

A Spectral Sequence for the K -theory of Tiling Spaces

Jean Bellissard
Jean Savinien

Vienna, Preprint ESI 1919 (2007)

May 19, 2007

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via <http://www.esi.ac.at>

A Spectral Sequence for the K -theory of Tiling Spaces*

Jean Bellissard^{1,2}, Jean Savinien¹

¹ Georgia Institute of Technology, School of Mathematics, Atlanta GA 30332-0160

² Institut Universitaire de France

Abstract

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with finite local complexity. We present a spectral sequence that converges to the K -theory of \mathcal{T} with page-2 isomorphic to the Pimsner cohomology of \mathcal{T} . It is a generalization of Serre spectral sequence to a class of spaces which are not fibered. The Pimsner cohomology of \mathcal{T} generalizes the cohomology of the base space of a fibration with local coefficients in the K -theory of its fiber. We prove that it is isomorphic to the Čech cohomology of the hull of \mathcal{T} (the closure for an appropriate topology of the family of its translates).

1 Main Results

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with finite local complexity (definition 3). The hull Ω is a compactification, for an adequate topology, of the family of translates of \mathcal{T} by vectors of \mathbb{R}^d (definition 4). The tiles of \mathcal{T} are given a simplicial decomposition, with each simplex punctured, and the simplicial transversal Ξ_s is the subset of Ω corresponding to translates of \mathcal{T} having the puncture of one of those simplices at the origin $0_{\mathbb{R}^d}$. The prototile space \mathcal{B}_0 (definition 9) is built out of the prototiles of \mathcal{T} (translational equivalence classes of tiles) by gluing them together according to the local configurations of their representatives in the tiling.

The hull is given a dynamical system structure with the homeomorphic action of the group \mathbb{R}^d by translation [13]. The C^* -algebra of the hull is the crossed-product C^* -algebra $C(\Omega) \rtimes \mathbb{R}^d$.

There is a projection (proposition 1)

*Work supported by the NSF grants no. DMS-0300398 and no. DMS-0600956.

$$\begin{array}{ccc} \Xi_s & \longrightarrow & \Omega \\ & & \downarrow \mathfrak{p}_0 \\ & & \mathcal{B}_0 \end{array}$$

which thanks to a lamination structure on Ω (remark 2) resembles (although is *not*) a fibration with base space \mathcal{B}_0 and “fiber space” Ξ_s (remark 4). The Pimsner cohomology H_P^* of the tiling (definition 15) is a cohomology of the base space \mathcal{B}_0 with “local coefficients” in the K -theory of the “fiber” Ξ_s (remark 5).

Theorem 1 *There is a spectral sequence that converges to the K -theory of the C^* -algebra of the hull*

$$E_2^{rs} \Rightarrow K_{r+s+d}(C(\Omega) \rtimes \mathbb{R}^d),$$

and whose page-2 is given by

$$E_2^{rs} \cong H_P^r(\mathcal{B}_0; K^s(\Xi_s)).$$

By an argument using the Thom-Connes isomorphism [21] the K -theory of $C(\Omega) \rtimes \mathbb{R}^d$ is isomorphic to the topological K -theory of Ω (with a shift in dimension by d), and the above theorem can thus be seen formally as a generalization of Serre spectral sequence [67] for a certain class of laminations $\Xi_s \rightarrow \Omega \rightarrow \mathcal{B}_0$, which are foliated spaces [52] but not fibrations.

This result generalizes a previous result by Hunton and Forrest [27] valid for \mathbb{Z}^d -actions on a Cantor set. In general, thanks to a result of Sadun and Williams [64], the hull of a repetitive tiling with finite local complexity is homeomorphic to a fiber bundle over a torus with fibers the Cantor set. These two results are sufficient to get the K -theory of the hull which, thanks to the Thom-Connes theorem [21], gives also the K -theory of the C^* -algebra of the tiling $C(\Omega) \rtimes \mathbb{R}^d$. However, the construction of the hull through an inverse limit of branched manifolds, initiated by Anderson and Putnam [1] for the case of substitution tilings and generalized in [13] to all repetitive tilings with finite local complexity, suggests a different construction and other equivalent definitions of the cohomology that might be useful for practical calculations. Remark also that the recent discovery [29] of torsion in some 3D-tilings has contradicted the claim made in [27] that the cohomology of the tiling is a free group, so that in general the above spectral sequence does not collapse on page-2.

One dimensional repetitive tilings with finite local complexity are all Morita equivalent to a \mathbb{Z} -action on a Cantor set. The Pimsner-Voiculescu exact sequence [55] is then sufficient to compute the K -theory of the hull [9]. In the late nineties, before the paper by Forrest and Hunton was written, Mihai Pimsner suggested to one of the authors¹ a method to generalize the theorem to \mathbb{Z}^d -actions. This spectral sequence was used and described already in [12] and is a special case of the Kasparov spectral sequence [40] for KK -theory.

¹J.B. is indebted to M. Pimsner for this suggestion

Theorem 2 *Let \mathcal{A} be a C^* -algebra endowed with a \mathbb{Z}^d action α by $*$ -automorphisms. The Pimsner complex is defined as $K_*(\mathcal{A}) \otimes \Lambda\mathbb{Z}^d \xrightarrow{d_P} K_*(\mathcal{A}) \otimes \Lambda\mathbb{Z}^d$ with $d_P = \sum_{i=1}^d (\alpha_{e_i} - \mathbf{1}) \otimes e_i \wedge$, where $\{e_1, \dots, e_d\}$ is the canonical basis of \mathbb{Z}^d , $\alpha_i = \alpha_{e_i}$ is the restriction of α to the i -th component of \mathbb{Z}^d , whereas $x \wedge$ is the exterior multiplication by $x \in \mathbb{Z}^d$. There is a spectral sequence converging to the K -theory of \mathcal{A}*

$$E_2^{r,s} \Rightarrow K_{r+s+d}(\mathcal{A} \rtimes_{\alpha} \mathbb{Z}^d),$$

with page-2 isomorphism to the cohomology of the Pimsner complex.

A proof of this result can be found in the Appendix. The crucial step consists in replacing the crossed product $\mathcal{A} \rtimes_{\alpha} \mathbb{Z}^d$ by the *mapping torus* $M_{\alpha}\mathcal{A}$. A more topological expression of this theorem consists in replacing \mathbb{Z}^d by its classifying space, the torus \mathbb{T}^d , with a *CW-complex decomposition* given by an oriented (open) d -cube and all of its (open) faces in any dimension. Then the Pimsner complex can be proved to be isomorphic to the following complex: the cochains are given by covariant maps $\varphi(e) \in K_*(\mathcal{A})$, where e is a cell of \mathbb{T}^d and $\varphi(\bar{e}) = -\varphi(e)$ if \bar{e} is the face e with opposite orientation. The covariance means that if two cells e, e' differ by a translation $a \in \mathbb{Z}^d$ then $\varphi(e') = \alpha^a \varphi(e)$. The differential is the usual one, namely $d\varphi(e) = \sum_{e' \in \partial e} \varphi(e')$. If $H_P^*(\mathbb{T}^d; K_*(\mathcal{A}))$ denotes the corresponding cohomology this gives

Corollary 1 *The Pimsner cohomology group for the crossed product $\mathcal{A} \rtimes_{\alpha} \mathbb{Z}^d$ is isomorphic to $H_P^*(\mathbb{T}^d; K_*(\mathcal{A}))$.*

The spectral sequence used by Forrest and Hunton in [27] in the case $\mathcal{A} = \mathcal{C}(X)$ where X is the Cantor set coincides with the Pimsner spectral sequence. The present paper generalizes this construction for tilings by replacing the classifying space \mathbb{T}^d , by the prototile space \mathcal{B}_0 .

The hull can also be built out of a *box decomposition* [13]. Namely a *box* is a local product of the transversal (which is a Cantor set) by a polyhedron in \mathbb{R}^d called the *base* of the box. Then the hull can be shown to be given by a finite number of such boxes together with identifications on the boundary of the bases. This is an extension of the mapping torus, for which there is only one box with base given by a *cube* [67]. Since \mathcal{A} in the present case is the space of continuous functions on the Cantor set X , $K_0(\mathcal{A})$ is isomorphic to $\mathcal{C}(X, \mathbb{Z})$ whereas $K_1(\mathcal{A}) = 0$, leading to the present result. In Section 4.3 an example of an explicit calculation of the Pimsner cohomology is proposed illustrating the way it can be used. However, as it turns out (see section 4.1), the Pimsner cohomology is isomorphic to other cohomologies used so far on the hull, such as the Čech cohomology [1, 63], the group cohomology [27] or the pattern equivariant cohomology [42, 43, 66].

Acknowledgements. It is a pleasure for J. Savinien to thank the Erwin Schrödinger Institute (Vienna, Austria) that supported him as a Junior Research Fellow in Summer 2006; most of this work was done there.

2 Historic Background

Spectral sequences were introduced by Leray [48] during WWII, as a way to compute the cohomology of a sheaf. It was later put in the framework used today by Koszul [44]. One of the first applications of this method was performed by Borel and Serre [16]. Later, Serre, in his Ph. D. Thesis [67] defined the notion of fibration and could build a spectral sequence to compute their singular homology or cohomology. This led him to calculate the rational homotopy of spheres.

In the early fifties, Hirzebruch made an important step in computing the Euler characteristic of various complex algebraic varieties and complex vector bundles over them [35]. He showed that this characteristic can be computed from the Chern classes of the tangent bundle and of the vector bundle through universal polynomials [36] which coincides with the Todd genus in the case of varieties. It allowed him to show that the Euler characteristic is additive for extensions namely if E, E', E'' are complex vector bundles and if $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ is an exact sequence, then $\chi(E) = \chi(E') + \chi(E'')$. This additivity property led Grothendieck to define axiomatically an additive group characterizing this additivity relation, which he called the K -group [32]. It was soon realized by Atiyah and Hirzebruch [2] that the theory could be extended to topological spaces X and they defined the topological K -theory as a cohomology theory without the axiom of dimension. The K_0 -group is the set of stable equivalence classes of complex vector bundles over X , while K_1 (and more generally K_n) is the set of stable equivalence classes of complex vector bundles over the suspension (respectively the n -th suspension) of X . The Bott periodicity theorem reduces the number of groups to two only. Moreover, the Chern character was shown to define a natural map between the K -group and the integer Čech cohomology of X , and that it becomes an isomorphism when both groups are rationalized. In this seminal paper [2] Atiyah and Hirzebruch define a spectral sequence that will be used in the present work. It is a particular case of Serre spectral sequence for the trivial fibration of X by itself with fiber a point: it converges to the K -theory of X and its page-2 is isomorphic to the cohomology of X .

Almost immediately after this step, Atiyah and Singer extended the work of Hirzebruch to the so-called *Index theorem* [3, 4] for elliptic operators. Such an operator is defined between two vector bundles, it is unbounded, in general and, with the correct domains of definition defines a Fredholm operator. The index can be interpreted as an element of the K -group and is calculated through a formula which generalizes the results of Hirzebruch for algebraic varieties. Eventually, Atiyah and Singer extended the theory to the covariant K -theory valid if a compact Lie group G acts on the vector bundle. If P is an elliptic operator commuting with G its index gives an element of the covariant K -theory. In a programmatic paper [71], Singer proposed various extensions to elliptic operators with coefficients depending on parameters. As an illustration of such a program, Coburn, Moyer and Singer [19] gave an index theorem for elliptic operators with almost periodic coefficients (see also [73]). Eventually the index theorem became the cornerstone

of Connes' program to build a *Noncommutative Geometry*. In a seminal paper [20] he defined the theory of *noncommutative integration* and showed that its first application was an index theorem for elliptic operators on a foliation (see also [22, 24]).

In the late seventies, it was realized that the Atiyah-Hirzebruch K -theory could be expressed in algebraic terms through the C^* -algebra $\mathcal{C}(X)$ of continuous functions on the space X . The definition of the K -group requires then to consider matrix valued continuous functions $M_n(\mathcal{C}(X))$ instead for all n . The smallest C^* -algebra containing all of them is $\mathcal{C}(X) \otimes \mathcal{K}$, where \mathcal{K} denotes the C^* -algebra of compact operators on a separable Hilbert space. Therefore, since this later algebra is non commutative, all the construction could be used for any C^* -algebra. Then Kasparov [39, 40] defined the notion of KK -theory generalizing even more the K groups to correspondences between two C^* -algebras.

The problem investigated in the present paper is the latest development of a program that was initiated in the early eighties [38, 5] when the first version of the *gap labeling theorem* was proved (see [6, 9, 13] for later developments). At that time the problem was to compute the spectrum of a Schrödinger operator H in an aperiodic potential. Several examples were discovered of Schrödinger operators with a Cantor-like spectrum [37, 53]. Labeling the infinite number of gaps per unit length interval, was a challenge. It was realized that the K -theory class of the spectral projection on spectrum below the gap was a proper way of doing so [5]. The first calculation was made on the Harper equation and gave an explanation for a result already obtained by Claro and Wannier [18], a result eventually used in the theory of the quantum Hall effect [75]. This problem was motivated by the need for a theory of aperiodic solids, in particular their electronic and transport properties. With the discovery of *quasicrystals* in 1984 [69], this question became crucial in Solid State Physics. The construction of the corresponding C^* -algebra became then the main issue and led to the definition of the *Hull* [6] and it was proved that it is a compact metrizable set Ω endowed with an action of \mathbb{R}^d via homeomorphisms. Then it was proved in [6, 9], that the resolvent of the Schrödinger operator H belongs to the C^* -algebra $\mathcal{A} = \mathcal{C}(\Omega) \rtimes \mathbb{R}^d$. With each \mathbb{R}^d -invariant probability measure \mathbb{P} on Ω is associated a trace $\mathcal{T}_{\mathbb{P}}$ on this algebra. Following an argument described in [5], it was proved that a gap could be labeled by the value of the density of state, and that this value belongs to the image by the trace $\mathcal{T}_{\mathbb{P}}$ of the group $K_0(\mathcal{A})$. During the eighties several results went on to compute the set of gap labels [56, 6]. In one dimension, detailed results could be proved (see [9] for a review). The most spectacular result was given in the case of a discretized Laplacian with a potential taking on finitely many values (called *letters*): in such a case the set of gap labels is the \mathbb{Z} -module generated by the occurrence probability of all possible finite words found in the sequence defined by the potential. If this sequence is given by a *substitution*, these occurrence probabilities can be computed explicitly in terms of the incidence matrix of the substitution and of the associated substitution induced on the set of words with two letters [58, 9]. The key property in proving such results was the use of the Pimsner-Voiculescu exact sequence [55]. Soon after, A. van Elst [26] extended these

results to the case of 2D-potentials, using the same method.

In the French version [23] of his book on *Noncommutative geometry* [24], Connes showed how the general formalism he contributed to develop could be illustrated with the special case of the Penrose tiling. Using the substitution rules for its construction, he introduced a C^* -algebra, which turns out to be AF, and computed its ordered K_0 -group, using the classification of AF-algebras obtained by Bratteli in 1972 [17]. This result was an inspiration for Kellendonk, who realized that, instead of looking at the inflation rule as a source of noncommutativity, it was actually better to consider the space translations of the tiling [41], in the spirit of the formalism developed in [5, 6, 9] for aperiodic solids. He extended this latter construction of the Hull to tilings and then this Hull is called *tiling space*. In this important paper, Kellendonk introduced the notion of *forcing the border* for a general tiling, which appears today as an important property for the calculation of the K-groups and cohomology of a tiling space. This work gave a strong motivation to prove the *gap labeling theorem in higher dimension* and to compute the K -group of the Hull. It was clear that the method used in one dimension, through the Pimsner-Voiculescu exact sequence could only be generalized through a spectral sequence. The first use of spectral sequences in computing the set of gap labels on the case of the 2D-octagonal tiling [10] was followed by a proof of the gap labeling theorem for 3D-quasicrystals [12]. Finally, the work by Forrest and Hunton [27] used a classical spectral sequence to compute the full K -theory of the C^* -algebra for an action of \mathbb{Z}^d on the Cantor set. It permitted to compute the K -theory and the cohomology of the Hull for quasicrystals in two and three dimensions [28, 29]. Other examples of tilings followed later [65].

In addition to Kellendonk's work, several important contributions helped build tools to prove the higher dimensional version of the gap labeling theorem. Among the major contributions was the work of Lagarias [45, 46, 47], who introduced a geometric and combinatoric aspect of tiling through the notion of a Delone set. This concept was shown to be conceptually crucial in describing aperiodic solids [11]. Through the construction of Voronoi, Delone sets and tilings become equivalent concepts, allowing for various intuitive point of views to study such problems.

Another important step was performed in 1998 by Anderson and Putnam [1], who proposed to build the tiling space of a substitution tiling, through a CW-complex built from the prototiles of a tiling (called prototile space in this paper). The substitution induces a map from this CW-complex into itself and they showed that the inverse limit of such system becomes homeomorphic to the tiling space. A similar construction was proposed independently by Gambaudo and Martens in 1999 to describe dynamical system. This latter case corresponds to 1D repetitive tilings with finite local complexity, so that this latter construction goes beyond the substitution tilings. One interesting outcome of this work was a systematic construction of minimal dynamical systems, uniquely ergodic or not and with positive entropy² (see [30]). Eventually, the Gambaudo-Martens construc-

²This result circulated as a preprint in 1999 but was published only in 2006

tion led to the construction of the Hull as an inverse limit of compact oriented branched flat Riemannian manifolds [13]³ used in the present paper (although only their topological CW -complex structure is needed here). Equivalently, the Hull can be seen as a *lamination* [31] of a *foliated space* [52]. The extension of this construction to include tilings without finite local complexity, such as the pinwheel model [60], was performed by Gambaudo and Benedetti [14] using the notion of solenoids (introduced by Williams in the seventies [76]).

3 A Mathematical Reminder

The preliminary definitions and results required for stating theorem 1 are presented here. They are taken mostly from previous work of the first author in [11, 13] on tilings and Delone sets and the reader is referred to those papers for complete proofs. Let \mathbb{R}^d denote the usual Euclidean space of dimension d with euclidean norm $\|\cdot\|$. First of all the definition of a repetitive tiling of \mathbb{R}^d with finite local complexity is recalled, and then the hull and transversal of such a tiling are introduced. The connection with Delone sets is briefly mentioned as well as the definition of the groupoid of the transversal of a Delone set. The prototile space of a tiling and the ring of functions on the transversal are then built, and finally the Pimsner cohomology defined.

