle}(p)(x)\,u(x,z)\,dx \;\;= \\ &=\; I(g,z) \;+\; \int_{a_\sigma}^{b} e^{-z(b-x)}g(x)\,(u(x,z)-1)\,dx \;+\; J(\widetilde{p}_{\langle\nu\rangle},z) \qquad (18)\end{aligned}$$

and evaluate separately the summands of this decomposition, where

$$g(x) \;:=\; (x-a)^{\mu-\langle\nu\rangle}p_{\langle\nu\rangle}(x)(b-x)^{\nu-\langle\nu\rangle}\in(\mu-\langle\nu\rangle\,|\,1\,|\,\theta)$$

and $g(x)(b-x)^{-\theta}$ is continuously differentiable on $(a,b]$ since $p_{\langle\nu\rangle}\in C^1[a,b]$. By Proposition 3 one immediately sees that

$$\begin{aligned} I(g,z) \;&=\; \Gamma(\nu+1)(b-a)^{\mu}p_0(b)\,z^{-\nu+\langle\nu\rangle-1} \;+ \\ &\qquad\qquad +\; O\left(z^{-1}\sigma^{-\{\nu-0\}}\right) \;+\; O\left(\sigma^{\langle\nu\rangle}\,e^{-\sigma(b-a)}\right)\end{aligned}$$

as $\sigma\to\infty$. Further, the second summand in (18) can be estimated (by the use of Proposition 1 and Lemma 1) as follows

$$\begin{aligned} &\left|\int_{a_\sigma}^{b} e^{-z(b-x)}g(x)\,(u(x,z)-1)\,dx\right| \;\leqslant \\ &\qquad\leqslant\; \|u(\,\cdot\,,z)-1\|\int_{a_\sigma}^{b} e^{-\sigma(b-x)}|g(x)|\,dx \;=\; O\left(z^{-1}\sigma^{-\{\nu-0\}}\right), \quad \sigma\to\infty.\end{aligned}$$

Finally, due to Lemma 4 inclusion $\widetilde{p}_{\langle\nu\rangle}\in(\,\mu-\langle\nu\rangle+1\,|\,2\,|\,\theta+2\,)$ implies the asymptotic estimate

$$\int_{a_\sigma}^{b} e^{-z(b-x)}\widetilde{p}_{\langle\nu\rangle}(x)u(x,z)\,dx \;=\; O(z^{-2}) \;+\; O\big(\sigma^{\langle\nu\rangle}\,e^{-\sigma(b-a)}\big)$$

valid as $\sigma \to \infty$, that completes the proof of the fist assertion of Proposition 4 concerning the case $\mathcal{N} \geqslant \langle\nu\rangle + 1$.

To prove the second assertion the above arguments need to be modified. Following the same lines as before let us decompose the integral

$$\begin{aligned} J(l^{\langle\nu\rangle}(p), z) \; &= \; \int_{a_\sigma}^{b} e^{-z(b-x)}\, l^{\langle\nu\rangle}(p)(x)\, u(x,z)\; dx \; = \\ &= \; I(g,z) \; + \; \frac{I(\widetilde{g},z)}{z} \; + \; \int_{a_\sigma}^{b} e^{-z(b-x)} g(x) U(x,z)\; dx \; + \; J(\widetilde{p}_{\langle\nu\rangle}, z), \end{aligned} \tag{19}$$

making use of the formulas (17) and (11), where $\widetilde{g}(x) := g(x)\displaystyle\int_a^x p(t)dt$. Since $\mathcal{N} \geqslant \langle\nu\rangle + 2$ then $p_{\langle\nu\rangle} \in C^2[a,b]$ and it follows that $g(x) \in (\mu - \langle\nu\rangle \,|\, 2 \,|\, \theta)$ while $g(x)(b-x)^{-\theta}$ is twice continuously differentiable on $(a,b]$. Consequently by Proposition 3 the first summand in (19) admits a representation

$$\begin{aligned} I(g,z) \; = \; \Gamma(\nu{+}1)(b-a)^{\mu} p_0(b)\, z^{-\nu+\langle\nu\rangle-1} \; + \\ + \; O\left(z^{-\nu+\langle\nu\rangle-2}\right) \; + \; O\left(\sigma^{\langle\nu\rangle}\, e^{-\sigma(b-a)}\right). \end{aligned}$$

