ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Pasteurgasse 6/7
A-1090 Wien, Austria

Berger's Isoperimetric Problem and Minimal Immersions of Surfaces

Nikolai Nadirashvili

Vienna, Preprint ESI 193 (1995)

January 31, 1995

Supported by Federal Ministry of Science and Research, Austria
Available via WWW.ESI.AC.AT

BERGER'S ISOPERIMETRIC PROBLEM AND MINIMAL IMMERSIONS OF SURFACES

Nikolai Nadirashvili

International Erwin Schrödinger Institute for Mathematical Physics

February 13, 1995

Abstract. We establish an isoperimetric inequality for the first non-zero eigenvalue of the Laplace operator on the surface of genus one and of a fixed area.

1. Introduction

In this paper we establish inequalities between eigenvalues of the Laplacian of a compact surface and geometrical characteristics of this surface. Although this subject is classical there are still some open problems.

Let M be a compact two-dimensional smooth manifold and g be a smooth Riemannian metric on M. Let $\Delta = \Delta_g$ be the Beltrami-Laplace operator on (M, g). We denote by $\lambda_1(g)$ the first non-zero eigenvalue of Δ_g, by $V(g)$ we denote area of the surface (M, g). We define

$$\Lambda(M) = \sup_g \lambda_1(g)V(g), \tag{1.1}$$

where the supremum is taken over all smooth Riemannian metrics g on M. If there exists a smooth metric g' on M such that $\lambda_1(g') = \Lambda(M)$, we call (M, g') a λ_1-maximal manifold. It is well known that $\Lambda(M) < \infty$ and furthermore, for an orientable surface of genus γ the following inequality holds, [YY]:

$$\Lambda(M) \leq 8\pi(\gamma + 1). \tag{1.2}$$

Earlier, for the surface of genus zero Hersch, [H], proved that

$$\Lambda(S^2) = 8\pi$$

and (S^2, can) is a λ_1-maximal manifold.

Supported by the Federal Ministry of Science and Research, Austria. This work was done while the author was visiting IHES, Bures-sur-Yvette and ESI, Vienna. I thank these Institutes for their hospitality.
AMS subject classification: 35P15, 58G25, 58E12, 49Q05.

In the nonorientable case the following equality for the projective plane is due to Li and Yau, [LY]:

$$\Lambda(\mathbb{R}P^2) = 12\pi \tag{1.3}$$

and $(\mathbb{R}P^2, can)$ is a λ_1-maximal manifold.

Let us consider now the case of an orientable surface of genus one, T^2. It can be shown, (cf. [B1]), that in the class of flat tori of the fixed area the eigenvalue λ_1 attains its supremum on the equilateral torus, i.e. on the torus $\mathbb{R}^2/\Gamma$, where Γ is the lattice generated by (1.0) and $(1/2, \sqrt{3/2})$. In [B1] Berger raised the following question: is it true that the flat equilateral torus is λ_1-maximalΓ The main aim of our paper is to give the affirmative answers to Berger's problem. We prove.

Theorem 1.

$$\Lambda(T^2) = 8\pi^2/\sqrt{3}, \tag{1.4}$$

and the flat equilateral torus is the only λ_1-maximal manifold.

Earlier the inequality $\lambda_1(g)V(g) \leq 8\pi^2/\sqrt{3}$ was proved for certain conformal classes of (T^2, g), [LY], [MR]. For instance for conformal flat Riemannian metrics on the flat square torus Li and Yau proved, [LY], that $\lambda_1(g)V(g)$ reached its supremum on a flat square torus.

Let us discuss some consequences of the inequalities above. Denote by $\varphi(M, g)$ the systole of the manifold (M, g), i.e. the shortest geodesic loop on (M, g) in the non-trivial homotopical class. Let $S(M, g)$ be the length of $\varphi(M, g)$. The following isoperimetric inequality is due to Loewner (cf. [B2]): for any Riemannian metric g on T^2

$$Vol(T^2, g)/S^2(T^2, g) \geq 2/\sqrt{3},$$

with equality only for the flat equilateral torus. Combining (1.4) and (1.5), we recover the isoperimetric inequality of Berger, [B1]:

$$\lambda_1(g)S^2(T^2, g) \leq 16\pi^2/3$$

Similar inequalities are also true on $\mathbb{R}P^2$:

$$Vol(\mathbb{R}P^2, g)/S^2(\mathbb{R}P^2, g) \geq 2\pi$$

for any Riemannian metric g on $\mathbb{R}P^2$, $[P]$. As a consequence of (1.3) we have

$$\lambda_1(g)S^2(\mathbb{R}P^2, g) \leq 6\pi^2.$$

In our proof of Theorem 1 we use certain properties of minimal surfaces in the n-dimensional sphere. Let $S^n_r \subset \mathbb{R}^{n+1}$ be the sphere of radius r centred in the origin. Let

$$\psi : (M, g) \to S^n_r \subset \mathbb{R}^n$$

be an isometric immersion. It is well known. (cf. [4]), that ψ is a minimal immersion if and only if

$$\Delta\psi = -\lambda\psi.$$

If, in addition λ is the first eigenvalue of Δg, i.e. $\lambda_1(g) = \lambda$, then the manifold (M, g) is called a λ_1-minimal. We prove

Theorem 2. *Any λ_1-maximal manifold is a λ_1-minimal.*

If one has the existence of a λ_1-maximal manifold of genus 1, then Theorem 1 immediately follows from Theorem 2 and the following theorem due to Montiel and Ros [MR]: any conformal automorphism of a λ_1-minimal manifold is isometric. In Section 4 we prove the existence of a λ_1-maximal Riemannian metric on the torus. Unfortunately, it is technically the most difficult part of the work.

In Sections 5 and 6 we study inequalities for λ_1 on other surfaces.

Theorem 3. *Let K^2 be a Klein bottle. Then there exists a real-analytic Riemannian metric g on K^2, such that (K^2, g) is a λ_1-maximal manifold. Moreover, the multiplicity of the eigenvalue $\lambda_1(g)$ is equal to 5 and $4\pi^2 < \Lambda(K^2) < 8\pi^2/\sqrt{3}$.*

Remark. The λ_1-maximal manifold (K^2, g) is not flat.

Let us consider the λ_1-maximal sphere, projective plane, torus and the Klein bottle. The multiplicities of λ_1 on these surfaces will be correspondingly 3,4,6,5. It is interesting to note that these multiplicities are also the maximal possible ones on these surfaces for any Riemannian metric on them, [C], [B], [CV], [N].

I would like to thank M. Gromov, Y. Colin de Verdière and T. Hoffmann-Ostenhof for fruitful discussions.

2. Preliminaries

Let (M, g) be a two-dimensional Riemannian manifold. In local coordinates (x^1, x^2) the metric g has the form $\sum g_{ij} dx^i dx^j$. Then

$$\Delta = \Delta g = \frac{1}{\sqrt{|g|}} \sum \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}, \right)$$

where $(g^{ij}) = (g_{kl})^{-1}$, $|g| = \det(g_{ij})$.