3.1 Tilings and their hulls

Definition 1 (i) A tile of \mathbb{R}^d is a compact subset of \mathbb{R}^d which is homeomorphic to the unit ball.

(ii) A punctured tile is an ordered pair consisting of a tile and one of its points.

(iii) A tiling of \mathbb{R}^d is a covering of \mathbb{R}^d by a family of tiles whose interiors are pairwise disjoint. A tiling is said to be punctured if its tiles are punctured.

(iv) A prototile of a tiling is a translational equivalence class of tiles (including the puncture).

(v) The first corona of a tile in a tiling \mathcal{T} is the union of the tiles of \mathcal{T} intersecting it.

(vi) A collared prototile of \mathcal{T} is the subclass of a prototile whose representatives have the same first corona up to translation.

³This paper was posted on [arXiv.com](http://arxiv.com) [math.DS/0109062](https://arxiv.org/abs/math/0109062) in its earlier version in 2001 but was eventually published in a final form in 2006 only.

A collared prototile is a prototile where a local configuration of its representatives has been specified: each representative has the same neighboring tiles.

In the sequel it is implicitly assumed that tiles and tilings are punctured. All tiles are assumed to be finite Δ -complexes which are particular (simplicial) CW -complex structures (see section 4.2). They are also required to be compatible with the tiles of their first coronas, *i.e.* the intersection of any two tiles is itself a sub- Δ -complex of both tiles. In other words, tilings considered here are assumed to be Δ -complexes of \mathbb{R}^d .

Definition 2 *Let \mathcal{T} be a tiling of \mathbb{R}^d .*

- (i) *A patch of \mathcal{T} is a finite union of neighboring tiles in \mathcal{T} that is homeomorphic to a ball. A patch is punctured by the puncture of one the tiles it contains. The radius of a patch is the radius of the smallest ball that contains it.*
- (ii) *A pattern of \mathcal{T} is a translational equivalence class of patches of \mathcal{T} .*
- (iii) *The first corona of a patch of \mathcal{T} is the union of the tiles of \mathcal{T} intersecting it.*
- (iv) *A collared pattern of \mathcal{T} is the subclass of a pattern whose representatives have the same first corona up to translation.*

The following notation will be used in the sequel: prototiles and patterns will be written with a hat, for instance \hat{t} or \hat{p} , to distinguish them from their representatives. Often the following convention will be implicit: if \hat{t} is a prototile and \hat{p} a pattern, then t and p will denote their respective representatives that have their punctures at the origin $0_{\mathbb{R}^d}$.

The results in this paper are valid for the class of tilings that are *repetitive* with *finite local complexity*.

Definition 3 *Let \mathcal{T} be a tiling of \mathbb{R}^d .*

- (i) *\mathcal{T} has Finite Local Complexity (FLC) if for any $R > 0$ the set of patterns of \mathcal{T} whose representatives have radius less than R is finite.*
- (ii) *\mathcal{T} is repetitive if for any patch of \mathcal{T} and every $\epsilon > 0$, there is an $R > 0$ such that for every x in \mathbb{R}^d there exists modulo an error ϵ with respect to the Hausdorff distance, a translated copy of this patch belonging to \mathcal{T} and contained in the ball $B(x, R)$.*
- (iii) *\mathcal{T} is aperiodic if there is no $x \neq 0$ in \mathbb{R}^d such that $\mathcal{T} + x = \mathcal{T}$, where $\mathcal{T} + x = \{t + x : t \in \mathcal{T}\}$.*

For tilings with FLC the repetitivity condition (ii) above can be stated more precisely: *A tiling \mathcal{T} with FLC is repetitive if given any patch there is an $R > 0$ such that for every x in \mathbb{R}^d there exists an exact copy of this patch in \mathcal{T} contained in the ball $B(x, R)$.*

The class of repetitive tilings satisfying the FLC property is very rich and has been investigated for decades. It started in the 70's with the work of Penrose [57] and Meyer [51] and went on both from an abstract mathematical level and with a view towards applications, in particular to the physics of quasicrystals [45, 46]. It contains an important subclass of the class of *substitution tilings* that was reinvestigated in the 90's by Anderson and Putnam in [1], and also the class of tilings obtained by the *cut-and-projection* method, for which a comprehensive study by Hunton, Kellendonk and Forrest can be found in [28], and more generally it contains the whole class of *quasiperiodic tilings*, which are models for quasicrystals.

The Ammann aperiodic tilings are examples of substitution tilings (see [33] chapter 10, Ammann original work does not appear in the literature). In addition other examples include the octagonal tiling as well as the famous Penrose “kite and darts” [57] tilings which are both substitution and cut-and-projection tilings (see [33] chapter 10, and [60] chapter 4). However the so-called Pinwheel tiling (see [59] and [60] chapter 4) is a substitution tiling but does not satisfy the FLC property given here since prototiles are defined here as equivalence classes of tiles under only translations and not more general isometries of \mathbb{R}^d like rotations.

The first author has proposed in [11] a topology that applies to a large class of tilings (for which there exist an $r_0 > 0$ such that all tiles contain a ball of radius r_0). In the case considered here, where tiles are assumed to be finite *CW*-complexes, this topology can be adapted as follows.

Let \mathcal{F} be a family of tilings whose tiles have compatible *CW*-complex structures and contain a ball of a fixed radius $r_0 > 0$. Given an open set O in \mathbb{R}^d with compact closure and an $\epsilon > 0$, a neighborhood of a tiling T in \mathcal{F} is given by

$$U_{O,\epsilon}(T) = \left\{ T' \in \mathcal{F} : \sup_{0 \leq k \leq d} h_k(O \cap T^k, O \cap T'^k) < \epsilon \right\},$$

where T^k and T'^k are the k -skeletons of T and T' respectively and h_k is the k -dimensional Hausdorff distance.

Let \mathcal{T} be a tiling of \mathbb{R}^d . The group \mathbb{R}^d acts on the set of all translates of \mathcal{T} , the action (translation) is denoted τ^a , $a \in \mathbb{R}^d$: $\tau^a \mathcal{T} = \mathcal{T} + a = \{t + a : t \in \mathcal{T}\}$.

Definition 4 (i) *The hull of \mathcal{T} , denoted Ω , is the closure of $\tau^{\mathbb{R}^d} \mathcal{T}$.*

(ii) *The canonical transversal, denoted Ξ , is the subset of Ω consisting of tilings that have the puncture of one of their tiles at the origin $0_{\mathbb{R}^d}$.*

The hull of a tiling is seen as a dynamical system $(\Omega, \mathbb{R}^d, \tau)$ which, for the class of tilings considered here, has interesting properties that are now stated (see [13] section 2.3).

Theorem 3 [13, 45] *Let \mathcal{T} be a tiling of \mathbb{R}^d .*

- (i) \mathcal{T} is repetitive if and only if the dynamical system of its hull $(\Omega, \mathbb{R}^d, \mathbb{T})$ is minimal.
- (ii) If \mathcal{T} has FLC, then its transversal Ξ is totally disconnected.
- (iii) If \mathcal{T} is aperiodic, repetitive and has FLC, then Ξ is a Cantor set (perfect and totally disconnected).

The minimality of the hull allows one to see any of its points as just a translate of \mathcal{T} .

Remark 1 A metric topology for tiling spaces has been used in the literature for historical reasons. Let \mathcal{T} be a repetitive tiling of \mathbb{R}^d with FLC. The orbit space of \mathcal{T} under translation by vectors of \mathbb{R}^d , $\mathbb{T}^{\mathbb{R}^d}\mathcal{T}$, is endowed with a metric as follows (see [13] section 2.3). For T and T' in $\mathbb{T}^{\mathbb{R}^d}\mathcal{T}$, let A denote the set of ε in $(0, 1)$ such that there exist x and x' in $B(0, \varepsilon)$ for which $\mathbb{T}^x T$ and $\mathbb{T}^{x'} T'$ agree on $B(0, \frac{1}{\varepsilon})$, *i.e.* their tiles whose punctures lie in the ball are matching, then

$$\delta(T, T') = \begin{cases} \inf A & \text{if } A \neq \emptyset, \\ 1 & \text{if } A = \emptyset. \end{cases}$$

Hence the diameter of $\mathbb{T}^{\mathbb{R}^d}\mathcal{T}$ is bounded by 1 and the action of \mathbb{R}^d is continuous. For the class of repetitive tilings with FLC, the topology of the hull given in definition 4 is equivalent to this δ -metric topology [13].

3.2 Delone sets and groupoid of the transversal

The notions for tilings given in the previous subsection can be translated for the set of their punctures in terms of Delone sets.

Definition 5 Let \mathcal{L} be a discrete subset of \mathbb{R}^d .

- (i) Given $r > 0$, \mathcal{L} is r -uniformly discrete if any open ball of radius r in \mathbb{R}^d meets \mathcal{L} in at most one point.
- (ii) Given $R > 0$, \mathcal{L} is R -relatively dense if any open ball of radius R in \mathbb{R}^d meets \mathcal{L} in at least one point.
- (iii) \mathcal{L} is an (r, R) -Delone set if it is r -uniformly discrete and R -relatively dense.
- (iv) \mathcal{L} is repetitive if given any finite subset $p \subset \mathcal{L}$ and any $\epsilon > 0$, there is an $R > 0$ such that the intersection of \mathcal{L} with any closed ball of radius R contains a copy (translation) of p modulo an error of ϵ (w.r.t. the Hausdorff distance)

(v) A patch of radius $R > 0$ of \mathcal{L} is a subset of \mathbb{R}^d of the form $(\mathcal{L} - x) \cap B(0, R)$, for some $x \in \mathcal{L}$. If for all $R > 0$ the set of its patches of radius R is finite then \mathcal{L} has finite local complexity.

(vi) \mathcal{L} is aperiodic if there is no $x \neq 0$ in \mathbb{R}^d such that $\mathcal{L} - x = \mathcal{L}$.

Condition (iv) above is equivalent to saying that $\mathcal{L} - \mathcal{L}$, the set of vectors of \mathcal{L} , is discrete. Given a punctured tiling \mathcal{T} if there exists $r, R > 0$ such that each of its tiles contains a ball a radius r and is contained in a ball of radius R , then its set of punctures $\mathcal{L}_{\mathcal{T}}$ is an (r, R) -Delone set, and it is repetitive and has FLC if and only if the tiling is repetitive and has FLC. Conversely, given a Delone set the Voronoi construction below gives a tiling, and they both share the same repetitivity or FLC properties.

Definition 6 Let \mathcal{L} be an (r, R) -Delone set of \mathbb{R}^d . The Voronoi tile at $x \in \mathcal{L}$, is defined by

$$T_x = \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - x'\|, \forall x' \in \mathcal{L}\},$$

with puncture the point x . The Voronoi tiling \mathcal{V} associated with \mathcal{L} is the tiling of \mathbb{R}^d whose tiles are the Voronoi tiles of \mathcal{L} .

The tiles of the Voronoi tiling of a Delone set are (closed) convex polytopes that touch on common faces.

The hull and transversal of a Delone set are defined as follows [11].

Definition 7 Let \mathcal{L} be a Delone set of \mathbb{R}^d . The hull Ω of \mathcal{L} is the closure in the weak-* topology of the set of translations of the Radon measure that has a Dirac mass at each point of \mathcal{L}

$$\Omega = \overline{\{T^a \mu; a \in \mathbb{R}^d\}}^{w*}, \quad \mu = \sum_{x \in \mathcal{L}} \delta_x,$$

and the transversal Ξ of \mathcal{L} is the subset

$$\Xi = \{\omega \in \Omega; \omega(\{0\}) = 1\}.$$

Given a general Delone set this topology of its hull is strictly coarser than the δ -metric topology for its Voronoi tiling given in remark 1, but if it is repetitive and has FLC then they are equivalent [13]. The weak-* topology is here also equivalent to the local Hausdorff topology on the set $T^{\mathbb{R}^d} \mathcal{L}$ of translates of \mathcal{L} : given an open set O in \mathbb{R}^d with compact closure and an $\epsilon > 0$, a neighborhood of ℓ is given by

$$U_{O, \epsilon}(\ell) = \{\ell' \in T^{\mathbb{R}^d} \mathcal{L} : h_d(\ell \cap O, \ell' \cap O) < \epsilon\},$$

where h_d is the d -dimensional Hausdorff distance.

The hull of a Delone set is also seen as a dynamical system under the homeomorphic action of \mathbb{R}^d by translation. Just as in proposition 3 the hull and transversal of repetitive Delone sets with FLC have similar interesting properties.

Theorem 4 [11] *Let \mathcal{L} be a Delone set of \mathbb{R}^d .*

- (i) \mathcal{L} is repetitive if and only if the dynamical system of its hull $(\Omega, \mathbb{R}^d, \mathbb{T})$ is minimal.
- (ii) If \mathcal{L} has FLC, then its transversal Ξ is totally disconnected.
- (iii) If \mathcal{L} is aperiodic, repetitive and has FLC, then Ξ is a Cantor set (perfect and totally disconnected).

The two approaches of tilings or Delone sets which are repetitive and have FLC are thus equivalent. Such tilings give rise to Delone sets (their sets of punctures), and conversely such Delone sets give rise to tilings (their Voronoi tilings) and their respective hulls share the same properties.

Recall that a groupoid is a small category in which every morphism is invertible [20, 61].

Definition 8 *Let \mathcal{L} be a Delone set of \mathbb{R}^d . The groupoid of the transversal is the groupoid Γ whose set of objects is the transversal: $\Gamma^0 = \Xi$, and whose set of arrows is*

$$\Gamma^1 = \{(\xi, x) \in \Xi \times \mathbb{R}^d : \mathbb{T}^{-x}\xi \in \Xi\}.$$

Given an arrow $\gamma = (\xi, x)$ in Γ^1 , its source is the object $s(\gamma) = \mathbb{T}^{-x}\xi \in \Xi$ and its range the object $r(\gamma) = \xi \in \Xi$.

The Delone sets considered here are repetitive with FLC and the groupoids of their transversals are *etale*, i.e. given any arrow γ with range (or source) object x , there exists an open neighborhood O_x of x in Γ^0 and a homeomorphism $\varphi : O_x \rightarrow \Gamma^1$ that maps x to γ . This implies that the sets of arrows having any given object as a source or range are discrete. This follows because for such Delone sets their sets of vectors are discrete and given any fixed $l > 0$ their sets of vectors of length less than l are finite.

3.3 Prototile space of a tiling

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC and assume its tiles are compatible finite CW-complexes (the intersection of two tiles is a subcomplex of both). A finite CW-complex \mathcal{B}_0 , called prototile space, is built out of its prototiles by gluing them along their boundaries according to all the local configurations of their representatives in \mathcal{T} .

Definition 9 *Let \hat{t}_j , $j = 1, \dots, N_0$, be the prototiles of \mathcal{T} . Let t_j denote the representative of \hat{t}_j that has its puncture at the origin. The prototile space of \mathcal{T} , $\mathcal{B}_0(\mathcal{T})$, is the quotient CW-complex*

$$\mathcal{B}_0(\mathcal{T}) = \coprod_{j=1}^{N_0} t_j / \sim,$$

where two n -cells $e_i^n \in t_i^n$ and $e_j^n \in t_j^n$ are identified if there exists $u_i, u_j \in \mathbb{R}^d$ for which $t_i + u_i$ and $t_j + u_j$ are tiles of \mathcal{T} such that $e_i^n + u_i$ and $e_j^n + u_j$ coincide on the intersection of their n -skeletons.

The collared prototile space of \mathcal{T} , $\mathcal{B}_0^c(\mathcal{T})$, is built similarly from the collared prototiles of \mathcal{T} : $t_i^c, i = 1, \dots, N_0^c$.

The images in $\mathcal{B}_0(\mathcal{T})$ or $\mathcal{B}_0^c(\mathcal{T})$ of the tiles $t_j^{(c)}$'s will be denoted τ_j and still be called tiles.

Proposition 1 *There is a continuous map $\mathfrak{p}_{0,\mathcal{T}}^{(c)} : \Omega \rightarrow \mathcal{B}_0^{(c)}(\mathcal{T})$ from the hull onto the (collared) prototile space.*

Proof. Let $\lambda_0^{(c)} : \coprod_{j=1}^{N_0^{(c)}} t_j \rightarrow \mathcal{B}_0^{(c)}(\mathcal{T})$ be the quotient map. And let $\rho_0^{(c)} : \Omega \times \mathbb{R}^d \rightarrow \coprod_{j=1}^{N_0^{(c)}} t_j^{(c)}$ be defined as follows. If x belongs to the intersection of k tiles $t^{\alpha_1}, \dots, t^{\alpha_k}$, in ω , with $t^{\alpha_l} = t_{j_l}^{(c)} + u_{\alpha_l}(\omega)$, $l = 1, \dots, k$, then $\rho_0^{(c)}(\omega, x) = \coprod_{l=1}^k x - u_{\alpha_l}(\omega)$ and lies in the disjoint union of the $t_{j_l}^{(c)}$'s.

The projection $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is defined as

$$\mathfrak{p}_{0,\mathcal{T}}^{(c)} : \begin{cases} \Omega & \rightarrow \mathcal{B}_0^{(c)}(\mathcal{T}) \\ \omega & \mapsto \lambda_0^{(c)} \circ \rho_0^{(c)}(\omega, 0_{\mathbb{R}^d}) \end{cases} .$$

The map $\rho_0^{(c)}(\cdot, 0_{\mathbb{R}^d})$ sends the origin of \mathbb{R}^d , that lies in some tiles of ω , to the corresponding tiles $t_j^{(c)}$'s at the corresponding positions.

In $\mathcal{B}_0^{(c)}(\mathcal{T})$, points on the boundaries of two tiles $\tau_j^{(c)}$ and $\tau_{j'}^{(c)}$ are identified if there are neighboring copies of the tiles $t_j^{(c)}$ and $t_{j'}^{(c)}$ somewhere in \mathcal{T} such that the two associated points match. This ensures that the projection $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is well defined, for if in \mathbb{R}^d tiled by ω , the origin $0_{\mathbb{R}^d}$ belongs to the boundaries of some tiles, then the corresponding points in $\coprod_{j=1}^{N_0^{(c)}} t_j^{(c)}$ given by $\rho_0^{(c)}(\omega, 0_{\mathbb{R}^d})$ are identified by $\lambda_0^{(c)}$.