To deal with the second summand note that $\widetilde{g}(x) = g(x)\big[p^{(-1)}(x) - p^{(-1)}(a)\big]$, where $p^{(-1)} \in (\mu|\mathcal{N}|\nu+1)$ and so $p^{(-1)}g \in (\mu - \langle\nu\rangle \,|\, 1 \,|\, \theta+1)$, $\langle\theta+1\rangle = 1$. Applying now Lemma 1 to estimate $I(p^{(-1)}g, z)$ we get

$$I(\widetilde{g},z) \; = \; I(p^{(-1)}g, z) \; - \; p^{(-1)}(a) I(g,z) \; = \; O\left(z^{-\nu+\langle\nu\rangle-1}\right) \; + \; O\left(\sigma^{\langle\nu\rangle}\, e^{-\sigma(b-a)}\right)$$

as $\sigma \to \infty$. The third summand on the r.h.s. of (19) is evaluated as follows

$$\begin{aligned} \left| \int_{a_\sigma}^{b} e^{-z(b-x)} g(x) U(x,z)\; dx \right| \; \leqslant \\ \leqslant \; \|U(\,\cdot\,,z)\| \int_{a_\sigma}^{b} e^{-\sigma(b-x)} |g(x)|\; dx \; = \; O\left(z^{-2}\right), \quad \sigma \to \infty, \end{aligned}$$

where the estimate $\|U(\,\cdot\,,z)\| = O(z^{-2})$ from Proposition 1 and the first assertion of Lemma 1 are used. The last summand $J(\widetilde{p}_{\langle\nu\rangle}, z)$ can be treated in the same manner as above and the proof is complete.

5. Location of the spectrum

In Section 2 it was shown that the spectrum of the problem (1)-(2) is discrete, i.e. consists of isolated eigenvalues of finite multiplicity, which coincide (counting the multiplicities) with the zeroes of a characteristic determinant corresponding to a certain fundamental set of solutions to equation (1). It was also established there that the lower half-plane $\mathbb{C}_-$ contains at most finite (possibly empty) set of eigenvalues.

Let us consider now the portion of the spectrum of the problem (1)-(2) located in the upper half-plane $\mathbb{C}_+$ and show that the points that are sought for, i.e. the zeroes of the characteristic determinant $\Delta(k)$, split into two series of (asymptotically simple) eigenvalues $k_n^\pm$ of the form (3). Beforehand it is relevant to search for (and useful to locate) the spectrum-free regions.

After the change of the variable $z = -2ik$ equation $\Delta(k) = 0$ takes the form

$$e^{z(b-a)}\, u_1'(b,z)\, u_2'(b,z) \;=\; z^2, \tag{20}$$

while $u_1(x,z)$ and $u_2(x,z)$ are the solutions to integral equations

$$u_j(x,z) \;=\; 1 + \frac{1}{z}\int_a^x (1-e^{-z(x-t)})p_j(t)u_j(t,z)\,dt, \quad j=1,2$$

with coefficients $p_1(x) = q(x)$ and $p_2(x) = q(a+b-x)$. Note that $v_1(x,k) = u_1(x,-2ik)$ and $v_2(x,k) = u_2(a+b-x,-2ik)$ turn out to be the solutions to equations (4) and (5) respectively. Under the assumptions imposed on the potential $q(x)$ in Section 1 the function $p_1(x)\,(p_2(x))$ vanishes at points a and b at the tangency orders α and β (respectively β and α), namely

$$\begin{aligned} p_1(x) \;&=\; (x-a)^\alpha q_0(x)(b-x)^\beta \\ p_2(x) \;&=\; (x-a)^\beta q_0(a+b-x)(b-x)^\alpha, \end{aligned}$$

where $q_0(a)\cdot q_0(b) \neq 0,\ q_0 \in C^{\mathcal{N}}[a,b]$ and $\mathcal{N} \geqslant \max\{\alpha,\,\beta\}+1$. Below these requirements are assumed to be fulfilled without saying.