Let $g_0 = a(x)g$, $a > 0$, be a metric conformally equivalent to the metric g. We have

$$\Delta g_0 = \frac{1}{a} \Delta g.$$

Let $u \in H^1(M, g)$. We denote by $D(u)$ the Dirichlet integral of the function u:

$$D(u) = D_g(u) := \int_M |\nabla u|^2 dA_g$$

where dA_g is the element of area related to the metric g. The Quadratic form $D(u)$ in the space $L_2(M, g)$ determine the eigenvalues λ_i and the eigenfunctions u_i of the Laplace operator: $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \ldots, \lambda_i := \lambda_i(g)$. We denote by $R(u)$ the Rayleigh quotient:

$$R(u) = R_g(u) = D_g(u) / \int_M u^2 dA_g$$

We have

$$\lambda_1(g) = \inf R_g(u)$$

where inf is subject to all smooth functions u which are orthogonal to 1.

3. Extremals of eigenvalues of Laplacians.

Let M be a compact smooth surface. Denote by U the set of Riemannian metrics on M with total area equal to 1. Let us consider $\lambda_k(g), k \geq 1$, as a functional on U. In general λ_k is a Lipschitz function on U. We say that $g_0 \in U$ is an extremal of λ_k if for all one-parametric families of metrics $g_\tau \in U$ smoothly depending on the parameter $\tau \in \mathbb{R}$ the following holds: either

(3.1) $$\lambda_k(g_\tau) \leq \lambda_k(g_0) + o(\tau)$$

as $\tau \to 0$, or

$$\lambda_k(g_\tau) \geq \lambda_k(g_0) + o(\tau)$$

as $\tau \to 0$.

Theorem 5. *Let $g_0 \in U$ be an extremal of $\lambda_k, k \geq 1$. Let E be the eigenspace of the Laplacian on M, g_0 corresponding to the eigenvalue $\lambda_k(g_0)$. Then there exist orthogonal vectors $W_1, \ldots, W_l \in E$ such that*

$$\sum W_i^2 \equiv 1 \text{ on } M,$$
$$2\sum (dW_i)^2(X) \equiv \lambda_k(g_0) g_0(X,X), X \in TM.$$

Proof. Let Q be the space of quadratic forms on $\mathbb{R}^2$. Then $Q = Q_1 \oplus Q_2$, where $Q_1(X) = Q_1(*X)$, $Q_2(X) = -Q_2(*X)$, $Q_1 \simeq \mathbb{R}^1$, $Q_2 \simeq \mathbb{R}^2$. Let $\rho_1, \rho_2 \in Q$, we set

$$< \rho_1, \rho_2 >= \int_{|x|=1} \varphi_1(x)\varphi_2(x) d\theta$$

We define maps $p_1 : \mathbb{R} \to Q_1$, $p_1(a) = |ax|^2$ and $p_2 : \mathbb{R} \to Q_2$ such that $-q(x) = < p_2(x), q >$ for all $q \in Q_2$, $\quad x \in \mathbb{R}^2$. Let x_1, x_2 be coordinates on $\mathbb{R}^2$. Evidently $(x_1, x_2) = F(x_1^2, x_2^2, x_1 x_2)$ for some linear map

(3.2) $$F : \mathbb{R}^3 \to Q.$$

We denote by $\mathfrak{S}M$ the space of fields of quadratic forms on TM. We define a map

$$N : f \ni C^\infty(M, g) \to \rho \in \mathfrak{S}M,$$
$$\rho(X) = p_1(f) + p_2(\pi df),$$

for all $X \in TM$, where $\pi : T^*M \to TM$ is the standard projection. Let $h \in \mathfrak{S}M$, denote $g_\tau^h = g_0 + \tau h, \tau \in \mathbb{R}$. If $u \in C^\infty(M)$ satisfies

$$Dg_0^h(u) = \int_M u^2 dA_{g_0} = 1$$

then a trivial computation gives us

(3.3) $$\frac{\partial}{\partial \tau} R_{g_\tau^h}|_{\tau=0} = - < h, Nu >$$

We define $I = |X|^2$, $X \in TM$. Then we have

$$\frac{\partial}{\partial \tau} Vol(M, g_\tau^h)|_{\tau=0} =< I, h > . \tag{3.4}$$

To be definite we assume from now on that the extremal condition on λ_k is given by inequality (3.1). Let K be the convex envelope of the set $N(E)$. Then K is a convex cone in $\mathfrak{S}M$. Let us assume first that $I \notin K$. Then by the Hahn-Banach theorem there exists a vector $e \in \mathfrak{S}M$ such that

$$< I, e >> 0 \tag{3.5}$$

and for any $\varphi \in K$.

$$< e, \varphi >< 0. \tag{3.6}$$

Let $\dim E = m$. We may assume without loss of generality that $\lambda_k(g_0) = 1$. Let $\lambda_n(g_0) = \lambda_{n+1}(g_0) = \cdots = \lambda_{n+m}(g_0) = 1$. We denote by D_τ the restriction of the quadratic form Dg_τ^e on the subspace $E \subset L_2(M, g_\tau^e)$. Let $\mu_1(\tau) \leq \cdots \leq \mu_m(\tau)$ be the eigenvalues of D_τ. Let $S \subset E$ be the unite sphere in $L_2(M, g_0)$ norm. From (3.3) and (3.4) it follows that for every $u \in S$

$$\frac{\partial}{\partial \tau} R_{g_\tau^e}(u)|_{\tau=0} > 0.$$

Consequently $\partial \mu_1(0)/\partial \tau > 0$. The last inequality implies, (cf. [K]) that

$$\frac{\partial}{\partial \tau} \lambda_n(g_\tau^e)|_{\tau=0} > 0.$$

Since from (3.4) and (3.5) it follows that

$$\frac{\partial}{\partial \tau} Vol(M, g_\tau^e)|_{\tau=0} > 0$$

we get a contradiction to (3.1). Hereby the extremal property of $\lambda_k(g_0)$ implies that $I \in k$.

Since (3.2) is a linear map, it follows that $\dim K \leq 9m^2 + 1$. By Carathéodory's theorem, (cf. [R]) the vector I can be represented as a linear combination of at most $9m^2$ extremal points of K. Since the extremal set of K is in the cone $N(E)$ there exist $V_1, \ldots, V_n \in E$, $n \leq 9m^2$ such that

$$I = \sum_{i=1}^{n} N(V_i).$$

The last equality implies,

$$\sum V_i^2 \equiv 1 \text{ on } M,$$

and for any $X \in M$ there is a constant $C > 0$ such that

$$\sum (dV_i(x))^2(X) = Cg_0(X, X), \quad X \in T_xM.$$

If we apply the Laplace operator to the identity (3.7) we obtain

$$\sum V_i \Delta V_i + \sum |\nabla V_i|^2 \equiv 0 \text{ on } M$$

or

$$\sum |\nabla V_i|^2 \equiv \sum V_i^2 \text{ on } M,$$

recall that we have assumed that $\lambda_k(g_0) = 1$. From the identities (3.8), (3.9) it follows that

$$2 \sum (dV_i)^2(X) \equiv g_0(X, X) \text{ on } TM$$

To finish the proof of Theorem 4 we only need to prove that the vectors V_i can be chosen to be orthogonal. The equalities

$$-\Delta V_i = V_i$$

and (3.7), (3.10) imply that the map

$$(M, g) \to (\sqrt{2}V_1, \dots, \sqrt{2}V_n) \in S^{n-1}_{\sqrt{2}} \subset \mathbb{R}^n$$

is a minimal isometric immersion, ([L]). Since $V_i \in E$ the image of the last immersion is in a subspace $H \subset \mathbb{R}^n$ and $\dim H \leq m$. Let $\{W_i\}$ be an orthonormal basis in H such that the quadratic form

$$P(U, V) = \int_M UV \, dAg_0$$

has a diagonal form in the basis $\{W_i\}$. Since the map

$$(M, g) \to (W_1, \dots, W_m) \subset S^{m-1}_{\sqrt{2}}$$

is a minimal isometric immersion, the W_i are as required. Theorem 5 is proved.

