Extremal Basis, Geometrically Separated Domains and Applications

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EXTREMAL BASIS, GEOMETRICALLY SEPARATED DOMAINS AND APPLICATIONS

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ABSTRACT. In this paper we introduce the notion of extremal basis of tangent vector fields at a boundary point of finite type of a pseudo-convex domain in $\mathbb{C}^n$, $n \geq 3$. Using this notion we define the class of geometrically separated domains at a boundary point and we give a description of their complex geometry. Examples of such domains are given, for instance, by locally convex domains, domains with locally diagonalizable Levi form at a point or domains for which the Levi form have comparable eigenvalues near a point and moreover we show that geometrically separated domains can be localized. Next we define what we call "adapted pluri-subharmonic function and give sufficient conditions, related to extremal basis, for their existence. Then, for these domains, when such functions exist, we prove global and local sharp estimate for the Bergman and Szegö projections. As an application, we strengthen a result by C. Fefferman, J. J. Kohn and M. Machedon ([FKM90]) for the local Hölder estimate of the Szegö projection removing the arbitrary small loss in the Hölder index and giving a stronger non-isotropic estimate.

1. INTRODUCTION

The study of the regularity with sharp estimates for the Bergman and Szegö projections for pseudo-convex domains in $\mathbb{C}^n$ became very active for domains of finite type when D. Catlin proved his fundamental characterization of subelliptic estimates ([Cat87]).

Quite quickly, the case of domains in $\mathbb{C}^2$ was completely solved by D. Catlin in [Cat89], A. Nagel, J.-P. Rosay, E. M. Stein and S. Wainger in [NRSW89], M. Christ in [Chr88] and by C. Fefferman and J. J. Kohn in [FK88] and J. Mc Neal in [McN89].

In higher dimensions, the situation is more complicated and, until now, there are only partial results. One of the main difficulties is the description of the geometry of the domain: there are some special basis of the complex tangent space at the boundary playing an important role in this description and also in the Lipschitz estimates of the projectors. Thus the first results concern domains for which these basis are more or less evident. For example, the class of domains for which the Levi form have rank larger than $n-2$ was studied by M. Machedon in [Mac88] (see also S. Cho [Cho94, Cho96], [AC99]) and, even in that case, the situation is not so simple. An other example is given by decoupled domains, treated by several authors (see for example [McN91], [CG94]).

A typical example where the choice of the special basis is essential, and not evident, is the case of convex domains in $\mathbb{C}^n$. In [McN94, MN02] J. Mc Neal introduced some special basis (called ε-extremal in [BCD98]) and gave a description of the complex geometry with the construction of a pseudo-distance near the boundary related to these basis. With that geometry, and a construction of a "good" pluri-subharmonic function, he proved sharp pointwise estimates for the Bergman kernel and its derivatives. Always using the geometry related to these basis J. Mc Neal and E. M. Stein ([MS94] and [MS97]) proved all sharp estimates for the Bergman and Szegö projections.

More recently similar results were obtained, when the Levi form have comparable eigenvalues, by K. Koenig in [Koe02] and S. Cho in [Cho03], [Cho02b].

In [FKM90] C. L. Fefferman, J. J. Kohn and M. Machedon studied the case where the Levi form is locally diagonalizable near a point p_0 of the boundary. They solved the $\overline{\partial}_b$-Neuman problem and deduced that if f is a $L^2(\partial\Omega)$ function which is locally in the classical Lipschitz space Λ_α (near p_0) then, for all $\varepsilon > 0$ it's Szegö projection Sf is locally (near p_0) in $\Lambda_{\alpha-\varepsilon}$ (an application of our theory will remove the loss of ε in this estimate and get, in fact, a better non-isotropic estimate).

The main idea of the present paper is to introduce a general notion of "extremal basis" of the complex tangent space at a boundary point of a pseudo-convex domain in $\mathbb{C}^n$, $n \geq 3$, generalizing the ε-extremal basis of the convex case. With this notion we define a class of pseudo-convex domains, containing all previously studied classes, called "geometrically separated", for which a good family of extremal basis exist near a point of the boundary. The fundamental properties of extremal basis allow one to prove that, for these domains, there exists an associated structure of homogeneous space on the boundary (and an extension of that structure inside the domain) which describes the complex geometry of the domain. An important property of domains which are geometrically separated at a boundary point is that this structure can be nicely localized (see the end of Section 2 for more details).

Moreover, when special pluri-subharmonic functions (called "adapted pluri-subharmonic functions" in this paper) exist, this structure is used to obtain sharp global and local estimates for classical analytic objects as Bergman kernel, Bergman and Szegö projection and invariant metrics. The existence of such adapted pluri-subharmonic functions for geometrically separated domains is not evident in general. For example, if the domain is locally convex, this is done using special support

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functions (see [DF99, MN02]) which cannot exist, in general, without convexity. Here we prove their existence, under an additional condition (which is satisfied, for example, when the Levi form is locally diagonalizable) on the extremal basis, for the domain and also for the localized one (see the end of Section 2 for more details).

2. NOTATIONS AND ORGANIZATION OF THE PAPER

In all the paper, $\Omega = \{\rho < 0\}$ denotes a bounded domain in $\mathbb{C}^n$, $n \geq 3$, with a $\mathscr{C}^\infty$ boundary, and $\rho \in \mathscr{C}^\infty(\mathbb{C}^n)$ is a defining function of Ω such that $|\nabla\rho| = 1$ on $\partial\Omega$. We denote by $N = \frac{1}{|\nabla\rho|}\sum \frac{\partial\rho}{\partial\bar{z}_i}\frac{\partial}{\partial z_i}$ the unitary complex normal vector field to ρ (i.e. $N\rho \equiv 1$ and $\|N\| \equiv 1$).

For each point p of the boundary let us denote $T_p^{1,0}(\partial\Omega)$ the subbundle of $T_p(\partial\Omega)$ of tangential complex vectors and $T_p^{0,1}(\partial\Omega)$ its conjugate. As usual, we will say that a family $(L_i)_{1\leq i\leq n-1}$ of $\mathscr{C}^\infty$ vector fields is a basis of the complex tangent space at $\partial\Omega$ in a open neighborhood $V \subset \partial\Omega$ of a point p_0 in $\partial\Omega$ if it is a basis of sections of $T^{1,0}(\partial\Omega)$ in V (i.e. $L_i(\rho) \equiv 0$ in V, a condition which is independent of the defining function).

Clearly, all $\mathscr{C}^\infty$ vector field L in a open neighborhood $V \subset \partial\Omega$ can be extend in a open neighborhood $V(p_0) \subset \mathbb{C}^n$ so that $L(\rho) \equiv 0$ on $V(p_0)$. Of course this extension depends on the defining function ρ, but all the results we will state will be independent of such a choice. Thus, in all the paper, the tangent vector fields considered in $V(p_0)$ are always supposed to annihilate ρ in $V(p_0)$, and we will use the terminology of “vector fields tangent to ρ” for this property.

Let L and L' be two $(1,0)$ vector fields tangent to ρ. The bracket $[L,\overline{L'}]$ being tangent to ρ, it can be written

$$[L,\overline{L'}] = 2\sqrt{-1}c_{LL'}T + L''$$

where T is the imaginary part of N and $L'' \in T_p^{1,0}(\partial\Omega) \oplus T_p^{0,1}(\partial\Omega)$. Thus $c_{LL'} = [L,\overline{L'}](\partial\rho) = \left\langle \partial\rho; [L,\overline{L'}]\right\rangle$. The Levi form of $\partial\Omega$ at p is defined as the hermitian form whose value at $(L,\overline{L'})$ is the number $c_{LL'}$. The pseudo-convexity of Ω means that this hermitian form is non-negative. If $(L_i)_{1\leq i\leq n-1}$ is a local basis of $(1,0)$ vector fields tangent to ρ, $(c_{L_iL_j})_{i,j}$ is then the matrix of the Levi form in the given basis. This basis will be generally denoted $(c_{ij})_{i,j}$.

Let $p_0 \in \Omega$ and $V(p_0)$ be a neighborhood of p_0 in $\mathbb{C}^n$. If W is a set of $\mathscr{C}^\infty(V(p_0))$ $(1,0)$ complex vector fields, $\mathscr{L}(W)$ denotes the set of all lists $\mathscr{L} = \left(L^1,\ldots,L^k\right)$ such that $L^j \in W \cup \overline{W}$, and, for $l \in \mathbb{N}$, $\mathscr{L}_l(W)$ denotes the set of such lists $\mathscr{L}$ of length $|\mathscr{L}| = k \in \{0,1,\ldots,l\}$. Moreover, if $|\mathscr{L}| = k \geq 2$, we denote

$$\mathscr{L}(\partial\rho) = L^1\ldots L^{k-2}\left(\left\langle \partial\rho, \left[L^{k-1},L^k\right]\right\rangle\right).$$

Let L be a $\mathscr{C}^\infty(V(p_0))$ $(1,0)$ complex vector field tangent to ρ and $M \geq 2$ be an integer. We define the weight $F_M(L,p,\delta) = F(L,p,\delta) = F(L)$ associated to L at the point $p \in V(p_0)$ and to $\delta > 0$ by

$$F(L,p,\delta) = \sum_{\mathscr{L}\in\mathscr{L}_M(L)} \left|\frac{\mathscr{L}(\partial\rho)(p)}{\delta}\right|^{2/|\mathscr{L}|}.$$

Moreover, for the complex normal direction N we define $L_n = N$ and $F(N,p,\delta) = \delta^{-2}$.

Note that, with the conditions on ρ, the functions $\mathscr{L}(\partial\rho)$ restricted to $\partial\Omega$ does not depend on the choice of the defining function ρ. Thus the defining function ρ of Ω is suppose to be fixed and the number M also. When we say that some number depend on “ϑ” and on “the data”, we mean that it depends on “ϑ”, n, M, and ρ but neither on the point p in $V(p_0)$ nor on $\delta \leq \delta_0$.

If $\mathscr{B} = \{L_1,\ldots,L_{n-1}\}$ is a $\mathscr{C}^\infty$ basis of (1.0) vector fields tangent to ρ in $V(p_0)$, and $\mathscr{L} \in \mathscr{L}(\mathscr{B}\cup\{N\})$, we denote

$$F(p,\delta)^{\mathscr{L}/2} = \prod_{i=1}^{n} F(L_i,p,\delta)^{l_i/2},$$

where $l_i = l_i(\mathscr{L})$ is the number of times L_i or $\overline{L_i}$ appears in $\mathscr{L}$, $i \leq n-1$, and $l_n = l_n(\mathscr{L})$ the number of times N or $\overline{N}$ appears in $\mathscr{L}$ (and thus $|\mathscr{L}| = k = \sum_{i=1}^n l_i$).

The organization of the paper is as follows:

In Section 3 we define the notion of extremal basis and give some examples. Then we give their basic properties and, in Section 3.3 we prove the following fundamental property of a extremal basis at a point of finite type: under this hypothesis there exists a coordinate system which is adapted to that basis in the sense that all the derivatives of the matrix of the Levi form in that basis are controlled by the weights attached to the extremal basis. We give also some sufficient conditions of extremality for a given basis, useful for some examples. Finally, in Section 3.5 we show how the existence of extremal basis can be localized in the sense that, near a boundary point p_0 of Ω of finite type, if there exists extremal basis at every boundary points near p_0, then one can construct a small pseudo-convex domain D of finite type inside the original domain, containing a piece of the boundary of Ω in its boundary such that there exists extremal basis a every point of the boundary of D.

In Section 4 we define the notion of geometrically separated domains at a point p_0 of its boundary and give examples. Then we show that a geometrically separated domain is automatically equipped with a local structure of homogeneous space on its boundary. In Section 4.3 we prove that the structure of geometrically separated domain can always be localized (in the sense described above).

In Section 5 we study the existence of a pluri-subharmonic functions adapted to a given geometrically separated domain. In particular, we prove their existence when the domain is "strongly" geometrically separated at a point p_0 of its boundary, and we prove that, in this case, such function exists for the localized domain at every point of its boundary.

In the last Section (6) we show that all the sharp global and local results for Bergman kernel, Bergman and Szegö projections and invariant metrics can be established for geometrically separated domains when there exists adapted pluri-subharmonic functions. The local sharp estimate of the Szegö projection when the Levi form is locally diagonalizable is an example of these results.

3. EXTREMAL BASIS

3.1. **Definition and examples.**

Definition 3.1. Let Ω and $V(p_0)$ defined on Section 2. Let $\mathscr{B} = \{L_1, \dots, L_{n-1}\}$ be a $\mathscr{C}^\infty$ basis of $(1,0)$ vector fields tangent to ρ in $V(p_0)$ and M an integer. Let $p \in V(p_0)$ and $0 < \delta$. We say that $\mathscr{B} = \{L_1, \dots, L_{n-1}\}$ is (M, K, p, δ)-extremal (or simply (K, p, δ)-extremal or K-extremal) if the $\mathscr{C}^{2M}$ norms, in $V(p_0)$, of the L_i are bounded by K, the Jacobian of $\mathscr{B}$ is bounded from below by $1/K$ on $V(p_0)$, and the two following conditions are satisfied:

EB$_1$ For any vector field L of the form $L = \sum_{i=1}^{n-1} a_i L_i$, $a_i \in \mathbb{C}$, we have

$$\frac{1}{K} \sum_{i=1}^{n-1} |a_i|^2 F(L_i, p, \delta) \leq F(L, p, \delta) \leq K \sum_{i=1}^{n-1} |a_i|^2 F(L_i, p, \delta).$$

EB$_2$ For all index i, j, k such that $i, j < n$, $k \leq n$ and all list $\mathscr{L}$ of $\mathscr{L}_M(\mathscr{B} \cup \{N\})$,

$$F(L_k, p, \delta)^{1/2} \left| \mathscr{L} a^{\overset{(-)}{k}}_{\overset{(-)}{i},\overset{(-)}{j}}(p) \right| \leq K F(p, \delta)^{\mathscr{L}/2} F(L_i, p, \delta)^{1/2} F(L_j, p, \delta)^{1/2},$$

where $a^{\overset{(-)}{k}}_{\overset{(-)}{i},\overset{(-)}{j}}$ is the coefficient of the bracket $\left[\overset{(-)}{L}_i, \overset{(-)}{L}_j\right]$ in the direction $\overset{(-)}{L}_k$ (with $L_n = N$), and $\overset{(-)}{L}_i$ means L_i or$\overline{L_i}$.

Remark. In general this Definition depends of the choice of the defining function ρ. But note that, for $p \in \partial\Omega$, it does not and depends only on the restriction of $\mathscr{B}$ to $\partial\Omega \cap V(p_0)$.

Example 3.1.

(1) *Locally convex domains.* A first example of extremal basis concerns the case of a locally convex domain near a point of finite type: it can be easily shown, using the work of Mc Neal [McN94], that if Ω is convex near a point of finite type $p_0 \in \partial\Omega$, if the canonical coordinate system is chosen so that the last coordinate is the complex normal at p_0, and, if P is the projection onto the complex tangent space of the defining function of Ω parallel to the last coordinate, then for each point p in a small neighborhood of p_0, and each $\delta \leq \delta_0$, the P-projection of the first $n-1$ vectors of the Mc Neal δ-extremal basis at p (c.f. [BCD98, McN94]) is (K, p, δ)-extremal in our sense for a constant K depending only on the data.

(2) *Levi form with comparable eigenvalues.* A second example is given by a pseudo-convex domain having a point of finite type $p_0 \in \partial\Omega$ where the eigenvalues of the Levi form are comparable (see [Koe02, Cho02b, Cho03, Cho02a]). Indeed, in [Cho03] it is proved that any (normalized) basis of the complex tangent space is K-extremal for a well controlled constant K.

(3) *Locally diagonalizable Levi form.* In Section 3.4 we will show that if at a point of finite type $p_0 \in \partial\Omega$ the Levi form is locally diagonalizable then the basis diagonalizing the Levi form is K-extremal for a constant K depending only on the data (in fact, this basis is K-strongly-extremal (see Definition 3.5) for every constant $\alpha > 0$ with $\delta \leq \delta_0$, δ_0 small depending on α).

(4) *Localization.* An other important example will be given in Section 3.5: for any $\tau > 0$ there exists $M(\tau)$ such that if a family of $(M(\tau), K, p, \delta)$-extremal basis exists in a neighborhood of a boundary point p_0, of finite type τ, of Ω then one can construct a small smooth pseudo-convex domain D containing a neighborhood of p_0 in $\partial\Omega$ in its boundary and for which there exists $(M(\tau), K', q, \delta)$-extremal basis at every points $q \in \partial D$.