Proposition 5. *There exits $\sigma_0 > 0$ such that the region*

$$\{\, z = \sigma + i\tau \mid \sigma > \sigma_0,\ |z|^\varkappa \,\geqslant\, e^{\sigma(b-a)} \,\}, \quad \varkappa = \max\{\alpha,\beta\}+2,$$

is free of the roots of equation (20)*, while the zone $\{0 \leqslant \sigma \leqslant \sigma_0\}$ contains but finite number of them.*

Proof. Without loss of generality let us assume that $\alpha \leqslant \beta$. Since $\mathcal{N} \geqslant \langle\alpha\rangle$ then according to Lemma 3 for a certain $M > 0$ the inequalities

$$|u_j'(b,z)| \;\leqslant\; M|z|^{-\alpha}, \quad j=1,2 \tag{21}$$

are valid provided $|z| \geqslant 4(b-a)\|q\|$ and $\sigma \geqslant 0$. Thus, given a solution $z = \sigma + i\tau$ to equation (20) such that $\sigma > \sigma_1 := \max\{4(b-a)\|q\|, M^{2/\alpha}\}$ one has

$$e^{-\sigma\,(b-a)} \;=\; \left|\frac{u_1'(b,z)\,u_2'(b,z)}{z^2}\right| \;\leqslant\; M^2\,|z|^{-2\alpha-2} \;<\; |z|^{-\alpha-2}$$

and, therefore, the zone $\left\{\, \sigma > \sigma_1,\ |z|^{\alpha+2} \geqslant e^{\sigma(b-a)} \,\right\}$ contains no roots of equation (20).

Further, due to the condition $\mathcal{N} \geqslant \langle\beta\rangle + 1$, Proposition 4 can be applied to obtain the asymptotic representations

$$\begin{aligned} u_1'(b,z) \;&=\; \mathcal{C}_1 z^{-\beta-1} \;+\; \Phi_1(z) \;+\; \Psi_1(z), \quad \mathcal{C}_1 = q_0(b)(b-a)^\alpha\Gamma(\beta+1),\\ u_2'(b,z) \;&=\; \mathcal{C}_2 z^{-\alpha-1} \;+\; \Phi_2(z) \;+\; \Psi_2(z), \quad \mathcal{C}_2 = q_0(a)(b-a)^\beta\Gamma(\alpha+1), \end{aligned}$$

where $\Phi_1(z) = o(z^{-\beta-1})$, $\Phi_2(z) = o(z^{-\alpha-1})$ and $\Psi_{1,2}(z) = O(e^{-\sigma(b-a)})$. Hence the function

$$\psi(z) \;:=\; \left(\mathcal{C}_1 + z^{\beta+1}\Phi_1(z)\right)\left[\frac{1}{\mathcal{C}_2 + z^{\alpha+1}\left(\Phi_2(z) + \Psi_2(z)\right)} - \frac{e^{z(b-a)}\Psi_1(z)}{z^{\alpha+3}}\right]^{-1}$$

is uniformly bounded in the domain $\left\{|z|^{\alpha+2} < e^{\sigma(b-a)}\right\}$ provided σ is large enough. Let $\sigma_0 \geqslant \sigma_1$ be chosen so that the inequality $|\psi(z)| < |z|^{\alpha+2}$ is satisfied when $|z|^{\alpha+2} < e^{\sigma(b-a)}$ and $\sigma > \sigma_0$.

Consider now a solution $z = \sigma + i\tau$ to equation (20) from the half-plane $\sigma > \sigma_0$. According to the remarks above one has $|z|^{\alpha+2} < e^{\sigma(b-a)}$ and, moreover, relationship (20) written in the form

$$e^{-z(b-a)} \;=\; z^{-\alpha-\beta-4}\psi(z),$$

implies (by the choice of σ_0) the following inequality

$$e^{-\sigma(b-a)} \;=\; |z|^{-\alpha-\beta-4}|\psi(z)| \;<\; |z|^{-\beta-2}.$$

This proves actually that the region $\left\{\sigma > \sigma_0,\ |z|^{\beta+2} \geqslant e^{\sigma(b-a)}\right\}$ does not contain the roots of equation (20).

Finally note that by the estimate (21) the l.h.s. of (20) is bounded as $|z| \to \infty$ provided $0 \leqslant \sigma \leqslant \sigma_0$, while the r.h.s. is not. Hence the strip $\left\{0 \leqslant \sigma \leqslant \sigma_0\right\}$ contains at most finite number of solutions to equation (20) that completes the proof of the Proposition.