Remark. Extremals of other functionals related to the spectrum of Riemannian surfaces was considered in [CV2] and [OPS]. It is interesting to compare the results of these papers with Theorem 5.

We also prove, in relation to Theorem 5, propositions for Schrödinger operators. Let $X \in (M, g)$. We denote by $B^x_r \subset (M, g), r > 0$, the geodesic disk on (M, g) of radius r centred at x. Let $E \subset (M, g)$ be a Borel set. We denote by $ess\partial E$ (essential boundary) the set of all points $x \in \bar{E}$ such that for every $\varepsilon > 0$ $meas(B^x_\varepsilon \cap E) > 0$, $meas(B^x_\varepsilon \cap \complement E) > 0$. By $essE$ we denote the set of all points $x \in \bar{E}$ such that for any $\varepsilon > 0$ $meas(B^x_\varepsilon \cap E) > 0$. We define $LT := \{f \in L_\infty(M), 0 \leq f \leq T\}$.

We consider the Schrödinger operator $L = -\Delta - V$ with a potential $V \in LT$. Let λ_k be eigenvalues of L. We set

$$X_1 = \{x \in M, 0 < V(x) < T\},$$
$$X_2 = \{x \in M, V(x) = T\},$$
$$X_3 = \{x \in M, V(x) = 0\}.$$

We assume that there is an eigenvalue λ_k such that for every $f \in L_\infty(M)$ satisfying $V + f \in LT$, $\int_M f dA_g = 0$ the inequality

$$\lambda_k \geq \lambda_k(\tau, f) + o(\tau) \tag{3.11}$$

holds as $\tau \to 0$, $\tau > 0$. Here $\lambda_k(\tau, f)$ are eigenvalues of the operator $L - \tau f$. Let E be the eigenspace of λ_k.

Lemma 3.1. *Let $z_1, \dots, z_n \in (essX_1 \cup ess\partial X_2 \cup ess\partial X_3)$. Then there exists $W_1, \dots, W_k \in E$, $k \leq (\dim E)^2$ such that*

$$\sum_{i=1}^{k} W_i^2(z_j) = 1, i \leq j \leq n$$

Proof. For $u \in H^1(M, g)$ we define

$$r_V(u) := (D_g(u) - \int_M u^2 V dA_g) / \int_M u^2 dA_g.$$

Let $\varphi \in L_\infty(M)$, we set

$$V_\tau = V + \tau\varphi$$

Then

$$\frac{\partial}{\partial\tau} r_{V_\tau}(u)|_{\tau=0} = -\int_M u^2 \varphi dA_g / \int_M u^2 dA_g \tag{3.12}$$

We set $Z = z_1 \cup \dots \cup z_n$. Let $u \in E$. We define a map $u : Z \to (u(z_1), \dots, u(z_n)) \in \mathbb{R}^n$, then $E(Z)$ is a subspace in $\mathbb{R}^n$. We can also define a map $Q : \mathbb{R}^n \to \mathbb{R}^n$, $Q(x_1, \dots, x_n) = (x_1^2, \dots, x_n^2)$. We denote by K the convex envelope of the set $QE(Z)$. Set $I = (1, \dots, 1) \in \mathbb{R}^n$. If $I \in K$ then the existence of the required functions W_i immediately follows, as in the proof of Theorem 4. If $I \in K$ then there is a vector $e = (e_1, \dots, e_n) \in \mathbb{R}^n$ such that $< I, e >= 0$ and for every $x \in K, x \neq 0, < x, e >< 0$. Since E is a finite dimensional space of continuous functions on M there exists a sufficiently small $\varepsilon > 0$ such that the following holds: if $z_i' \in B_\varepsilon^{z_i}, Z' = z_1' \cup \dots \cup z_n'$, and K' is the convex envelope of $QE(Z')$ then for any $x \in Kl', x \neq 0, < x, e >< 0$. We also assume that the disks $B_\varepsilon^{z_i}$ have no mutual intersections. Let $A_i \subset B_\varepsilon^{z_i}$ be closed sets satisfying the following requirement: for any $i, 1 \leq i \leq n$, $Vol_g(A_i) = \delta$ and if $e_i \geq 0$ then $V|_{A_i} \leq T - \delta$, if $e_i < 0$ then $V|_{A_i} \geq \delta$, for some positive constant δ. By our assumption for sufficiently small

$\delta > 0$ the required sets A_i exist. Let $\chi(\cdot)$ be the characteristic function of a set. We define

$$f = \frac{\delta}{|e|}\sum_{i=1}^{n} e_i\chi(A_i)$$

Then $\int_M f dA_g = 0$, $V + f \in LT$ and for any $u \in E, u \neq 0$, $\int_M u^2 f dA_g < 0$. Now, as in the proof of Theorem 4, from (3.12) a contradiction with (3.11) follow. Lemma 3.1 is proved. □

Lemma 3.2. *Let $z_1 \in essX_3$ and $z_2, \ldots, z_n \in (essX_1 \cup ess\partial X_2 \cup ess\partial X_3)$. Then there exists $W_1, \ldots, W_k \in E$, $k \leq (\dim E)^2$ such that*

$$\sum_{i=1}^{k} W_i^2(z_j) = 1$$

Proof. We use the notations of Lemma's 3.1 proof. If $I \in K$ the statement follows. Let us assume now that $I \notin K$. If $e_1 \geq 0$ there exists $A_1 \in X_3 \cap B_\varepsilon^{z_1}$, $Vol_g(A_1) = \delta$ and then the contradiction follows as above. Hence $e_1 < 0$. By Lemma 3.1 there exists $W_1, \ldots, W_k \in E$ such that (3.13) holds. Define a vector $h = (\sum W_i^2(z_i), 1, \ldots, 1) \in K$. Now (3.14) follows from the inequalities $e_1 < 0, < e, h >< 0, < e, I >= 0$. Lemma 3.2 is proved. □

Since E is a finite dimensional space of continuous functions on M we have as an immediate consequence of Lemma 3.2 the following:

Proposition 3.1. *For any point $z \in essX_3$ there exists $W_1, \ldots, W_k \in E$, $k \leq (\dim E)^2$ such that*

$$\sum_{i=1}^{k} W_i^2 = 1 \text{ on } essX_1 \cup ess\partial X_2 \cup ess\partial X_3$$

and

$$\sum_{i=1}^{k} W_i^2(z) \geq 1.$$

Lemma 3.3. *Let $z_1, \ldots, z_2 \in essX_2$ and $z_{n+1}, \ldots, z_n \in (essX_1 \cup ess\partial X_2 \cup ess\partial X_3)$. Then there exist $W_1, \ldots, W_k, k \leq (\dim E)^2$ such that*

$$\sum_{i=1}^{k} W_i^2(z_j) = 1, \quad m < j \leq n, \tag{3.15}$$

and

$$\sum_{j=1}^{m}\sum_{i=1}^{k} W_i^2(z_j) \leq m \tag{3.16}$$

Proof. We use the notation from the proof of Lemma 3.1. Define a linear map $H : \mathbb{R}^n \to \mathbb{R}^{n-m+1}$

$$H : (x_1, \ldots, x_n) \to (\sum_{i=1}^{m} x_i/m, x_{m+1}, \ldots, x_n)$$

Let K_0 be the convex envelope of the set $HQE(Z)$. Set $I_0 = (1, \ldots, 1) \in \mathbb{R}^{n-m+1}$. If $I_0 \in K_0$ the statement follows. So, let us assume that $I_0 \notin K_0$. Then there is a vector $y = (y_1, \ldots, y_{n-m+1})$ such that $< I_0, y >= 0$ and for any $x \in K_0, x \neq 0$, $< x, y >< 0$. Let us choose sets $A_i \subset B_\varepsilon^{z_i} \cap X_2, i, \ldots, m$, $Vol_g(A_i) = \delta$ sets $A_{m+1}, \ldots, A_n$ as they are defined in the proof of Lemma 3.1. in correspondence with the signs of $y_z, \ldots, y_{n-m+1}$. We define

$$f = \frac{\delta}{|y| + m}(\sum_{i=1} y_1 \chi(A_i) + \sum_{i=m+1}^{n} y_{i-m+1}\chi(A_i))$$

If $y_1 \leq 0$ the contradiction follows as above. Hence $y_1 > 0$. By Lemma 3.1 there exists $W_1, \ldots, W_k \in E$ such that (3.15) holds. Define a vector $h \in K_0$,

$$h = (\sum_{j=1}^{m} \sum_{i=1}^{k} W_i^2(z_j)/m, 1, \ldots, 1)$$

Now (3.16) follows from the inequalities $y_1 > 0, < h, y >< 0, < I_0, y >= 0$. Lemma 3.3 is proved. $\square$

Since E is a finite dimensional space of continuous functions on M we have as an immediate consequence of Lemma 3.3 the following:

Proposition 3.2. *There exists $W_1, \ldots, W_k \in E$, $k \leq (\dim E)^2$ such that*

$$\sum_{i=1}^{k} W_i^2 \equiv 1 \text{ on } essX_1 \cup ess\partial X_2 \cup ess\partial X_3$$

and

$$\int_{X_2} \sum W_i^2 dA_g \leq Vol_g(X_2).$$

4. Proof of Theorem 1.

(1) Let (T^2, g) be an orientable compact surface of genus one. Then (T^2, g) is conformally equivalent to a flat torus $(\mathbb{R}^2/\Gamma, (dx)^2)$ where Γ is a 2-dimensional lattice and $(dx)^2$ is the metric on $\mathbb{R}^2/\Gamma$ induced from the Euclidian metric on $\mathbb{R}^2$. It is well-known also that each flat torus is isometric, up to dilations, to a flat torus $T(a,b) = \mathbb{R}^2/\Gamma(a,b)$ where $\Gamma(a,b)$ is the lattice generated by $\{(1,0),(a,b)\}$ with $0 \leq a \leq 1/2, b > 0, a^2 + b^2 \geq 1$ (see [BGM]). Let us denote $t : g \to (a,b), t_1 : g \to b$.

(2) Let g_n be a sequence of Riemannian metrics on T^2 such that $\lambda_1(g_n)V(g_n) \to \Lambda(T^2)$ as $n \to \infty$. We prove that $t_1(g_n) < C < \infty$.

Without loss of generality we may assume that $V(g_n) = 1$ for all n. Denote $(a_n, b_n) := t(g_n)$. Assume by contradiction that $b_n \to \infty$ as $n \to \infty$.

Let h_n be a conformally flat metric on $\mathbb{R}^n$ with the lattice of periods $\Gamma(a_n, b_n)$ such that the torus $(\mathbb{R}^2, h_n)/\Gamma(a_n, b_n)$ is isometric to (T^2, g_n). Let x^1, x^2 be coordinates in $\mathbb{R}^2$. Denote by $T_n \subset \mathbb{R}^2$ the fundamental parallelogram of the group $\Gamma(a_n, b_n)$ with the centre at 0. Without loss we may assume that

$$\int_{T_n} \sin 2\pi x^2/b_n dV_{h_n} = 0$$

Then

$$\int_{T_n} |\nabla \sin 2\pi x^2/b_n|^2 dx = \pi/b_n \tag{4.1}$$

Denote

$$A_N = \{(x^1, x^2) \in T_n, -N < x^2 < N\},$$
$$B_N = \{(x^1, x^2) \in T_n, -b_{n/2} + N < x^2 < b_{n/2} - N\}$$

Since $\lambda_1(g_n) \to \Lambda(T^2) \geq 8\pi^2/\sqrt{3}$ as $n \to \infty$, it follows from (4.1) that for any $\varepsilon > 0$ there is $N(\varepsilon) > 0$ such that for all sufficiently big n

$$V_{h_n}(A_n \cup B_n) > 1 - \varepsilon$$

Define $\zeta_n^+ = \max(\cos 2\pi x_2^2/b_n, 0)$, $\zeta_n^- = \min(\cos 2\pi x^2/b_n, 0)$. Then

$$\int_{T_n} |\nabla \zeta^{\pm}|^2 dx = \pi/2bn,$$
$$\int_{T_n} (\zeta_n^+)^2 dV_{h_n} > V_{h_n}(A_N) - \varepsilon_n,$$
$$\int_{T_n} (\zeta_n^-)^2 dV_{h_n} > V_{h_n}(B_N) - \varepsilon_n,$$

where $\varepsilon_n \to 0$ as $n \to \infty$. If $\inf(V_{h_n}(A_N), V_{h_n}(B_N)) > C > 0$ for all n then $\lambda_1(g_n) \to 0$ as $n \to \infty$. Therefore we may assume without loss that $V_{h_n} A_N > 1 - 2\varepsilon$ for all sufficiently big n. Let us fix $n, \varepsilon > 0$ such that $b_n > 5N(\varepsilon)$.

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, π_i be the projection of $\mathbb{R}^3$ on the coordinate axis x^i. By the result of Hersch [H], see also [G], there exists a conformal map

$$\varphi : A_{2N} \to S^2$$

such that function $u = \pi_1 \circ \varphi$ has the following properties

$$\int_{A_{2N}} u dV_{h_n} = 0, \tag{4.2}$$

$$\int_{A_{2N}} u^2 dV_{h_n} \geq \frac{1}{3} V_{h_n} A_{2N}, \tag{4.3}$$

$$R_{(A_{2N}, h_n)} u \leq \Lambda(S^2)/V_{h_n} A_{2N} \tag{4.4}$$

Let $\alpha < \beta$, denote

$$A_\alpha^\beta = \{(x^1, x^2) \in T_n, \alpha < x^2 < \beta\}$$

¿From (4.3), (4.4) it follows that there are $\alpha \in (-2N, N)$, $\beta \in (N, 2N)$ such that

$$\int_{\partial A_\alpha^\beta} (\partial u/\partial x^1)^2 dx < 8/N$$

We denote $B = T_n \backslash A_\alpha^\beta$. Let v be a solution of the Dirichlet problem

$$\Delta v = 0 \text{ on } B, v = 0 \text{ on } \partial B$$

Since $|v| < 1$ in B it follows from (4.5) that

$$\int_B |\nabla v|^2 dx < CN$$

and

$$\int_B v dV_{h_n} = \delta, \tag{4.5}$$

$|\delta| < 2\varepsilon$. We define the function w on T_n as follows

$$w = \begin{cases} u - \delta \text{ on } A_\alpha^\beta \\ v - \delta \text{ on } B \end{cases}$$

¿From (4.2), (4.6) it follows that

$$\int_{T_n} w dV_{h_n} = 0$$

and $D_{h_n} w < \Lambda(S^2) - \varepsilon'_n$ where $\varepsilon'_n \to 0$, as $n \to \infty$. Hence

$$\lim_{n \to \infty} \lambda_1(g_n) \leq \Lambda(S^2).$$

Since $\Lambda(S^2) = 8\pi < 8\pi^2/\sqrt{3} \leq \Lambda(T^2)$ we obtain a contradiction. Therefore the b_n are uniformly bounded.