3.2. **Basic properties of extremal basis.**

The first property states that an extremal basis at p can be orthogonalized at the point p:

Proposition 3.1. *For any K there exists a constant K' depending only on K and the data such that, if $\mathscr{B}$ is a basis of complex $(1,0)$ vector fields tangent to ρ in an open set $V(p_0)$ which is (K, p, δ)-extremal, there exists a basis $\mathscr{B}'$, orthonormal at p which is (K', p, δ)-extremal.*

Proof. We can suppose that the vector fields L_i of $\mathscr{B}$ are ordered such that $F(L_{i+1}, p, \delta) \leq F(L_i, p, \delta)$, for $i < n-1$. Then, using the Graam-Schmidt process, we first define a basis $\mathscr{B}_1$ by decreasing induction, $L_i^1 = \sum_{j=i}^{n-1} \alpha_i^j L_j$, $\alpha_i^j \in \mathbb{C}$, and $\sum \left| a_i^j \right|^2 =$

1. The determinant condition implies that there exists $c>0$ such that $\left|\alpha_i^i\right|>c$. Then

$$F(L_i^1,p,\delta)\simeq_K \sum_{j\geq i}\left|\alpha_i^j\right|^2 F(L_j,p,\delta)\simeq_K F(L_i,p,\delta).$$

Now, let $L=\sum_i a_i L_i^1$ be a linear combination, with constant coefficients, of the L_i^1. Then

$$F(L,p,\delta)\simeq_K \sum_k \left|\sum_{i\leq k} a_i a_i^k\right|^2 F(L_k,p,\delta)\simeq_K \sum |a_k|^2 F(L_k,p,\delta),$$

using $\left|\sum_{i\leq k} a_i a_i^k\right|\geq c\,|a_k|-\sum_{i<k}|a_i|$ and the fact that the $F(L_k,p,\delta)$ are decreasing. This proves EB$_1$ for $\mathscr{B}_1$.

Note now that property EB$_2$ for $\mathscr{B}$ trivially implies the same property for $\mathscr{B}_1$ because L_i^1 involves only fields L_j^1 for $j\geq i$ (and the decreasing property).

Finally, define $\mathscr{B}'$ by $L_i'=L_i^1/\left\|L_i^1\right\|$. The condition on the $\mathscr{C}^{2M}$ norm of the vectors L_i immediately implies the result. □

Let us now prove that the mixed derivatives of the Levi form in the directions of an extremal basis are controlled by the pure ones, that is by the weights associated to the vector fields of the basis:

Proposition 3.2. *Let $\mathscr{B}=\{L_i,\,1\leq i\leq n-1\}$ be a $\mathscr{C}^\infty$ basis of complex $(1,0)$ vector fields tangent to ρ in $V(p_0)$ which is (K,p,δ)-extremal for a fixed $\delta>0$. Let $\mathscr{L}$ be a list of vector fields belonging to $\mathscr{L}_M(\mathscr{B}\cup\{N\})$. Then there exits a constant $C>0$ depending only on Ω and K such that $|\mathscr{L}(\partial\rho)(p)|\leq C\delta F^{\mathscr{L}/2}(p,\delta)$.*

Proof. Recall the notation notations $c_{ij}=\left\langle\partial\rho,\left[L_i,\overline{L_j}\right]\right\rangle$.

Lemma 3.2.1. *With the previous notations (and the definition of the coefficients a_{ij}^s given in Definition 3.1):*

$$L_j c_{ik}=L_i c_{jk}+\sum a_{k\bar i}^{\bar s}c_{js}-\sum a_{ij}^s c_{sk}-\sum a_{j\bar k}^{\bar s}c_{is},$$

$$\overline{L_j}c_{ik}=\overline{L_k}c_{ij}+\sum a_{i\bar k}^{s}c_{sj}+\sum a_{j\bar i}^{s}c_{sk}-\sum a_{k\bar j}^{\bar s}c_{is}.$$

Proof. The first formula is simply obtained considering the coefficient of $\Im\mathrm{m}N$ in Jacobi's identity applied to the bracket $\left[L_j,\left[L_i,\overline{L_k}\right]\right]$, and the second using $\left[\overline{L_j},\left[L_i,\overline{L_k}\right]\right]$. □

The proof of *Proposition 3.2* is done by induction on the length of the lists. Suppose first $|\mathscr{L}|=2$. Hypothesis EB$_1$ imply that, for all numbers a and b and all index i and j,

$$\left||a|^2c_{ii}+|b|^2c_{jj}+a\bar{b}c_{ij}+\bar{a}bc_{ji}\right|\lesssim\delta\left(|a|^2F_i+|b|^2F_j\right).$$

Suppose both F_i and F_j non zero. Taking $a=F_j^{1/2}F_i^{-1/2}\lambda$ and $b=\mu$, $|\lambda|$ and $|\mu|$ less than 1, the equivalence of norms in finite dimensional spaces gives the result. If $F_i=0$ or $F_j=0$ a similar argument gives $c_{ij}=c_{ji}=0$.

Now we use the following notation: if $\mathscr{L}\in\mathscr{L}(\mathscr{B}\cup\{N\})$, we denote by l_i^1 (resp. l_i^2) the number of times L_i (resp. $\overline{L_i}$) appears in $\mathscr{L}$ (thus $l_i=l_i^1+l_i^2$).

For lists of greater length, we prove, at the same time, by induction the estimate and the following Lemma:

Lemma 3.2.2. *Let $\mathscr{L}$ and $\mathscr{L}'$ be two lists of $\mathscr{L}_M(\mathscr{B}\cup\{N\})$, $\mathscr{L}(\partial\rho)=\mathscr{L}_1c_{ij}$ and $\mathscr{L}'(\partial\rho)=\mathscr{L}_1'c_{kl}$, such that $l_i^1={l'}_i^1$, $l_i^2={l'}_i^2$,. Then $\mathscr{L}(\partial\rho)\simeq\mathscr{L}'(\partial\rho)$ in the sense that*

$$\mathscr{L}(\partial\rho)-\mathscr{L}'(\partial\rho)=\sum_{\left|\widetilde{\mathscr{L}}\right|<|\mathscr{L}|}a_{\widetilde{\mathscr{L}}}\widetilde{\mathscr{L}}(\partial\rho),$$

where $a_{\widetilde{\mathscr{L}}}$ satisfy, $\forall\mathscr{L}''\in\mathscr{L}_M(\mathscr{B}\cup\{N\})$, $F^{\widetilde{\mathscr{L}}/2}\left|\mathscr{L}''a_{\widetilde{\mathscr{L}}}\right|\lesssim\delta F^{(\mathscr{L}+\mathscr{L}'')/2}$, the constant depending only on K and the data.

Suppose thus the estimates and the Lemma proved for all list of length less or equal to N.

First, we prove Lemma 3.2.2 for lists of length $N+1$. Let us write $\mathscr{L}(\partial\rho)=\mathscr{L}_1c_{ij}$ and $\mathscr{L}'(\partial\rho)=\mathscr{L}_2c_{kl}$. Then three cases can happen:

(1) $(i,j)=(k,l)$;
(2) $i\neq k$, $j\neq l$;
(3) $i\neq k$ and $j=l$ or $i=k$ and $j\neq l$.

The first case is a trivial consequence of EB$_2$. For the second, the hypothesis on the length and case (1) imply that there exists a list $\widetilde{\mathscr{L}}$ such that $\mathscr{L}_1c_{ij}\simeq\widetilde{\mathscr{L}}L_k\overline{L_l}c_{ij}$ and $\mathscr{L}_2c_{kl}\simeq\widetilde{\mathscr{L}}L_i\overline{L_j}c_{kl}$, in the sense of Lemma 3.2.2. By Lemma 3.2.1 and EB$_2$, $\overline{L_l}c_{ij}\simeq\overline{L_j}c_{il}$. The result is obtained using an other time EB$_2$, Lemma 3.2.1 and the induction hypothesis. The third case is similar.

Now we prove the estimate of the Proposition for lists of length $N+1$. Suppose that the vector fields are ordered so that there exists an integer $n_0\in\{0,\dots,n-1\}$ such that, for $k\leq n_0$, $F_k\neq0$, and, for $n-1\geq k>n_0$, $F_k=0$. Let $L=\sum a_jL_j$,

$a_j = \varepsilon\lambda_j F_{n_0}^{1/2} F_j^{-1/2}$ if $j \leq n_0$ and $a_j = \lambda_j$ if $j > n_0$, with $\left|\lambda_j\right| \leq 1$. If we apply the extremality property to $F(L)$, we obtain, for example, for all $k \leq N-1$,

$$\sup_{\left|\lambda_j\right|\leq 1} \left|L^k \overline{L}^{N-k-1} c_{LL}\right| \lesssim \delta\varepsilon^{(N+1)/2} F_{n_0}^{(N+1)/2}$$

with the convention $F_{n_0} = 0$ if $n_0 = 0$. Writing $L^k \overline{L}^{N-k-1} c_{LL} = \sum C_{\alpha\beta}\lambda^\alpha \overline{\lambda}^\beta$, the equivalence of norms in finite dimensional spaces gives, when $\varepsilon \rightarrow 0$,

$$\begin{array}{ll} C_{\alpha\beta} = 0 & \text{if there exits } j > n_0 \text{ such that } \alpha_j + \beta_j > 0, \\ \left|C_{\alpha\beta}\right| \lesssim \delta F^{(\alpha+\beta)/2} & \text{otherwise.} \end{array} \tag{3.1}$$

Let $\mathscr{E}_{\alpha\beta}$ be the set of lists $\mathscr{L}$ such that $l_i^1(\mathscr{L}) = \alpha_i$ and $l_i^2 = \beta_i$. Then $C_{\alpha\beta} = \sum_{\mathscr{L}\in\mathscr{E}_{\alpha\beta}} \mathscr{L}(\partial\rho)$. Then, Lemma 3.2.2 and the induction hypothesis give the expected estimation for each list in $\mathscr{E}_{\alpha\beta}$ and finishes the proof of the Proposition. ☐

3.3. **Adapted coordinates system for points of finite 1-type.**

3.3.1. *Definition of an adapted coordinate system and statement of the main result.* Let $p_0 \in \partial\Omega$ and $V(p_0)$ a neighborhood of p_0 in $\mathbb{C}^n$.

Definition 3.2. A basis $\mathscr{B} = (L_1, \dots, L_{n-1})$ of sections of $(1,0)$ complex tangent vector fields to ρ in $V(p_0)$ and a coordinate system in $\mathbb{C}^n$, $z = \Phi_p^\delta(Z)$, are called (M,K,δ)adapted (or simply (K,δ) *adapted) at the point* p in $V(p_0)$ if Φ_p^δ and $(\Phi_p^\delta)^{-1}$ are polynomial maps (of degree less than $(2M)^{n-1}$) diffeomorphisms of $\mathbb{C}^n$ centered at p (i.e. $\Phi_p^\delta(p) = 0$) satisfying (with the notation $F_i = F_i(p,\delta) = F(L_i, p, \delta)$):

1. The coefficients of the polynomials of Φ_p^δ and $(\Phi_p^\delta)^{-1}$ (and the Jacobians of Φ_p^δ and $(\Phi_p^\delta)^{-1}$) are bounded by K;
2. For all $|\alpha| \leq 2M$, $\frac{\partial^\alpha\left(\rho\circ(\Phi_p^\delta)^{-1}\right)(0)}{\partial z'^\alpha} = \frac{\partial^\alpha\left(\rho\circ(\Phi_p^\delta)^{-1}\right)(0)}{\partial \overline{z'}^\alpha} = 0$, $z' = (z_1, \dots, z_{n-1})$;
3. If $L_i = \sum a_i^j \frac{\partial}{\partial z_i}$, then $a_i^j(0) = \delta_{ij}$ and for all $\mathscr{L} \in \mathscr{L}_M(\mathscr{B}\cup\{N\})$,
$$\left|\mathscr{L}a_i^j\right| \leq K \text{ in } \Phi_p(V(p_0)) \text{ and } F_j^{1/2}\left|\mathscr{L}a_i^j(0)\right| \leq KF_i^{1/2}F^{\mathscr{L}/2};$$
4. For all (α,β), $|\alpha+\beta| \leq M$, $\left|\frac{\partial^{\alpha\beta}\left(\rho\circ(\Phi_p^\delta)^{-1}\right)(0)}{\partial z^\alpha \partial \overline{z}^\beta}\right| \leq K \min\left\{\delta F^{(\alpha+\beta)/2}, 1\right\}$;

One of our main goals is to prove the following existence Theorem:

Theorem 3.1. *Suppose* p_0 *is of finite* 1*-type* τ*, and choose an integer* M *larger than* $2\left(\frac{2\left(\frac{\tau}{2}\right)^{n-1}+1}{2}\right)^{n-1}$. *For any positive constant* K*, there exists a constant* $\delta_0 > 0$*, a neighborhood* $V(p_0)$*, both depending on the data, and a constant* K' *depending on* K *and the data such that if* $\mathscr{B} = \{L_i, 1 \leq i \leq n-1\}$ *is a* $\mathscr{C}^\infty$ *basis of* $(1,0)$ *complex vector fields tangent to* ρ *in* $V(p_0)$ *which is* (M,K,p,δ) *extremal at a point* $p \in V(p_0)\cap\partial\Omega$*, then there exists a coordinate system* $(z_i)_{1\leq i\leq n}$ *centered at* p *which is* (K',δ)*-adapted to* $\mathscr{B}$*.*

To proof is divided in two steps: in the next Section we work without the assumption of finite type and construct an adapted coordinate system using modified weights; then in Section 3.3.3 we use the finite type hypothesis to deduce the Theorem.

3.3.2. *Construction of an adapted coordinate system.* In this Section we suppose that the integer M is fixed. Let $p \in V(p_0)$ and $\delta > 0$. Suppose $\mathscr{B} = (L_1, , L_{n-1})$ is a basis of $(1,0)$ vector fields tangent to ρ in $V(p_0)$, satisfying the following properties:

(A) The $\mathscr{C}^{2M}(V(p_0))$ norms of the L_i are bounded by K and $\mathscr{B}$ is ordered so that $F(L_{i+1}, p, \delta) \leq F(L_i, p, \delta)$.
(B) Let $p \in W(p_0) \Subset V(p_0)$ and $\delta > 0$. Denoting $\widetilde{F_i} = F_i + 1 = F(L_i, p, \delta) + 1$:
  (B$_1$) For all list $\mathscr{L} \in \mathscr{L}_M(\mathscr{B}\cup\{N\})$, $|\mathscr{L}(\partial\rho)(p)| \leq K\delta\widetilde{F}(p,\delta)^{\mathscr{L}/2}$;
  (B$_2$) $\mathscr{B}$ satisfies condition EB$_2$ of Definition 3.1 with the $F(L_s, p, \delta)$ replaced by the $\widetilde{F_s}$.

Then under these hypothesis, we have:

Proposition 3.3. *There exists a constant* K' *depending on* K*,* M *and the data (but neither on* p *nor on* δ*) such that there exists a* (M,K',δ)*-adapted coordinate system to* $\mathscr{B}$ *at* p *in the sense of Definition 3.2, the weights* $F(L_i,p,\delta)$ *being replaced by* $\widetilde{F_i}$*.*

Proof. In [CD06b] (Prop 3.2, p. 85) we proved that hypothesis (A) implies the existence of a coordinate system $\Phi_{p,\delta}$ satisfying conditions (1) and (2) of Definition 3.2 and

$$\left\{\begin{array}{c} \text{For } j < i < n, \text{ and } \alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{N}^{n-1} \text{ such that } |\alpha| \leq M,\ \alpha_p = 0 \text{ if } p > i \text{ or } p \leq j, \\ \frac{\partial^\alpha a_i^j(0)}{\partial z'^\alpha} = 0. \end{array}\right. \tag{3.2}$$

We now prove that under condition (B) the two last properties of Definition 3.2 (with the $\widetilde{F_i}$) are satisfied. This follows quite closely the ideas of p. 87-90 of [CD06b], but, as the context here is more general and as it is a fundamental tool, we write it completely.

Let $\mathscr{L} \in \mathscr{L}(\mathscr{B} \cup \{N\})$ considered as a differential operator. Denoting $D^{\alpha\beta}$ the derivative $\frac{\partial^{\alpha+\beta}}{\partial z^{\alpha}\partial \bar{z}^{\beta}}$ in the coordinate system $z = \Phi_p^{\delta}$, it is easy to see that, if $|\mathscr{L}| = S$,

$$\mathscr{L} = \sum_{\substack{m\in\mathbb{N}^n \\ 1\leq|m|\leq S}} \sum_{\alpha_i+\beta_i=m_i} c_{\alpha\beta}^{\mathscr{L}} D^{\alpha\beta}$$

where

$$c_{\alpha\beta}^{\mathscr{L}} = c_{\alpha\beta} = \sum_{p=1}^{S} \sum * a_{j_1}^{\overset{(-)}{i_1}} \cdots a_{j_p}^{\overset{(-)}{i_p}} \prod_{k=p+1}^{S} D^{s_k}\left(a_{j_k}^{\overset{(-)}{i_k}}\right)$$

where the summation in the second formula is taken over the derivatives associated to the multiindex s_k satisfying $\sum_{k=p+1}^{S} s_k + (m_1, \cdots, m_n) = \sum_{k=1}^{S} \chi(i_k)$, $\sum_{k=1}^{S} \chi(j_k) = (l_1, \cdots, l_{n-1}, l_n)$ and the coefficients $*$ are absolute constants. The following Lemma is then easily established:

Lemma 1. *If for all $s \in \mathbb{N}^n$, $|s| \leq S$, we have $\left|D^s a_j^i(0)\right| \lesssim_{K_1} \widetilde{F}^{s/2}\widetilde{F}_j^{1/2}\widetilde{F}_i^{-1/2}$, then we have*

$$\left|c_{\alpha\beta}(0)\right| \lesssim \widetilde{F}^{\mathscr{L}/2}\widetilde{F}^{-\frac{\alpha+\beta}{2}}. \tag{3.3}$$

To fix notations, recall that if f is a $\mathscr{C}^2$ function and L and L' two vector fields, then $\left\langle \partial\bar{\partial} f; L, \overline{L}\right\rangle = \overline{L'}Lf + \left[L, \overline{L'}\right](\partial f)$, and, in particular, if $L\rho = 0$, $\left\langle \partial\bar{\partial}\rho; L, \overline{L}\right\rangle = \left[L, \overline{L}\right](\partial\rho) = c_{LL}$, where c_{LL} is the coefficient of the Levi form in the direction L. In all the proof that follows, we denote $\left[L_i, \overline{L_j}\right](\partial\rho) = c_{ij}$.