Let x be a point in $\mathcal{B}_0^{(c)}(\mathcal{T})$, and O_x and open neighborhood of x . Say x belongs to the intersection of some tiles $\tau_{j_1}^{(c)}, \dots, \tau_{j_k}^{(c)}$. Let ω be a preimage of x : $\mathfrak{p}_{0,\mathcal{T}}^{(c)}(\omega) = x$. The preimage of O_x is the set of tilings ω' 's for which the origin lies in some neighborhood of tiles that are translates of $t_{j_1}^{(c)}, \dots, t_{j_k}^{(c)}$, and this defines a neighborhood of ω in the hull.

Therefore $\mathfrak{p}_{0,\mathcal{T}}^{(c)}$ is continuous. \square

For simplicity, the prototile space $\mathcal{B}_0(\mathcal{T})$ is written \mathcal{B}_0 , and the projection $\mathfrak{p}_{0,\mathcal{T}}$ is written \mathfrak{p}_0 .

The lift of the puncture of the tile τ_j in \mathcal{B}_0 , denoted $\Xi(\tau_j)$, is a subset of the transversal called the *acceptance zone* of the prototile \hat{t}_j . It consists of all the tilings that have the puncture of a representative of \hat{t}_j at the origin. The $\Xi(\tau_j)$'s for $j = 1, \dots, N_0$, form a clopen partition of the transversal, because any element of Ξ has the puncture of a unique tile

at the origin which corresponds to a unique prototile. The $\Xi(\tau_j)$'s are thus Cantor sets like Ξ .

Remark 2 Although those results will not be used here, it has been proven in [13] that the prototile space \mathcal{B}_0 (as well as the patch spaces \mathcal{B}_p defined similarly in the next section, definition 10) has the structure of a flat oriented Riemannian branched manifold. Also, the hull Ω can be given a *lamination* structure as follows: the lifts of the interiors of the tiles $B_{0j} = \mathfrak{p}_0^{-1}(\tau_j^\circ)$ are boxes of the lamination which are homeomorphic to $t_j^\circ \times \Xi(\tau_j)$ via the maps $(x, \xi) \mapsto \mathbb{T}^{-x}\xi$ which read as local charts.

3.4 The hull as an inverse limit of patch spaces

As in the previous section, let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC, and assume its tiles are compatible finite *CW*-complexes (the intersection of two tiles is a subcomplex of both). Let $\mathcal{L}_{\mathcal{T}}$ denote the Delone set of punctures of \mathcal{T} , it is repetitive and has FLC. Let $\mathcal{P}_{\mathcal{T}}$ denote the set of patterns of \mathcal{T} . As \mathcal{T} has finite local complexity (hence finitely many prototiles) the set $\mathcal{P}_{\mathcal{T}}$ is countable, and for any given $l > 0$ the set of patterns whose representatives have radius less than l is finite.

A finite *CW*-complex \mathcal{B}_p , called a patch space, associated with a pattern \hat{p} in $\mathcal{P}_{\mathcal{T}}$ is built from the prototiles of an appropriate subtiling of \mathcal{T} , written \mathcal{T}_p below, in the same way that \mathcal{B}_0 was built from the prototiles of \mathcal{T} in definition 9. The construction goes as follows. Let \hat{p} in $\mathcal{P}_{\mathcal{T}}$ be a pattern of \mathcal{T} .

- (i) Consider the sub-Delone set \mathcal{L}_p of $\mathcal{L}_{\mathcal{T}}$ consisting of punctures of all the representative patches in \mathcal{T} of \hat{p} . \mathcal{L}_p is repetitive and has FLC.
- (ii) The Voronoi tiling \mathcal{V}_p of \mathcal{L}_p is built and each point of $\mathcal{L}_{\mathcal{T}}$ is assigned to a unique tile of \mathcal{V}_p as explained below.
- (iii) Each tile v of \mathcal{V}_p is replaced by the patch p_v of \mathcal{T} made up of the tiles whose punctures have been assigned to v . This gives a repetitive tiling with FLC, \mathcal{T}_p , whose tiles are those patches p_v 's.
- (iv) \mathcal{B}_p is built out of the collared prototiles of \mathcal{T}_p , by gluing them along their boundaries according to the local configurations of their representatives in \mathcal{T}_p .

The second point needs clarifications since the tiles of \mathcal{V}_p are Voronoi tiles (convex polytopes, see definition 6) of \mathcal{L}_p and not patches of \mathcal{T} . If a point of $\mathcal{L}_{\mathcal{T}}$ (a puncture of a tile of \mathcal{T}) lies on the boundary of some (Voronoi) tiles of \mathcal{V}_p , a criterion for assigning it to a specific one is required. To do so, let u be a vector of \mathbb{R}^d that is not colinear to any of the faces of the tiles of \mathcal{V}_p (such a vector exists since \mathcal{V}_p has FLC, hence finitely

many prototiles). A point x is said to be u -interior to a subset X of \mathbb{R}^d if there exist an $\epsilon > 0$ such that $x + \epsilon u$ belongs to the interior of X . Since u is not colinear to any of the faces of the tiles of \mathcal{V}_p , if a point x belongs to the intersection of the boundaries of several (Voronoi) tiles of \mathcal{V}_p , it is u -interior to only one of them. This allows as claimed in (ii) to assign each point of \mathcal{L}_T to a unique tile of \mathcal{V}_T . Now as explained in (iii), each Voronoi tile v can then be replaced by the patch of \mathcal{T} which is the union of the tiles of \mathcal{T} whose punctures are u -interior to v .

Each patch p_v is considered a tile of \mathcal{T}_p and punctured by the puncture of the Voronoi tile v which is by construction the puncture of some representative patch of \hat{p} in \mathcal{T} . As patches of \mathcal{T} , the p_v 's are also compatible finite CW -complexes as they are made up of tiles of \mathcal{T} which are.

The prototiles of \mathcal{T}_p are actually patterns of \mathcal{T} , and thus \mathcal{T}_p is considered a subtiling of \mathcal{T} . From this remark it can be proven that there is a homeomorphism between the hull of \mathcal{T}_p and Ω , that conjugates the \mathbb{R}^d -action of their associated dynamical systems (see [13] section 2).

Definition 10 *The patch space \mathcal{B}_p is the collared prototile space of \mathcal{T}_p (definition 9):*

$$\mathcal{B}_p = \mathcal{B}_0^c(\mathcal{T}_p).$$

The images in \mathcal{B}_p of the patches p_j 's (tiles of \mathcal{T}_p) will be denoted π_j and still be called patches. The projection $\mathfrak{p}_{0, \mathcal{T}_p}^c : \Omega \rightarrow \mathcal{B}_p$, built in proposition 1, is denoted \mathfrak{p}_p .

This construction of \mathcal{B}_p from \mathcal{T}_p is essentially the same as the construction of \mathcal{B}_0 from \mathcal{T} given in definition 9, the only difference being that collared prototiles of \mathcal{T}_p (collared patches of \mathcal{T}) are used here instead. Such patch spaces are said to *force their borders*. This condition was introduced by Kellendonk in [41] for substitution tilings and was required in order to be able to recover the hull as the inverse limit of such spaces. It was generalized in [13] for branched manifolds of repetitive tilings with FLC and coincides here with the above definition.

The map $f_p : \mathcal{B}_p \rightarrow \mathcal{B}_0$ defined by $f_p = \mathfrak{p}_0 \circ \mathfrak{p}_p^{-1}$ is surjective and continuous. It projects \mathcal{B}_p onto \mathcal{B}_0 in the obvious way: a point x in \mathcal{B}_p belongs to some patch π_j , hence to some tile, and f_p sends x to the corresponding point in the corresponding tile $\tau_{j'}$. More precisely, if \tilde{x} in p_j is the point of \mathbb{R}^d corresponding to x in π_j , then \tilde{x} belongs to some tile, which is the translate of some $t_{j'}$ and $f_p(x)$ is the corresponding point in $\tau_{j'}$ in \mathcal{B}_0 . If x belongs to the boundaries of say $\pi_{j_1}, \dots, \pi_{j_k}$, in \mathcal{B}_p , then there are corresponding points $\tilde{x}_{j_1}, \dots, \tilde{x}_{j_k}$, in p_{j_1}, \dots, p_{j_k} , which are then on the boundaries of the copies of some tiles $t_{j'_1}, \dots, t_{j'_k}$. The boundaries of those tiles are identified by the map ρ_0 in the definition of \mathfrak{p}_0 and $f_p(x)$ is the corresponding point on the common boundaries of the $\tau_{j'_1}, \dots, \tau_{j'_k}$.

Recall the convention stated after definition 2: if \hat{p} is a pattern, then p denotes it representative that has its puncture at the origin. Given two patterns \hat{p} and \hat{q} with $q \subset p$, the map $f_{qp} : \mathcal{B}_p \rightarrow \mathcal{B}_q$ defined by $f_{qp} = f_q^{-1} \circ f_p = \mathfrak{p}_q \circ \mathfrak{p}_p^{-1}$ is continuous and surjective.

And given three patterns \hat{p}, \hat{q} and \hat{r} with $r \subset q \subset p$ the following composition rule holds: $f_{rq} \circ f_{qp} = f_{rp}$. Moreover given two arbitrary patterns \hat{q} and \hat{r} , there exists another pattern \hat{p} such that p contains q and r (since \mathcal{L}_q and \mathcal{L}_r are repetitive sub-Delone sets of $\mathcal{L}_\mathcal{T}$). Hence the index set of the maps f_p 's is a directed set and (\mathcal{B}_p, f_{qp}) is a projective system. As shown in [13], the hull Ω can be recovered from the inverse limit $\varprojlim(\mathcal{B}_p, f_{qp})$ under some technical conditions that are for convenience taken here to be directly analogous to those given in [13] section 2.6. Namely: the patch spaces \mathcal{B}_p are required to *force their borders*, and only maps f_{qp} between patch spaces that are *zoomed out* of each other are allowed (definition 11 below). The first condition, as mentioned above, is fulfilled here by the very definition of patch spaces given above, because they are built out of *collared patterns* as can be checked from the more general definition of a branched manifold that forces its border given in [13] definition 2.43. The second condition is stated as follows.

Definition 11 *Given two patterns \hat{p}, \hat{q} in $\mathcal{P}_\mathcal{T}$, with $q \subset p$, \mathcal{B}_p is said to be zoomed out of \mathcal{B}_q if the following two conditions hold.*

- (i) *For all $i \in \{1, \dots, N_p\}$, the patch p_i is the union of some copies of the patches q_j 's.*
- (ii) *For all $i \in \{1, \dots, N_p\}$, the patch p_i contains in its interior a copy of some patch q_j .*

The first condition is equivalent to requiring that the tiles of \mathcal{L}_p are patches of \mathcal{L}_q .

Given a patch space \mathcal{B}_q it is always possible to build another patch space \mathcal{B}_p that is zoomed out of \mathcal{B}_q : it suffices to choose a pattern \hat{p} such that $p \supset q$ with a radius large enough. This can be done by induction for instance: building \mathcal{B}_p from patches of \mathcal{L}_q (which are patches of \mathcal{T}) the same way \mathcal{B}_q was built from patches of \mathcal{T} ; if the radius of q is large enough, then each patch of \mathcal{T} that \mathcal{B}_q is composed of will contain a tile of \mathcal{T} in its interior and so \mathcal{B}_q will be zoomed out from \mathcal{B}_0 .

Definition 12 *A proper sequence of patch spaces of \mathcal{T} is a projective system $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ where, for all $l \geq 1$, \mathcal{B}_l is a patch space associated with a pattern \hat{p}_l of \mathcal{T} and $f_l = f_{p_{l-1}p_l}$, such that \mathcal{B}_l is zoomed out of \mathcal{B}_{l-1} , with the convention that $f_0 = f_{p_0}$ and \mathcal{B}_0 is the prototile space of \mathcal{T} .*

Note that the first patch space in a proper sequence can be chosen to be the prototile space of the tiling (*i.e.* made of uncollared prototiles), all that matters for recovering the hull by inverse limit as shown in the next theorem, is that the next patch spaces are zoomed out of each other, and built out of collared patches (*i.e.* force their borders).

Theorem 5 *The inverse limit of a proper sequence $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ of patch spaces of \mathcal{T} is homeomorphic to the hull of \mathcal{T} :*

$$\Omega \cong \varprojlim(\mathcal{B}_l, f_l).$$

Proof. The homeomorphism is given by the map $\mathbf{p} : \Omega \rightarrow \varprojlim (\mathcal{B}_l, f_l)$, defined by $\mathbf{p}(\omega) = (\mathbf{p}_0(\omega), \mathbf{p}_1(\omega), \dots)$, with inverse $\mathbf{p}^{-1}(x_0, x_1, \dots) = \cap \{\mathbf{p}_l^{-1}(x_l), l \in \mathbb{N}\}$.

The map \mathbf{p} is surjective, because all the \mathbf{p}_l 's are. For the proof of injectivity, consider $\omega, \omega' \in \Omega$ with $\mathbf{p}(\omega) = \mathbf{p}(\omega')$. For simplicity the metric δ defined in remark 1 is used in this proof. Fix $\epsilon > 0$. To prove that $\delta(\omega, \omega') < \epsilon$ it suffices to show that the two tilings agree on a ball of radius $\frac{1}{\epsilon}$. For each l in \mathbb{N} , $\mathbf{p}_l(\omega) = \mathbf{p}_l(\omega')$ in some patch $\pi_{l,j}$ in \mathcal{B}_l . This means that the tilings agree on some translate of the patch $p_{l,j}$ that contains the origin. The definition of patch spaces made up from collared patterns (condition of *forcing the border* in [13]) implies that the tilings agree on the ball $B(0_{\mathbb{R}^d}, r_l)$, where r_l is the parameter of uniform discreteness of the Delone set \mathcal{T}_{p_l} . The assumption that the patch spaces are zoomed out of each other (condition (ii) in definition 11) implies that $r_l > r_{l-1} + r$, where r is the parameter of uniform discreteness of $\mathcal{L}_{\mathcal{T}}$. Hence $r_l > (l+1)r$, and choosing l bigger than the integer part of $\frac{1}{r\epsilon}$ concludes the proof of the injectivity of \mathbf{p} .

Condition (i) in definition 11 implies that for every $l \geq 1$, $\mathbf{p}_l^{-1}(x_l) \subset \mathbf{p}_{l-1}^{-1}(x_{l-1})$. The definition of \mathbf{p}^{-1} then makes sense by compactness of Ω because any finite intersection of some $\mathbf{p}_l^{-1}(x_l)$'s is non empty and closed.

A neighborhood of $x = (x_0, x_1, \dots)$ in $\varprojlim (\mathcal{B}_l, f_l)$ is given by $U_n(x) = \{y = (y_0, y_1, \dots) : y_i = x_i, i \leq n\}$ for some integer n . If ω is a preimage of x , $\mathbf{p}(\omega) = x$, then the preimage of $U_n(x)$ is given by all tilings ω' such that $\mathbf{p}_n(\omega') = \mathbf{p}_n(\omega)$, *i.e.* those tilings agree with ω on some patch p_n around the origin; they form then a neighborhood of ω in Ω . This proves that that \mathbf{p} and \mathbf{p}^{-1} are continuous. \square

Another important construction which allows one to recover the hull by inverse limit has been given by Gähler in an unpublished work as a generalization of the construction of Anderson and Putnam in [1]. Instead of gluing together patches to form the patch spaces \mathcal{B}_p 's, Gähler's construction consists in considering "multicollared prototiles" and the graph that link them according to the local configurations of their representatives in the tiling. This construction keeps track of the combinatorics of the patches that those "multicollared tiles" represent and thus is enough to recover the hull topologically, which is sufficient for topological concerns (for cohomology or K -theory in particular). The construction given here, and taken from [13] where those patch spaces are proven to be branched manifolds, takes more structure into account and the homeomorphism between the hull and the inverse limit of such branched manifolds which is built in [13] is not only a topological conjugacy but yields a conjugacy of the dynamical systems' actions.

4 Cohomology of Tiling Spaces

4.1 Tiling Cohomologies

Various cohomologies for tiling spaces have been used in the literature. For the class of tilings considered here (aperiodic, repetitive with FLC) they are all isomorphic.

First, the Čech cohomology of the hull was introduced for instance in [1] for substitution tilings (and in full generality for tilings on Riemannian manifolds in [63]). Hulls of such tilings are obtained by inverse limit of CW -complexes, and their Čech cohomology are defined by direct limit.

For repetitive tilings with FLC, if a lamination structure is given to the hull as in [13], the cohomology of the hull is defined by direct limit of the simplicial cohomologies of branched manifolds that approximate the hull by inverse limit (a generalisation of theorem 5). This cohomology is isomorphic the Čech cohomology of the hull, using the natural isomorphism between Čech and simplicial cohomologies that holds for such branched manifolds (which are CW -complexes) and passing to direct limit. It has been used to prove that the generators of the d -th cohomology group are in one to one correspondance with invariant ergodic probability measures on the hull.

Another usefull cohomology, the Pattern-Equivariant (PE) cohomology, has been proposed by Kellendonk and Putnam in [42, 43] for real coefficients and then generalized to integer coefficients by Sadun [68]. This cohomology has been used for proving that the Ruelle-Sullivan map (associated with an ergodic invariant probability measure on the hull) from the Čech cohomology of the hull to the exterior algebra of the dual of \mathbb{R}^d is a ring homomorphism. Let \mathcal{T} be a repetitive tiling with FLC. Assume its tiles are compatible CW -complexes, so that it gives a CW -complex decomposition of \mathbb{R}^d . The group of PE n -cochains C_{PE}^n is a subgroup of the group of integer singular n -cochains of \mathbb{R}^d that satisfy the following property: an n -cochain φ is said to be PE if there exists a patch p of \mathcal{T} such that $\varphi(\sigma_1) = \varphi(\sigma_2)$, for 2 n -simplices σ_1, σ_2 with image cells e_1, e_2 , whenever there exists $x \in e_1, y \in e_2$ such that $\mathbf{p}_p(\tau^{-x}\mathcal{T}) = \mathbf{p}_p(\tau^{-y}\mathcal{T})$. The simplicial coboundary of a PE cochain is easily seen to be PE (possibly with respect to a patch of larger radius), and this defines the complex for integer PE cohomology. A PE cochain is a cochain that agree on points which have the same local environments in \mathcal{T} or equivalently is the pull back of a cochain on some patch space \mathcal{B}_p . Hence PE cohomology can be seen as the direct limit of the singular cohomologies of a proper sequence of patch spaces. Using the natural isomorphisms between cellular and Čech cohomologies that holds for those CW -complexes and taking direct limit, the integer PE cohomology turns out to be isomorphic to the Čech cohomology of the hull.