(3) We may assume that $t(g_n) \to (\alpha, \beta)$. We also assume that the sequence of the probability measures V_{h_n} is converging weekly to the probability measure μ on $T(\alpha, \beta)$.

We prove that the support of the measure μ is different from a point. Let us assume by contradiction that $\sup \mu = \{0\} \in T$, where T is the fundamental parallelogram of the group $\Gamma(\alpha,\beta)$ and T is centred in the origin $\mathbb{R}^2$. Denote $D_r = \{x \in \mathbb{R}^2, |x| < r\}$. and $D = D_{r_0} \subset T$ for some fixed $r_0 > 0$. By our assumption for every $\varepsilon > 0$

$$V_{h_n} D_\varepsilon \to 1 \tag{4.7}$$

as $n \to \infty$. By Hersch's result (see Assertion (2)) there exists a conformal map

$$\varphi_n : (D, h_n) \to S^2 \subset \mathbb{R}^3$$

such that

$$\int_D \pi_i \circ \varphi_n dV_{h_n} = 0, \tag{4.8}$$

$i = 1,2,3$ and $R_{(D,h_n)}(\pi_1 \circ \varphi_n) \le \Lambda(S^2)/V_{h_n}(D)$. Denote $B_n = S^2 \backslash \varphi_n(D)$ and let ρ_n be the radius of the geodesical circle $B_n \subset S^2$. From (4.7) and (4.8) it follows that $\rho_n \to 0$ as $n \to \infty$. Let us choose a sequence of numbers $\rho'_n > 0$ such that $\rho'_n \to 0$, $\rho_n/\rho'_n \to 0$ as $n \to \infty$. Let $B'_n \subset S^2$ be the circle of radius ρ'_n concentric to B_n. Let u_n be a continuous function on S^2 defined by the following:

$$\begin{aligned} u_n(x) &= \pi_1(x) \text{ for } x \in S^2 \backslash B'_n, \\ u_n(x) &= 0 \text{ for } x \in B_n, \\ \Delta u_n(x) &= 0 \text{ for } x \in B'_n \backslash B. \end{aligned}$$

Since the capacity of a point on the plane is equal to zero we conclude that

$$\int_{S^2} |\nabla u_n|^2 ds \to \int_{S^2} |\nabla \pi_1(x)|^2 ds \tag{4.9}$$

as $n \to \infty$. Denote

$$\int_D u_n \circ \varphi_n dV_{h_n} = \delta_n.$$

By (4.9) we obtain that $|\delta_n| \to 0$ as $n \to \infty$. Set $w_n = u_n - \delta_n$. From (4.7), (4.9) it follows that

$$R_{h_n}(w_n) < \Lambda(S^2) + o(n)$$

as $n \to \infty$ and since $\Lambda(S^2) < \Lambda(T^2)$ we obtained a contradiction.

(4) Let $E \subset T$ be a closed set. We prove that equality cap $E = 0$ implies equality $\mu(E) = 0$. Let us assume by contradiction that $\mu(E) > 0$. Then by Assertion (3) there are two closed sets $E_1, E_2 \subset E, E_1 \cap E_2 = \emptyset$, $\mu(E_i) > 0, i = 1,2$. Let d be the distance between the sets E_1 and E_2 in the Euclidian metric, Ω_i be $(d/2)$-neighbourhood of E_i. Then by the definition of the capacity for any $\varepsilon > 0$ there are the functions, $f_i \in C_0^\infty(\Omega_i)$ such that $f_i \ge 1$ on E_i and $D(f_i) < \varepsilon$. Since

$$\int_T f_i^2 dV_{h_n} \to \int_T f_i^2 d\mu$$

as $n \to \infty$, $i = 1, 2$ we obtain that $\lambda_1(h_n) \to 0$ and this is in contradiction with our assumption on h_n.

(5) We say that a Riemannian metric on torus $T(\alpha, \beta)$ is in the class G_N if $V_g(T^2) = 1$ and the metric g has a form $g = a(x)(sx)^2$, where the function $a(x)$ satisfies the inequality $0 < a(x) \leq N$. Denote

$$\Lambda_N = \sup_{g \in G_N} \lambda_1(g)$$

Evidently $\Lambda_N \to \Lambda(T^2)$ as $N \to \infty$. Let $\lambda_1(g_n) \to \Lambda_N$ as $n \to \infty$, $g_n \in G_N$, $g_n = a_n(x)(dx)^2$ on $T(\alpha, \beta)$. The Beltrami-Laplace operator Δ_{g_n} we can represent in the form

$$\Delta_{g_n} = \frac{1}{a_n}\Delta,$$

here Δ is the Laplace operator in the Euclidian metric $(dx)^2$. Therefore eigenfunctions u_n of Δ_{g_n} with the eigenvalues $\lambda_1(g_n)$ are solutions of the Schrödinger equation

$$L_n u_n = -\Delta u_n - \lambda_1(g_n) a_n(x) u_n = 0$$

and the Schrödinger operator L_n has exactly one bounded state.

Without loss of generality we may assume that $a_n \to b_N$ weekly as $n \to \infty$. Then $0 \leq b_N \leq N$. Let us consider the Schrödinger operator

$$P_N = -\Delta - V_N,$$

$V_N = \Lambda_N b_N$. It is well known that the weak convergence of uniformly bounded potentials of Schrödinger operators on a compact manifold implies the convergence of its eigenvalues. Let us denote by E_N the eigenspace of P_N of the eigenvalue 0,

$$X_1 = \{x \in T(\alpha, \beta), 0 < b_N(x) < N\},$$
$$X_2 = \{x \in T(\alpha, \beta), b_N(x) = N\},$$
$$X_3 = \{x \in T(\alpha, \beta), b_N(x) = 0\}.$$

By Proposition 3.1, for every point $z \in essX_3$ there are $u_1, \dots, u_k \in E_N$, $u := \sum u_i^2$ such that

$$u = 1 \text{ on } essX_1 \cup ess\partial X_2 \cup ess\partial X_3$$

and $u(z) \geq 1$.