To state the second Lemma let us introduce the notation $\tilde{\rho} = \rho \circ (\Phi_p^{\delta})^{-1}$):

Lemma 2. (1) *For every multiindex l, $|l| \leq 2M$, we have $\left|D^l\tilde{\rho}(0)\right| \lesssim \delta\widetilde{F}^{l/2}$, where D^l is any derivative $\frac{\partial^{|l|}}{\partial z^{\alpha}\partial\bar{z}^{\beta}}$ with $|\alpha + \beta| = l$.*

(2) *For every multiindex $m \neq (0, \cdots, 0)$, $|m| < M$, and every i, j, $\left|D^m a_i^{\overset{(-)}{j}}(0)\right| \lesssim \widetilde{F}^{m/2}\widetilde{F}_i^{1/2}\widetilde{F}_j^{-1/2}$.*

Proof. Note first that, for (2), it suffices to get the estimate for $D^m a_i^j(0)$ and that the estimate (1) (resp. (2)) is trivial if $l_n > 0$ (resp. $m_n > 0$) (recall $F_n = \delta^{-2}$ and the fact that the fields L_i are of $\mathscr{C}^{2M}$ norms controlled). We then suppose $l_n = m_n = 0$. The proof is done by induction: The induction hypothesis $\mathscr{P}_{k_0}$ is the two conclusions of the proposition for $|l| \leq k_0$ and $|m| < k_0$.

Remark first that $\mathscr{P}_{k_0}$ and the first property of $\mathscr{P}_{k_0+1}$ imply the second property of $\mathscr{P}_{k_0+1}$ for $j = n$: this is evident if $i = j = n$ and, if $i < j = n$, $L_i r \equiv 0$ implies

$$a_i^n = \left(\frac{\partial\tilde{\rho}}{\partial z_n}\right)^{-1} \sum_{k=1}^{n-1} a_i^k \frac{\partial\tilde{\rho}}{\partial z_k},$$

and the result is clear because $\frac{\partial\tilde{\rho}}{\partial z_k}(0) = 0$ for $k < n$.

Moreover, note also that, the weights $\widetilde{F}_i$, $i \leq n-1$, being "decreasing", the second inequality of $\mathscr{P}_{k_0}$ is trivial if $i \leq j < n$ and if $i = n$. Thus it suffices to prove this inequality when $j < i < n$.

Let us now prove $\mathscr{P}_{k_0}$ by induction. The case $k_0 = 1$ is trivial. Let us study first the case $k_0 = 2$. By definition of the coordinate system, $\frac{\partial^2\tilde{\rho}}{\partial z_i\partial z_j}(0) = 0$, and, using the notations and remarks stated before the statement of the Lemma, we have

$$a_i^i\overline{a_j^j}\frac{\partial^2\tilde{\rho}}{\partial z_i\partial\overline{z_j}} = c_{i\bar{j}} - \sum_{(k,p)\neq(i,j)} a_i^k\overline{a_j^p}\frac{\partial^2\tilde{\rho}}{\partial z_k\partial\overline{z_p}} \tag{3.4}$$

which implies $\frac{\partial^2\tilde{\rho}}{\partial z_i\partial\overline{z_j}}(0) = c_{i\bar{j}}(0)$ and gives the first inequality by definition of F. To prove the second inequality, let us look at the definition of the functions $a^{\overset{(-)}{k}}_{\overset{(-)}{i}\overset{(-)}{j}}$. Writing the bracket $[L_i, \overline{L_p}]$ with the coordinate system and taking the component of $\frac{\partial}{\partial z_j}$, we get

$$\sum_{k'=1}^{n-1} a_{i\bar{p}}^{k'} a_{k'}^j = -\sum_{k=1}^{n} \overline{a_p^k}\frac{\partial}{\partial\overline{z_k}}\left(a_i^j\right) - c_{i\bar{p}}a_n^j. \tag{3.5}$$

Extracting the term $\frac{\partial}{\partial\overline{z_p}}\left(a_i^j\right)$ and taking all at zero we obtain $\frac{\partial}{\partial\overline{z_p}}\left(a_i^j\right)(0) = a_{i\bar{p}}^j(0)$ and the inequality follows (B_2) hypothesis.

We have now to consider $\frac{\partial a_i^j}{\partial z_q}$. If $q \leq j$, the inequality comes from the decreasing property of the $\widetilde{F}_k$, and if $j < q \leq i$, this derivative is zero at the origin by the properties of the coordinate system. Suppose then $j < i < q$. Looking at the Lie bracket $[L_i, L_q]$ and taking the component of $\frac{\partial}{\partial z_j}$, we obtain

$$a_i^i\frac{\partial}{\partial z_i}\left(a_q^j\right) - a_q^q\frac{\partial}{\partial z_q}\left(a_i^j\right) = \sum_{k\neq q} a_q^k\frac{\partial}{\partial z_k}\left(a_i^j\right) - \sum_{k\neq i} a_i^k\frac{\partial}{\partial z_k}\left(a_q^j\right) + \sum_{p=1}^{n-1} a_{iq}^p a_p^j, \tag{3.6}$$

and then, at the origin, $\frac{\partial}{\partial z_q}\left(a_i^j\right)(0) = \frac{\partial}{\partial z_i}\left(a_q^j\right)(0) - a_{iq}^j(0) = -a_{iq}^j(0)$, by the properties of the coordinate system, and the conclusion comes again from (B$_2$). This proves $\mathscr{P}_2$.

Let us now suppose $\mathscr{P}_{k_0}$ verified ($k_0 < 2M$). Let $D^{\tilde{l}}$ be a derivative of order k_0+1. If $D^{\tilde{l}}$ is purely holomorphic or anti-holomorphic, $D^{\tilde{l}}\tilde{\rho}(0) = 0$. Then we suppose $D^{\tilde{l}} = D^l \frac{\partial}{\partial z_i}\frac{\partial}{\partial \overline{z_j}}$, and we denote by $\widetilde{\mathscr{L}} = \mathscr{L} L_i \overline{L_j}$ a list of vectors fields associated to $D^{\tilde{l}}$ (in the obvious sense that, if $\partial/\partial z_i$ (resp. $\partial/\partial \overline{z_i}$) appears l_i (resp $\overline{l_i}$) times in D^l then L_i (resp. $\overline{L_i}$) appears l_i (resp $\overline{l_i}$) times in $\mathscr{L}$). Applying (3.4), we get

$$\begin{aligned}
D^l\left(\frac{\partial^2\tilde{\rho}}{\partial z_i \partial \overline{z_j}}\right)(0) \quad &= \quad \mathscr{L} c_{i\bar{j}}(0) - \sum_{\substack{l_1 \neq 0 \\ l_1+l_2=l}} * D^{l_1}\left(a_i^i \overline{a_j^j}\right) D^{l_2}\left(\frac{\partial^2\tilde{\rho}}{\partial z_i \partial \overline{z_j}}\right)(0) \\
&\quad - \sum_{(k,p)\neq(i,j)} D^l\left(a_i^k \overline{a_j^p} \frac{\partial^2\tilde{\rho}}{\partial z_k \partial \overline{z_p}}\right)(0) \\
&\quad - \sum_{|\alpha'|+|\beta'|<k_0-1} c_{\alpha'\beta'} D^{\alpha'\beta'}(c_{i\bar{j}})(0),
\end{aligned} \tag{3.7}$$

with $* = 0$ or 1. The first term of the second member of (3.7) satisfies the wright inequality (i.e. $\lesssim \delta \widetilde{F}^{l/2}\widetilde{F}_i^{1/2}\widetilde{F}_j^{1/2}$ in modulus) by (B$_1$). For the second, l_1 being non 0, we can apply the induction hypothesis to $D^{l_2}\left(\frac{\partial^2\tilde{\rho}}{\partial z_i \partial \overline{z_j}}\right)(0)$ to get the wright estimate. The third term is of the same nature because, for $(k,p) \neq (i,j)$, $a_i^k \overline{a_j^p}(0) = 0$. If we replace $c_{i\bar{j}}$ by its expression in (3.4), the induction hypothesis $\mathscr{P}_{k_0}$ implies directly (for $s < k_0 - 1$):

$$\left|D^s c_{i\bar{j}}(0)\right| \lesssim \delta F^{s/2} F_i^{1/2} F_j^{1/2},$$

and then, using Lemma 1 for $S = k_0$ (whose hypothesis are also verified by the induction hypothesis $\mathscr{P}_k$), we prove that the last term in (3.7) satisfies also the wright estimate.

We finish now proving the second inequality of $\mathscr{P}_{k_0+1}$. It suffices to consider the case $j < i < n$. Let us first look at a derivative D^m of the form $D^m = D^s \frac{\partial}{\partial z_p}$, $|s| = k_0 - 1$. Using formula (3.5), we can write

$$D^m a_i^j = D^s\left(\sum_{t=1}^{n-1} \circledast a_{i\bar{p}}^t a_t^j - \sum_{t\neq p} \circledast \overline{a_p^t} \frac{\partial}{\partial \overline{z_t}}(a_i^j) + \circledast c_{i\bar{p}} a_n^j\right) = D^s(A) - D^s(B) + D^s(C),$$

where $\circledast$ is equal to $\frac{1}{a_p^p}$. In $D^s(B)$, to get a non zero term at 0, $\overline{a_p^t}$ must be derivated because $p \neq t$; this gives derivatives of $\frac{\partial}{\partial \overline{z_k}}(a_i^j)$ of order $< k_0 - 1$ which are well controlled by the induction hypothesis and then $|D^s(B)(0)| \lesssim \widetilde{F}^{m/2}\widetilde{F}_i^{1/2}\widetilde{F}_j^{1/2}$.

Consider now the terms $D^s\left(\circledast a_{i\bar{p}}^t a_k^j\right)$.

Claim. For $|l| \leq k$, $D^l\left(a_{i\overset{(-)}{p}}^t\right) \lesssim F_i^{1/2} F_p^{1/2} F_t^{-1/2} F^{l/2}$.

Proof of the Claim. We do it by induction on $|l|$. (B$_2$) proves the result for $|l| = 0$. Suppose the claim proved for $|l| < k' \leq k_0 - 1$ and suppose $|l| = k'$. Then,

$$D^l a_{i\overset{(-)}{p}}^t(0) = \mathscr{L}^l a_{i\overset{(-)}{p}}^t(0) + \sum_{|s'|<l} c_{s'}(0) D^{s'} a_{i\overset{(-)}{p}}^t(0).$$

But, by (B$_2$),

$$\left|\mathscr{L}^l a_{i\bar{p}}^t(0)\right| \lesssim F^{l/2} F_i^{1/2} F_p^{1/2} F_t^{-1/2},$$

and for the second term of the previous identity, we have $|s'| < l$ and we can apply the induction hypothesis and Lemma 1 whose hypothesis are satisfied, using $\mathscr{P}_{k_0}$, because $|l| \leq k_0$. □

Then the estimate of $D^s\left(\circledast a_{i\bar{p}}^k a_k^j\right)$ follows the induction hypothesis $\mathscr{P}_{k_0}$ because $|s| < k_0$. Thus we get $|D^s(A)(0)| \lesssim \widetilde{F}^{m/2}\widetilde{F}_i^{1/2}\widetilde{F}_j^{1/2}$.

Finally, the terms $D^s\left(\circledast c_{i\bar{p}} a_n^j\right)$ satisfies also the good estimates because $a_n^j(0) = 0$ and, for $|s'| < k_0 - 1$, we have seen that $\left|D^{s'}(c_{i\bar{p}})(0)\right| \lesssim \delta \widetilde{F}^{s'/2}\widetilde{F}_i^{1/2}\widetilde{F}_p^{1/2}$, and, the derivatives of a_n^j are controlled by the induction hypothesis $\mathscr{P}_{k_0}$.

To finish, we have to consider the case where D^m is a holomorphic derivative. Note that the inequality is trivial if $i \leq j$ or if there exists $k \leq j$ such that $m_k \neq 0$. Suppose then, for all $k \leq j$, $m_k = 0$ and $j < i < n$. Let q the largest index such that $m_q > 0$. If $q \leq i$, we have $D^m a_i^j(0) = 0$ by the properties of the coordinate system. If $q > i$, write $D^m = D^s \frac{\partial}{\partial z_q}$. To conclude it suffices then to use (3.6), the first Claim and the fact that $D^s \frac{\partial}{\partial z_i}\left(a_q^j\right)(0) = 0$ also by the properties of the coordinates system. This completes the proof of the Lemma. □

To finish the proof of Proposition 3.3, it suffices to note that, in addition to the estimates of the coefficients $c_{\alpha\beta}^{\mathscr{L}}$ given by Lemma 1, we also have, for $|\alpha+\beta| \leq 2M$,

$$D^{\alpha\beta} = \sum_{1 \leq |\mathscr{L}| \leq |\alpha+\beta|} d_{\mathscr{L}}^{\alpha\beta} \mathscr{L},$$

with $\left| d_{\mathscr{L}}^{\alpha\beta}(0) \right| \lesssim \widetilde{F}^{(\alpha+\beta)/2}(p,\delta) \widetilde{F}^{-\mathscr{L}/2}$. ☐

For the case of extremal basis we have thus proved (using Proposition 3.2):

Corollary. *If $\mathscr{B}$ is (M,K,p,δ)-extremal, for δ small enough, there exists a coordinate system $(M,K'(K),\delta)$-adapted to $\mathscr{B}$ in the sense of Definition 3.2with the weights F_i replaced by $\widetilde{F_i} = F_i + 1$.*

3.3.3. *Proof of Theorem 3.1.* If p_0 is a point of finite 1-type τ, then, by a Theorem of D'Angelo (see [D'A82, Cat87]) there exists a neighborhood $U(p_0)$ such that, if $p \in \partial\Omega \cap U(p_0)$, then p is of finite 1-type less than $\tau' = 2\left(\frac{\tau}{2}\right)^{n-1}$. We suppose $V(p_0) \subset U(p_0)$. Then, if $\mathscr{B}$ is a (M,K,p,δ)-extremal basis, by the Corollary of Proposition 3.3 we have a coordinate system $\Phi_{p,\delta}$ adapted to $\mathscr{B}$ in terms of the $\widetilde{F_i}$. Suppose M larger than $2\left(\frac{\tau'}{2}\right)^{n-1}$. Then, considering the manifold $\zeta \mapsto (0,\ldots 0,\zeta,0,\ldots,0)$, $|\zeta| \leq \sigma$, Theorem 3.4 of [Cat87] (applied with a suitable constant σ) gives us a derivative of $\tilde{\rho} = \rho \circ \Phi_{p,\delta}$ which is bounded from below by a constant depending only on the data. The last property of Definition 3.2 shows thus that $\widetilde{F_i}(p,\delta) \gtrsim \delta^{-2/M}$ with a constant depending only on the data, and, of course, the same is true for $F_i(p,\delta)$.

This proves the following essential Proposition:

Proposition 3.4. *Let $p_0 \in \partial\Omega$ be a point of finite 1-type τ. Let $M = M(\tau) = \left[2\left(\frac{\tau}{2}\right)^{n-1}\right] + 1$. Then for all integer K there exists a real number $\delta_0 > 0$ and a constant C, depending on K and the data, such that, if there exists a coordinate system (M,K,δ)-adapted to a basis $\mathscr{B} = (L_1,\ldots,L_{n-1})$ at p_0, then $F_M(L_i,p_0,\delta) \geq C\delta^{-2/M}$. In particular, if $\tau' = 2\left(\frac{\tau}{2}\right)^{n-1}$ and $M' = M'(\tau) = \left[2\left(\frac{\tau'}{2}\right)^{n-1}\right] + 1$, for all integer K there exists a neighborhood $V(p_0)$ a real number $\delta_0 > 0$ and a constant C (depending on τ, Ω and K) such that, for $p \in V(p_0) \cap \partial\Omega$ and $0 < \delta \leq \delta_0$, if there exists a coordinate system (M',K,δ)-adapted to a basis $\mathscr{B} = (L_1,\ldots,L_{n-1})$ at p, then $F(L_i,p,\delta) \geq C\delta^{-2/M'}$.*

This proves completely Theorem 3.1.

Remark 3.2. Note that the proofs show that if a basis $\mathscr{B}$ satisfies only properties (A) and (B) of the beginning of Section 3.3.2, then, under the assumption of finite 1-type, the conclusions of Proposition 3.4 and Theorem 3.1 are still valid.