In the rest of this section the location of the roots of equation (20) in the domain

$$\Omega \;:=\; \left\{z = \sigma + i\tau \mid \sigma > \sigma_0,\ |z|^{\varkappa} < e^{\sigma(b-a)}\right\}$$

will be studied. To this end let us introduce function

$$\varphi(z) \ := \ (\mathcal{C}_1\mathcal{C}_2)^{-1}\, z^{\alpha+\beta+2}\, u_1'(b,z)\, u_2'(b,z) \ = \ 1 \ + \ O\big(\mu(z)\big)$$

analytic in Ω and having (by Proposition 4) the indicated asymptotic behavior as $|z| \to \infty$, where

$$\mu(z) \ := \ \begin{cases} z^{m-1}\sigma^{-m} & \text{in case } A:\ \mathcal{N} = \max\{\langle\alpha\rangle, \langle\beta\rangle\} + 1, \\ z^{-1} & \text{in case } B:\ \mathcal{N} \geqslant \ \max\{\,\alpha,\, \beta\,\} \ + \ 2, \end{cases}$$

and $m := \max\big\{\{\alpha - 0\}, \{\beta - 0\}\big\}$. Let σ_0 be enlarged (if necessary) in such a way that $|\varphi(z) - 1| < 1$ for $z \in \Omega$. Set $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2$, $\gamma = \alpha+\beta+4$ and rewrite (20) in an equivalent form

$$\exp\Big\{z(b-a) \ - \ \gamma \ln z \ + \ \ln \mathcal{C} \ + \ \ln \varphi(z)\Big\} \ = \ 1\,.$$

Thus $z \in \Omega$ is a root of equation (20) iff the equality

$$z \ - \ \lambda \ln z \ + \ \delta(z) \ = \ \omega_n, \tag{9}$$

is satisfied for a certain $n \in \mathbb{Z}$, where $\lambda = \dfrac{\gamma}{b-a}$, $\omega_n = \dfrac{2\pi i n - \ln \mathcal{C}}{b-a}$ and $\delta(z) := \dfrac{\ln \varphi(z)}{b-a} = O(\mu(z))$. The l.h.s. of (9) is analytic in the domain Ω and increases unboundedly as $|z| \to \infty$. Hence equation (9) (provided n is fixed) can have at most finite number of solutions in Ω.

The proof of the Theorem formulated in Section 1 is thus completed by

Lemma 5. *There exist $C > 0$ and $\widetilde{N} > 0$ such that for every $n \in \mathbb{Z}$, $|n| \geqslant \widetilde{N}$, domain Ω contains a unique root z_n of the corresponding equation* (9) *and, moreover*

$$|z_n - \omega_n - \lambda \ln \omega_n| \ \leqslant \ C \begin{cases} |n|^{m-1}(\ln |n|)^{-m} & \textit{in case } A, \\ |n|^{-1} \ln |n| & \textit{in case } B, \end{cases}$$

where $m \ = \ \max\big\{\{\alpha - 0\}, \{\beta - 0\}\big\}$.

Proof. The function $z - \lambda \ln z + \delta(z)$ is bounded on any compact subset in Ω, while $|\omega_n| \to \infty$ as $|n| \to \infty$, hence for arbitrary $R > 0$ there exists $N(R) > 0$ such that the intersection $\Omega \cap \big\{|z| \leqslant R\big\}$ contains no roots of equation (9) if $|n| \geqslant N(R)$. After being divided by z equation (9) takes the form

$$1 \ - \ \frac{\omega_n}{z} \ = \ \lambda \frac{\ln z}{z} \ - \ \frac{\delta(z)}{z}\,,$$

where the r.h.s. vanishes as $|z| \to \infty,\ z \in \Omega$. Therefore, given $r>0$ one can find $R(r)>1/r$ such that the set $\{z \in \Omega,\ |z/\omega_n - 1| \geqslant r,\ |z| > R(r)\}$ does not contain the roots of equation (9) as well. So, for $n \in \mathbb{Z}$, $|n| \geqslant N(R(r))$, the roots of equation (9) in Ω (if any) are necessary located in the domains

$$\Omega_n(r) := \{z \in \Omega,\ |z/\omega_n - 1| < r,\ |z| > R(r)\}.$$

Let us now choose and fix $\rho > 0$ so that an inequality $|\lambda \ln (z/\omega_n)| + |\delta(z)| \leqslant 1/2$ will be valid for $z \in \Omega_n(\rho)$. Substitution $z(\zeta) = \omega_n + \lambda \ln \omega_n + \zeta$ reduces equation (9) to the form

$$\zeta = \lambda \ln \left(\frac{z(\zeta)}{\omega_n} \right) - \delta(z(\zeta)). \tag{22}$$