Let us assume first that the set $essX_3$ contains an open component Ω and $z \in \Omega$. Then we have: u_i are harmonic functions on Ω. Hence the u_i^2 are sub-harmonic functions on Ω. $u = 1$ on $\partial\Omega$ and since u is sub-harmonic on Ω from the inequality $u(z) \geq 1$ it follows that $u = 1$ on Ω. Therefore $0 = \Delta u = \Delta \sum u_i^2 = 2\sum u_i \Delta u_i + 2\sum |\nabla u_i|^2 = 2\sum |\nabla u_i|$ on Ω and hence $u_i \equiv const$ on Ω for all $i = 1, \dots, k$. We may assume without loss that all u_i are linear independent. Now we consider three cases:

(i) $k \geq 2$. Then for some constant $C \in \mathbb{R}$, $u_1 + Cu_2 \equiv 0$ on Ω and hence by the Carleman theorem on unique continuation (see [Ho]) $u_1 + Cu_2 \equiv 0$ on M.

(ii) $k = 1$ and for all $a, b \in \mathbb{R}$, $a^2 + b^2 \neq 0$ the sum $v = v_{a,b} = a\partial u_1/\partial x^1 + b\partial u_2/\partial x^2 \not\equiv 0$.

We have: $\Delta v = -N\Lambda_N v$ on X_2, since $v \in C^1$ and $u_1 \equiv 1$ on $\complement essX_2$ it follows that $v = 0$ on $\partial essX_2$ and hence $r_{V_N}(v) < 0$. Therefore the operator P_N has as least two bound states.

(iii) $k = 1$ and in the orthogonal coordinates y^1, y^2 we have $\partial u_1/\partial y^1 \equiv 0$. Then $u_1 = u_1(y^2)$ and we may assume without loss that S_2 is a strip $0 < y_2 < \delta$. Since $u_1 \equiv 1$ on $\complement X_3, u_1 = \cos 2\pi y^2/\delta$, and $\delta < 1/\sqrt{N}$ so we conclude that $D(u_1) \to \infty$ as $N \to \infty$. Therefore $\Lambda_N \to \infty$ as $N \to \infty$.

So, in each of the possible cases we obtained a contradiction. Hence the set $essX_3$ has an empty interior.

By Proposition 3.2 there are $v_{1,N\ldots}, v_{k,N} \in E_N$, $v := \sum v_{i,N}^2, k \leq (\dim E_N)^2$, such that

$$v \equiv 1 \text{ on } essX_1 \cup ess\partial X_2 \cup ess\partial X_3$$

and

$$\int_{X_2} v dx \leq \int_{X_2} dx$$

Since $essX_3$ has an empty interior it implies that there is an open, maybe empty set $\Omega_N \subset essX_2$ such that $v \equiv 1$ on $\complement\Omega_N$ and

$$\int_{\Omega_N} v dx \leq \int_{\Omega_N} dx$$

Let us denote by L_N the k-dimensional subspace of E_N spanned by the vectors $v_1, \ldots, v_k$.

(6) We prove that $\dim E_N \leq 6$. For the Schrödinger operator with a smooth potential on the two-dimensional torus the multiplicity of the second eigenvalue is no more that 6, [N], (see also [B], [CV]). In [N] we used smoothness of the potential only in the proof of the following proposition.

Let u be a solution of the Schrödinger equation $\Delta u = Vu$ defined in $\Omega \subset \mathbb{R}^2$, $0 \in \Omega, V \in C^\infty(\Omega)$. Then there exists a homogeneous harmonic polynomial p_k of degree k such that $u = p_k + o(|x|^k)$ as $|x| \to 0$ and there is C^1 diffeomorphism $d : \Omega \to \Omega$, $d(0) = 0$ such that the nodal sets of functions $u \circ d$ and p_k coincide in a neighbourhood of 0.

It was proved in [HO] that the above proposition is valid for the Schrödinger operator with a potential $V \in L_\infty$. Therefore the multiplicity of the eigenvalue 0 of the operator P_N is at the most 6.

(7) We may assume, if necessary, choosing a subsequence L_{n_i}, that $L_n \to L$ weekly in $H^1(T(\alpha, \beta), dx)$. By Assertion (6) $\dim L \leq 6$. Let $T \subset \mathbb{R}^2$ be a fixed fundamental domain of $\Gamma(\alpha, \beta)$. We also consider functions defined on $T(\alpha, \beta)$ as functions defined on T. Since L is a finite dimensional subspace $bH^1(T)$, for any $\varepsilon > 0$ there exists an open set $\omega(\varepsilon) \subset T$ and a subsequence L_{n_i} such that $cap \quad \omega(\varepsilon) < \varepsilon$ and $L_{n_i} \to L$ uniformly in $C(T\backslash\omega(\varepsilon))$ ([VGR]). From Assertion (4) it follows that for any $\delta > 0$ there exists $\varepsilon_0 > 0, N_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0, N > N_0$ then

$$\int_{\omega(\varepsilon)} V_N dx < \delta \tag{4.10}$$

we may assume that $v_{i,N} \to v_i$ weakly in H^1. From Assertion (5) and (4.10) it follows that we can define

$$\int_T v_i^2 d\mu := \lim_{N\to\infty} \int_T v_{i,N}^2 V_N dx/\Lambda_N \tag{4.11}$$

If the convergence $v_{i,N} \to v_i$ is not a strong convergence in H^1 then

$$\overline{\lim_{n\to\infty}} D(v_{n,i}) > D(v_i)$$

and hence

$$D(v_i)/\int_T v_i^2 d\mu < \Lambda(T^2)$$

Since the last inequality via (4.11) implies a contradiction, we obtain

$$v_{n,i} \to v_i \text{ in } H^1, \tag{4.12}$$

as $n \to \infty$. From Assertion (5) it follows that

$$\sum v_i^2 \equiv 1 \text{ on } T$$

For any $w \in H^1(T(\alpha, \beta))$ we have an integral identity

$$\int \nabla v_{i,N} \nabla w dx = \Lambda_N \int v_{i,N} w b_N dx$$

Thus for any $w \in H^1(T(\alpha, \beta)) \cap C(T(\alpha, \beta))$

$$\int \nabla v_i \nabla w dx = \Lambda(T^2) \int v_i w d\mu \tag{4.13}$$

¿From the identity

$$\sum_{i=1}^{k} v_{i,N}^2 \equiv 1 \text{ on } \complement\Omega_N$$

we obtain on $\complement\Omega_N$

$$0 = \Delta \sum_i v_{i,N}^2 = 2 \sum_i v_{i,N} \Delta v_{i,N} + 2 \sum_i |\nabla v_{i,N}|^2$$

Thus

$$\sum |\nabla v_{i,N}|^2 = V_N \sum v_{i,N}^2 = V_N = \Lambda_N b_N \tag{4.14}$$

Let us decompose b_N. We set $b_N = b_N^1 + b_N^2$ where $b_N^1(x) = b_N(x)$ if $x \in \complement\Omega_N$ and $b_N^1(x) = 0$ if $x \in \Omega_N$, $b_N^2 = b_N - b_N^1$. From (4.12) and (4.14) it follows that $b_N^1 \to b \in L_1(T)$ in L_1 as $N \to \infty$. Since $Vol_{dx^2}(\Omega_N) \le 1/N$ it follows that $b_N^2 dx$ converges to a singular measure μ_0 on T.