A simple consequence (which will be used in Section 3.5) of the minoration of the weights F_i is the following:

Lemma 3.1. *Suppose the point p_0 of finite 1-type τ. For any K, there exists two constants C and δ_0, depending only on K, τ and the data, such that if $\mathscr{B} = \{L_i^{p,\delta}, i < n\}$ is (K,p,δ)-extremal, $p \in WV(p_0) \cap \partial\Omega$, and (α_i) is a family of $\mathscr{C}^\infty$ functions, of $\mathscr{C}^{2M}$ norm $\leq K$ and $1/K \leq |\alpha_i| \leq K$, then the basis $\mathscr{B}_1 = \{L_i\}$, where $L_i = \frac{1}{\alpha_i} L_i^{p,\delta}$, is (C,p,δ)-extremal, and, moreover, $F\left(\sum a_i L_i, p, \delta\right) \simeq_C F\left(\sum a_i L_i^{p,\delta}, p, \delta\right)\Big)$, $a_i \in \mathbb{C}$.*

3.3.4. *Associated polydiscs and pseudo-balls for finite type points.* In this Section we suppose p_0 of finite 1-type τ and we choose $M = M'(\tau)$. Now we will associate to an adapted coordinate system some special "polydiscs" and give some related properties.

Definition 3.3. Let $W(p_0) \Subset V(p_0)$ small enough. Suppose that for some point $p \in W(p_0) \cap \partial\Omega$ and $0 < \delta$ there exists a basis $\mathscr{B}(p,\delta) = \left\{L_i^{p,\delta}\right\}$ of $(1,0)$ vector fields tangent to ρ in $V(p_0)$ satisfying conditions (A) and (B) (of Section 3.3.2) and let $\Phi_p^\delta = \Phi_p$ the coordinate system which is (K,δ)-adapted to $\mathscr{B}(p,\delta)$. Then the functions $F(L_i,p,\delta) = F_i(p,\delta)$ does not vanish and, for $0 < c < 1$, we denote

$$\Delta_c(p,\delta) = \{z \in \mathbb{C}^n \text{ such that } |z_i| < cF_i^{-1/2}, 1 \leq i \leq n\},$$

and

$$B^c(p,\delta) = \Phi_p^{-1}(\Delta_c(p,\delta)) \cap V(p_0).$$

Taylor's formula, Proposition 3.2 and Theorem 3.1 lead easily to the following properties (denoting $L_i = L_i^{p,\delta}$):

Proposition 3.5. *There exists three constants c_0, K_0 and δ_0, depending only on K and the data, such that the following properties hold:*

(1) *If $L_i = \sum a_i^j \frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial z_j} = \sum b_j^i L_i$, $|\alpha+\beta| \leq M$, for $z \in \Delta_{c_0}(p,\delta)$,*

$$\begin{aligned} \left| D^{\alpha\beta} a_i^j(z) \right| &\leq K_0 F^{(\alpha+\beta)/2}(p,\delta) F_i^{1/2}(p,\delta) F_j^{-1/2}(p,\delta), \\ \left| D^{\alpha\beta} b_i^j(z) \right| &\leq K_0 F^{(\alpha+\beta)/2}(p,\delta) F_i^{1/2}(p,\delta) F_j^{-1/2}(p,\delta). \end{aligned}$$

(2) *If $\mathscr{L} \in \mathscr{L}_M(\mathscr{B}(p,\delta)\cup\{N\})$, $|\mathscr{L}| = S$, and D^T is a derivative in the coordinate system (z) with $|T| \leq M$, then $\mathscr{L} = \sum_{|s|\leq S} c_s D^s$, $D^T = \sum_{|\mathscr{L}'|\leq|T|} d_{\mathscr{L}'}\mathscr{L}'$, and, for $z \in \Delta_{c_0}(p,\delta)$ and $q = \Phi_p(z)$ we have*

$$\begin{array}{rcl} c_s(z) & \leq & K_0 F^{(\mathscr{L}-s)/2}(p,\delta), \\ d_{\mathscr{L}'}(q) & \leq & K_0 F^{(\mathscr{L}-\mathscr{L}')/2}(p,\delta). \end{array}$$

(3) *For $L = \sum a_i L_i$, $a_i \in \mathbb{C}$, for all $q \in B^{c_0}(p,\delta)$, $\frac{1}{2}F(L,p,\delta) \leq F(L,q,\delta) \leq 2F(L,p,\delta)$.*

(4) *For all list $\mathscr{L}$, $|\mathscr{L}| \leq M$ belonging to $\mathscr{L}_M(\mathscr{B})$ and all point $q \in B^c(p,\delta)$,*
   (a) $|\mathscr{L}(\partial\rho)(q)| \leq K_0 \delta F(p,\delta)^{\mathscr{L}/2}$,
   (b) *with the notation introduced in EB_2 in Definition 3.1,*

$$\left|\mathscr{L} a^{\overset{(-)}{k}}_{\overset{(-)}{i}\overset{(-)}{j}}(q)\right| \leq K_0 F^{\mathscr{L}/2}(p,\delta) F_i^{1/2}(p,\delta) F_j^{1/2}(p,\delta) F_k^{-1/2}(p,\delta).$$

(5) $\rho(B^c(p,\delta)) \subset [-\frac{1}{2}\delta, \frac{1}{2}\delta]$.

The proofs are almost straightforward calculus.

In Section 4 we will need to use two other kind of “pseudo-balls” and we will prove that they are closely related to the “polydisc” B^c:

Definition 3.4. Suppose that $\mathscr{B} = (L_1,\ldots,L_{n-1})$ is a basis satisfying conditions (A) and (B) (at a point of finite 1-type).

(1) Denote $\mathscr{Y}_i = \Re\mathrm{e} L_i$ and $\mathscr{Y}_{i+n} = \Im\mathrm{m} L_i$, $1 \leq i \leq n$ (recall $L_n = N$). Then we denote by $B^c_{\mathscr{C}}(\mathscr{B},p,\delta)$ the set of points $q \in V(p_0)$ for which there exists a piecewise $\mathscr{C}^1$ curve $\varphi : [0,1] \to V$ such that $\varphi(0) = p$, $\varphi(1) = q$ and $\varphi'(t) = \sum a_i \mathscr{Y}_i(\varphi(t))$, with $|a_i|$ and $|a_{i+n}| \leq cF^{-1/2}(L_i,p,\delta)$, $0 < c < 1$.

(2) $\exp_p$ denoting the exponential map based at p associated to the vector fields $\mathscr{Y}_i$ (defined in (1)), we put

$$B^c_{\exp}(p,\delta) = \left\{q = \exp_p(u_1,\ldots,u_{2n}),\ \text{such that}\ \max(|u_i|,|u_{i+n}|) \leq cF_i(p,\delta)^{-1/2}\right\} \cap V(p_0).$$

The terminology used in Definition 3.1 is justified by the following property:

Proposition 3.6. *Let $\mathscr{B} = \{L_1,\ldots,L_{n-1}\}$ be a basis (of $(1,0)$ complex vector fields, tangent to ρ in $V(p_0)$) satisfying conditions (A) and (B) (for example if it is K-extremal) at $p \in W(p_0) \cap \partial\Omega$. Let $\mathscr{B}^1 = \{L_1^1,\ldots,L_{n-1}^1\}$ be an other basis in $V(p_0)$ such that, for all i, $L_i^1 = \sum a_i^j L_j$, $a_i^j \in \mathbb{C}$, $\sum |a_i|^2 = 1$. Then there exists a constant A depending only on K, τ and the dimension n such that $B^c_{\mathscr{C}}(\mathscr{B}^1,p,\delta) \subset B^{Ac}_{\mathscr{C}}(p,\delta)$.*

The proof of this Proposition is immediate following property (B).

3.4. **Sufficient conditions of extremality.**

In this Section we always suppose that p_0 is a point of finite 1-type τ and choose $M = M(\tau)$.

Here and in Section 5.2 we will need a stronger control on certain derivatives of the coefficients of the Levi form. Thus we introduce the following condition: suppose $\mathscr{B}$ is a basis of $(1,0)$ vector fields tangent to ρ in $V(p_0)$. We say that it satisfy to condition B(α), $\alpha > 0$, if for all list $\mathscr{L} \in \mathscr{L}_{M-2}(\mathscr{B})$ we have

B(α) for $i \neq j$, $1 \leq i,j \leq n-1$, $\left|\mathscr{L}c_{ij}(p)\right| \leq \alpha\delta F(p,\delta)^{\mathscr{L}/2} F(L_i,p,\delta)^{1/2} F(L_j,p,\delta)^{1/2}$.

Note that B(α) together with conditions (A) and (B) implies a new condition on the brackets of the vector fields:

Lemma 3.2. *Suppose $\mathscr{B}$ satisfies conditions (A) and (B). Then there exists two constants $K_1 = K_1(K,M,n)$ and $\delta_0$depending on K, α and the data such that, for all $i \neq k$, $i,k < n$, $j \leq n$ and all $\mathscr{L} \in \mathscr{L}_M(\mathscr{B}\cup\{N\})$, if $\mathscr{B}$ satisfy B(α) at p,δ), $p \in W(p_0)$, $0 < \delta \leq \delta_0$,*

$$\left|\mathscr{L} a^{\overset{(-)}{j}}_{\overset{(-)}{i}\overset{(-)}{k}}(p)\right| \leq K_1 \alpha F(p,\delta)^{\mathscr{L}/2} F_i(p,\delta)^{1/2} F_k(p,\delta)^{1/2} F_j(p,\delta)^{-1/2}.$$

Proof. To simplify the notations we write the proof for $a^{\bar{j}}_{j\bar{k}}$. Choose $\delta_0$so that $C\delta_0^{-2/M} > \alpha^{-1}$, where C is the constant of Proposition 3.4. Note that the property is trivial if $l_n \neq 0$ or if $l_n = 0$ and $j = n$ ($a^{\bar{n}}_{ik} = \frac{1}{2}c_{ik}$ and $a^{\bar{n}}_{ik} = 0$), thus we suppose $l_n = 0$ and $j < n$. As the property is also trivial if j or k is $\geq i$, we have to study only the case when $j < \min(i,k)$.

To simplify the notations, we introduce the following spaces of functions:

$$\underset{0}{\ast} = \{\varepsilon,\ \varepsilon a^{\overset{(-)}{k}}_{\overset{(-)}{i}\overset{(-)}{j}},\ \varepsilon c_{\overset{(-)}{i}\overset{(-)}{j}},\ \text{where}\ \varepsilon \in \{-1,0,1,-\sqrt{-1},\sqrt{-1}\}\},$$

and

$$\underset{k+1}{\widetilde{\ast}} = \bigcup_i \overset{(-)}{L}_i(\underset{k}{\ast}) \cup \underset{k}{\ast}\ \text{and}\ \underset{k+1}{\ast} = \left\{\sum_{i=1}^{3} f_i,\ f_i \in \underset{k+1}{\widetilde{\ast}}\right\}.$$

The elements of $\underset{k}{\ast}$ will be generically denoted by $*_k$.

The Jacobi identity applied to the bracket $\left[L_j,\left[L_i,\overline{L_k}\right]\right]$ imply

$$a_{i\bar{k}}^{\bar{j}}c_{jj}+L_jc_{ik}+\sum_{p\neq j}a_{j\bar{k}}^{\bar{p}}c_{jp}-a_{j\bar{k}}^{\bar{i}}c_{ii}-L_ic_{jk}-\sum_{p\neq i}a_{j\bar{k}}^{\bar{p}}c_{ip}-a_{ij}^{k}c_{kk}-\sum_{p\neq k}a_{ji}^{p}c_{pk}=0$$

which we write $a_{i\bar{k}}^{\bar{j}}c_{jj}=*_0c_{ii}+*_0c_{kk}+h$. Then, by induction on the length l of a list $\mathscr{L}\in\mathscr{L}_M(L_j)$, it is easy to show that

$$a_{i\bar{k}}^{\bar{j}}\mathscr{L}c_{jj}=\mathscr{L}h+\sum_{\mathscr{L}'\in\mathscr{L}_{|\mathscr{L}|}(L_j)}\left(*_l\mathscr{L}'c_{ii}+*_l\mathscr{L}'c_{kk}\right)+\sum_{\mathscr{L}'\in\mathscr{L}_{|\mathscr{L}|-1}(L_j)}*_l\mathscr{L}'c_{jj},$$

and choosing $\mathscr{L}$ so that $\left|\mathscr{L}c_{jj}(p)\right|\gtrsim\delta F(p,\delta)^{(|\mathscr{L}|+2)/2}$, the Lemma is easily proved using the control on the lists and the hypothesis. □

Now we first prove that conditions B(α), (A) and (B) imply the extremality of the basis and then that Lemma 3.2implies a better control on mixed lists, result that will be important in Section 5.

Lemma 3.3. *Suppose that $\mathscr{B}=(L_1,\ldots,L_{n-1})$ is a basis of $(1,0)$ vector fields in $V(p_0)$ satisfying conditions (A) and (B) at a point $p\in V(p_0)\cap\partial\Omega$ for a fixed δ. Then there exists a function $\alpha(K)$, depending on K and the data, such that, if $\mathscr{B}$ satisfy B(α) for $\alpha\leq\alpha(K)$, there exists a constant K_1, depending on K, M and n, such that, if $\mathscr{L}^0\in\mathscr{L}_M(\mathscr{B})$satisfies $\left|\mathscr{L}^0c_{ii}(p)\right|\geq\frac{1}{K}\delta F_i(p,\delta)F(p,\delta)^{\mathscr{L}/2}$ then there exists k_0, $2k_0+2\leq|\mathscr{L}|$, such that $\mathfrak{Re}\left(\left(L_i\overline{L_i}\right)^{k_0}c_{ii}\right)(p)>\frac{1}{K_1}\delta F_i(p,\delta)^{(2k_0+2)/2}$. In particular,*

$$F_i(p,\delta)\geq\frac{1}{K'}\sum_{\substack{\mathfrak{Re}\left(\left(L_i\overline{L_i}\right)^kc_{ii}\right)(p)>0\\2k+1\leq M}}\left[\frac{\mathfrak{Re}\left(\left(L_i\overline{L_i}\right)^kc_{ii}\right)(p)}{\delta}\right]^{\frac{2}{2k+2}},$$

where K' is a constant depending only on K and the data.

Proof. We know that there exists a coordinate system Φ_p^δ adapted to $\mathscr{B}$. These new coordinates are denoted (z_i). The derivatives $D^{\alpha\beta}$ are the derivatives with respect to (z_i), and if $\mathscr{L}$ is a list of vector fields then $D^{\mathscr{L}}$ is the derivative $\frac{\partial^{|\alpha+\beta|}}{\partial z^\alpha\partial\bar{z}^\beta}$ with $\alpha_i=l_i^1(\mathscr{L})$ and $\beta_i=l_i^2(\mathscr{L})$ (notation of Lemma 3.2.2). In the proof we will use a general result on derivatives of positives function proved in Section 7.

Suppose $\mathscr{L}\in\mathscr{L}_M(\mathscr{B})$ is such that $\mathscr{L}(\partial\rho)=\mathscr{L}^0c_{ii}$ and $|\mathscr{L}(\partial\rho)(p)|\gtrsim_K\delta F_i(p,\delta)F(p,\delta)^{\mathscr{L}/2}$. Then we can write

$$\mathscr{L}(\partial\rho)=D^{\mathscr{L}^0}c_{ii}+\sum_{|\alpha+\beta|<\left|\mathscr{L}^0\right|}c_{\alpha\beta}D^{\alpha\beta}c_{ii}$$

with $\left|c_{\alpha\beta}\right|\lesssim_K F^{\mathscr{L}^0/2}F^{-(\alpha+\beta)/2}$.

Thus there exists a derivative $D^{\alpha\beta}$ satisfying $\left|D^{\alpha\beta}c_{ii}(0)\right|\gtrsim_K\delta F_iF^{(\alpha+\beta)/2}$ and $|\alpha+\beta|\leq\left|\mathscr{L}^0\right|$ and $\alpha_n+\beta_n=0$ (indeed, if $\alpha_n+\beta_n\geq1$, $\left|c_{\alpha\beta}(0)\right|\lesssim_K F^{\mathscr{L}^0/2}F^{-(\alpha+\beta)/2}\leq\delta F^{\mathscr{L}^0/2}$, and as $\left|D^{\alpha\beta}c_{ii}\right|\lesssim_K 1$, $\left|c_{\alpha\beta}D^{\alpha\beta}c_{ii}\right|\ll\delta F^{\mathscr{L}/2}$). Then applying Lemma 7.1 to the function $g(z)=\delta F_i^{-1}(p,\delta)c_{ii}\circ\Phi_{p,\delta}^{-1}(z')$, where $z'=\left(cF_1^{-1/2}z_1,\ldots,cF_{n-1}^{-1/2}(p,\delta)z_{n-1},0\right)$ with $c\leq c_0$, c_0 given by Proposition 3.5, we conclude that there exists a derivative $D^{\alpha^1\beta^1}$, satisfying $\alpha_j^1=\beta_j^1$, $\forall j$, $\alpha_n^1=\beta_n^1=0$, such that $D^{\alpha^1\beta^1}c_{ii}(0)\geq_K F_iF^{(\alpha^1+\beta^1)/2}$.