The Pimsner cohomology described in the next section 4.2 is also isomorphic to the Čech cohomology of the hull (see theorem 7).

4.2 The Pimsner cohomology

The definition of a Δ -complex structure is first recalled, following the presentation of Hatcher in his classical book on Algebraic Topology [34] section 2.1.

Given $n + 1$ points v_0, \dots, v_n , in \mathbb{R}^m , $m > n$, that are not colinear, let $[v_0, \dots, v_n]$ denote the n -simplex with vertices v_0, \dots, v_n . Let Δ^n denote the standard n -simplex

$$\Delta^n = \left\{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^n x_i = 1 \text{ and } x_i \geq 0 \text{ for all } i \right\},$$

whose vertices are the unit vectors along the coordinate axis. An ordering of those vertices is specified and this allows to define a canonical linear homeomorphism between Δ^n and any other n -simplex $[v_0, \dots, v_n]$, that preserves the order of the vertices, namely, $(x_0, x_1, \dots, x_n) \mapsto \sum x_i v_i$.

If one of the $n + 1$ vertices of an n -simplex $[v_0, \dots, v_n]$ is deleted, then the remaining n vertices span an $(n - 1)$ -simplex, called a face of $[v_0, \dots, v_n]$. And by convention the vertices of any subsimplex spanned by a subset of the vertices are ordered according to their order in the larger simplex.

The union of all the faces of Δ^n is the boundary of Δ , written $\partial\Delta^n$. The open simplex $\overset{\circ}{\Delta}^n$ is $\Delta^n \setminus \partial\Delta^n$, the interior of Δ^n .

A Δ -complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \rightarrow X$, with n depending on the index α , such that

- (i) The restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$ is injective, and each point of X is in the image of exactly one such restriction.
- (ii) Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \rightarrow X$. The face of Δ^n is identified with Δ^{n-1} by the canonical linear homeomorphism between them that preserves the ordering of the vertices.
- (iii) A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α .

A Δ -complex X can be built as a quotient space of a collection of disjoint simplices by identifying various subsimplices spanned by subsets of the vertices, where the identifications are performed using the canonical linear homeomorphism that preserves orderings of the simplices. It can be shown that X is a Hausdorff space. Then condition (iii) implies that each restriction $\sigma_\alpha|_{\overset{\circ}{\Delta}^n}$ is a homeomorphism onto its image, which is thus an open simplex in X . These open simplices $\sigma_\alpha(\overset{\circ}{\Delta}^n)$ are cells e_α^n of a CW -complex structure on X with the σ_α 's as characteristic maps.

It is now assumed that the tiles of \mathcal{T} are compatible finite Δ -complexes: the intersection of two tiles is a sub- Δ -complex of both. In addition each cell $e_\alpha^n = \sigma_\alpha(\overset{\circ}{\Delta}^n)$ of each tile

is punctured, by say the image under σ_α of the barycenter of Δ^n . Hence \mathcal{T} can be seen as a Δ -complex decomposition of \mathbb{R}^d , and this Δ -complex structure gives a “refinement” of the tiling (each tile being decomposed into the union of the closures of the cells it contains). If \mathcal{T} is the Voronoi tiling of a Delone set its tiles can be split into d -simplices since they are polytopes: this gives a Δ -complex structures for which the maps σ_α are simply affine maps. Abusing notation, the maps $\sigma_\alpha : \Delta^n \rightarrow \mathcal{B}_0$ of the Δ -complex structure of \mathcal{B}_0 , will be called n -simplices on \mathcal{B}_0 .

By construction (definition 9) as a quotient CW -complex, the prototile space \mathcal{B}_0 is a finite Δ -complex.

Definition 13 *The simplicial transversal, denoted Ξ_s , is the subset of Ω consisting of tilings that have the puncture of one of their cells (of one of their tiles) at the origin $0_{\mathbb{R}^d}$.*

Since \mathcal{T} has finitely many prototiles and they have finite CW -complex structures, the set of the punctures of the cells of the tiles of \mathcal{T} is a Delone set. This Delone set will be called the *simplicial Delone set* of \mathcal{T} , and denoted \mathcal{L}_s . The simplicial transversal is thus the transversal of \mathcal{L}_s .

The simplicial transversal is not immediately related to the canonical transversal. Indeed the lift of the puncture of a cell does not belong to the transversal in general, unless this puncture coincides with the puncture of the tile of \mathcal{B}_0 that contains that cell.

The simplicial transversal is the lift of the punctures of the cells in \mathcal{B}_0 . It is partitioned by the lift of the punctures of the n -cells, denoted Ξ_s^n which is the subset of Ω consisting of tilings that have the puncture of an n -cell at the origin. As for the transversal, the simplicial transversal is a Cantor set, and the Ξ_s^n 's give a partition by clopen sets. The ring of continuous integer-valued functions on the simplicial transversal, $C(\Xi_s, \mathbb{Z})$, is thus the direct sum of the $C(\Xi_s^n, \mathbb{Z})$'s for $n = 0, \dots, d$.

Given a simplex σ on \mathcal{B}_0 , let $\Xi_s(\sigma)$ denote the lift of the puncture of its image cell in \mathcal{B}_0 , and χ_σ its characteristic function. $\Xi_s(\sigma)$ is called the *acceptance zone* of σ . Since continuous integer-valued functions on a totally disconnected space are generated by characteristic functions of clopen sets, χ_σ belongs to $C(\Xi_s, \mathbb{Z})$.

Consider an n -simplex σ in \mathcal{B}_0 with image cell e , and τ a face of σ with image cell f . The cells e and f in \mathcal{B}_0 , are contained in some tile τ_j . Viewing e and f as subset of the tile t_j in \mathbb{R}^d , it is possible to define the vector $x_{\sigma\tau}$ that joins the puncture of f to the puncture of e . Notice that since \mathcal{B}_0 is a *flat* branched manifold (remark 2, and [13]), the vector $x_{\sigma\tau}$ is also well defined in \mathcal{B}_0 as a vector in a region containing the image cell e of σ .

Definition 14 *Let σ and τ be simplices on \mathcal{B}_0 . The operator $\theta_{\sigma\tau}$, on $C(\Xi_s, \mathbb{Z})$, is defined by*

$$\theta_{\sigma\tau} = \begin{cases} \chi_\sigma T^{x_{\sigma\tau}} \chi_\tau & \text{if } \tau \subset \partial\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

The function ring of the transversal \mathcal{A}_{Ξ_s} is the ring (finitely) generated by the operators $\theta_{\sigma\tau}$'s and their adjoints $\theta_{\sigma\tau}^ = \chi_\tau T^{-x_{\sigma\tau}} \chi_\sigma$, over all simplices σ and τ on \mathcal{B}_0 .*

The operators $\theta_{\sigma\tau}$'s satisfy the following properties:

$$\theta_{\tau\sigma} = \theta_{\sigma\tau}^*, \quad (1a)$$

$$\theta_{\sigma\tau}\theta_{\tau\sigma} = \chi_\sigma \quad \text{if } \tau \subset \partial\sigma, \quad (1b)$$

$$\sum_{\sigma: \partial\sigma \supset \tau} \theta_{\tau\sigma}\theta_{\sigma\tau} = \chi_\tau. \quad (1c)$$

The $\theta_{\sigma\tau}$'s are thus *partial isometries*. The ring \mathcal{A}_{Ξ_s} is unital and its unit is the characteristic function of the simplicial transversal $1_{\mathcal{A}_{\Xi_s}} = \chi_{\Xi_s}$. The ring of continuous integer-valued functions on the simplicial transversal $C(\Xi_s, \mathbb{Z})$ is a left \mathcal{A}_{Ξ_s} -module. There is a graduation of \mathcal{A}_{Ξ_s} by the $\mathcal{A}_{\Xi_s}^n$, $n = 1, \dots, d$, generated by the $\theta_{\sigma\partial\sigma}$'s for all n -simplices σ .

Remark 3 It is important to notice that the operators $\theta_{\sigma\partial\sigma}$'s are in one to one correspondance with a set of arrows that generate Γ_s^1 , the arrows of the groupoid Γ_s of the simplicial transversal (groupoid of the transversal of the Delone set \mathcal{L}_s , see definition 8). The set of arrows Γ_s^1 is indeed in one-to-one correspondance with the set of vectors joining points of \mathcal{L}_s , and it is generated by the set of vectors joining a point of \mathcal{L}_s to a ‘‘nearest neighbour’’ (a puncture of a face of the cell of \mathcal{T} whose puncture is that point, or of a cell containing it on its boundary). Each vector can be decomposed into a sum of such generating vectors, and each arrow can be decomposed into a composition of such generating arrows (corresponding to those generating vectors). This generating set is finite, since \mathcal{T} has FLC (in particular there are finitely many prototiles), and each of its vectors is an $x_{\sigma\partial\sigma}$ and thus corresponds to a unique operator $\theta_{\sigma\partial\sigma}$.

A ‘‘representation by partial isometries’’ of the set of generators of Γ_s^1 is given by

$$\theta(\gamma) = \chi_{\mathbf{p}_0 \circ \mathbf{p}_0^{-1}(s(\gamma))} \mathbb{T}(\gamma) \chi_{\mathbf{p}_0 \circ \mathbf{p}_0^{-1}(r(\gamma))},$$

where, if $\gamma = (\xi, x)$, $\mathbb{T}(\gamma)$ is the translation operator \mathbb{T}^x .

Let \mathcal{S}_0^n denote the set of n -simplices on \mathcal{B}_0 . The set of singular n -chains on \mathcal{B}_0 , $C_{0,n}$, is the free abelian group with basis \mathcal{S}_0^n , the set of n -simplices on \mathcal{B}_0 .

Definition 15 *The Pimsner cohomology of the hull of \mathcal{T} is the homology of the complex $\{C_P^n, d_P^n\}$, where:*

(i) *the Pimsner cochain groups are the groups of continuous integer valued functions on Ξ_s^n : $C_P^n = C(\Xi_s^n, \mathbb{Z})$ for $n = 0, \dots, d$,*

(ii) *the Pimsner differential, d_P , is the element of \mathcal{A}_{Ξ_s} given by the sum over $n = 1, \dots, d$, of the operators*

$$d_P^n : \begin{cases} C_P^{n-1} \longrightarrow C_P^n \\ d_P^n = \sum_{\sigma \in \mathcal{S}_0^n} \sum_{i=0}^n (-1)^i \theta_{\sigma\partial\sigma} \end{cases} \quad (2)$$

The “simplicial form” of d_P^n makes it clear that $d_P^{n+1} \circ d_P^n = 0$ for $n = 1, \dots, d-1$. We shall call alternatively the Pimsner cohomology of the hull of \mathcal{T} simply the *Pimsner cohomology of \mathcal{T}* .

Remark 4 Thanks to the lamination structure on Ω described in remark 2 the “projection” \mathfrak{p}_0

$$\begin{array}{ccc} \Xi_s & \longrightarrow & \Omega \\ & & \downarrow \mathfrak{p}_0 \\ & & \mathcal{B}_0 \end{array}$$

looks very much like a fibration of Ω with base space \mathcal{B}_0 and “fiber” Ξ_s . However because of the branched structure there are paths in \mathcal{B}_0 that cannot be lifted to any leaf of Ω . Hence this is not a fibration. Nevertheless the result in the present paper (theorem 1) gives a spectral sequence analogous to Serre spectral sequence for a fibration.

Remark 5 This Pimsner cohomology of \mathcal{T} is formally analogous to a cohomology with local coefficients (see [34] chapter 3.H.) but in a more general setting. Let G_x be the group $C(\mathfrak{p}_0^{-1}(x), \mathbb{Z})$ for x in \mathcal{B}_0 . As $\mathfrak{p}_0^{-1}(x)$ is a Cantor set, it is actually its K^0 -group: $G_x = K^0(\mathfrak{p}_0^{-1}(x))$. The family of groups $(G_x)_{x \in \mathcal{B}_0}$ is analogous to a local coefficient system, with the operators $\theta_{\sigma\tau}$ ’s in the role of group isomorphisms between the fibers, but they are not isomorphisms here (as they come from a groupoid). If σ in C_0^n is an n -chain, and φ in C_P^{n-1} is a Pimsner $(n-1)$ -cochain, then the differential

$$d_P^n \varphi(\sigma) = \sum_{i=0}^n (-1)^i \theta_{\sigma \partial_i \sigma} \varphi(\partial_i \sigma),$$

takes the restrictions of φ to the faces $\partial_i \sigma$, which are elements of the groups $G_{x_{\partial_i \sigma}}$ ($x_{\partial_i \sigma}$ being the puncture of the image cell of $\partial_i \sigma$), and pull them to σ with the operators $\theta_{\sigma \partial_i \sigma}$ ’s to get an element of the group G_{x_σ} (x_σ being the puncture of the image cell of σ).

A rigorous formulation of the points in this remark as well as in the previous remark 3 will be investigated in further research.

4.3 An example: Pimsner cohomology in 1-dimension

This section presents the explicit calculation of Pimsner cohomology of some 1-dimensional tilings obtained by “cut and projection”.

Let \mathbb{R}^2 be the usual Euclidean plane with basis vectors e_1 and e_2 . Let \mathbb{Z}^2 be the lattice of points of integer coordinates of the plane. Let $C = [0, 1) \times [0, 1)$ be the fundamental domain of the lattice. Fix an irrational number α in $\mathbb{R}_+ \setminus \mathbb{Q}$. Let L_α be line of slope α (through the origin) and p_α the orthogonal projection onto L_α . Let L_α^\perp be the orthogonal

complement of L_α and $p_\alpha^\perp = 1 - p_\alpha$. Let $W_\alpha = p_\alpha^\perp([0, 1] \times [0, 1])$ be the orthogonal projection of the unit square onto L_α^\perp . Let $\Sigma = \{a \in \mathbb{Z}^2 : p_\alpha^\perp(a) \in W_\alpha\}$. Therefore a lattice point $a = (a_1, a_2)$ in \mathbb{Z}^2 belongs to Σ if and only if $-\alpha \leq a_2 - a_1\alpha < 1$. The projection of Σ onto L_α defines a quasiperiodic tiling \mathcal{T}_α whose tiles' edges are the projections of points of Σ . The tiling \mathcal{T}_α has two types of tiles that correspond to the projections of the vertical and horizontal of the fundamental domain C . In particular \mathcal{T}_α is repetitive with FLC [11, 45, 46, 47]. The transversal of the hull of \mathcal{T}_α can be represented as the projection of Σ onto W_α : $\Xi = p_\alpha^\perp(\Sigma)$, and for computational convenience it will be rescaled and represented on the unit circle \mathbb{S}_α^1 , endowed with a Cantor topology with countable basis given by the closed and open (clopen) intervals $i_{lm} = [l\frac{\alpha}{1-\alpha}, m\frac{\alpha}{1-\alpha}) \pmod 1$ for integers l, m .

In the special case when $\alpha = \sigma = \frac{\sqrt{5}-1}{2}$ is the gold mean, T_σ is the Fibonacci tiling, and is also a substitution tiling with substitution rules $a \rightarrow ab$ and $b \rightarrow a$ [7, 74]. If α is quadratic irrational (a root of a polynomial of degree 2 with integer coefficients) then \mathcal{T}_α is also a substitution tiling [74].

Assuming an orientation of L_α has been chosen, say in the direction of the first quadrant, the tiles are punctured for convenience by their left vertices (not their barycenters). The prototile space $\mathcal{B}_0 = \mathbb{S}^1 \vee \mathbb{S}^1$ is the wedge sum of two circles corresponding to the two prototiles with their vertices identified. The difference with section 4.2 is that the Ξ_s^n 's do not partition Ξ_s since there is only one vertex here, see definition 13. Consequently the transversal and simplicial transversal are identical here. The transversal $\Xi = p_\alpha^\perp(\Sigma)$ is seen as the interval $(\frac{-\alpha}{\sqrt{1+\alpha^2}}, \frac{1}{\sqrt{1+\alpha^2}}]$ on the line L_α^\perp . The interval $(\frac{-\alpha}{\sqrt{1+\alpha^2}}, 0]$ is the acceptance zone Ξ_a of the prototile a whose representatives are the projections of horizontal intervals of Σ , and $(0, \frac{1}{\sqrt{1+\alpha^2}}]$ is the acceptance zone Ξ_b of the prototile whose representatives are the projections of vertical intervals of Σ . The vectors $x_{\sigma\tau}$'s in the definitions of the operators $\theta_{\sigma\tau}$'s are here the projections $x_a = p_\alpha^\perp(e_1) = \frac{-1}{\sqrt{1+\alpha^2}}u_\alpha^\perp$ and $x_b = p_\alpha^\perp(e_2) = \frac{\alpha}{\sqrt{1+\alpha^2}}u_\alpha^\perp$, where $u_\alpha^\perp = \frac{1}{\sqrt{1+\alpha^2}}(-e_1 + \alpha e_2)$ is a unit vector along L_α^\perp . Once rescaled to the unit circle \mathbb{S}_α , the operator T^{x_a} becomes the translation by $\frac{-1}{1+\alpha}$, and T^{x_b} the translation by $\frac{\alpha}{1+\alpha}$, and thus T^{x_a} and T^{x_b} are equal modulo 1. The Pimsner complex reads here simply

$$0 \longrightarrow C(\mathbb{S}_\alpha^1, \mathbb{Z}) \xrightarrow{d_P = \text{id} - \theta_\alpha} C(\mathbb{S}_\alpha^1, \mathbb{Z}) \longrightarrow 0, \quad (3)$$

where θ_α is the translation by $\frac{\alpha}{1-\alpha} \pmod 1$ on \mathbb{S}_α^1 , and is unitary.