Thus the measure can be decomposed in the form $d\mu = bdx + d\mu_0$ where $b = \sum |\nabla v_i|^2/\Lambda(T^2)$

(8) Let $z \in \mathbb{R}^2, B_r^z$ be the disk $|x-z| < r, r > 0$. Let $x \in T$. We prove that for all sufficiently small $\varepsilon > 0$

$$\inf_{u \in C_0^1(B_\varepsilon^x)} D(u)/\int_{B_\varepsilon^x} u^2 d\mu \geq \Lambda(T^2).$$

Let us assume by contradiction that for every $\varepsilon > 0$ there is a function $u \in C_0^1(B_\varepsilon^x)$ such that $D(u)/\int u^2 d\mu < \Lambda(T^2)$. Since an isolated point has a zero capacity for any $\varepsilon > 0$ there is a $\delta, 0 < \delta < \varepsilon$ such that there is a $v \in C_0^1(B_x^\varepsilon \backslash B_x^\delta)$, with $D(v)/\int v^2 d\mu < \Lambda(T^2)$. Hence there exists a pair of functions $v_1 \in C_0^1(B_x^\varepsilon \backslash B_x^\delta)$, $v_2 \in C_0^1(B_x^\delta)$ such that $D(v_i)/\int v_i^2 d\mu < \Lambda(T^2) i = 1,2$. Evidently the last inequalities imply a contradiction.

(9) Let $y \in T, \varepsilon > 0$ be such that statement (8) holds. Assume also that $\mu(\partial B_\varepsilon^y) = 0$. We consider a map

$$\Phi : B_\varepsilon^y \to (v_1, \ldots, v_k) \in S^{k-1} \subset \mathbb{R}^n.$$

We prove that Φ minimizes the Dirichlet integral on the set of all vectors $z \in H^1$, $z : B_\varepsilon^y \to S^{k-1}$ with fixed boundary values:

$$z|_{\partial B_\varepsilon^y} = \Phi|_{\partial B_\varepsilon^y}$$

Let us assume by contradiction that there exists $z = (z_1, \ldots, z_k)$ such that (4.15) holds and $D(z) < D(\Phi)$. Without loss we may assume also that z is continuous in the open disk B_ε^y. Since $\sum z_i^2 \equiv 1$, $\sum v_i^2 \equiv 1$

$$\sum_{i=1}^k \int_{B_\varepsilon^y} z_i^2 d\mu = \sum_{i=1}^k \int_{B_\varepsilon^y} v_i^2 d\mu.$$

Since $\mu(\partial B_\varepsilon^y) = 0$ the integrals on the left are correctly defined. Thus there is a component z_j such that

$$D(z_j) - \Lambda(T^2) \int z_j^2 d\mu < D(v_j) - \Lambda(T^2) \int v_j^2 d\mu = 0$$

Define $w(x) = z_j(x) - v_j(x), x \in B_\varepsilon^y$ and $w(x) = 0$, $x \in T \backslash B_\varepsilon^y$. Then

$$D(w + v_j) - \Lambda(T^2) \int (w + v_j)^2 d\mu < 0$$

thus

$$\int (|\nabla w|^2 + |\nabla v_j|^2 + 2 < \nabla v_j, \nabla w >) dx - \Lambda(T^2) \int (w^2 + v_j^2 + 2wv_j) d\mu < 0.$$

By identity (4.13) we conclude that

$$D(w) - \int w^2 d\mu < 0.$$

The last inequality contradicts Assertion (8) and hence the statement is proved.

(10) By (9) and Morrey's regularity theorem ([M], Theorem 9.4.2) it follows that the v_i are real-analytic functions.

We prove now that the measure μ has no singular part.

Let us fix $\rho(x) \in C^\infty(\mathbb{R}^2)$ such that $\rho \geq 0$, $\rho(x) \equiv 1$ if $x \in B_1^0$, $\rho(x) \equiv 0$ if $x \in \mathbb{R}^2 \backslash B_2^0$. Let $\rho_\varepsilon(x) = \rho(x/\varepsilon)$. Let $z \in T, \varepsilon > 0$. By identity (4.13) we have

$$\int \nabla v_i \nabla \rho_\varepsilon(x-z)dx = \Lambda(T^2) \int v_i(x)\rho_\varepsilon(x-z)d\mu.$$

Thus

$$\int v_i(x)\rho_\varepsilon(x-z)d\mu = -\int \rho_\varepsilon(x-z)\Delta v_i dx/\Lambda(T^2).$$

Since the function v_i is smooth and $v_i(z) > 0$ it follows that

$$\int_{B_\varepsilon^z} d\mu \leq C\varepsilon^2$$

for some positive constant C independent of z and ε. Therefore the singular part of the measure μ is equal to zero.

(11) Thus the Riemannian metric

$$g = \sum |\nabla v_i|^2 (dx)^2/\Lambda(T^2)$$

is real-analytic and Λ_1-maximal metric on $T(\alpha,\beta)$. By Theorem 5 there are $v_i \in E$ such that the map

$$\Phi : (T(\alpha,\beta), g) \to (v_1, \dots, v_k) \in S^{k-1}$$

is a minimal isometric immersion in S^{k-1}.

(12) Since the map Φ is λ_1-minimal immersion then by the Montiel-Ross theorem, [MR], (see also [ESI]), we conclude that the group of isometrics of $T(\alpha,\beta)$ is $\mathbb{R}^2$. Therefore g is a flat metric.

For the flat metric g we have an inequality $\lambda_1(g)V_g(T^2) \leq \Lambda(T^2)$, $[B1]$. Thus Theorem 1 is proved.

5. Proof of Theorem 3.

(1) Let K^2 be a Klein bottle, g be a smooth Riemannian metric on K^2. Then (K^2, g) has a conformal type of a flat Klein bottle of a form $\mathbb{R}^2/\Gamma_a, a > 0, \Gamma_a$ is a discrete group generated by $(x,y) \to (x+a, y)$ and $(x,y) \to (x, y+1)$.

Denote $\tau : g \to a$.

(2) Let $\pi : T^2 \to (K^2, g)$ the standard two-sheets covering of the Klein bottle, g^* be a pull-back metric on T^2. Let u be an eigenfunction of Laplacian on (K^2, g),

u^* be the pull-back of u on T^2. Then u^* is an eigenfunction of the Laplacian on (T^2, g^*). Thus $\lambda_1(\mathbb{R}^2/\Gamma_1) = 4\pi^2$ and consequently

$$\Lambda(K^2) \geq 4\pi^2 > 12\pi = \Lambda(\mathbb{R}P^2) > \Lambda(S^2) \tag{5.1}$$

(3) Let g_n be a sequence of Riemannian metrics on K^2 such that $\lambda(g_n)V(g_n) \to \Lambda(K^2)$ as $n \to \infty$. Then for some constant $C > 0$

$$C^{-1} < \tau(g_n) < C$$

The Assertion (5.2) follows from (5.1) via the reasoning of Section 4, Assertion (2) in just the same way. (To exclude the case $\tau(g_n) \to 0$ we utilize instead of the Hersh's theorem the corresponding Li-Yau result on $\mathbb{R}P^2$, see [LY]).

(4) There exists a real-analytic λ_1-maximal Riemannian metric g on K^2.

Since the conformal class of K^2 is bounded the proof of the proposition is just the same as in Section 4, Assertions (3)-(10).