Writing $\mathscr{L}'=\left(\overline{L_i}L_i\right)^{\alpha_i^1}\prod_{j\neq i,\,j<n}\left(\overline{L_j}L_j\right)^{\alpha_j^1}$ and $\mathscr{L}'c_{ii}=D^{\alpha^1\beta^1}c_{ii}+\sum_{|\alpha+\beta|<|\mathscr{L}'|}c_{\alpha\beta}D^{\alpha\beta}c_{ii}$, by induction we conclude that there exists a differential operator $\mathscr{L}^1$ of the form $\mathscr{L}^1=\left(\overline{L_i}L_i\right)^{\alpha_i}\prod_{j\neq i,\,j<n}\left(\overline{L_j}L_j\right)^{\alpha_j}$ such that $\mathfrak{Re}\left(\mathscr{L}^1c_{ii}\right)(p)\gtrsim_K\delta F^{\mathscr{L}^1/2}F_i$. Suppose there exists $j\neq i$ such that $\alpha_j\neq0$. Then

$$\mathscr{L}^1c_{ii}=\mathscr{L}'\overline{L_j}L_jc_{ii}=\mathscr{L}'\overline{L_j}\left(-\gamma_i^jc_{jj}+L_kc_{jk}+\left(a_{jk}^i-a_{ij}^{\bar{i}}\right)c_{ii}-\sum_{p\neq i}\left(a_{ij}^{\bar{p}}c_{ip}-a_{ji}^pc_{pi}\right)+\sum_{p\neq j}\gamma_i^{\bar{p}}c_{ip}\right).$$

The controls of the coefficients a_{ij}^p and of the lists $\mathscr{L}c_{kp}$, $k\neq p$ (by condition (B)), implies, for α sufficiently small (depending only on K), that

$$\left|\mathscr{L}'\overline{L_j}c_{jj}\right|\gtrsim_K\delta F^{\mathscr{L}'/2}F_j^{3/2}\text{ and }\left|\gamma_i^j\right|\gtrsim_K F_iF_j^{-1/2}.$$

Repeating the initial procedure, we conclude that there exists a list $\mathscr{L}''\in\mathscr{L}(\mathscr{B})$, “completely even”, $|\mathscr{L}''|\leq|\mathscr{L}'|$ such that $\left|\mathscr{L}''c_{jj}\right|\gtrsim_K\delta F^{\mathscr{L}''/2}F_j$. Consider then

$$\mathscr{L}''\overline{L_j}c_{ii}=\mathscr{L}''\left(-\gamma_i^jc_{jj}+L_kc_{jk}+\left(a_{jk}^i-a_{ij}^{\bar{i}}\right)c_{ii}-\sum_{p\neq i}\left(a_{ij}^{\bar{p}}c_{ip}-a_{ji}^pc_{pi}\right)+\sum_{p\neq j}\gamma_i^{\bar{p}}c_{ip}\right).$$

Then $\left|\mathscr{L}''c_{jj}\gamma_i^j\right| \gtrsim \delta F^{\mathscr{L}''/2}F_j^{1/2}F_i$, and, by similar arguments, for α sufficiently small, we conclude that there exists a list $\mathscr{L}^2$, $\left|\mathscr{L}^2\right| < \left|\mathscr{L}^0\right|$such that $\mathscr{L}^2 c_{ii} \gtrsim_K \delta F^{\mathscr{L}^2/2}F_i$, and we can repeat the procedure. The Lemma is thus proved by induction. □

Proposition 3.7. *There exists constants α_0 and K' depending on K and the data such that if the basis $\mathscr{B}$ satisfies (A), (B) and B(α) for $\alpha \leq \alpha_0$ at (p,δ), $p \in V(p_0)$, then $\mathscr{B}$ is (K',p,δ)-extremal.*

Proof. We may suppose the basis ordered so that the weights $F_i = F(L_i,p,\delta)$ are ordered decreasingly. Let $L = \sum_{i=1}^{n-1} a_i L_i$, $a_i \in \mathbb{C}$, $\sum |a_i|^2 = 1$ so that $c_{LL} = \sum_{i=1}^{n-1}\left|a_i^2\right| c_{ii}$. Denote $F(L) = F(L,p,\delta)$. By hypothesis (B) it is clear that $F(L) \lesssim_K \sum |a_i|^2 F_i$.

To show the converse inequality, we prove the following assertion:

Claim. For every constant $K > 0$, there exists a constant K_1, depending on K and the data, such that:

if $i_0 \in \{1,\dots,n-1\}$ and $k_0 \in \{1,\dots,M\}$ are such that $\left|a_{i_0}\right|^2 F_{i_0}(p) \geq \frac{\sum|a_i|^2F_i(p)}{K}$ and $\Re\mathrm{e}\left(L_{i_0}\overline{L_{i_0}}\right)^{k_0} c_{i_0i_0}(p) > \delta\frac{F_{i_0}^{k_0+1}(p)}{K}$, then:

- either $\Re\mathrm{e}\,(L\overline{L})^{k_0} c_{L\overline{L}} > \delta\frac{\left(\sum|a_i|^2F_i(p)\right)^{k_0+1}}{K_1}$,
- or there exist i_1 and $k_1 < k_0$ such that $|a_{i_1}|^2 F_{i_1}(p) \geq \frac{\sum|a_i|^2F_i(p)}{K_1}$ and $\Re\mathrm{e}\left(L_{i_1}\overline{L_{i_1}}\right)^{k_1} c_{i_1i_1}(p) > \delta\frac{F_{i_1}^{k_1+1}(p)}{K_1}$.

Proof of the Claim. We have

$$(L\overline{L})^{k_0} c_{L\overline{L}} = \sum |a_i|^{2k_0+2}\left(L_i\overline{L_i}\right)^{k_0} c_{ii} + \sum \alpha_{\mathscr{L}}\mathscr{L}(\partial\rho), \tag{3.8}$$

where the second sum contains lists of length $2k_0+2$ containing L_i or $\overline{L_i}$ for, at least, two different values of i. As

$$\left|a_{i_0}\right|^{2k_0+2}\Re\mathrm{e}\left(L_{i_0}\overline{L_{i_0}}\right)^{k_0} c_{i_0i_0}(p) > \delta\frac{\left(\sum|a_i|^2F_i(p)\right)^{k_0+1}}{K^{k_0+2}},$$

the conclusion is clear except in the two following cases: in the second member of (3.8), there is a term in the first sum which is $< -A = -\delta\frac{\left(\sum|a_i|^2F_i(p)\right)^{k_0+1}}{CK^{k_0+2}}$, or a term in the second sum which is, in modulus, bigger than A, with a constant C depending only on M and the coefficients a_i.

Suppose first that there exists an index $i \neq i_0$ such that $|a_i|^{2k_0+2}\Re\mathrm{e}\left(L_i\overline{L_i}\right)^{k_0} c_{ii}(p) < -A$. This implies first $|a_i|^2 F_i(p) \geq \frac{\sum|a_i|^2F_i(p)}{K_1'}$ and secondly $\Re\mathrm{e}\left(L_i\overline{L_i}\right)^{k_0} c_{ii}(p) < -\delta\frac{1}{K_1''}F_i^{k_0+1}$. By Lemma 3.3 there exists $k_1 < k_0$ such that $\Re\mathrm{e}\left(L_i\overline{L_i}\right)^{k_1} c_{ii}(p) > \delta\frac{1}{K_1'''}F_i^{k_1+1}$. Thus the second assertion of the Claim is verified.

Suppose now that there is a term $\alpha_{\mathscr{L}}\mathscr{L}(\partial\rho)$ in the second sum of (3.8) satisfying $|\alpha_{\mathscr{L}}\mathscr{L}(\partial\rho)| > A$. Denote by l_i the number of times the vector fields L_i and $\overline{L_i}$ appear in $\mathscr{L}$. If $l_k \neq 0$, hypothesis (B) implies immediately $|a_k|^2 F_k \gtrsim \sum|a_i|^2 F_i$ and $|\mathscr{L}|(\partial\rho) \gtrsim \delta\prod F_i^{l_i/2}$. □

□

Corollary. *Suppose that $p_0 \in \partial\Omega$ is a point of finite type τ where the Levi form is locally diagonalizable. Then there exists a neighborhood $V(p_0)$ of p_0 and constants K and $\delta_0 > 0$ such that at every point p of $V(p_0)\cap\partial\Omega$ and for every $0 < \delta \leq \delta_0$, the basis diagonalizing the Levi form is (M,p,δ)-extremal (with $M = M'(\tau)$).*

Proof. Properties (A) and (B) where proved in [CD06b], and, by definition the basis diagonalizing the Levi form satisfy B(α) for all $\alpha > 0$. □

Definition 3.5. $\mathscr{B}$ is called (K,α,p,δ)-*strongly-extremal* if it is (K,p,δ)-extremal and, if, it satisfies B(α) at (p,δ).

Note that the first part of Proposition 3.2 say that every (K,p,δ)-extremal basis is (K,α,p,δ)-strongly-extremal for some large positive number α depending on K and Ω. Thus this is an extra hypothesis only for small α.

The next Proposition shows that for a strongly extremal basis some derivatives of the diagonal terms of the Levi matrix satisfy a better control:

Proposition 3.8. *Suppose p_0 of finite 1-type τ and let $M = M'(\tau)$. There exists a neighborhood $V(p_0)$ of p_0 such that, for $\alpha > 0$, there exists constants $\delta_0 = \delta_0(\alpha, \mathrm{data})$ and $K' = K'(K, \mathrm{data})$ such that if $\mathscr{B}$ is a (K,α,p,δ)-strongly-extremal basis then for all lists $\mathscr{L} \in \mathscr{L}_{2M}(\mathscr{B})$ such that there exists $j > i$ with $l_j \neq 0$ (here we suppose $\mathscr{B}$ ordered so that the F_i are decreasing) we have $|\mathscr{L}c_{ii}(p)| \leq K'\alpha F(p,\delta)^{\mathscr{L}/2}F_i(p,\delta)$.*

Proof. Let $\mathscr{L} = \mathscr{L}'\overset{(-)}{L}_j\overset{(-)}{L}_p\mathscr{L}''$ with $j \leq i$ and write

$$\mathscr{L}c_{ii} = \mathscr{L}'\overset{(-)}{L}_p\overset{(-)}{L}_j\mathscr{L}''c_{ii} + \sum\mathscr{L}'\left(a^k_{\overset{(-)}{j}\overset{(-)}{p}}L_k + a^{\overline{k}}_{\overset{(-)}{j}\overset{(-)}{p}}\overline{L_k}\right)\mathscr{L}''c_{ii}.$$

Then successive application of the Lemma 3.2 show that there exists a list $\widetilde{\mathscr{L}} = \widetilde{\mathscr{L}}' L_j$ such that, for all k, $\widetilde{l_k} = l_k$ and $\left|\widetilde{\mathscr{L}} c_{ii} - \mathscr{L} c_{ii}\right| \leq K_2 \alpha F^{\mathscr{L}/2} F_i$.

Now the result is trivial applying once again Lemma 3.2, Lemma 3.2.1 and the hypothesis $\mathrm{B}(\alpha)$. $\square$

Proposition 3.9. *If the basis $\mathscr{B}$ is (K, α, p, δ)-strongly extremal, the conclusion of Proposition 3.8 is still valid at each points $q \in B^{c_0}(p, \delta)$ with α replaced by 2α for $\delta \leq \delta(\alpha)$ ($\delta(\alpha)$ depending on α, K and the data).*

3.5. **Localization of extremal basis.**

3.5.1. *Definition of the local domain.*

Definition 3.6. Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$. Suppose that P_0 is a boundary point of Ω and $W(P_0) \Subset V(P_0)$ are neighborhood of P_0. Let O be a point of the real normal to $\partial\Omega$ at P_0 and denote by d the distance from O to P_0. Let us denote by $(z_i)_{1 \leq i \leq n}$ the coordinate system obtained translating the origin at O.

Let $\mu > 0$ and $\psi(z) = \varphi\left(|z|^2\right)$ where

$$\varphi(x) = \begin{cases} 0 & \text{if } x \leq \mu^2, \\ K_0 e^{-1/(x-\mu^2)} & \text{if } x \geq \mu^2, \end{cases}$$

with $\frac{4}{3}d \leq \mu \leq 2d$.

Let us denote $r(z) = \rho(z) + \psi(z)$. Then d is chosen small enough and K_0 large enough such that, in particular:

- $D = \{r(z) < 0\} \subset W(P_0)$ and r is a defining function of D;
- D have a $\mathscr{C}^\infty$ boundary and is pseudo-convex;
- At each point of $\partial\Omega \setminus \partial D$, the boundary of D is strictly pseudo-convex;
- In the closure of $B(0, 2\mu)$ the vector z (in the coordinate system centered at 0) is not tangent to ρ (i.e. $\sum_{i=1}^n \frac{\partial\rho}{\partial z_i} z_i \neq 0$ everywhere in $B(0, 2\mu)$).

The fact that such a domain always exits for any $d > 0$ small and $K_0 > 0$ large is based on the construction of R. Gay and A. Sebbar in [GS85] (Théorème 2.1). Simply, note that, on $\partial D \setminus \partial\Omega$, the function r is strictly pluri-subharmonic if K_0 is large enough and μ small enough (the hessian of ρ is $O\left(\varphi\left(|z|^2\right)\right)$). Moreover, if P_0 is of finite type, then all the boundary points of D are of finite type because the order of contact of $\partial\Omega$ with ∂D is infinite at the points of $\partial\left(\partial\Omega \cap \partial D\right)$.

The goal of this Section is to prove the following:

Theorem 3.3. *Suppose that P_0 is a point of finite 1-type τ of $\partial\Omega$ and choose $M'(\tau)$ (c. f. Proposition 3.4). Let $\delta > 0$ and $K > 0$. If at every point of $\partial\Omega \cap V(P_0)$ there exists a (K, p, δ)-extremal basis then one can construct the domain D contained in $V(P_0)$ so that, at every point p' of its boundary there exists a (K', p', δ)-extremal basis with K' depending only on K and the data.*

3.5.2. *Preliminary remarks.* We fix now some general notations.

Let π be the $\mathscr{C}^\infty$ projection of $V(P_0) \cap \overline{\Omega}$ onto $\partial\Omega$ defined with the integral curves of the real normal to ρ. We can suppose $V(P_0)$ small enough such that π can be considered as a $\mathscr{C}^\infty$ diffeomorphism of $\partial D \cap V(P_0)$ onto $\partial\Omega \cap V(P_0)$.

If L is a $\mathscr{C}^\infty$ vector field, defined on an open set U of $\partial D \cap V(p_0)$, tangent to ∂D, we associate to it a vector field L^ρ, defined in the open set $\pi(U) \subset \partial\Omega$, tangent to $\partial\Omega$ using π as follows: if $L = \sum a_i \frac{\partial}{\partial z_i}$, considering it as an application of U into $\mathbb{C}^n$, we denote by $L \circ \pi^{-1}$ the vector field in $\pi(U)$ defined by $L \circ \pi^{-1} = \sum a_i \circ \pi^{-1} \frac{\partial}{\partial z_i}$, and

$$L^\rho = L \circ \pi^{-1} - \beta N, \tag{3.9}$$

where N is the complex unitary normal to ρ and $\beta = L \circ \pi^{-1}(\rho)$.

Clearly, $L \mapsto L^\rho$ is an isomorphism from $T^{1,0}_{\partial D \cap U}$ onto $T^{1,0}_{\partial\Omega \cap \pi(U)}$ ($V(P_0)$ sufficiently small), and thus, we also consider L associated to L^ρ by $L = L^\rho \circ \pi + (\beta \circ \pi) N \circ \pi$ and, as L is tangent to ∂D and $(L^\rho \circ \pi)(\rho)$ is identically zero on $\partial\Omega$, we have

$$\beta \circ \pi(z) = \frac{-\left\langle L^\rho \circ \pi, z\right\rangle \varphi'(|z|^2)}{(N \circ \pi)(\rho) + \left\langle N \circ \pi, z\right\rangle \varphi'(|z|^2)} + k,$$

where k is a $\mathscr{C}^\infty$ function whose derivatives of order less than M are $O(\varphi(|z|^2))$, with constant controlled by the $\mathscr{C}^{2M}$ norm of L, and, if $L = \sum a_i \frac{\partial}{\partial z_i}$ (in the coordinate system of Definition 3.6), $\langle L, z\rangle$ denotes the usual scalar product $\sum a_i \overline{z_i}$, and $\langle L, L'\rangle = \sum a_i a_i'$.

With the previous notations, let P be a point of ∂D such that $\psi(P) = 0$ (thus $P \in \partial D \cap \partial\Omega$) and $V(P)$ a neighborhood of P such that π is a diffemorphism of $V(P) \cap \partial D$ onto $V(P) \cap \partial\Omega$.

Let $p \in \partial D \cap V(P)$. Essentially, the construction of the extremal basis $\mathscr{B}$ at p for D is done using a suitable basis $\mathscr{B}^\rho$ of the tangent space of $\partial\Omega$ near the point $\pi(p)$ translated at p (using π) then projected onto the tangent space of ∂D, to get a basis $\widetilde{\mathscr{B}}$ from which the basis $\mathscr{B}$ is defined. Now, we only look at the relation between the weights of the basis $\widetilde{\mathscr{B}}$ and $\mathscr{B}^\rho$.