Proposition 2 *The Pimsner cohomology of (the hull Ω_α of) \mathcal{T}_α is given by*

$$\begin{cases} H_P^0(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \cong \mathbb{Z} \\ H_P^1(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \cong \mathbb{Z} \oplus \frac{\alpha}{1-\alpha}\mathbb{Z} \end{cases} .$$

Proof. A function $f \in C(\mathbb{S}_\alpha^1, \mathbb{Z})$ reads as a finite sum $f = \sum n\chi_{I_n}$ where n is an integer and χ_{I_n} is the characteristic function of the clopen set $I_n = f^{-1}(n)$. The 0-th cohomology

group is the set of invariant functions under θ_α . Each I_n is a finite disjoint union of base clopen sets i_{lm} 's. Given two base clopens $i_{lm} \subset I_n$ and $i_{l'm'} \subset I_{n'}$, $\theta_\alpha^{m-m'} i_{l'm'} = [(l' + m - m') \frac{\alpha}{1-\alpha}, m \frac{\alpha}{1-\alpha}] \pmod 1$, so that $i_{lm} \cap \theta_\alpha^{m-m'} i_{l'm'} \neq \emptyset$. Hence if f is invariant under θ_α , then n must equal n' and therefore f must be constant.

The calculation of the 1-th cohomology group, the group of coinvariants under θ_α , relies upon the fact that any function $f \in C(\mathbb{S}_\alpha^1, \mathbb{Z})$ can be written as $n_f \chi_{i_{01}} + m_f \chi_{\mathbb{S}_\alpha^1}$ modulo $(1 - \theta_\alpha)C(\mathbb{S}_\alpha^1, \mathbb{Z})$, for some integers n_f, m_f , that are uniquely determined by the class of f . This technical point is tedious but elementary (it uses an encoding of the real numbers from the partial fraction decomposition of $\frac{\alpha}{1-\alpha}$). The integration, $\int_{\mathbb{S}_\alpha^1}$, then provides a group isomorphism with $\mathbb{Z} \oplus \frac{\alpha}{1-\alpha} \mathbb{Z}$. \square

Remark 6 (i) If α is an irrational quadratic number, then the 1-th cohomology group is isomorphic to the quotient ring of $\mathbb{Z}[x]$ by the ideal generated by the minimal polynomial for α . In the case of the Fibonacci tiling, where $\alpha = \sigma$ is the gold mean, then $H_P^1(\mathcal{B}_0; \mathcal{C}_{\mathbb{S}_\alpha^1}) \cong \mathbb{Z}[x]/(x^2 + x - 1)$.

(ii) The above calculation yields also the K -groups of the tiling space:

$$\begin{cases} K^0(\Omega) \cong \check{H}^0(\Omega_\alpha; \mathbb{Z}) \cong H_P^0(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \\ K^1(\Omega) \cong \check{H}^1(\Omega_\alpha; \mathbb{Z}) \cong H_P^1(\mathcal{B}_0; C(\mathbb{S}_\alpha^1, \mathbb{Z})) \end{cases}$$

The isomorphism between Čech and Pimsner cohomologies for hull of tilings is proven in theorem 7. The isomorphisms between K -theory and cohomology of the hull comes from the natural isomorphisms between K -theory and cohomology of CW -complexes of dimension less than 2 (a proof can be found in [1], proposition 6.2), and the fact that Ω_α is the inverse limit of such space (theorem 5).

(iii) With the previous remark, the Pimsner complex (3) can be viewed as the Pimsner-Voiculescu exact sequence [55] for the K -theory of the C^* -algebra $C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}$:

$$\begin{array}{ccccc} K_0(C(\mathbb{S}_\alpha^1)) & \xrightarrow{\text{id}-\theta_\alpha} & K_0(C(\mathbb{S}_\alpha^1)) & \longrightarrow & K_0(C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}) & \longleftarrow & K_1(C(\mathbb{S}_\alpha^1)) & \xleftarrow{\text{id}-\theta_\alpha} & K_1(\mathbb{S}_\alpha^1) \end{array} .$$

Indeed $C(\mathbb{S}_\alpha^1) \rtimes_{\theta_\alpha} \mathbb{Z}$ is the C^* -algebra of the groupoid of the transversal, which is Morita equivalent to the C^* -algebra of the hull of \mathcal{T}_α [6]. Also $K_0(C(\mathbb{S}_\alpha^1)) \cong C(\mathbb{S}_\alpha^1, \mathbb{Z})$ and $K_1(C(\mathbb{S}_\alpha^1)) \cong 0$ [8].

5 Proof of theorem 1

The proof of theorem 1 follows from the following two theorems that will be proved separately.

Let \mathcal{T} be an aperiodic and repetitive tiling of \mathbb{R}^d with FLC (definition 3). Let Ω be its hull (definition 4) and Ξ_s its simplicial transversal (definition 13).

Theorem 6 *There is a spectral sequence that converges to the K-theory of the C^* -algebra of the hull*

$$E_2^{rs} \Rightarrow K_{r+s+d}(C(\Omega) \rtimes \mathbb{R}^d),$$

and whose page-2 is isomorphic to the integer Čech cohomology of the hull

$$E_2^{rs} \cong \begin{cases} \check{H}^r(\Omega; \mathbb{Z}) & s \text{ even,} \\ 0 & s \text{ odd.} \end{cases}$$

Theorem 7 *The integer Čech cohomology of the hull is isomorphic to the Pimsner cohomology of the hull*

$$\check{H}^*(\Omega; \mathbb{Z}) \cong H_P^*(\mathcal{B}_0; C(\Xi_s, \mathbb{Z})).$$

The proofs rely essentially on theorem 5 which expresses Ω as an inverse limit of patch spaces \mathcal{B}_p 's (definition 10). Some abstract results on spectral sequences are also needed to establish that the page-2 of the spectral sequence of theorem 6 is the cohomology of the hull, in particular some properties of the Atiyah-Hirzebruch spectral sequence [2]. For the convenience of the reader, those known results are grouped together and proven in a separate section, section 6.

5.1 Proof of theorem 7

The proof of theorem 7 follows from propositions 3 and 4 below. First a Pimsner cohomology of \mathcal{B}_p , written $H_P^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$, is defined. It is proven to be isomorphic to the simplicial cohomology of \mathcal{B}_p in proposition 3, and then proposition 4 establishes that the Pimsner cohomology of the hull is isomorphic to the direct limit of the Pimsner cohomologies of a proper sequence of patch spaces.

Let \mathcal{B}_p^n be the n -skeleton of \mathcal{B}_p , \mathcal{S}_p^n the set of its n -simplices, and \mathcal{S}_p the (disjoint) union of the \mathcal{S}_p^n 's. The set of singular n -chains on \mathcal{B}_p , $C_{p,n}$, is the free abelian group with basis \mathcal{S}_p^n , the set of n -simplices on \mathcal{B}_p .

Remark 7 The map $f_p : \mathcal{B}_p \rightarrow \mathcal{B}_0$ preserves the orientations of the images of the simplices. Let $f_{p\#} : C_{p,n} \rightarrow C_{0,n}$ denote the map induced by f_p on the singular chain groups. If a simplex σ_p on \mathcal{B}_p is in the preimage of a simplex σ on \mathcal{B}_0 , $f_{p\#}(\sigma_p) = \sigma$, then the ordering of their vertices is the same.

Given a simplex σ_p on \mathcal{B}_p , let $\Xi_{p,s}(\sigma_p)$ denote the lift of the puncture of its image in \mathcal{B}_p and χ_{σ_p} its characteristic function. $\Xi_{p,s}(\sigma_p)$ is called the *acceptance zone* of σ_p ; it is a clopen set in Ξ_s .

Lemma 1 *Given a simplex σ on \mathcal{B}_0 , its acceptance zone is partitioned by the acceptance zones of its preimages σ_p 's on \mathcal{B}_p*

$$\Xi_s(\sigma) = \coprod_{\sigma_p \in f_p^{-1}(\sigma)} \Xi_{p,s}(\sigma_p).$$

Proof. The union of the $\Xi_{p,s}(\sigma_p)$'s is equal to $\Xi_s(\sigma)$, because each lift of the image cell of a simplex σ_p corresponds to the lift of the image cell of σ that has a given local configuration (it is contained in some patch of \mathcal{T} or some tile of \mathcal{T}_p , see section 3.4.). If σ_p and σ'_p are two distinct simplices in $f_p^{-1}(\sigma)$, then $\mathfrak{p}_p(\Xi_{p,s}(\sigma_p)) \cap \mathfrak{p}_p(\Xi_{p,s}(\sigma'_p))$ is empty, and thus so is $\Xi_{p,s}(\sigma_p) \cap \Xi_{p,s}(\sigma'_p)$. \square

The simplices in $f_p^{-1}(\sigma)$ are thus in one to one correspondance with the set of atoms of the partition of $\Xi_s(\sigma)$ by the $\Xi_{p,s}(\sigma_p)$'s, and the union over σ in \mathcal{S}_0^n of the $f_p^{-1}(\sigma)$'s is just \mathcal{S}_p^n .

The simplicial n -cochain group C_p^n is $Hom(C_{p,n}, \mathbb{Z})$, the dual of the n -chain group $C_{p,n}$. It is represented faithfully on the group $C(\Xi_s, \mathbb{Z})$ of continuous integer valued functions on the simplicial transversal by

$$\rho_{p,n} : \begin{cases} C_p^n & \longrightarrow C(\Xi_s^n, \mathbb{Z}) \\ \psi & \longmapsto \sum_{\sigma_p \in \mathcal{S}_p^n} \psi(\sigma_p) \chi_{\sigma_p} \end{cases} . \quad (4)$$

The image of $\rho_{p,n}$ will be written $C(\Sigma_p^n, \mathbb{Z})$, to remind the reader that this consists of functions on the “discrete transversal” Σ_p^n which corresponds to the atoms of the partitions of the $\Xi_s(\sigma)$'s by the $\Xi_{p,s}(\sigma_p)$'s for $\sigma_p \in f_p^{-1}(\sigma)$. The representation $\rho_{p,n}$ is a group isomorphism onto its image $C(\Sigma_p^n, \mathbb{Z})$, its inverse is defined as follows: given $\varphi = \sum_{\sigma_p \in \mathcal{S}_p^n} \varphi_{\sigma_p} \chi_{\sigma_p}$, where φ_{σ_p} is an integer, $\rho_p^{n-1}(\varphi)$ is the group homomorphism from $C_{p,n}$ to \mathbb{Z} whose value on the basis simplex σ_p is φ_{σ_p} .

Consider an n -simplex σ_p in \mathcal{B}_p . Its image cell e in \mathcal{B}_p is contained in some tile π_j . Viewing e_p as a subset of the patch p_j in \mathbb{R}^d , it is possible to define the vector $x_{\sigma_p, \partial_i \sigma_p}$, for some i in $1, \dots, n$, that joins the puncture of the i -th face $\partial_i e_p$ (the image cell of $\partial_i \sigma_p$ in \mathcal{B}_p) to the puncture of e_p . As a consequence of remark 7, those vectors $x_{\sigma_p, \partial_i \sigma_p}$'s are identical for all σ_p 's in the preimage of a simplex σ on \mathcal{B}_0 , and equal to the vector $x_{\sigma, \partial_i \sigma}$ which defines the operator $\theta_{\sigma, \partial_i \sigma}$ in definition 14. By analogy, let $\theta_{\sigma_p, \partial_i \sigma_p}$ be the operator $\chi_{\sigma_p} \mathbb{T}^{x_{\sigma_p, \partial_i \sigma_p}} \chi_{\partial_i \sigma_p}$. With the relation $\mathbb{T}^a \chi_\Lambda = \chi_{\mathbb{T}^a \Lambda} \mathbb{T}^a$ for $\Lambda \subset \Omega$, $a \in \mathbb{R}^d$, and lemma 1 it is easily seen that $\theta_{\sigma, \partial_i \sigma}$ is the sum of the $\theta_{\sigma_p, \partial_i \sigma_p}$'s over all $\sigma_p \in f_p^{-1}(\sigma)$. Hence Pimsner differential given in equation (2) can be written

$$d_P^n = \sum_{\sigma_p \in \mathcal{S}_p^n} \sum_{i=0}^n (-1)^i \theta_{\sigma_p} \partial_{\sigma_p} \quad (5)$$

and is then well defined as a differential from $C(\Sigma_p^{n-1}, \mathbb{Z})$ to $C(\Sigma_p^n, \mathbb{Z})$.

Definition 16 Let $C_P^n(p) = C(\Sigma_p^n, \mathbb{Z})$, for $n = 0, \dots, d$. The Pimsner cohomology of the patch space \mathcal{B}_p , denoted $H_P^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$, is the homology of the complex $\{C_P^n(p), d_P^n\}$.

The notation $H_P^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z}))$ requires some comments. The map $f_p : \mathcal{B}_p \rightarrow \mathcal{B}_0$ is a ‘‘branched covering’’ with discrete ‘‘fibers’’, the \mathcal{S}_p^n ’s, that correspond to the ‘‘discrete transversals’’ Σ_p^n ’s. In analogy with remark 5, the Pimsner cohomology of \mathcal{B}_p is analogous to a cohomology of the base space \mathcal{B}_0 with local coefficients in the K -theory of the ‘‘fiber’’.

Proposition 3 The Pimsner cohomology of the patch space \mathcal{B}_p is isomorphic to its integer simplicial cohomology: $H_P^*(\mathcal{B}_0; C(\Sigma_p, \mathbb{Z})) \cong H^*(\mathcal{B}_p; \mathbb{Z})$.

Proof. As mentioned earlier, $\rho_{p,n}$ in equation (4) defines a group isomorphism between the simplicial cochain group C_P^n and the Pimsner cochain group $C_P^n(p)$. Here they are both isomorphic to the direct sum $\mathbb{Z}^{|\mathcal{S}_p^n|}$, where $|\mathcal{S}_p^n|$ is the cardinality of \mathcal{S}_p^n , i.e. the number of n simplices on \mathcal{B}_p .

The differential of $\varphi \in C_P^{n-1}(p)$, evaluated on an n -simplex σ_p , reads using (5) $d_P^n \varphi(\sigma_p) = \sum_{\sigma_p \in \mathcal{S}_p^n} (d_P^n \varphi)_{\sigma_p} \chi_{\sigma_p}$, with $(d_P^n \varphi)_{\sigma_p} = \sum_{i=1}^n (-1)^i \varphi_{\partial_i \sigma_p}$. On the other hand the simplicial differential of $\psi \in C_p^{n-1}$ reads $\delta^n \psi(\sigma_p) = \sum_{i=1}^n (-1)^i \psi(\partial_i \sigma_p)$, and therefore $d_P^n \circ \rho_{p,n-1} = \rho_{p,n} \circ \delta^n$, $n = 1, \dots, d$. Hence the $\rho_{p,n}$ ’s give a chain map and conjugate the differentials, and thus yield isomorphisms $\rho_{p,n}^*$ ’s between the n -th cohomology groups. \square

Lemma 2 Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . The following holds: $\Xi_s \cong \varprojlim (\mathcal{S}_l, f_l)$, and $C(\Xi_s, \mathbb{Z}) \cong \varinjlim (C(\Sigma_l, \mathbb{Z}), f^l)$, where f^l is the dual map to f_l .

Proof. This is a straightforward consequence of theorem 5. \square

Proposition 4 Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . There is an isomorphism:

$$H_P^*(\mathcal{B}_0; C(\Xi_s, \mathbb{Z})) \cong \varinjlim (H_P^*(\mathcal{B}_0; C(\Sigma_l, \mathbb{Z})), f_l^*) .$$

Proof. By the previous lemma 2 the cochain groups C_P^n of the hull are the direct limits of the cochain groups $C_P^n(l)$ of the patch spaces \mathcal{B}_l . Let $f_l^\# : C_P^n(l) \rightarrow C_P^n(l)$ denote the map induced by f_l on the Pimsner cochain groups. Since the Pimsner differential d_P is the same for the complexes of each patch space \mathcal{B}_l , it suffices to check that the following diagram is commutative

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_P^{n-1}(l) & \xrightarrow{d_P^n} & C_P^n(l) & \longrightarrow & \dots \\ & & \downarrow f_l^\# & & \downarrow f_l^\# & & \\ \dots & \longrightarrow & C_P^{n-1}(l+1) & \xrightarrow{d_P^n} & C_P^n(l+1) & \longrightarrow & \dots \end{array}$$

which is straightforward using the relation $(f_l^\# \varphi)_{\partial_j \sigma_l} = \varphi_{f_l \# (\partial_j \sigma_l)}$. \square

Proof of theorem 7. By standard results in algebraic topology [34, 72], the Čech cohomology of the hull is isomorphic to the direct limit of the Čech cohomologies of the patch spaces \mathcal{B}_i , $\check{H}^*(\Omega; \mathbb{Z}) \cong \varinjlim (\check{H}^*(\mathcal{B}_i; \mathbb{Z}), f_l^*)$. By the natural isomorphism between Čech and simplicial cohomologies for CW-complexes [72], $\check{H}^*(\mathcal{B}_i; \mathbb{Z}) \cong H^*(\mathcal{B}_i; \mathbb{Z})$, and by proposition 3, $H^*(\mathcal{B}_i; \mathbb{Z}) \cong H_P^*(\mathcal{B}_0; C(\Sigma_i, \mathbb{Z}))$, so that $\check{H}^*(\mathcal{B}_i; \mathbb{Z}) \cong H_P^*(\mathcal{B}_0; C(\Sigma_i, \mathbb{Z}))$. By proposition 4, the direct limit of the $H_P^*(\mathcal{B}_0; C(\Sigma_i, \mathbb{Z}))$'s is the Pimsner cohomology of the hull. Therefore the integer Čech cohomology of the hull is isomorphic to the Pimsner cohomology of the hull: $\check{H}^*(\Omega; \mathbb{Z}) \cong H_P^*(\mathcal{B}_0; C(\Xi_s, \mathbb{Z}))$. \square

5.2 Proof of theorem 6

The proof of theorem 6 follows from propositions 5 and 6 below. First by Thom-Connes isomorphism [21], $K_{*+d}(C(\Omega) \rtimes \mathbb{R}^d) \cong K_*(C(\Omega))$, and therefore it suffices to build a spectral sequence that converges to the K -theory of the C^* -algebra $C(\Omega)$. This is done by constructing a Schochet spectral sequence [70], $\{E_*^{r,s}\}$, associated with an appropriate filtration of $C(\Omega)$. It is shown in proposition 5 that this spectral sequence is the direct limit of spectral sequences, $\{E_*^{r,s}(l)\}$, for the K -theory of the C^* -algebras $C(\mathcal{B}_l)$ of continuous functions on the patch spaces of a proper sequence. Then $\{E_*^{r,s}(l)\}$ is shown to be isomorphic to the Atiyah-Hirzebruch spectral sequence [2] for the topological K -theory of \mathcal{B}_l in proposition 6.