We set $\tau(g) = a$. Let $(\mathbb{R}^2/\Gamma_a, f(dx)^2)$ be isometric to (K^2, g). Then by [MR] (see also Section 4, Assertion (12)) it follows that $f(x,y) \equiv f(y)$.

(5) Let $g^* = \pi^* g$ be the pull-back of a g on T^2. Then the operator Δ_{g^*} admits a separation of variables x, y. If u is an eigenfunction of Δ_{g^*} with an eigenvalue λ then u is a linear combination of the following functions: $\varphi_k(y)\cos 2\pi kx/a$, $\varphi_k(y)\sin 2\pi kx/a$, $\psi_k(y)\cos(2k-1)\pi x/a$, $\psi_k(y)\sin(2k-1)\pi x/a$, $k = 0, 1, \ldots$ and φ_k, ψ_K satisfy the following Sturm- Liouville equations: $\varphi_k'' = (\lambda/f - 4k^2\pi^2/a^2)\varphi_k$, $\psi_k'' = (\lambda/f - (2k-1)^2\pi^2(a^2)\psi_k)$ and $\varphi_k(y) \equiv \varphi_k(y+1)$, $\psi_k(y) \equiv \Psi_k(y+1)$. Consequently if E is the λ-eigenspace of Laplacian on $(\mathbb{R}^2/\Gamma_a, f(dx)^2)$ any $u \in E$ is a linear combination of the following functions: $u_1 = \varphi_1(y)\cos 2\pi x/a$, $u_2 = \varphi_1(y)\sin 2\pi x/a$, $u_3 = \psi_1(y)\cos \pi x/a$, $u_4 = \psi_1(y)\sin \pi x/a$, $u_5 = \varphi_0(y)$.

By Courant's theorem on nodal domains, [CH] each of the functions φ_0, ψ_1 has 2 zeros on $\mathbb{R}/\mathbb{Z}$ and $\varphi_1 > 0$.

Denote by U the space spanned by $u_1, \ldots, u_5$.

(6) By Theorem 4 there are $v_i, \ldots, v_k, k \leq 5$ such that the map

$$\Phi : (K^2, g) \to (v_1, \ldots, v_k) \in S^{k-1} \subset \mathbb{R}^k$$

is a minimal isometry. Denote by V the space spanned by $v_1, \ldots, v_k$.

It is required to prove that $k = 5$. Evidently the inclusion of one of the two functions u_1, u_2 in V implies the inclusion of the other and inclusion of any of u_3, u_4 in V also implies the inclusion of the other. Since $\dim V \geq 4$ then either $U = V$, or V spanned by u_1, u_2, u_3, u_4. Let us assume by contradiction that $\dim V = 4$. Since u_i are orthogonal there are exist $\alpha_i > 0, i = 1, \ldots, 4$ such that the map

$$\Phi_0 : (K^2, g) \to (\alpha_1 u_1, \ldots, \alpha_4 u_4) \in S^3$$

is a minimal isometry. Let us assume that in point y_0 function ψ_1 has a supremum. Sine $\varphi_1 > 0$ from the identity

$$\sum_{i=1}^{4} \alpha_i^2 u_i^2 \equiv 1$$

it follows that φ_1 has an infimum at y_0. Thus $\psi_1'(y_0) = \varphi_1'(y_0) = 0$. Therefore $\partial u_i(\cdot, y_0)/\partial y = 0$, $i = 1, \ldots, 4$. Since Φ_0 is an isometry we obtained a contradiction.

(7) Denote by Γ the lattice generated by $(0,1), (a,0)$. Then u_5 is an eigenfunction of Laplacian on the torus $(\mathbb{R}^2/\Gamma, f(ds)^2)$. Hence $\Lambda(k^2) < \Lambda(T^2)$. Theorem 3 is proved.

References

[B1] Berger, M., *Sur les premières valeurs propres des variétés Riemanniennes*, Composito Math **26** (1973), 129-149.

[B2] Berger, M., *Systoles et applications selon Gromov*, Seminaire Bourbaki **es.771** (1992/93).

[BGM] Berger, M., Ganduchon, P., Mazet, E., *Le spectre d'une variété Riemannienne*, Lect. Notes Math., v. 194, Springer, 1971.

[B] Besson, G., *Sur la multiplicité de la prmière valeur propre des surfaces Riemanniennes*, Ann.Inst. Fourier **30** (1980), 109-128.

[C] Cheng, S.Y., *Eigenfunctions and nodal sets*, Comment. Math. Helv. **51** (1976), 43-55.

[CV1] Colin de Verdière, Y., *Sur la multiplicité de la première valeur propre non nulle du Laplacien*, Comment. Math. Helv. **61** (1986), 254-270.

[CV2] Colin de Verdière, Y., *Sur une hypothèse de transversalité d'Arnold*, Comment Math. Helv. **63** (1988), 184-193.

[CH] Courant, R., Hilbert, D., *Methoden der Mathematishen Physik, I*, Springer, 1968.

[ESI] El Soufi, A., Ilias, S., *Imersions minimales, première valeur propre du Laplacien et volume conforme*, Math. Ann. **275** (1986), 257-267.

[G] Gromov, M., *Metric invariants of Kähler manifolds. Proceedings in Differential Geometry and Topology, Alghero, Italy*, World Scientific (1992), 90-116.

[H] Hersch, J.,, *Quatre propriétés isopérimétriques de membranes sphérique homogènes*, C.R.Ac. Paris **270** (1970), 1645-1648.

[HO] Hoffmann-Ostenhof, M., Hoffmann-Ostenhof, T., *Local properties of solutions of Schrödinger equations*, Comm. PDE **17** (1992), 491-522.

[Ho] Hörmander, L., *Linear Partial Differential Operators*, Springer, 1963.

[K] Kato, T., *Perturbation Theory for Linear Operators*, Springer, 1976.

[L] Lawson, H.B., *Lectures on minimal submanifolds*, v.1. Math. Lecture Series 9, Perish Inc., Berkeley, 1080.

[LY] Li, P., Yau, S-T., *A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces*, Invent. Math. **69** (1982), 269-291.

[MR] Montiel, S., Ros, A., *Minimal immersion of surfaces by the first eigenfunctions and conformal area*, Invent. Math. **83** (153-166), 1986.

[M] Morrey C.B, *Multiple integrals in the calculus of variations*, Springer, 1966.

[N] Nadirashvili, N., *Multiple eigenvalues of Laplace operator*, Math. USSR Sbornik **61** (1988), 225-238.

[OPS] Osgood, B., Phillips, R.S., Sarnak, P., *Extremals of determinants of Laplaciens*, J. Funk. Anal. **80** (1988), 148-211.

[P] Pu, P., *Some inequalities in certain nonoreintable manifolds*, Pacific J. Math. **2** (1952), 55-71.

[R] Rockafellar, R.T., *Convex analysis*, Princeton Univ. Press, Princeton, 1970.

[S] Springer, G., *Introduction to Riemann surfaces*, Addison-Wesley Publ. Comp, 1957.

[VGR] Vodopianov, S.K., Goldstein, V.U., Resetnajak, Yu.G., *On geometric properties of functions with generalized first derivatives*, Russian Math. Surveys **34** (1979), 19-74.

[YY] Yang, P., Yau, S-T., *Eigenvalues of the Laplacian of compact Riemann surfaces and minimal submanifolds*, Ann.Sc. Sup. Pisa **7** (1980), 55-63.