Thus, if $\widetilde{\mathscr{B}}=\{\widetilde{L_1},\dots,\widetilde{L_{n-1}}\}$ is a basis of $T^{1,0}_{\partial D}$ in $V(P)\cap\partial D$, with our notations, the basis $\mathscr{B}^\rho=\{L^\rho_1,\dots,L^\rho_{n-1}\}$ of $T^{1,0}_{\partial\Omega}$, in $V(P)\cap\partial\Omega$, is given by

$$L^\rho_i=\widetilde{L}_i\circ\pi^{-1}-\beta_iN, \tag{3.10}$$

with $\beta_i=\widetilde{L}_i\circ\pi^{-1}(\rho)$, and

$$\widetilde{L}_i=L^\rho_i\circ\pi+(\beta_i\circ\pi)N\circ\pi. \tag{3.11}$$

with

$$\beta_i\circ\pi=\frac{-\left\langle L^\rho_i\circ\pi,z\right\rangle\varphi'(|z|^2)}{(N\circ\pi)(\rho)+\langle N\circ\pi,z\rangle\,\varphi'(|z|^2)}+k. \tag{3.12}$$

Let us calculate the weights $F(\widetilde{L}_i,p,\delta)$ in terms of the weights $F(L^\rho_i,\pi(z),\delta)$ and the derivatives of φ. We suppose that the L^ρ_i are normalized. Writing $\widetilde{c_{ij}}=\left[\widetilde{L}_i,\overline{\widetilde{L}_j}\right](\partial r)$ and $c^\rho_{ij}=\left[L^\rho_i,\overline{L^\rho_j}\right](\partial\rho)$, using that $N\circ\pi$ is identically 1 on $\partial\Omega$, a simple calculus shows

$$\begin{aligned}\widetilde{c_{ij}} &= c^\rho_{ij}\circ\pi+\left\langle L^\rho_i\circ\pi,L^\rho_j\circ\pi\right\rangle\varphi'(|z|^2)+\left\langle L^\rho_i\circ\pi,z\right\rangle\overline{\left\langle L^\rho_j\circ\pi,z\right\rangle}\varphi''(|z|^2)+\\ &\quad+\varphi'(|z|^2)\textstyle\sum_{k=1}^{n-1}\left(*\left\langle L^\rho_k\circ\pi,z\right\rangle+*\left\langle\overline{L^\rho_k}\circ\pi,z\right\rangle\right)+k,\\ &= c^\rho_{ij}\circ\pi+\varphi'(|z|^2)\left(\left\langle L^\rho_i\circ\pi,L^\rho_j\circ\pi\right\rangle+h\right)+\left\langle L^\rho_i\circ\pi,z\right\rangle\overline{\left\langle L^\rho_j\circ\pi,z\right\rangle}\varphi''(|z|^2)+k,\end{aligned} \tag{3.13}$$

where all the derivatives of k are $O(\varphi(|z|^2)$ and the functions $*$ have a bounded $\mathscr{C}^M$ norm the constants depending only on Ω and the $\mathscr{C}^{2M}$ norms of the $\widetilde{L}_i$.

As the L^ρ_i are normalized, we also have

$$\begin{aligned}\widetilde{c_{ii}} &= c^\rho_{ii}\circ\pi+\varphi'(|z|^2)+\left|\left\langle L^\rho_i\circ\pi,z\right\rangle\right|^2\varphi''(|z|^2)+\varphi'(|z|^2)\textstyle\sum_{k=1}^{n-1}\left(*\left\langle L^\rho_k\circ\pi,z\right\rangle+*\left\langle\overline{L^\rho_k}\circ\pi,z\right\rangle\right)+k\\ &= c_{ii}\circ\pi+\varphi'(|z|^2)(1+h)+\left|\left\langle L^\rho_i\circ\pi,z\right\rangle\right|^2\varphi''(|z|^2)+k\end{aligned} \tag{3.14}$$

and d is chosen small enough such that the $\mathscr{C}^M$ norm of h is small.

Now we need to introduce a new notation. Let L be a $\mathscr{C}^\infty(\partial D\cap V(P))$ vector field tangent to ∂D. For $z\in\partial D\cap V(P)$ let us define

$$\widetilde{F^\varphi}(L,z,\delta)=\sum_{k=1}^{M/2}\left(\frac{\varphi^{(k)}(|z|^2)}{\delta}\right)^{1/k}+|\langle L^\rho\circ\pi(z),z\rangle|^2\sum_{2}^{M}\left|\frac{\varphi^{(k)}(|z|^2)}{\delta}\right|^{2/k}+\delta^{-1/M}.$$

Lemma 3.4. *We have, for δ and $V(P)$ small enough, for $z\in\partial D\cap V(P)$,*

$$\widetilde{F^\varphi}(L,z,\delta)\simeq\frac{\varphi'(|z|^2)}{\delta}+|\langle L^\rho\circ\pi,z\rangle|^2\frac{\varphi''(|z|^2)}{\delta}+\delta^{-1/M}.$$

Proof. It suffices to consider the case when $|z|^2=\mu^2+x>\mu^2$. Note that, for $V(P)$ small, $\varphi^{(k)}(\mu^2+x)\simeq Ke^{-1/x}x^{-2k}$ and $\left(\frac{1}{x}\right)^{2k}\leq e^{1/Mx}$, for $k\leq M$.

Suppose $\left(\frac{\varphi^{(k)}(\mu^2+x)}{\delta}\right)^{1/k}>\delta^{-1/M}$ and $e^{-1/x}<\delta$. Then

$$\left(\frac{\varphi^{(k)}(\mu^2+x)}{\delta}\right)^{1/k}\simeq\left(\frac{K_0e^{-1/x}}{\delta}\right)^{1/k}\frac{1}{x^2}\lesssim K_0^{1/2}\delta^{-1/kM}\leq\delta^{-1/M},$$

for δ small. Thus, for $\delta\leq\delta_0(K_0)$, $\left(\frac{\varphi^{(k)}(\mu^2+x)}{\delta}\right)^{1/k}>\delta^{-1/M}$ implies $e^{-1/x}>\delta$ and $\sum_1^{M/2}\left(\frac{\varphi^{(k)}(\mu^2+x)}{\delta}\right)^{1/k}\simeq\frac{\varphi'(\mu^2+x)}{\delta}$.

Similarly, $\left(\frac{\varphi^{(k)}(\mu^2+x)}{\delta}\right)^{2/k}>\delta^{-1/M}$ implies $e^{-1/x}>\delta$ and $\sum_2^{M}\left(\frac{\varphi^{(k)}(\mu^2+x)}{\delta}\right)^{2/k}\simeq\frac{\varphi''(\mu^2+x)}{\delta}$. □

Thus, we denote

$$F^\varphi(L,z,\delta)=\frac{\varphi'(|z|^2)}{\delta}+|\langle L^\rho\circ\pi,z\rangle|^2\frac{\varphi''(|z|^2)}{\delta}+\delta^{-1/M},$$

and $F^\varphi_i=F^\varphi_i(z,\delta)=F^\varphi(\widetilde{L}_i,z,\delta)$, $1\leq i\leq n-1$. Define again $F^\varphi_n=\delta^{-2}$. Let $\widetilde{L}_n$ denotes the unitary complex normal to r, the defining function of D, and L^ρ_n the unitary complex normal to ρ.

Proposition 3.10. *Let $\widetilde{\mathscr{L}}$ be a list of $\mathscr{L}_M\left(\widetilde{\mathscr{B}}\cup\left\{\widetilde{L}_n\right\}\right)$ and $\mathscr{L}^\rho$ be the list obtained replacing $\overset{(-)}{\widetilde{L}_i}$ in $\mathscr{L}$ by $\overset{(-)}{L^\rho_i}$. Then, reducing $V(P)$ if necessary, on $\partial D\cap V(P)$ we have ($\tilde{l}_i$ denoting the number of times the vector fields $\widetilde{L}_i$ or $\overline{\widetilde{L}_i}$ appears in $\widetilde{\mathscr{L}}$):*

(1) $\left|\widetilde{\mathscr{L}}(c^\rho_{ij}\circ\pi)-(\mathscr{L}^\rho c^\rho_{ij})\circ\pi\right|\lesssim\delta\prod_{k=1}^n\left(F^\varphi_k\right)^{\tilde{l}_k/2}$, *for* $\left|\widetilde{\mathscr{L}}\right|\geq2$,

(2) $\left|\widetilde{\mathscr{L}}\varphi(|z|^2)\right| \lesssim \delta\prod_{i=1}^{n}\left(F_i^{\varphi}\right)^{\tilde{l}_i/2}$, $\left|\widetilde{\mathscr{L}}\right| \geq 2$,

the constants depending only on Ω and the $\mathscr{C}^{M+2}$ norms of the $\widetilde{L}_i$.

Proof. These properties are trivially satisfied if $\tilde{l}_n \neq 0$, thus we suppose $\tilde{l}_n = 0$. Using (3.13) and the fact that if f is a $\mathscr{C}^\infty$ function on $\partial\Omega\cap V(P)$ and $L^\rho\rho\equiv 0$ then $(L^\rho\circ\pi)(f\circ\pi)-(L^\rho f)\circ\pi = O_f(\varphi)$ on $\partial D\cap V(P)$, the Proposition is an easy consequence of (3.12) and the the following Lemma:

Lemma 3.5. *Let $\mathscr{L}^{\rho,\pi}$ be a list of $\mathscr{L}_M\left\{L_i^\rho\circ\pi,\, i\leq n-1\right\}$ of length ≥ 1, then $|\mathscr{L}^{\rho,\pi}\psi| \lesssim \delta\prod_{i=1}^{n-1}\left(F_i^{\varphi}\right)^{l_i/2}$.*

Proof of Lemma 3.5. By induction, we have

$$\mathscr{L}^{\rho,\pi}\psi = \mathscr{L}^{\rho,\pi}\left(\varphi\left(|z|^2\right)\right) = \sum_{l=1}^{\left[\frac{m-1}{2}\right]} *\varphi^{(l)}\left(|z|^2\right) + \sum_{l=\left[\frac{m+1}{2}\right]}^{m} \alpha_l\varphi^{(l)}\left(|z|^2\right),$$

where

$$\alpha_{m-k} = \sum_{\substack{\mathscr{L}^*=\{W_1^*,\ldots,W_{m^*}^*\}\subset\mathscr{L}^{\rho,\pi}\\ m^*\leq m-2k}} *\prod_{W_i^*\in\mathscr{L}^*}\left\langle W_i^*,\overset{(-)}{z}\right\rangle,$$

where $\left\langle W_i^*,\overset{(-)}{z}\right\rangle$ denotes $\langle W_i^*,z\rangle$ if W_i^* is of type $(0,1)$ and $\langle W_i^*,\overline{z}\rangle$ if not, and the functions $*$ have a $\mathscr{C}^M$norm controlled by the $\mathscr{C}^{2M}$ norms of the vector fields $\widetilde{L}_i$. Now, the proof of Lemma 3.4 shows

$$\sum_{l=1}^{\left[\frac{m-1}{2}\right]}\frac{*\varphi^{(l)}\left(|z|^2\right)}{\delta} \lesssim \left(\delta^{-1/M}+\frac{\varphi'\left(|z|^2\right)}{\delta}\right)^{m/2},$$

and it is enough to see that $\left|\alpha_l\varphi^{(l)}\left(|z|^2\right)\right| \lesssim \delta\left(F^\varphi\right)^{\mathscr{L}/2}$, for $l\in\left\{\left[\frac{m+1}{2}\right],\ldots,m\right\}$. If $l=m$, this follows Lemma 3.4; suppose $l=m-k$, $k\geq 1$.

Suppose $\left|\frac{\varphi^{(m-k)}\left(|z|^2\right)}{\delta}\right|^{2/(m-k)} \geq \delta^{-1/M}$. By Lemma 3.4 $\left|\frac{\varphi^{(m-k)}\left(|z|^2\right)}{\delta}\right|^{2/(m-k)} \leq \frac{\varphi''\left(|z|^2\right)}{\delta}$. Let $\mathscr{L}^*\subset\mathscr{L}$ of length $m^* = m-2k = \sum_{i=1}^{n-1} l_i^*$. The corresponding term in α_{m-k} is bounded by

$$\begin{aligned} *\left(\frac{\varphi''\left(|z|^2\right)}{\delta}\right)^{(m-k)/2}\prod_i\left|\left\langle L_i^\rho\circ\pi,z\right\rangle\right|^{l_i^*} &= *\left(\frac{\varphi''\left(|z|^2\right)}{\delta}\right)^{k/2}\prod_{i=1}^{n-1}\left(\frac{\varphi''\left(|z|^2\right)}{\delta}\left|\left\langle L_i^\rho\circ\pi,z\right\rangle\right|^2\right)^{l_i^*/2}\\ &\lesssim *\left(\frac{\varphi'\left(|z|^2\right)}{\delta}\right)^{k}\prod_{i=1}^{n-1}\left(F_i^\varphi\right)^{l_i^*/2}, \end{aligned}$$

because the hypothesis implies $\left(\frac{\varphi''\left(|z|^2\right)}{\delta}\right)^{1/2} \lesssim \frac{\varphi'\left(|z|^2\right)}{\delta}$. ☐

To finish the proof of Proposition 3.10 note that, for $\left|\widetilde{\mathscr{L}}\right|\geq 1$,

$$\left|\widetilde{\mathscr{L}}\left(\beta_i\circ\pi\right)(z)\right| \leq F^\varphi(z,\delta)^{\widetilde{\mathscr{L}}/2}F_i^\varphi(z,\delta)^{1/2},$$

and use (3.12). ☐

Finally the relations between the weights associated to $\widetilde{\mathscr{B}}$ and to $\mathscr{B}^\rho$are as follows.

Let $\widetilde{L}$ a holomorphic vector field on ∂D tangent to ∂D near p and L^ρ the associated vector field tangent to $\partial\Omega$. Then

Proposition 3.11. *For V sufficiently small, we have, if $\frac{1}{K}\leq\left\|\widetilde{L}\right\|\leq K$,*

$$F\left(\widetilde{L},z,\delta\right)\simeq F\left(L^\rho,\pi(z),\delta\right)+F^\varphi\left(\widetilde{L},z,\delta\right),$$

with constants depending on the $\mathscr{C}^{2M}$ norm of $\widetilde{L}$, K and the data.

Proof. From Proposition 3.10 it easily follows that $F\left(\widetilde{L},z,\delta\right)\lesssim F\left(L^\rho\circ\pi,z,\delta\right)+F^\varphi\left(\widetilde{L},z,\delta\right)$. Let us then see that there exists a list $\widetilde{\mathscr{L}}$ composed of $\widetilde{L}$ and $\overline{\widetilde{L}}$ such that $\widetilde{\mathscr{L}}c_{\widetilde{L}\overline{\widetilde{L}}}\simeq\delta\left(F\left(L^\rho\circ\pi,z,\delta\right)+F^\varphi\left(\widetilde{L},z,\delta\right)\right)^{\left(\left|\widetilde{\mathscr{L}}\right|+2\right)/2}\overset{\text{def}}{=}\delta F^{\left(\left|\widetilde{\mathscr{L}}\right|+2\right)/2}$. If $\frac{\varphi'}{\delta}+\left|\left\langle L^\rho\circ\pi(z),z\right\rangle\right|^2\frac{\varphi''}{\delta}\simeq F$, $c_{\widetilde{L}\overline{\widetilde{L}}}$ do it. Suppose $\frac{\varphi'}{\delta}+\left|\left\langle L^\rho\circ\pi(z),z\right\rangle\right|^2\frac{\varphi''}{\delta}\ll F$, there exists a list $\mathscr{L}^\rho$such that $\left|\mathscr{L}^\rho c_{L^\rho \overline{L^\rho}}(\pi(z))\right|\simeq\delta F^{\left(\left|\widetilde{\mathscr{L}}\right|+2\right)/2}$. Then calculating $\widetilde{\mathscr{L}}c_{\widetilde{L}\overline{\widetilde{L}}}$ in term of $\mathscr{L}^\rho\left(c_{L^\rho \overline{L^\rho}}\right)\circ\pi$, the result follows Proposition 3.10, (3.14) and the properties of the functions h and k.

$\square$

3.5.3. *Extremal basis on D.* In this Section, we suppose always that p_0 is of finite type τ, $M = M'(\tau)$ and that at all points q of $V(P_0) \cap \partial\Omega$ and for all $\delta > 0$, $0 < \delta \leq \delta_0$, there exists a (K, q, δ)-extremal basis, and we show that at all points p of ∂D and for all $\delta > 0$ there exists a (K', p, δ)-extremal basis (for D) with a constant K' controlled by K and the data.

If P is a point of ∂D such that $|P| > \mu$ then ∂D is strictly pseudoconvex near P and the construction of extremal basis in $V(P) \cap \partial D$ is trivial (for $V(P)$ small). If $|P| < \tau$ then $V(P) \cap \partial D$ is contained in $\partial\Omega$ and the existence of extremal basis is the hypothesis. Thus, we have only to consider neighborhood of points $P \in \partial D$ such that $|P| = \mu$ (that is points P in the boundary of $\partial\Omega \cap \partial D$).