For basic definitions, terminology and results on spectral sequences, the reader is referred to [50]. In section 6 some technical results on exact couples used here are provided.

For $s = 0, \dots, d$, let $\Omega^s = \mathfrak{p}_0^{-1}(\mathcal{B}_0^s)$ be the lift of the s -skeleton of the prototile space, and let $I_s = C_0(\Omega \setminus \Omega^s)$. I_s is a closed two-sided ideal of $C(\Omega)$, it consists of functions that vanish on the faces of dimension s of the boxes of the hull (see section 3.3). This gives a filtration of $C(\Omega)$:

$$\{0\} = I_d \hookrightarrow I_{d-1} \hookrightarrow \dots \hookrightarrow I_0 \hookrightarrow I_{-1} = C(\Omega). \quad (6)$$

Let $Q_s = I_{s-1}/I_s$, which is isomorphic to $C_0(\Omega^s \setminus \Omega^{s-1})$. Let $K(I)$ and $K(Q)$ be the respective direct sums of the $K_\epsilon(I_s)$ and $K_\epsilon(Q_s)$ over $\epsilon = 0, 1$, and $s = 0, \dots, d$. The short exact sequences $0 \rightarrow I_s \xrightarrow{i_s} I_{s-1} \xrightarrow{\pi_s} Q_s \rightarrow 0$, lead, through long exact sequences in K -theory, to the exact couple $\mathfrak{P} = (K(I), K(Q), i, \pi, \partial)$, where i and π are the induced maps and ∂ the boundary map in K -theory. Its associated Schochet spectral sequence [70], $\{E_*^{r,s}\}$, converges to the K -theory of $C(\Omega)$:

$$\begin{cases} E_1^{r,s} & \Rightarrow & K_{r+s}(C(\Omega)) \\ E_1^{r,s} & = & K_{r+s}(Q_s) \end{cases} . \quad (7)$$

Let \mathcal{B}_p be a patch space associated with a pattern \hat{p} of \mathcal{T} . Consider the filtration of the C^* -algebra $C(\mathcal{B}_p)$ by the closed two sided-ideals $I_s(p) = C_0(\mathcal{B}_p \setminus \mathcal{B}_p^s)$:

$$\{0\} = I_d(p) \hookrightarrow I_{d-1}(p) \hookrightarrow \cdots \hookrightarrow I_0(p) \hookrightarrow I_{-1}(p) = C(\mathcal{B}_p). \quad (8)$$

Let $Q_s(p) = I_{s-1}(p)/I_s(p)$, which is isomorphic to $C_0(\mathcal{B}_p^s \setminus \mathcal{B}_p^{s-1})$. Let $K(I(p))$ and $K(Q(p))$ be the respective direct sums of the $K_\epsilon(I_s(p))$ and $K_\epsilon(Q_s(p))$ over $\epsilon = 0, 1$, and $s = 0, \dots, d$.

The short sequences $0 \rightarrow I_s(p) \xrightarrow{i_{p,s}} I_{s-1}(p) \xrightarrow{\pi_{p,s}} Q_s(p) \rightarrow 0$, lead to the exact couple $\mathfrak{P}(p) = (K(I(p)), K(Q(p)), i_p, \pi_p, \partial)$, and its associated Schochet spectral sequence [70], $\{E_*^{r,s}(p)\}$, converges to the K -theory of $C(\mathcal{B}_p)$:

$$\begin{cases} E_1^{r,s}(p) & \Rightarrow & K_{r+s}(C(\mathcal{B}_p)) \\ E_1^{r,s}(p) & = & K_{r+s}(Q_s(p)) \end{cases} . \quad (9)$$

Proposition 5 *Given a proper sequence $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ of patch spaces of \mathcal{T} , the following holds:*

$$\{E_*^{r,s}\} \cong \varinjlim (\{E_*^{r,s}(l)\}, f_{l*}) ,$$

where $\{E_*^{r,s}(l)\}$ is the Schochet spectral sequence (9) for \mathcal{B}_l corresponding to the patch $p = p_l$.

Proof. The map $f_l : \mathcal{B}_l \rightarrow \mathcal{B}_{l+1}$ induces a morphism of exact couples from $f_{l*} : \mathfrak{P}(l) \rightarrow \mathfrak{P}(l+1)$. Indeed, consider the following diagram:

$$\begin{array}{ccccccc} K_\epsilon(I_s(l)) & \xrightarrow{i_{l,s}} & K_\epsilon(I_{s-1}(l)) & \xrightarrow{\pi_{l,s}} & K_\epsilon(Q_s(l)) & \xrightarrow{\partial} & K_{\epsilon+1}(I_s(l)) \\ \downarrow f_{l*} & & \downarrow f_{l*} & & \downarrow f_{l*} & & \downarrow f_{l*} \\ K_\epsilon(I_s(l+1)) & \xrightarrow{i_{l+1,s}} & K_\epsilon(I_{s-1}(l+1)) & \xrightarrow{\pi_{l+1,s}} & K_\epsilon(Q_s(l+1)) & \xrightarrow{\partial} & K_{\epsilon+1}(I_s(l+1)) \end{array}$$

The left and middle squares are easily seen to be commutative. To check the commutativity of the right square, recall that given a short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/J \rightarrow 0$ where A is a C^* -algebra and J a closed two-sided ideal, the boundary map of an element $[x] \in K_\epsilon(A/J)$ is computed via a lift $z \in A \otimes \mathcal{K}$ of x . Let $[x] \in K_\epsilon(Q_s(l))$. If $z \in I_{s-1}(l) \otimes \mathcal{K}$ is a lift of x , then $f_{l*}z \in I_{s-1}(l+1) \otimes \mathcal{K}$ is a lift of $f_{l*}x$ and the commutativity of the right square follows.

As a consequence of theorem 5, $I \cong \varinjlim \{I(l), f_l^\#\}$ and $Q \cong \varinjlim \{Q(l), f_l^\#\}$, therefore $K(I) \cong \varinjlim \{K(I(l)), f_l^*\}$ and $K(Q) \cong \varinjlim \{K(Q(l)), f_l^*\}$. Hence $\{\mathfrak{P}(l), f_l^*\}$ is a direct system of exact couples and by lemma 4 $\mathfrak{P} \cong \varinjlim \{\mathfrak{P}(l), f_l^*\}$, and the same result on their associated spectral sequences follows by corollary 3. \square

Given a finite CW -complex X , the Atiyah-Hirzebruch spectral sequence [2] for the topological K -theory of X is a particular case of Serre spectral sequence [25, 50] for the trivial

fibration of X by itself with fiber a point:

$$\begin{cases} E_{AH,2}^{r,s} & \Rightarrow K^{r+s}(X) \\ E_{AH,2}^{r,s}(p) & \cong H^r(X; K^s(\cdot)) \end{cases} , \quad (10)$$

where \cdot denotes a point, so $K^s(\cdot) \cong \mathbb{Z}$ if s is even, and 0 if s is odd.

It is defined on page-1 by $E_{AH,2}^{r,s} = K^r(X^s, X^{s-1})$ in [2], and then proven that the page-2 is isomorphic to the cellular cohomology of X . If X is a locally compact Hausdorff space, the spectral sequence can be rewritten algebraically, using the isomorphisms $K^*(Y) \cong K_*(C_0(Y))$ and $K^*(Y, Z) \cong K_*(C_0(Y)/C_0(Z))$ for locally compact Hausdorff spaces Y, Z . Consider the Schochet spectral sequence [70] for the K -theory of the C^* -algebra $C_0(X)$ associated with its filtration by the ideals $I_s = C_0(X, X^s)$ of functions vanishing on the s -skeleton. Then the spectral sequence associated with the cofiltration of $C_0(X)$ by the ideals $F_s = C_0(X)/I_s \cong C_0(X^s)$ turns out to be this algebraic form of Atiyah-Hirzebruch spectral sequence.

Proposition 6 *The Schochet spectral sequence $\{E_*^{r,s}(p)\}$ for the K -theory of the C^* -algebra $C(\mathcal{B}_p)$ is isomorphic to the Atiyah-Hirzebruch spectral sequence for the topological K -theory of \mathcal{B}_p .*

Proof. By theorem 9 (section 6) the Schochet spectral sequences built from the filtration of $A(p) = C(\mathcal{B}_p)$ by the ideals $I_s(p)$, namely $\{E_*^{r,s}(p)\}$, and from the cofiltration of $A(p)$ by the quotients $F_s(p) = A(p)/I_s(p) \cong C_0(\mathcal{B}_p^s)$, are isomorphic. As remarked above this last spectral sequence is nothing but the Atiyah-Hirzebruch spectral sequence. \square

Proof of theorem 6. Let $\{\mathcal{B}_l, f_l\}_{l \in \mathbb{N}}$ be a proper sequence of patch spaces of \mathcal{T} . By proposition 6, the page-2 of Schochet spectral sequence $\{E_*^{r,s}(l)\}$ is isomorphic to the simplicial cohomology of \mathcal{B}_l : $E_2^{r,s}(l) \cong \check{H}^r(\mathcal{B}_l; \mathbb{Z})$ for s even and 0 for s odd. By the natural isomorphism between simplicial and Čech cohomologies for CW -complexes [72], it follows that $E_2^{r,s}(l) \cong \check{H}^r(\mathcal{B}_l; \mathbb{Z})$ for s even and 0 for s odd. By standard results in algebraic topology [34, 72] $\check{H}^*(\Omega; \mathbb{Z}) \cong \varinjlim (\check{H}^*(\mathcal{B}_l; \mathbb{Z}), f_l^*)$, and therefore by proposition 5 the page-2 of Schochet spectral sequence for the K -theory of $C(\Omega)$ is isomorphic to the integer Čech cohomology of the hull: $E_2^{r,s} \cong \check{H}^r(\Omega; \mathbb{Z})$ for s even and 0 for s odd. \square

6 Appendix: a reminder on spectral sequences

Some technical results about exact couples are recalled here. A few elementary definitions are given to set the terminology. The original reference on exact couples is the work of Massey [49]. The link with spectral sequences is only mentioned, and the reader is referred to [25, 50] for further details.

The formalism allows to state corollary 2 which leads to the isomorphism between Schochet spectral sequences associated with the filtration of a C^* -algebra and with its corresponding cofiltration in theorem 9.

An *exact couple* is a family $T = (D, E, i, j, k)$ where D and E are abelian groups and i, j, k are group homomorphisms making the following triangle exact

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ k \swarrow & & \searrow j \\ & E & \end{array} \quad (11)$$

In more generality, D and E can be graded modules over a ring, and i, j, k module maps of various degrees. A morphism α between two exact couples T and T' is a pair of group homomorphisms (α_D, α_E) making the following diagram commutative

$$\begin{array}{ccccccc} D & \xrightarrow{i} & D & \xrightarrow{j} & E & \xrightarrow{k} & D \\ \downarrow \alpha_D & & \downarrow \alpha_D & & \downarrow \alpha_E & & \downarrow \alpha_D \\ D' & \xrightarrow{i'} & D' & \xrightarrow{j'} & E' & \xrightarrow{k'} & D' \end{array} \quad (12)$$

The composition map $d = j \circ k : E \rightarrow E$ is called the *differential* of the exact couple. Since (11) is exact, it follows that $d^2 = j \circ (k \circ j) \circ k = 0$. Let then $H_d(E)$ be the homology of the complex $E \xrightarrow{d} E$. Then the following theorem holds.

Theorem 8 (i) *There is a derived exact couple*

$$\begin{array}{ccc} D^{(1)} = i(D) & \xrightarrow{i^{(1)}} & D^{(1)} = i(D) \\ k^{(1)} \swarrow & & \searrow j^{(1)} \\ & E^{(1)} = H_d(E) & \end{array} \quad (13)$$

defined by $i^{(1)} = i|_{i(D)}$, $j^{(1)}(i(x)) = j(x) + \text{Im}(d)$ and $k^{(1)}(e + \text{Im}(d)) = k(e)$.

(ii) *The derivation $T \rightarrow T^{(1)}$ is a functor on the category of exact couples (with morphisms of exact couples).*

Proof. This follows from the commutativity of diagram (12) and remark 8 below. \square

The n -th iterated derived couple of T is denoted $T^{(n)} = (D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)})$.

Remark 8 A morphism of exact couples $\alpha : T \rightarrow T'$ induces derived morphisms $\alpha^{(n)}$ between the derived couples: $\alpha_D^{(n)} = \alpha_D^{(n-1)}|_{D^{(n)}}$ is the restriction of $\alpha_D^{(n-1)}$ to $D^{(n)} = i^{(n-1)}(D^{(n-1)})$, and $\alpha_E^{(n)} = (\alpha_E^{(n-1)})_*$ is the induced map of $\alpha_E^{(n-1)}$ on homology. Also α_E conjugates the differentials, namely $\alpha_E^{(n)} \circ d^{(n)} = d'^{(n)} \circ \alpha_E^{(n)}$ for all n .

Definition 17 (i) *An exact couple $T = (D, E, i, j, k)$ is said to be trivial if $E = 0$.*

(ii) Two exact couples T and T' are said to be equivalent if there is a morphism $\alpha : T \rightarrow T'$ such that α_E is an isomorphism.

Definition 18 An exact couple is said to converge whenever there is an $L \in \mathbb{N}$ such that for $l \geq L$ the l -th derived couples are trivial.

The *spectral sequences* considered here come from exact couples. Given an exact couple (E, D, i, j, k) , its associated spectral sequence is the family $(E_l, d_l)_{l \in \mathbb{N}}$ where $E_l = E^{(l)}$ is the derived E -term, and called the E_l -page or simply *page- l* of the spectral sequence, and $d_l = d^{(l)}$ is its differential. As noted above those E -terms are generally (bi)graded modules. Such spectral sequences are said to converge whenever the exact couples they come from converge. A morphism of spectral sequence $\beta : (E_l, d_l)_{l \in \mathbb{N}} \rightarrow (E'_l, d'_l)_{l \in \mathbb{N}}$ is given by the α_E -morphism of the exact couples, and can be seen as a sequence of *chain maps* $\beta_l : E_l \rightarrow E'_l$ for each l , *i.e.* commute with the differentials: $d_l \beta_l = \beta_l d'_l$.

Remark 9 (i) If an exact couple is trivial, then it is of the form $(D, 0, \text{id}, 0, 0)$.

(ii) A trivial exact couple is identical to its derived couple.

(iii) An equivalence between two exact couples is equivalent to an isomorphism between their associated spectral sequences.

The first two remarks follow immediately from the above definition and the commutativity of diagram (12). Concerning the third, by Remark 8, if $\alpha_E^{(n)} : E^{(n)} \rightarrow E'^{(n)}$ is an isomorphism then it conjugates the differentials, namely $d'^{(n)} = \alpha_E^{(n)} d^{(n)} (\alpha_E^{(n)})^{-1}$, and thus $\alpha_E^{(n+1)} = (\alpha_E^{(n)})_*$ is an isomorphism that conjugates the $(n+1)$ -th differentials. By induction we therefore get isomorphisms between the $E^{(n)}$ -pages of the associated spectral sequences that conjugate the differentials.

Given a morphism of exact couples $\alpha : T \rightarrow T'$, its kernel $\text{Ker} \alpha = (\text{Ker} \alpha_D, \text{Ker} \alpha_E, i, j, k)$ and image $\text{Im} \alpha = (\text{Im} \alpha_D, \text{Im} \alpha_D, i', j', k')$ are 3-periodic complexes (by commutativity of diagram (12)), but no longer exact couples in general.

Lemma 3 Given an exact sequence of exact couples

$$\cdots \rightarrow T_{m-1} \xrightarrow{\alpha_{m-1}} T_m \xrightarrow{\alpha_m} T_{m+1} \xrightarrow{\alpha_{m+1}} T_{m+2} \rightarrow \cdots,$$

that is $\text{Im} \alpha_{m-1} = \text{Ker} \alpha_m$ for all m , the following holds: if T_{m-1} and T_{m+2} are trivial, then T_m and T_{m+1} are equivalent.

Proof. Using Remark 9 the couples can be rewritten $T_{m-1} = (D_{m-1}, 0, \text{id}, 0, 0)$ and $T_{m+2} = (D_{m+2}, 0, \text{id}, 0, 0)$, and therefore $\alpha_{m-1} = (\alpha_{D_{m-1}}, 0)$, and $\alpha_{m+1} = (\alpha_{D_{m+1}}, 0)$. The exact sequence for the E terms then reads $\cdots \rightarrow 0 \rightarrow E_m \xrightarrow{\alpha_{E_m}} E_{m+1} \rightarrow 0 \rightarrow \cdots$, and therefore α_{E_m} is an isomorphism and gives an equivalence between T_m and T_{m+1} . \square

Corollary 2 *Given an exact triangle of exact couples, one of which is trivial, then the two others are equivalent.*

Definition 19 *A direct system of exact couples $\{T_l, \alpha_{lm}\}_I$, is given by an directed set I , and a family of exact couples T_l 's and morphisms of exact couples $\alpha_{lm} : T_l \rightarrow T_m$ for $l \leq m$ (with α_{ll} the identity), such that given $l, m \in I$ there exists $n \in I$, $n \geq l, m$ with $\alpha_{ln} = \alpha_{mn} \circ \alpha_{lm}$.*

If T_l is written $(D_l, E_l, i_l, j_l, k_l)$, the definition of a direct system of associated spectral sequences is given similarly by keeping only the data of the E_l -terms, their differentials $j_l \circ k_l$ and the morphisms α_{E_l} .