Due to the choice of ρ the set $\{|\zeta| \geqslant 1,\ z(\zeta) \in \Omega_n(\rho)\}$ does not contain any roots of equation (22). Further, in the disc $|\zeta| \leqslant 1$ the two-sided estimates

$$|z(\zeta)| \asymp |n|, \quad \operatorname{Re} z(\zeta) \asymp \lambda \ln |n|, \quad e^{\operatorname{Re} z(\zeta)\,(b-a)} \asymp |n|^{\gamma}$$

are valid provided $|n|$ is sufficiently large. Hence there exists $\widetilde{N} \geqslant N(R(\rho))$ such that for $|n| \geqslant \widetilde{N}$ and $|\zeta| \leqslant 1$ the inequalities

$$|z(\zeta)/\omega_n - 1| < \rho, \quad |z(\zeta)| > R(\rho),$$

hold and, moreover,

$$\operatorname{Re} z(\zeta) > \sigma_0, \quad e^{\operatorname{Re} z(\zeta)\,(b-a)} > |z(\zeta)|^{\varkappa}.$$

Thus conditions $|\zeta| \leqslant 1$ and $|n| \geqslant \widetilde{N}$ guarantee that $z(\zeta) \in \Omega_n(\rho)$. Consequently, by the choice of ρ the inequality

$$|\,\zeta\,| > \left| \lambda \ln \left(\frac{z(\zeta)}{\omega_n} \right) - \delta(z(\zeta)) \right|$$

is satisfied on the circle $|\zeta| = 1$ provided $|n| \geqslant \widetilde{N}$.

According to Rouché theorem equation (22) for $|n| \geqslant \widetilde{N}$ possesses a single (simple) root ζ_n in the disc $|\zeta| \leqslant 1$. Moreover, by virtue of the estimate $\delta(z) = O(\mu(z))$ there exists $C > 0$ such that

$$|\zeta_n| \leqslant \max_{|\zeta| \leqslant 1} \left| \lambda \ln \left(1 + \frac{\lambda \ln \omega_n}{\omega_n} + \frac{\zeta}{\omega_n} \right) + \delta(z(\zeta)) \right| \leqslant$$
$$\leqslant C \begin{cases} |n|^{m-1} (\ln |n|)^{-m} & \text{in case } A, \\ |n|^{-1} \ln |n| & \text{in case } B, \end{cases}$$

if $|n| \geqslant \widetilde{N}$. Summing up we see that equation (9) for fixed $n \in \mathbb{Z}$, $|n| \geqslant \widetilde{N}$, has in Ω a unique root $z_n := z(\zeta_n) = \omega_n + \lambda \ln \omega_n + \zeta_n$ and the proof is complete.

Acknowledgment. The contemporary version of the manuscript was prepared during the visit to Erwin Schrödinder Institute in Vienna in April 2007. The author expresses gratitude to ESI for hospitality.

References

[1] De Alfaro V., Regge T.
Potential Scattering. Amsterdam, North-Holland, 1965.

[2] Baz A. I., Zel'dovich Ya. B., Perelomov A. M.
Scattering, Reaction and Decay in Non-Relativistic Quantum Mechanics. IPST Press, Jerusalem, 1969.

[3] Arsen'ev A. A.
Singular Potentials and Resonances. Moscow University Press, 1974.

[4] Melrose R.
Geometric Scattering Theory. Cambridge University Press, 1995.

[5] Zworski M.
Counting scattering poles. Spectral and Scattering Theory, Lecture Notes in Pure and Appl. Math., 1994, vol. 161, pp. 301-331.

[6] Sjöstrand J.
A trace formula and review of some estimates for resonances. Microlocal Analysis and Spectral Theory, Math. Phys. Sci., 1997, vol. 490, pp. 377-437.

[7] Zworski M.
Distribution of poles for scattering on the real line. J. Funct. Anal., 1987, vol. 73, no. 2, pp. 277-296.

[8] Fedoryuk M. V.
Asymptotic Analysis. Linear Ordinary Differential Equations. Springer-Verlag, Berlin, 1993.

Mathematics Department, Moscow State University, Moscow 119992, Russia

Institute of Mathematics, University in Bialystok, Akademicka 2, Bialystok, Poland