As we said before, the final extremal basis for D, at $p \in V(P) \cap \partial D$, will be obtained extending a basis $\widetilde{\mathscr{B}}$ defined on $V(P) \cap \partial D$ which is a projection onto the tangent space to r of a translation of a basis $\mathscr{B}^\rho$, at $\pi(p)$, tangent to ρ.

Formula (3.14) shows that the expressions $\left\langle L_i^\rho \circ \pi, z \right\rangle$ plays an important role: we have to take into account the vector fields which are orthogonal to z. In particular, to construct a extremal basis on ∂D, we cannot simply translate an extremal basis on $\partial\Omega$ and project it onto the tangent space to ∂D, because, even if the basis (L_i^ρ) is extremal, we can have $\left\langle L_i^\rho \circ \pi, z \right\rangle \neq 0$, for all i, and there are linear combinations of the $L_i^\rho \circ \pi$ which are orthogonal to z.

Now the point p and the positive number δ are fixed. We suppose we have a $(K, \pi(p), \delta)$-extremal (for ρ) basis $\mathscr{B}^\Omega = \{L_1^\Omega, \ldots, L_{n-1}^\Omega\}$ at the point $\pi(p)$ (the L_i^Ω being $\mathscr{C}^\infty$ in $V(P)$), such that the vectors $L_i^\Omega(\pi(p))$ are orthogonal (c.f. Proposition 3.1) and we construct the basis $\mathscr{B}^\rho = \{L_1^\rho, \ldots, L_{n-1}^\rho\}$ using it. The weight associated to $\mathscr{B}^\Omega$ are denoted $F_i^\Omega = F_i^\Omega(\pi(p), \delta) = F^\Omega(L_i^\Omega, \pi(p), \delta)$, and we suppose $F_{i+1}^\Omega \leq F_i^\Omega$, for $i \leq n-2$, changing the order of the L_i^Ω if necessary.

Recall that the canonical coordinate system is centered at the point O of Definition 3.6, thus $|z(P)| = \mu$.

For simplicity of notations, we denote $q = \pi(p)$ (thus $p = \pi^{-1}(q)$, π being considered as a diffeomorphism between open sets of the boundaries of Ω and D).

Let

$$\mathscr{H}_{n-1} = \left\{ W = \sum a_i L_i^\Omega,\ a_i \in \mathbb{C},\ \sum |a_i|^2 = 1, \text{ such that } \langle W(q), p \rangle = 0 \right\}.$$

Let $W_{n-1} = \sum a_i^{n-1} L_i^\Omega \in \mathscr{H}_{n-1}$ such that $\sum_{i=1}^{n-2} \left| a_i^{n-1} \right|^2 F(L_i^\Omega, q, \delta) = \inf_{W = \sum a_i L_i^\Omega \in \mathscr{H}_{n-1}} \sum_{i=1}^{n-2} |a_i|^2 F(L_i^\Omega, q, \delta)$, and define

$$L_{n-1}^\rho = \begin{cases} L_{n-1}^\Omega & \text{if } \sum_{i=1}^{n-2} \left| a_i^{n-1} \right|^2 F(L_i^\Omega, q, \delta) \geq \frac{\varphi''(|p|^2)}{\delta} \left| \left\langle L_{n-1}^\Omega(q), p \right\rangle \right|^2, \\ W_{n-1} & \text{otherwise.} \end{cases} \tag{3.15}$$

Suppose L_{n-l}^ρ defined for $1 \leq l \leq k-1 < n$. Let $\mathscr{H}_{n-k} = \mathscr{H}_{n-1} \cap \left[\mathscr{E}(L_{n-1}^\rho, \ldots, L_{n-k+1}^\rho) \right]^\perp$, $\mathscr{E}(L_{n-1}^\rho, \ldots, L_{n-k+1}^\rho)$ being the linear space span by L_{n-1}^ρ, ..., L_{n-k+1}^ρ, the orthogonality being taken at q. Let $W_{n-k} = \sum_{i=1}^{n-1} a_i^{n-k} L_i^\Omega$ a vector in $\mathscr{H}_{n-k}$ minimizing $\sum_{i=1}^{n-k-1} |a_i|^2 F(L_i^\Omega, q, \delta)$ for vectors $\sum_{i=1}^{n-1} a_i L_i^\Omega \in \mathscr{H}_{n-k}$. Let T_{n-k} a vector field, of norm 1 at q, in $\mathscr{E}(L_{n-1}^\Omega, \ldots, L_{n-k}^\Omega) \cap \left[\mathscr{E}(L_{n-1}^\rho, \ldots, L_{n-k+1}^\rho) \right]^\perp$. Then $L_{n-k}^\rho$is defined by

$$L_{n-k}^\rho = \begin{cases} T_{n-k} & \text{if } \sum_{i=1}^{n-k-1} \left| a_i^{n-k} \right|^2 F(L_i^\Omega, q, \delta) \geq \frac{\varphi''(|p|^2)}{\delta} \left| \langle T_{n-k}(q), p \rangle \right|^2, \\ W_{n-k} & \text{otherwise.} \end{cases}$$

Note that $\{L_i^\rho(q), 1 \leq i \leq n-1\}$ is orthonormal.

The two next Lemma prove some important properties of the vector fields L_i^ρ. Let us denote $\mathscr{B}^\rho = \left\{ L_i^\rho, i < n \right\}$ and L_n^ρ the unitary complex normal to ρ.

If $L = \sum_{i=1}^{n-1} a_i L_i^\rho$, $a_i \in \mathbb{C}$, let us denote

$$F^{\rho\varphi}(L) = F(L, q, \delta) + \frac{\varphi'\left(|p|^2\right)}{\delta} + |\langle L(q), p \rangle|^2 \frac{\varphi''\left(|p|^2\right)}{\delta},$$

and $F_i^{\rho\varphi} = F^{\rho\varphi}(L_i^\rho)$, $1 \leq i \leq n-1$, $F_n^{\rho\varphi} = \frac{1}{\delta^2}$ and $(F^{\rho\varphi})^{\mathscr{L}/2} = \prod_i \left(F_i^{\rho\varphi} \right)^{l_i/2}$, if $\mathscr{L}$ is a list of $\mathscr{L}_M(\mathscr{B}^\rho \cup \left\{ L_n^\rho \right\})$, with the usual notation for l_i.

Lemma 3.6. *There exits a constant K', depending only on Ω, such that $F_i^{\rho\varphi} \leq K' F_{i-1}^{\rho\varphi}$ and $F_i^{\rho\varphi} \geq \frac{1}{K'} F(L_i^\Omega, q, \delta)$, for $i < n$. Moreover, if $L = \sum a_i L_i^\rho$, $\sum |a_i|^2 = 1$ is orthogonal, at q, to the linear vector space generated by L_{n-k}^ρ,..., L_{n-1}^ρ, then $F^{\rho\varphi}(L) \geq \frac{1}{K'} F(L_{n-k-1}^\Omega, q, \delta)$.*

Proof. Note first that, the basis $\mathscr{B}^\Omega$ being extremal and the $F(L_i^\Omega, q, \delta)$, $i \leq n-1$, ordered decreasingly, it is clear that $F_{n-1}^{\rho\varphi} \gtrsim F(L_n^\Omega, q, \delta)$. Let us show now that if $L = \sum a_i L_i^\rho$, $\sum |a_i|^2 = 1$, we have $F^{\rho\varphi}(L) \gtrsim F_{n-1}^{\rho\varphi}$.

If $L_{n-1}^\Omega \in \mathscr{H}_{n-1}$, then $L_{n-1}^\rho = L_{n-1}^\Omega$ and $F_{n-1}^{\rho\varphi} = F(L_{n-1}^\Omega, q, \delta) + \frac{\varphi'(|p|^2)}{\delta}$ which gives the result. Suppose thus $L_{n-1}^\Omega \notin \mathscr{H}_{n-1}$. We separate the two cases of (3.15):

Suppose we are in the first case ($L_{n-1}^{\rho}=L_{n-1}^{\Omega}$). If $L\in\mathscr{H}_{n-1}$, the inequality is an immediate consequence of the extremality (EB$_1$) of $\mathscr{B}^{\Omega}$. Suppose $L\notin\mathscr{H}_{n-1}$. Then we can write $L=\alpha\left(L_{n-1}^{\rho}+\gamma H\right)$ with $H\in\mathscr{H}_{n-1}$. Writing $H=\sum a_i'L_i^{\Omega}$, we have

$$F^{\rho\varphi}(L)\simeq|\alpha|^2\left[\sum_{i=1}^{n-2}\left|\gamma a_i'\right|^2F(L_i^{\Omega},q,\delta)+\left|1+\gamma a_{n-1}'\right|^2F(L_{n-1}^{\Omega},q,\delta)\right]+\frac{\varphi'\left(|p|^2\right)}{\delta}+|\alpha|^2\,\frac{\varphi''\left(|p|^2\right)}{\delta}\left|\left\langle L_{n-1}^{\Omega}(q),p\right\rangle\right|^2,$$

and as $\sum_{i=1}^{n-2}|a_i'|^2F(L_i^{\Omega},q,\delta)\geq\sum_{i=1}^{n-2}\left|a_i^{n-1}\right|^2F(L_i^{\Omega},q,\delta)\geq\frac{\varphi''(|p|^2)}{\delta}\left|\langle L_{n-1}^{\Omega}(q),p\rangle\right|^2$, we obtain

$$F^{\rho\varphi}(L)\gtrsim|\alpha|^2\left(1+|\gamma|^2\right)\frac{\varphi''\left(|p|^2\right)}{\delta}\left|\left\langle L_{n-1}^{\Omega}(q),p\right\rangle\right|^2\gtrsim_K\frac{\varphi''\left(|p|^2\right)}{\delta}\left|\left\langle L_{n-1}^{\Omega}(q),p\right\rangle\right|^2,$$

because, by equivalence of norms in finite dimensional spaces, $|\alpha|^2\left(1+|\gamma|^2\right)\gtrsim_K 1$. The extremality of $\mathscr{B}^{\Omega}$ implying $F(L,q,\delta)\gtrsim F(L_{n-1}^{\Omega},q,\delta)$, and the inequality is proved.

The case $L_{n-1}^{\rho}=W_{n-1}$is obtained using the same method.

Thus, $F_{n-2}^{\rho\varphi}\gtrsim F(L_{n-1}^{\Omega},q,\delta)$, and for all $L=\sum a_iL_i^{\rho}$, $\sum|a_i|^2=1$ which is orthogonal, at p to L_{n-1}, we have $F^{\rho\varphi}(L)\gtrsim F(L_{n-2}^{\Omega},q,\delta)$.

The Lemma is then proved by a simple induction argument. □

We now estimate the brackets of the vector fields L_i^{ρ}, $i<n$, at the point q.

Lemma 3.7. *Let* $\left[\overset{(-)}{L_k^{\rho}},\overset{(-)}{L_s^{\rho}}\right]=\sum_{t=1}^n b^t_{\overset{(-)}{k}\overset{(-)}{s}}L_t^{\rho}+\sum_{t=1}^n b^{\bar t}_{\overset{(-)}{k}\overset{(-)}{s}}\overline{L_t^{\rho}}$. *For all list* $\mathscr{L}$, *of* $\mathscr{L}_M\left(\mathscr{B}^{\rho}\cup\left\{L_n^{\rho}\right\}\right)$, *we have*

$$\left|\mathscr{L}\left(b^{\overset{(-)}{t}}_{\overset{(-)}{k}\overset{(-)}{s}}\right)(q)\right|<K'\left(F^{\rho\varphi}\right)^{\mathscr{L}/2}\left(F_k^{\rho\varphi}\right)^{1/2}\left(F_s^{\rho\varphi}\right)^{1/2}\left(F_t^{\rho\varphi}\right)^{-1/2}$$

with K' depending only on K and the data.

Proof. Note that the Lemma is trivial if $l_n(\mathscr{L})\geq1$ and if $F_t^{\rho\varphi}\lesssim\frac{\varphi''(|p|^2)}{\delta}$ (because $F_k^{\rho\varphi}$ and $F_s^{\rho\varphi}$ are both $\geq$ to $\frac{\varphi'(|p|^2)}{\delta}$ and $\frac{\varphi''(|p|^2)}{\delta}\geq\delta^{-2/M}$ implies $\left|\frac{\varphi'(|p|^2)}{\delta}\right|^2\geq\left|\frac{\varphi''(|p|^2)}{\delta}\right|$). Moreover, we also have $F_t^{\rho\varphi}\lesssim F(L_t^{\rho},q,\delta)+\frac{\varphi''(|p|^2)}{\delta}$, and, if $L_t^{\rho}=T_t$, then, by definition of T_t and the extremality of $\mathscr{B}^{\Omega}$, $F(L_t^{\rho},q,\delta)\lesssim F(L_t^{\Omega},q,\delta)$, and, if $L_t=W_t$, $F(L_t^{\rho},q,\delta)\lesssim F(L_t^{\Omega},q,\delta)+\frac{\varphi''(|q|^2)}{\delta}$.

Thus, it suffices to prove that if $l_n=0$

$$\left|\mathscr{L}\left(b^t_{\overset{(-)}{k}\overset{(-)}{s}}\right)(q)\right|\lesssim\left(F^{\rho\varphi}\right)^{\mathscr{L}/2}\left(F_k^{\rho\varphi}\right)^{1/2}\left(F_s^{\rho\varphi}\right)^{1/2}\left(F(L_t^{\Omega},q,\delta)\right)^{-1/2}.$$

Let us write $L_k^{\rho}=\sum\alpha_k^iL_i^{\Omega}$ and $L_k^{\Omega}=\sum\beta_k^iL_i^{\rho}$. Using the notation $\left[\overset{(-)}{L_i^{\Omega}},\overset{(-)}{L_j^{\Omega}}\right]=\sum_{i=1}^n a^m_{\overset{(-)}{i}\overset{(-)}{j}}L_m^{\Omega}+\sum_{i=1}^n a^{\bar m}_{\overset{(-)}{i}\overset{(-)}{j}}\overline{L_t^{\Omega}}$, a calculus gives

$$b^t_{\overset{(-)}{k}\overset{(-)}{s}}=\sum_m\left(\sum_{i,j}\overset{(-)}{\alpha_k^i}\overset{(-)}{\alpha_s^j}a^m_{\overset{(-)}{i}\overset{(-)}{j}}\right)\beta_m^t,$$

with $\beta_m^t=\frac{1}{\det(\alpha)}\sum_\sigma\varepsilon_\sigma\prod_i\alpha_i^{\sigma(i)}$, where σ describes the set of permutations from $\{1,\dots,n-1\}\setminus\{t\}$ onto $\{1,\dots,n-1\}\setminus\{m\}$.

First, we prove that, if $t<m$, $|\beta_m^t|\lesssim\left(F_k^{\rho\varphi}\right)^{1/2}\left(F_s^{\rho\varphi}\right)^{1/2}\left(F(L_t^{\Omega},q,\delta)\right)^{-1/2}$ for any k and s. In that case, there exists an index $i>t$ such that $\sigma(i)\leq t$; if $L_i^{\rho}=T_i$ then $\alpha_i^{\sigma(i)}=0$, and if $L_i^{\rho}=W_i$ then

$$\left|\alpha_i^{\sigma(i)}\right|\leq\left[\frac{\varphi''(|p|^2)}{\delta}\left(F(L_{\sigma(i)}^{\Omega},q,\delta)\right)^{-1}\right]^{1/2}\leq\left(\frac{\varphi''(|p|^2)}{\delta}\right)^{1/2}\left(F(L_t^{\Omega},q,\delta)\right)^{-1/2}\leq\left(F_k^{\rho\varphi}\right)^{1/2}\left(F_s^{\rho\varphi}\right)^{1/2}\left(F(L_t^{\Omega},q,\delta)\right)^{-1/2},$$

because $F_m^{\rho\varphi}\gtrsim\delta^{-1/M}+\frac{\varphi'(|p|^2)}{\delta}$ and $\frac{\varphi''(|p|^2)}{\delta}\geq\delta^{-2/M}$ implies $\left(\frac{\varphi'(|p|^2)}{\delta}\right)^2\geq\frac{\varphi(|p|^2)}{\delta}$.

To finish the proof, it suffices to remark that the extremality of $\mathscr{B}^{\Omega}$ implies

$$\left|\alpha_k^i\right|\lesssim F(L_k^{\rho},q,\delta)^{1/2}F(L_i^{\Omega},q,\delta)^{-1/2},$$

and

$$\begin{aligned}\left|\mathscr{L}\left(a^m_{\overset{(-)}{i}\overset{(-)}{j}}\right)\right|&\lesssim\prod F(L_k^{\Omega},q,\delta)^{l_k/2}F(L_i^{\Omega},q,\delta)^{1/2}F(L_j^{\Omega},q,\delta)^{1/2}F(L_m^{\Omega},q,\delta)^{-1/2}\\&\lesssim\left(F^{\rho\varphi}\right)^{\mathscr{L}/2}F(L_i^{\Omega},q,\delta)^{1/2}F(L_j^{\Omega},q,\delta)^{1/2}F(L_t^{\Omega},q,\delta)^{-1/2},\end{aligned}$$

by Lemma 3.6, for $t\geq m$. □

Then, with the notations introduced before, we consider the basis at p (for D)

$$\widetilde{\mathscr{B}} = \{\widetilde{L_1}, \dots, \widetilde{L_{n-1}}\} \text{ with } \widetilde{L_i} = \frac{1}{\left\|L_i^\rho \circ \pi\right\|} \left(L_i^\rho \circ \pi + (\beta_i \circ \pi) N^\rho \circ \pi\right).$$

Note that Lemma 3.6 and Lemma 3.7 are proved for the vector fields L_i^ρ but it is easy to see that they are also valid for the vector fields $L_i^\rho / \left\|L_i^\rho\right\|$.