Let R be a ring. Recall that the direct limit of a directed system of R -modules, $\{M_l, \sigma_{lm}\}_I$ is given by the quotient of the direct product $\bigoplus_I M_l$ by the R -module generated by all elements of the form $a_l - \sigma_{lm}(a_l)$ for $a_l \in M_l$, where each R -module M_l is viewed as a submodule of $\bigoplus_I A_l$.

Lemma 4 *Let $\{T_l, \alpha_{lm}\}_I$ be a direct system of exact couples, and write the couples as $T_l = (D_l, E_l, i_l, j_l, k_l)$, and the morphisms as $\alpha_{lm} = (\alpha_{D_{lm}}, \alpha_{E_{lm}})$. There is a direct limit exact couple $T = \varinjlim \{T_l, \alpha_{lm}\}$, given by $T = (D, E, i, j, k)$ with $D = \varinjlim \{D_l, \alpha_{D_{lm}}\}$, $E = \varinjlim \{E_l, \alpha_{E_{lm}}\}$, and i, j, k the maps induced by the i_l, j_l, k_l 's.*

Proof. The direct limits D and E are well-defined and it suffices to give the expressions of the module homomorphisms i, j, k and show that T is exact. Let d be in D , i.e. it is the class $[d_l]$ for some $d_l \in D_l$, then $i(d) = [i_l(d_l)]$. If $d = [d_m]$ for some $m \geq l$ then $d_m = \alpha_{D_{lm}}(d_l)$ and $[i_m(d_m)] = [i_m \circ \alpha_{D_{lm}}(d_l)] = [\alpha_{D_{lm}} \circ i_l(d_l)]$ because $\alpha_{D_{lm}}$ is a morphism of exact couple (commutativity of diagram (12)), and therefore $[i_m(d_m)] = [i_l(d_l)]$ and i is well-defined. Similarly for $d = [d_l]$ in D , $j(d) = [j_l(d_l)]$ in E , and for $e = [e_l]$ in E , $k(e) = [k_l(e_l)]$ in D , and are well-defined.

This proves also that T is a 3-periodic complex since the compositions $j \circ i$, $k \circ j$ and $i \circ k$ involves the compositions of the i_l, j_l, k_l , and are thus zero. Let $d \in \text{Ker } j$ and $d = [d_l]$, then $j(d) = [j_l(d_l)] = 0$ and by exactness of T_l there exists $d'_l \in D_l$ such that $d_l = i_l(d'_l)$, let then $d' = [d'_l]$ in D to have $d = [i_l(d'_l)] = i(d')$ and therefore $\text{Ker } j = \text{Im } i$. The other two relations $\text{Ker } i = \text{Im } k$ and $\text{Ker } k = \text{Im } j$ are proven similarly, and this shows that T is exact. \square

Corollary 3 *The result of lemma 4 for direct limit of exact couples also holds for direct limit of associated spectral sequences.*

Let A be a C^* -algebra, and assume there is a finite filtration:

$$\{0\} = I_d \xrightarrow{i_d} I_{d-1} \xrightarrow{i_{d-1}} \cdots I_0 \xrightarrow{i_0} I_{-1} = A \quad (14)$$

by closed two-sided ideals I_p .

Let $F_p = A/I_p$ be the quotient C^* -algebra. There are short exact sequences:

$$0 \longrightarrow I_p \xrightarrow{i_p} I_{p-1} \xrightarrow{\pi_p} Q_p \longrightarrow 0 \quad (15a)$$

$$0 \longrightarrow Q_p \xrightarrow{j_p} F_p \xrightarrow{\rho_p} F_{p-1} \longrightarrow 0 \quad (15b)$$

$$0 \longrightarrow I_p \xrightarrow{l_p} A \xrightarrow{\sigma_p} F_p \longrightarrow 0 \quad (15c)$$

In (15a) i_p is the canonical inclusion, and π_p the quotient map $\pi_p(x) = x + I_p$. In (15c) $l_p = i_p \circ i_{p-1} \circ \cdots \circ i_0$ is the canonical inclusion, and σ_p the quotient map $\sigma_p(x) = x + I_p$. And in (15b) j_p is the canonical inclusion $j_p(x + I_p) = l_{p-1}(x) + I_p$, and ρ_p the quotient map $\rho_p(x + I_p) = x + I_{p-1}$.

Then there is an associated cofiltration of A by the quotients $F_p = A/I_p$:

$$A = F_d \xrightarrow{\rho_d} F_{d-1} \xrightarrow{\rho_{d-1}} \cdots F_0 \xrightarrow{\rho_0} F_{-1} = \{0\} \quad (16)$$

Out of the short exact sequences (15a) and (15b), the respective exact couples $T_I = (K(I), K(Q), i, \pi, \partial)$ and $T_F = (K(F), K(Q), \rho, \partial, j)$:

$$T_I : \begin{array}{ccc} K(I) & \xrightarrow{i} & K(I) \\ \partial \swarrow & & \swarrow \pi \\ & K(Q) & \end{array} \quad (17a)$$

$$T_F : \begin{array}{ccc} K(F) & \xrightarrow{\rho} & K(F) \\ j \swarrow & & \swarrow \partial \\ & K(Q) & \end{array} \quad (17b)$$

where we still denote by i , π , j and ρ the induced map on K-theory, and where ∂ is the boundary map of K-theory, and with $K(I)$, $K(F)$ and $K(Q)$ the direct sum of the $K_\epsilon(I_p)$, $K_\epsilon(F_p)$ and $K_\epsilon(Q_p)$, over $\epsilon = 0, 1$ and $p = -1, 0, \dots, d$.

Since the filtration (14) and the cofiltration (16) are finite, both exact couple converge to the K-theory of A . The spectral sequence associated with T_I is a Schochet spectral sequence [70].

Theorem 9 *The spectral sequences for the K-theory of a C^* -algebra A , associated with a filtration of A by ideals I_p as in (14), and with its corresponding cofiltration by quotients $F_p = A/I_p$ as in (16), are isomorphic.*

Proof. By remark 9 (iii) it is sufficient to prove that the exact couples T_I (17a) and T_F (17b) are equivalent. By corollary 2 it is also sufficient to prove that T_I and T_F fit into an exact triangle with a trivial exact couple.

Let T_A be the the trivial exact couple $T_A = (K(A), 0, \text{id}, 0, 0)$. We prove that the short exact sequences (15c) induce the following exact triangle

$$\begin{array}{ccc} T_I & \xrightarrow{l} & T_A \\ \partial \swarrow & & \swarrow \sigma \\ & T_F & \end{array} \quad (18)$$

The exactness of the triangle comes from the exactness of the K-theory functor (the short exact sequences (15c) induce long exact sequences in K-theory). We now verify that the applications l , σ and ∂ define morphisms between the exact couples T_I , T_A and T_F .

The commutativity of the following diagrams will be useful

$$\begin{array}{ccc} Q_p & \xrightarrow{j_p} & F_p \\ \uparrow \pi_p & & \uparrow \sigma_p \\ I_{p-1} & \xrightarrow{i_{p-1}} & A \end{array} \quad \text{and} \quad \begin{array}{ccc} F_p & \xrightarrow{\rho_p} & F_{p-1} \\ \uparrow \sigma_p & & \uparrow \sigma_{p-1} \\ A & \xrightarrow{\text{id}} & A \end{array}$$

which express the two identities

$$j_p \circ \pi_p = \sigma_p \circ l_{p-1}, \quad (19a)$$

$$\sigma_{p-1} = \rho_p \circ \sigma_p. \quad (19b)$$

Indeed, for $x \in I_{p-1}$, $j_p \circ \pi_p(x) = j_p(x + I_p) = l_{p-1}(x) + I_p$ which is equal to $\sigma_p \circ l_{p-1}(x)$, while for $x \in A$, $\rho_p \circ \sigma_p(x) = \rho_p(x + I_p) = x + I_{p-1}$ which is equal to $\sigma_{p-1}(x)$.

1. *Proof that $(l, 0) : T_I \rightarrow T_A$ is a morphism of exact couples*, where the application 0 is the induced quotient map of the trivial short exact sequence

$$0 \longrightarrow Q_p \xrightarrow{\text{id}} Q_p \xrightarrow{0} 0 \longrightarrow 0, \quad (20)$$

i.e. we prove that the following diagram is commutative (ε is 0 or 1):

$$\begin{array}{ccccccc} K_\varepsilon(I_p) & \xrightarrow{i_{p*}} & K_\varepsilon(I_{p-1}) & \xrightarrow{\pi_{p*}} & K_\varepsilon(Q_p) & \xrightarrow{\partial} & K_{\varepsilon+1}(I_p) \\ \downarrow l_{p*} & & \downarrow l_{p-1*} & & \downarrow 0 & & \downarrow l_{p*} \\ K_\varepsilon(A) & \xrightarrow{\text{id}_*} & K_\varepsilon(A) & \xrightarrow{0} & 0 & \xrightarrow{\partial} & K_{\varepsilon+1}(A) \end{array} \quad (21)$$

The middle square is commutative since the maps lead to 0. The commutativity of the left square comes from the functoriality of K-theory: since $l_p = l_{p-1} \circ i_p$, it follows that $\text{id}_* l_{p*} = l_{p*} = l_{p-1*} i_{p*}$. For the right square $l_{p*} = \text{id}_* l_{p*} = l_{p-1*} i_{p*}$ by commutativity of the left square, and thus $l_{p*} \partial = l_{p-1*} i_{p*} \partial = 0$ because $i_{p*} \partial = 0$ by exactness of the long exact sequence in K-theory induced by (15a).

2. *Proof that $(\sigma, \partial) : T_I \rightarrow T_A$ is a morphism of exact couples*, where the application ∂ is the K-theory boundary map in the long exact sequence induced by (20), *i.e.* we prove

that the following diagram is commutative (ε is 0 or 1):

$$\begin{array}{ccccccc}
K_\varepsilon(A) & \xrightarrow{\text{id}} & K_\varepsilon(A) & \xrightarrow{0} & 0 & \xrightarrow{\partial} & K_{\varepsilon+1}(A) \\
\downarrow \sigma_{p*} & & \downarrow \sigma_{p-1*} & & \downarrow \partial & & \downarrow \sigma_{p*} \\
K_\varepsilon(F_p) & \xrightarrow{\rho_{p*}} & K_\varepsilon(F_{p-1}) & \xrightarrow{\partial} & K_{\varepsilon+1}(Q_p) & \xrightarrow{j_{p*}} & K_{\varepsilon+1}(F_p)
\end{array} \tag{22}$$

The right square is commutative since the maps start from 0. The commutativity of the left square holds by (19b) and functoriality of K-theory: $\sigma_{p-1*}\text{id}_* = \sigma_{p-1*} = \rho_{p*}\sigma_{p*}$. For the middle square we have by commutativity of the left square $\sigma_{p-1*} = \sigma_{p-1*}\text{id}_* = \rho_{p*}\sigma_{p*}$, and thus $\partial\sigma_{p-1*} = \partial\rho_{p*}\sigma_{p*} = 0$ because $\partial\rho_{p*} = 0$ by exactness of the long exact sequence in K-theory induced by (15b).

3. *Proof that $(\partial, \text{id}) : T_F \rightarrow T_I$ is a morphism of exact couples*, where the application id is the induced map in K-theory of the identity map in (20), *i.e.* we prove that the following diagram is commutative (ε is 0 or 1):

$$\begin{array}{ccccccc}
K_\varepsilon(F_p) & \xrightarrow{\rho_{p*}} & K_\varepsilon(F_{p-1}) & \xrightarrow{\partial} & K_{\varepsilon+1}(Q_p) & \xrightarrow{j_{p*}} & K_{\varepsilon+1}(F_p) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \text{id}_* & & \downarrow \partial \\
K_{\varepsilon+1}(I_p) & \xrightarrow{i_{p*}} & K_{\varepsilon+1}(I_{p-1}) & \xrightarrow{\pi_{p-1*}} & K_{\varepsilon+1}(Q_p) & \xrightarrow{\partial} & K_\varepsilon(I_p)
\end{array} \tag{23}$$

The proofs of the commutativity of each square go along the same lines: the images of the boundary operator ∂ can be taken to be the “same”. Given a short exact sequence $0 \rightarrow I \rightarrow A \rightarrow A/J \rightarrow 0$ where A is a C^* -algebra and J a closed two-sided ideal, the boundary map of an element $[x] \in K_\varepsilon(A/J)$ is computed via a lift $z \in A \otimes \mathcal{K}$ of x . For the left square, if $[x] \in K_\varepsilon(F_p)$, then $\partial[x]$ is computed via a lift $z \in K_\varepsilon(A)$ of x , *i.e.* $\sigma_{p*}(z) = x$. But by (19b), $\sigma_{p-1*}(z) = \rho_{p*}\sigma_{p*}(z) = \rho_{p*}(x)$, thus z is also a lift of $\rho_{p*}(x)$, and thus $\partial\rho_{p*}[x]$ can be taken to be $i_{p*}\partial[x]$ - the image of $\partial[x]$ in $K_{\varepsilon+1}(I_{p-1})$, which proves that the left square commutes. For the middle square, given $[x] \in K_\varepsilon(F_{p-1})$ if z is a lift of x to $A \otimes \mathcal{K}$, then $\sigma_{p*}(z)$ is a lift of x to $F_p \otimes \mathcal{K}$ because $\rho_{p*}\sigma_{p*}(z) = \sigma_{p-1*}(z) = x$ by (19b). And the induced projection π_{p*} commutes with the boundary operator, and therefore the diagram commutes. Finally, for the right square, given $[x] \in K_{\varepsilon+1}(Q_p)$, if z is a lift of x to $I_{p-1} \otimes \mathcal{K}$, then $l_{p-1*}(z)$ is a lift of $j_{p*}(x)$ to $A \otimes \mathcal{K}$ because by (19a) $\sigma_{p*}l_{p-1*}(z) = j_{p*}\pi_{p*}(z) = j_{p*}(x)$. Hence the right square commutes. \square

7 Appendix: Pimsner spectral sequence

Let \mathcal{A} be a C^* -algebra endowed with a \mathbb{Z}^d action α by $*$ -automorphisms. The Pimsner spectral sequence of theorem 2 is built here. Its page-1 is identified with the Pimsner complex $K_*(\mathcal{A}) \otimes \Lambda\mathbb{Z}^d \xrightarrow{d_P} K_*(\mathcal{A}) \otimes \Lambda\mathbb{Z}^d$ with differential $d = \sum_{i=1}^d (\alpha_{i,*} - \mathbf{1}) \otimes e_i \wedge$, where $\{e_1, \dots, e_d\}$ is the canonical basis of \mathbb{Z}^d , $\alpha_i = \alpha_{e_i}$ is the restriction of α to the i -th component of \mathbb{Z}^d , whereas $x \wedge$ is the exterior multiplication by $x \in \mathbb{Z}^d$.

Let $M_\alpha(\mathcal{A})$ denote the mapping torus of \mathcal{A} , namely the set of functions $f : [0, 1]^d \rightarrow \mathcal{A}$ such that $f(t_1, \dots, t_{i-1}, 1, t_{i+1}, \dots, t_d) = \alpha_i f(t_1, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_d)$. The C^* -algebras $\mathcal{A} \rtimes_\alpha \mathbb{Z}^d$ and $M_\alpha(\mathcal{A}) \rtimes \mathbb{R}^d$ are Morita equivalent [62], thus have the same K -theory, and by Thom-Connes isomorphism [21] $K_*(M_\alpha(\mathcal{A}) \rtimes \mathbb{R}^d) \cong K_{*-d}(M_\alpha(\mathcal{A}))$. Hence it suffices to build a spectral sequence for the K -theory of the mapping torus: $E_2^{rs} \Rightarrow K_{r+s}(M_\alpha(\mathcal{A}))$.

The mapping torus is filtered by the ideals I_s , $s = 0, \dots, d$ (with $I_{-1} = M_\alpha(\mathcal{A})$) of functions that vanish on the faces of dimension s (or s -faces) of the cube $[0, 1]^d$. Let $Q_s = I_{s-1}/I_s$ be the quotient C^* -algebra, which is isomorphic to $C_0((0, 1)^s) \otimes \mathcal{A} \otimes \Lambda^s \mathbb{C}^d$, or $S^s \mathcal{A} \otimes \Lambda^s \mathbb{C}^d$, where $S^s \mathcal{A}$ is the s -th suspension of \mathcal{A} . The quotient Q_s can be viewed as the set of continuous \mathcal{A} -valued functions on the s -faces and vanishing on the $(s-1)$ -faces of the cube. Indeed, thanks to the identifications by α in $M_\alpha(\mathcal{A})$, a function on the s -faces of the cube is uniquely determined by its values on the s -faces that contain the origin. The subsets of $\{1, \dots, d\}$ of the form $I = \{i_1, \dots, i_s\}$ with $i_1 < \dots < i_s$ allow to label the s -faces that contain the origin as the subsets of the cube $\{t \in [0, 1]^d : t_i = 0, \forall i \notin I\}$, or equivalently by their normal vectors $e_I = e_{i_1} \wedge \dots \wedge e_{i_s}$ in $\Lambda^s \mathbb{C}^d$.

The Pimsner spectral sequence is the spectral sequence associated with the cofiltration of $M_\alpha(\mathcal{A})$ by the quotients $F_s = M_\alpha(\mathcal{A})/I_s$ which are isomorphic to the direct sums $\bigoplus M_{\alpha_I}(\mathcal{A}) \otimes e_I$ over all subsets $I = \{i_1, \dots, i_s\}$ of $\{1, \dots, d\}$ with $i_1 < \dots < i_s$, where α_I is the restriction of α to its components in I and $M_{\alpha_I}(\mathcal{A})$ the corresponding mapping torus. The quotient F_s can be viewed as the set of continuous \mathcal{A} -valued function on the faces of dimension s of the cube. Using the method of section 6, an exact pair is built out of the short exact sequences $0 \rightarrow Q_s \xrightarrow{j_s} F_s \xrightarrow{\rho_s} F_{s-1} \rightarrow 0$, where j_s is the inclusion, and the map ρ_s is given by $\rho_s(f_I \otimes e_I) = \sum_{i=1}^d \chi_I(i) ev_i(f_I) \otimes e_{I \setminus \{i\}}$ with χ_I the characteristic function of I and ev_i the evaluation at $t_i = 0$. The complex on page-1 of the associated spectral sequence then reads:

$$\begin{array}{ccccc} K_{\varepsilon+s}(Q_s) & \xrightarrow{j_{s*}} & K_{\varepsilon+s}(F_s) & \xrightarrow{\partial} & K_{\varepsilon+1}(Q_{s+1}) \\ \downarrow \simeq & & & & \downarrow \simeq \\ K_\varepsilon(\mathcal{A}) \otimes \Lambda^s \mathbb{Z}^d & \xrightarrow{d_P} & & & K_\varepsilon(\mathcal{A}) \otimes \Lambda^{s+1} \mathbb{Z}^d \end{array} \quad (24)$$

where $\varepsilon = 0$ or 1 . The above boundary map ∂ is calculated by taking a lift to $F_{s+1} \otimes \mathcal{K}$ of an element in $K_{\varepsilon+s}(F_s)$. In particular functions on the s -faces, the f_I for $|I| = s$, have to be extended to faces of dimension $s+1$, the $f_{I \cup \{i\}}$ for $i \notin I$. Those extensions can be done separately for each $i \notin I$. Also, since the functions in the quotients Q_s and F_s are uniquely determined by their values on the s -faces containing the origin, $f_I \otimes e_I$ and its extension $f_{I \cup \{i\}} \otimes e_{I \cup \{i\}}$ have the same sign (inserting e_i at position i in $e_I = e_{i_1} \wedge \dots \wedge e_{i_s}$). Hence it suffices to show that the differential corresponds to $(\alpha_{i_*} - \mathbf{1}) \otimes e_i \wedge$ for each $i \notin I$, which is a one dimensional problem. It therefore suffices to express the differential in the case of dimension 1.

In the case $d = 1$, the ideals I_s are $I_{-1} = M_\alpha(\mathcal{A})$, $I_0 = C_0((0, 1)) \otimes \mathcal{A} \cong S\mathcal{A}$ and $I_1 = \{0\}$, the quotients F_s are $F_{-1} = \{0\}$, $F_0 = \mathcal{A}$ and $F_1 = M_\alpha(\mathcal{A})$, and the quotients

Q_s are $Q_0 = F_0 = \mathcal{A}$, $Q_1 = S\mathcal{A}$. The exact pair is built from the two exact sequences $0 \longrightarrow Q_s \xrightarrow{j_s} F_s \xrightarrow{\rho_s} F_{s-1} \longrightarrow 0$, for $s = 0$ and $s = 1$, namely $0 \rightarrow \mathcal{A} \xrightarrow{j_0=\text{id}} \mathcal{A} \rightarrow 0 \rightarrow 0$ and $0 \rightarrow S\mathcal{A} \xrightarrow{j_1} M_\alpha(\mathcal{A}) \xrightarrow{\rho_1} \mathcal{A} \rightarrow 0$. The differential on page-1 then reads:

$$\begin{array}{ccccc} K_\varepsilon(\mathcal{A}) & \xrightarrow{\text{id}} & K_\varepsilon(\mathcal{A}) & \xrightarrow{\partial} & K_{\varepsilon+1}(S\mathcal{A}) \\ \downarrow = & & & & \uparrow \beta \\ K_\varepsilon(\mathcal{A}) & \xrightarrow{d_p} & & & K_\varepsilon(\mathcal{A}) \end{array} \quad (25)$$

where β is the Bott map, and $\varepsilon = 0$ or 1 . The reader is referred to [15] for the definition of the K -theory maps (Bott maps and boundary maps), and in particular to proposition 10.4.1 where this problem is treated briefly.

Consider the case $\varepsilon = 0$. Let $x = [p] - [p_l]$ in $K_0(\mathcal{A})$, where p is a projection in $M_n(\mathcal{A}^+)$, $p_l = \text{diag}(\mathbf{1}_k, 0)$ is a projection of rank $l \leq n$ and $p - p_l \in M_n(\mathcal{A})$. The map $h(s) = \text{diag}((1-s)p, s\alpha(p))$ is a lift of p to $M_{4n}(M_\alpha(\mathcal{A}))$, and $e^{2i\pi h(s)} = \text{diag}(e^{-2i\pi s p}, e^{2i\pi s \alpha(p)})$ which is homotope to $e^{2i\pi((\alpha(p)-p))} = \beta((\alpha - \mathbf{1})p)$. The boundary of x : $\partial x = [e^{2i\pi h(s)}]$ is thus $\beta[(\alpha - \mathbf{1})p]$, and therefore $d_p = \alpha_* - \mathbf{1}$. The case $\varepsilon = 1$ is obtained immediately by suspension.

References

- [1] Anderson J. E., Putnam I. F., "Topological invariants for substitution tilings and their associated C^* -algebra", in *Ergod. Th. & Dynam. Sys.*, **18** (1998), 509-537.
- [2] Atiyah M.F., Hirzebruch F., "Vector bundles and homogeneous spaces", *Proc. Symp. Pure Math.* **3** 7-38, Providence: Amer. Math. Soc. (1961).
- [3] Atiyah M.F., Singer I.M., "The index of elliptic operators on compact manifolds", *Bull. Amer. Math. Soc.* **69** (1963), 422-433.
- [4] Atiyah M.F., Segal G.B., Singer I.M. "The index of elliptic operators. I, II, III", *Ann. of Math. (2)* **87** (1968) 484-530, 531-445, 546-604.
- [5] Bellissard J., "Schrödinger's operators with an almost periodic potential : an overview", in *Lecture Notes in Phys.*, **153**, Springer Verlag, Berlin Heidelberg, New-York, (1982).
- [6] Bellissard J., "K-Theory of C^* -algebras in Solid State Physics", in *Statistical Mechanics and Field Theory, Mathematical Aspects*, T.C. Dorlas, M.N. Hugenholtz & M. Winnink, *Lecture Notes in Physics*, **257** (1986), 99-156.
- [7] Bellissard J., Iochum B., Scoppola E., Testard D., "Spectral Properties of One Dimensional Quasi-Crystals", *Commun. Math. Phys.* **125** (1989), 527-543.

- [8] Bellissard J., Bovier A., Ghez, J.M., “Gap Labelling Theorem for one Dimensional Schrödinger Operators”, *Rev. Math. Phys.*, **4** (1992), no. 1, 1-37.
- [9] Bellissard J., “Gap Labelling Theorems for Schrödinger’s Operators”, in *From Number Theory to Physics*, pp.538-630, Les Houches March 89, Springer, J.M. Luck, P. Moussa & M. Waldschmidt Eds., (1993).
- [10] Bellissard J., Contensou E., Legrand A., “ K -théorie des quasi-cristaux, image par la trace: le cas du réseau octogonal”, *C. R. Acad. Sci. Paris Sér. I Math.*, **326** (1998), 197-200.
- [11] Bellissard J., Hermmann D., Zarrouati M., “Hull of Aperiodic Solids and Gap labeling Theorems”, in *Directions in Mathematical Quasicrystals*, CRM Monograph Series, bf 13, 207-259, M.B. Baake & R.V. Moody Eds., AMS Providence, (2000).
- [12] Bellissard J., Kellendonk J., Legrand A., “Gap-labelling for three-dimensional aperiodic solids”, *C. R. Acad. Sci. Paris, Sér. I Math.*, **332** (2001), 521-525.
- [13] Bellissard J., Benedetti R., Gambaudo J.-M., “Spaces of Tilings, Finite Telescopic Approximations and Gap-labelling”, *Commun. Math. Phys.*, **261** (2006), 1-41.
- [14] Benedetti R., Gambaudo J.-M., “On the dynamics of \mathbb{G} -solenoids. Applications to Delone sets”, *Ergodic Theory Dynam. Systems*, **23** (2003), 673-691.
- [15] Blackadar B. *K-Theory for Operator Algebras*, 2nd Ed., Cambridge University Press, Cambridge (1998).
- [16] Borel A., Serre J.-P., “Impossibilité de fibrer un espace euclidien par des fibres compactes, *C. R. Acad. Sci. Paris*, **230** (1950), 2258-2260.
- [17] Bratelli O., “Inductive limits of finite dimensional C^* -algebras”, *Trans. Amer. Math. Soc.*, **171** (1972), 195-234.
- [18] Claro F. H., Wannier G. H., “Closure of Bands for Bloch Electrons in a Magnetic Field, *Phys. Stat. Sol. B*, **88** (1978), K147-K151.
- [19] Coburn L.A., Moyer R.D., Singer I.M., “ C^* -algebra of almost periodic pseudo-differential operators”, *Acta Math.*, **130** (1973), 279-307.
- [20] Connes A., “Sur la théorie non commutative de l’intégration”, *Algèbres d’opérateurs. (Sém. Les Plans-sur-Bex)* (1978), 19-143.
- [21] Connes A., “An analogue of the Thom isomorphism for crossed products of a C^* -algebra by an action of \mathbb{R} ”, *Adv. in Math.* **39** (1981), no. 1, 31-55.

- [22] Connes, A., “A survey of foliations and operator algebras. Operator algebras and applications, Part I”, (Kingston, Ont., 1980), pp. 521-628, *Proc. Sympos. Pure Math.*, **38**, Amer. Math. Soc., Providence, R.I., 1982.
- [23] Connes A., *Géométrie non commutative*, InterEditions, Paris, (1990).
- [24] Connes A., *Noncommutative Geometry*, Academic Press, San Diego, (1994).
- [25] Davis J.F., Kirk P., *Lecture Notes in Algebraic Topology*, AMS Graduate Studies in Mathematics **35** (2001)
- [26] Van Elst A., “Gap-labelling theorems for Schrödinger operators on the square and cubic lattice”. *Rev. Math. Phys.*, **6** (1994), 319-342.
- [27] Forrest A. H., Hunton J. R., “The cohomology and K -theory of commuting homeomorphisms of the Cantor set”, *Ergodic Theory Dynam. Systems*, **19** (1999), 611-625.
- [28] Forrest A. H., Hunton J. R., Kellendonk J., “Topological Invariants for Projection Method Patterns”, in *Memoirs of the American Mathematical Society*, **159** (2002), no. 758,
- [29] Gähler F., Hunton J., Kellendonk J., “Torsion in Tiling Homology and Cohomology”, at arXiv.com math-phys/0505048, (May 2005).
- [30] Gambaudo J.-M., Martens M., “Algebraic topology for minimal Cantor sets”. *Ann. Henri Poincaré*, **7** (2006), 423-446.
- [31] Ghys E., “Laminations par surfaces de Riemann”, in *Dynamique et géométrie complexes, Panoramas & Synthèses*, **8** (1999), 49-95.
- [32] Grothendieck A., “La théorie des classes de Chern”, *Bull. Soc. Math. France*, **86** (1958), 137-154.
- [33] B. Grünbaum, G.C. Shephard, *Tilings and Patterns*, W.H. Freeman and Co, New York, 1st Ed. (1987).
- [34] Hatcher, A., *Algebraic Topology*, 1st ed., Cambridge University Press, (2002), (and available online at <http://www.math.cornell.edu/hatcher/#ATI>
- [35] Hirzebruch F., “Arithmetic genera and the theorem of Riemann-Roch for algebraic varieties”, *Proc. Nat. Acad. Sci. U. S. A.*, **40** (1954), 110-114.
- [36] Hirzebruch F., *Neue topologische Methoden in der algebraischen Geometrie*, Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 9. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. *Topological Methods in Algebraic Geometry*, Reprint of the third Edition, Springer-Verlag, Berlin-Heidelberg-New-York, 1978.

- [37] Hofstadter D. R., “Energy levels and wave functions of Bloch electrons in a rational or irrational magnetic field”, *Phys. Rev. B*, **14** (1976) 2239-2249.
- [38] Johnson R., Moser, J., “The rotation number for almost periodic potentials”. *Comm. Math. Phys.*, **84** (1982), 403-438.
- [39] Kasparov, G. G., “The operator K -functor and extensions of C^* -algebras”, *Izv. Akad. Nauk SSSR*, **44** (1980), 571-636.
- [40] Kasparov G.G., “Equivariant KK -theory and the Novikov conjecture”, *Inv. Math.* **91** (1988), 147-201.
- [41] Kellendonk J., “Noncommutative geometry of tilings and gap labelling”. *Rev. Math. Phys.*, **7** (1995), 1133-1180.
- [42] Kellendonk J., “Pattern-equivariant functions and cohomology”, *J. Phys. A*, **36** (2003), 5765-5772.
- [43] Kellendonk J., Putnam I. F., “The Ruelle-Sullivan map for actions of \mathbb{R}^n ”, *Math. Ann.*, **334** (2006), 693-711.
- [44] Koszul J.-L., “Sur les opérateurs de dérivation dans un anneau”, *C. R. Acad. Sci., Paris*, **225** (1947), 217-219.
- [45] Lagarias J. C., “Geometric models for quasicrystals I. Delone sets of finite type”. *Discrete Comput. Geom.*, **21** (1999), 161-191.
- [46] Lagarias J. C., “Geometric models for quasicrystals. II. Local rules under isometries”. *Discrete Comput. Geom.*, **21** (1999), 345-372.
- [47] Lagarias J. C., Pleasants P. A. B., “Repetitive Delone sets and quasicrystals”. *Ergodic Theory Dynam. Systems*, **23** (2003), 831-867.
- [48] Leray J., “Structure de l’anneau d’homologie d’une représentation”, *C. R. Acad. Sci., Paris*, **222** (1946), 1419-1422.
- [49] Massey W. S., “Exact Couples in Algebraic Topology (Parts I & II)”, *Ann. Math.*, **56**, (1952), 363-396; “Exact Couples in Algebraic Topology (Parts III, IV & V)”, *Ann. Math.*, **57** (1953), 248-286.
- [50] McCleary J., *A User’s Guide to Spectral Sequences*, Cambridge Studies in Advanced Mathematics **58**, 2nd Ed., Cambridge University Press, Cambridge (2001).
- [51] Meyer Y., *Algebraic Number and Harmonic Analysis*, North-Holland, Amsterdam (1972).

- [52] Moore C. C., Schochet C., *Global Analysis on Foliated Spaces*, Math. Sci. Res. Inst. Publ. **No. 9**, New York; Springer-Verlag, (1988).
- [53] Moser J., “An example of a Schrödinger equation with almost periodic potential and nowhere dense spectrum”, *Comment. Math. Helv.*, **56** (1981), 198-224.
- [54] Palais R., *Seminar on the Atiyah-Singer index theorem*, Ann. of Math. Studies, no 57, Princeton Univ. Press, Princeton, N.J., 1965
- [55] Pimsner M., Voiculescu D., “Exact sequences for K -groups and Ext groups of certain cross-product C^* -algebras”. *J. Operator Theory.* **4** (1980), 93-118.
- [56] Pimsner M. V., “Ranges of traces on K_0 of reduced crossed products by free groups”, in *Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983)*, 374-408, *Lecture Notes in Math.*, **1132**, Springer, Berlin, (1985).
- [57] Penrose R., “The role of aesthetics in pure and applied mathematical research” *Bull. Inst. Math. Appl.* **10** (1974), 55-65.
- [58] Queffelec, M., *Substitution dynamical systems-Spectral analysis. Lecture Notes in Math.* Springer, Berlin Heidelberg New York (1987), vol. **1294**.
- [59] Radin C., “The pinwheel tiling of the plane”, *Ann. of Math. (2)* **139** (1994), 257-264.
- [60] Radin C., *Miles of Tiles*, AMS, Student Mathematical Library, vol. 1, (1999).
- [61] Renault J., *A groupoid approach to C^* -algebras* Lecture Notes in Math. **793** Berlin-Heidelberg-New-York: Springer-Verlag (1980).
- [62] Rieffel M.A., “Morita equivalence for operator algebras”, *Operator algebras and applications, Proc. Symposia Pure Math.* **32** (1982), Part 1, 285-298.
- [63] Sadun L., “Tiling spaces are inverse limits”, *J. Math. Phys.*, **44** (2003), 5410-5414.
- [64] Sadun L., Williams R. F., “Tiling spaces are Cantor set fiber bundles”, *Ergodic Theory Dynam. Systems*, **23** (2003), 307-316.
- [65] Sadun L., “Explicit computation of the cohomology for the chair tiling”, talk delivered at Banff Research Station, July 2005.
- [66] Sadun L., “Pattern-Equivariant Cohomology with Integer Coefficients”, at arXiv.com math.DS/0602066, Feb. 2006.
- [67] Serre J.-P., “Homologie singulière des espaces fibrés. Applications”, *Ann. of Math.*, **54** (1951), 425-505.

- [68] Sadun L., “Pattern-Equivariant Cohomology with Integer Coefficients”, to be published, available at <ftp://ftp.ma.utexas.edu/pub/papers/sadun/2006/integer4.pdf>
- [69] Shechtman, D., Blech, I., Gratias, D., Cahn, J.V., “Metallic phase with long range orientational order and no translational symmetry”. *Phys. Rev. Lett.*, **53** (1984), 1951-1953.
- [70] Schochet, C., “Topological methods for C^* -algebra I: Spectral sequences”, *Pacif. J. Math.* **96** (1981), no. 1, 193-211.
- [71] Singer I.M., “Future extensions of index theory and elliptic operators”, *Proc. Sympos., Princeton Univ., Princeton, N.J.*, (1970) 171-185
- [72] Spanier E., *Algebraic Topology*, McGraw-Hill, 1966 (reprinted by Springer-Verlag).
- [73] Subin M.A., “Spectral theory and the index of elliptic operators with almost-periodic coefficients” (Russian) *Uspeki Mat. Nauk* **34** (1979), no. 2(206), 95-135.
- [74] Süto, A.: “The spectrum of a quasi-periodic Schrödinger operator”, *Commun. Math. Phys.* **111** (1987), 409-415.
- [75] Thouless D. J., Kohmoto M., Nightingale M., den Nijs M., “Quantum Hall Conductance in Two Dimensional Periodic Potential”, *Phys. Rev. Lett.*, **49** (1982), 405-408.
- [76] Williams R.F., “Expanding attractors”, *Publ. IHES*, **43** (1974), 169-203.