To simplify the notations, in the remainder of the proof, the vector fields $\frac{L_i^\rho}{\|L_i^\rho\|}$ will be denoted by L_i^ρ, and the function $\frac{\beta_i}{\|L_i^\rho\|}$ will be denoted β_i so that $\widetilde{L_i} = \left(L_i^\rho \circ \pi + (\beta_i \circ \pi) N^\rho \circ \pi\right)$.

Proposition 3.12. *The basis $\widetilde{\mathscr{B}}$ is (K', p, δ)-extremal for a constant K' depending only on K and the data.*

Proof. We first prove condition EB_1, that is, if α_i are complex numbers then

$$F\left(\sum_{i=1}^{n-1} \alpha_i \widetilde{L_i}, p, \delta\right) \simeq \sum_{i=1}^{n-1} |\alpha_i|^2 F\left(\widetilde{L_i}, p, \delta\right).$$

By induction, it suffices to see that, for all k,

$$F\left(\sum_{i=1}^{n-k} \alpha_i \widetilde{L_i}, p, \delta\right) \simeq F\left(\sum_{i=1}^{n-k-1} \alpha_i \widetilde{L_i}, p, \delta\right) + |\alpha_{n-k}|^2 F\left(\widetilde{L_{n-k}}, p, \delta\right).$$

To simplify notations we write $\widetilde{X} = \sum_{i=1}^{n-k-1} \alpha_i \widetilde{L_i}$ and $X^\rho = \sum_{i=1}^{n-k-1} \alpha_i L_i^\rho$. By Proposition 3.11, we have to prove

$$\begin{aligned} (3.16)\quad & F(X^\rho + \alpha_{n-k} L_{n-k}^\rho, q, \delta) + \frac{\varphi''(|p|^2)}{\delta} \left|\left\langle \left(X^\rho + \alpha_{n-k} L_{n-k}^\rho\right) \circ \pi(p), p \right\rangle\right| + \frac{\varphi'(|p|^2)}{\delta} \\ & \simeq F(X^\rho, q, \delta) + |\alpha_{n-k}|^2 F(L_{n-k}^\rho, q, \delta) + \frac{\varphi''(|p|^2)}{\delta} \left(|\langle (X^\rho \circ \pi)(p), p \rangle|^2 + |\alpha_{n-k}|^2 \left|\left\langle (L_{n-k}^\rho \circ \pi)(p), p \right\rangle\right|^2 \right) + \frac{\varphi'(|p|^2)}{\delta}. \end{aligned}$$

Indeed, if $\beta(q) = \frac{\left\|\sum_{i=1}^t \alpha_i L_i^\rho\right\|(q)}{\left\|\sum_{i=1}^t \alpha_i L_i^\rho\right\|(p)}$, β^{-1} have a $\mathscr{C}^M$ norm controlled by K and $F\left(\beta^{-1} \sum_{i=1}^t \alpha_i \widetilde{L_i}, p, \delta\right) \simeq_K F\left(\sum_{i=1}^t \alpha_i \widetilde{L_i}, p, \delta\right)$.

Note that if that if Y and Z are two linear combinations (with constant coefficients) of the L_i^Ω, by extremality, $F(Y + Z, q, \delta) \leq K^2 \left[F(Y, q, \delta) + F(Z, q, \delta)\right]$, and then

$$F(Y + Z, q, \delta) \geq \frac{1}{K^2} F(Y, q\delta) - F(Z, q, \delta). \tag{3.17}$$

This implies that the first member of (3.16) is $\lesssim$ than the second one, and we have only to prove the converse inequality. To do it, we consider separately the two possibilities for L_{n-k}.

Suppose first $L_{n-k}^\rho = T_{n-k}$.

If the second member of (3.16) is equivalent to $F(X^\rho, q, \delta) + |\alpha_{n-k}|^2 F(L_{n-k}^\rho, q, \delta)$, by (3.17), we have only to consider the case when $F(X^\rho, q, \delta) \simeq |\alpha_{n-k}|^2 F(L_{n-k}^\rho, q, \delta)$. Using $F(T_{n-k}, q, \delta) \lesssim F(L_{n-k}^\Omega, q, \delta)$, Lemma 3.6 gives the result.

Suppose now that the second member of (3.16) is equivalent to

$$\frac{\varphi''(|p|^2)}{\delta} \left(|\langle (X^\rho \circ \pi)(p), p \rangle|^2 + |\alpha_{n-k}|^2 \left|\left\langle (L_{n-k}^\rho \circ \pi)(p), p \right\rangle\right|^2 \right).$$

Then, we only have to consider the case when $\langle (X^\rho \circ \pi)(p), p \rangle = -(1+\varepsilon) \alpha_{n-k} \left\langle (L_{n-k}^\rho \circ \pi)(p), p \right\rangle$, with ε small. Then if W is the vector field $X^\rho + (1+\varepsilon) \alpha_{n-k} L_{n-k}^\rho$ normalized at q, $W \in \mathscr{H}_{n-k}$ and thus $F(W, q, \delta) \geq \frac{\varphi''}{\delta} |\langle T_{n-k}(q), p \rangle|^2 = \frac{\varphi''}{\delta} \left|\left\langle L_{n-k}^\rho(q), p \right\rangle\right|^2$. Then $F(X^\rho, q, \delta) \gtrsim \frac{1}{K^2} \left(\frac{\varphi''}{\delta} \left|\left\langle L_{n-k}^\rho(q), p \right\rangle\right|^2 \right) - 2F(L_{n-k}^\rho, q, \delta)$, and the conclusion follows.

To finish suppose that $L_{n-k}^\rho = W_{n-k}$.

If the second member of (3.16) is equivalent to $\frac{\varphi''(|p|^2)}{\delta} \left(|\langle (X^\rho \circ \pi)(p), p \rangle|^2 + |\alpha_{n-k}|^2 \left|\left\langle (L_{n-k}^\rho \circ \pi)(p), p \right\rangle\right|^2 \right)$, there is nothing to do because $\left\langle L_{n-k}^\rho \circ \pi(p), p \right\rangle = 0$.

Suppose then that the second member of (3.16) is equivalent to $F(X^\rho, q, \delta) + |\alpha_{n-k}|^2 F(L_{n-k}^\rho, q, \delta)$. As before, the conclusion is evident except if $F(X^\rho, q, \delta) \simeq |\alpha_{n-k}|^2 F(L_{n-k}^\rho, q, \delta)$. Suppose

$$F(X^\rho + \alpha_{n-k} L_{n-k}^\rho, q, \delta) + \frac{\varphi'(|p|^2)}{\delta} \ll |\alpha_{n-k}|^2 F(W_{n-k}, q, \delta).$$

Note that $\langle T_{n-k}(q), p \rangle \neq 0$, and we can define $W = X^\rho + \alpha_{n-k} L_{n-k}^\rho + \mu T_{n-k}$ such that $\langle W(q), p \rangle = 0$. Then by Lemma 3.6, $|\alpha_{n-k}|^2 F(W_{n-k}, q, \delta) \gg F(L_{n-k}^\Omega, q, \delta)$, and (extremality of $\mathscr{B}^\Omega$) $|\langle T_{n-k}(q), p \rangle|^2 \frac{\varphi''\left(|p|^2\right)}{\delta} > \frac{1}{K} \left(F(W_{n-k}, q, \delta) - KF(L_{n-k}^\Omega, q, \delta)\right.$.

From this we deduce $|\mu| \ll |\alpha_{n-k}|$ and W is of norm almost 1 at q. Then

$$\begin{aligned} F(W,q,\delta) &\leq K^2\left(F(X^\rho + \alpha_{n-k}L^\rho_{n-k},q,\delta) + |\mu|^2 F(T_{n-k},q,\delta)\right) \\ &\ll |\alpha_{n-k}|^2\left(F(W_{n-k},q,\delta) + F(L^\Omega_{n-k},q,\delta)\right), \end{aligned}$$

because $T_{n-k} \in \mathscr{E}\left(L^\Omega_{n-k},\dots,L^\Omega_{n-1}\right)$, and thus $F(W,q,\delta) \ll F(W_{n-k},q,\delta)$ which contradicts the definition of W_{n-k}.

To see that $\widetilde{\mathscr{B}}$ satisfy EB$_2$, a simple calculus shows that it suffices to apply Lemma 3.7 and Proposition 3.10. □

Then, by Lemma 3.1 we conclude:

Proposition 3.13. *The basis $\mathscr{B}$ previously defined by $\mathscr{B} = \{L_i,\dots,l_{n-1}\}$, with $L_i = L^\rho_i \circ \pi + (\beta_i \circ \pi)N^\rho \circ \pi$ is (K',p,δ)-extremal for a constant K' depending on the constant K of extremality of $\mathscr{B}^\Omega$and the data.*

Now the proof of Theorem 3.3 is complete.

4. GEOMETRICALLY SEPARATED DOMAINS

4.1. **Definition and examples.**

Definition 4.1. Let $\Omega = \{\rho < 0\}$ be a bounded pseudo-convex domain with $\mathscr{C}^\infty$ boundary ($\nabla\rho \neq 0$ in a neighborhood of $\partial\Omega$). We call Ω K-*geometrically separated* at $p_0 \in \partial\Omega$ if p_0 is a point of finite 1-type τ and there exists two neighborhood of p_0, $W(p_0) \Subset V(p_0)$, a constant $\delta_0 > 0$, a constant K, an integer M larger than $\tau+1$ and a basis $\mathscr{B}^0 = \{L^0_1,\dots,L^0_{n-1}\}$ of $(1,0)$ vector fields tangent to ρ in $V(p_0)$, whose $\mathscr{C}^{2M}$ norm are bounded by K their "determinant" bounded from below by $1/K$ and a positive real number $\delta_0$such that:

For each point $p \in W(p_0)\cap\partial\Omega$ and each δ, $0<\delta<\delta_0$, there exits a (M,K,p,δ) extremal basis $\mathscr{B}(p,\delta) = \{L^{p,\delta}_1,\dots,L^{p,\delta}_{n-1}\}$ such that, for each i, the vector field $L^{p,\delta}_i$ can be written (on $V(p_0)$) $L^{p,\delta}_i = \sum_j a^j_i L^0_j$ with $a^j_i \in \mathbb{C}$, $\sum|a_i|^2 = 1$. In other words, the $L^{p,\delta}_i$ are normalized vector fields belonging to the vector space E_0 generated by $\mathscr{B}^0$.

A notable property (that will not be used later) of these domains is that the weights F_i satisfy a better estimate than the one given in Proposition 3.4:

Proposition. *Suppose Ω is geometrically separated at p_0 (of type τ). Then for $V(p_0)$ and δ_0 sufficiently small, there exists a constant $C>0$ depending only on K and Ω, such that the extremal basis $\mathscr{B}(p,\delta) = \left\{L^{p,\delta}_i, 1\leq i\leq n-1\right\}$, $p \in W(p_0)\cap\partial\Omega$, $0<\delta<\delta_0$, satisfy $F_M(L^{p,\delta}_i,p,\delta) \geq C\delta^{-2/\tau+1}$, for all i and all $\delta \in [0,\delta_0]$, with $M = [\tau]+1$.*

Proof. Suppose there exists a sequence of points p_m converging to p_0, a sequence δ_m in $]0,\delta_0[$ and an integer $i \leq n-1$ such that, denoting $\mathscr{B}(p_m,\delta_m) = \left(L^m_1,\dots,L^m_{n-1}\right)$ the (M,K,p_m,δ_m)-extremal basis at p_m, $\sum_{\mathscr{L}\in\mathscr{L}_M(L^m_i)}|\mathscr{L}(\partial\rho)(p_m)| \leq 1/m$. Then $L^m_i = \sum a^j_i(p_m)L^0_j$, $\sum\left|a^j_i(p_m)\right|^2 = 1$, and we may suppose that the sequences $n \mapsto a^j_i(p_m)$ converge to complex numbers a^j satisfying $\sum\left|a^j\right|^2 = 1$. Then, by uniform convergence, the vector field $L = \sum a^j L^0_j$ satisfies $F_M(L,p_0,\delta) = 0$, for all δ. But, we have $L = \sum b_k L^{p_0}_k$, $\sum|b_k|^2 \geq_K 1$, and, by extremality $F(L,p_0,\delta) \simeq_K \sum|b_k|^2 F_M(L^{p_0}_k,p_0,\delta)$, thus there exists k such that $F_M(L^{p_0}_k,p_0,\delta) = 0$, i. e. $\sum_{\mathscr{L}\in\mathscr{L}_M(L^{p_0}_k)}|\mathscr{L}(\partial\rho)(p_0)| = 0$. Then, by (4) of Definition 3.2 this contradicts the definition of the 1-type. □

Thus, in all the paper, for a geometrically separated domain at a boundary point p_0, the integer M could be changed to $[\tau]+1$. As this change gives no advantage, we will keep $M = M'(\tau)$ and then we can apply directly the results of the preceding Sections.

Remark 4.1. Suppose Ω is geometrically separated at $p_0 \in \partial\Omega$. Let p be a point of $\overline{\Omega}\cap W(p_0)$. If π is the projection onto $\partial\Omega$ defined in Section 3.5.2 let $q = \pi(p)$. Then, reducing $W(p_0)$ and δ_0 if necessary, if $-\frac{1}{3}\rho(p) < \delta < \delta_0$, the basis $B(q,\delta) = (L^{q,\delta}_1,\dots,L^{q,\delta}_{n-1})$ is clearly $(2K,p,\delta)$-extremal, and $F_M(L^{q,\delta}_i,p,\delta) \geq C'\delta^{-2/\tau+1}$ for a constant C' depending only on K and the data. *Thus we will always consider that a geometrically separated domain is equipped, by definition, with extremal basis of the form given in the definition at every point of $V(p_0)\cap\overline{\Omega}$ for $-\frac{1}{3}\rho(p) < \delta < \delta_0$.*

This is clear, because if $\mathscr{L} \in \mathscr{L}_M(\mathscr{B})$, then $|\mathscr{L}(\partial\rho)(p) - \mathscr{L}(\partial\rho)(\pi(p))| = O(\delta)$, where O depends only on K and Ω. Then EB$_1$ is satisfied because $F_i(p,\delta) \geq C\delta^{-2/M}$ with C depending only on Ω and EB$_2$ is also satisfied because $F_k(p,\delta) \leq \delta^{-2}$ (δ_0 small enough).

Example 4.1.

1. The three first examples of extremal basis given in Example 3.1 immediately show that, if p_0 is a point of finite type of $\partial\Omega$ then Ω is geometrically separated at p_0, under one of the three following conditions:
   (a) $\partial\Omega$ is convex near p_0;
   (b) The eigenvalues of the Levi form are comparable at p_0;

   (c) The Levi form is locally diagonalizable at p_0.

(2) Moreover, we will see in Section 4.3 that, if Ω is geometrically separated at p_0 then the local domain D defined in Section 3.5.1 is geometrically separated at every point of its boundary.

4.2. **Structure of homogeneous space.** First recall that we define in Section 3.3.4 the "polydisc" $B^c(\mathscr{B},p,\delta)$ (Definition 3.3) and the "pseudo-balls" $B^c_{\exp}(\mathscr{B},p,\delta)$ and $B^c_{\mathscr{C}}(\mathscr{B},p,\delta)$ (Definition 3.4).

In general, we will just denote by $B^c_{\exp}(p,\delta)$ and $B^c_{\mathscr{C}}(p,\delta)$ the pseudo-balls $B^c_{\exp}(\mathscr{B},p,\delta)$ and $B^c_{\mathscr{C}}(\mathscr{B},p,\delta)$ omitting $\mathscr{B}$, but recall that, if $\delta_1\neq\delta_2$, the balls $B^c_{\exp}(p,\delta_1)$ and $B^c_{\exp}(p,\delta_2)$ are not necessarily constructed with the same basis.

Then by the methods used in [CD06b] (based on the Campbell-Hausdorf formula and the ideas of [NSW85]), reducing $W(p_0)$ if necessary, one can prove the following properties of the balls:

Proposition 4.1. *There exists constants c_0, δ_0, α, β and γ such that, for $p\in W(p_0)\cap\partial\Omega$, $\delta\leq\delta_0$ and $c\leq c_0$, $B^{\alpha c}_{\exp}(p,\delta)\subset B^c(p,\delta)\subset B^{\beta c}_{\exp}(p,\delta)$ and $B^c_{\exp}(p,\delta)\subset B^c_{\mathscr{C}}(p,\delta)\subset B^{\gamma c}_{\exp}(p,\delta)$.*

The importance of this Proposition to construct the structure of homogeneous space is the following: to be able to use Taylor's formula, we have to work with a coordinates system, which is easy in the sets $B^c(p,\delta)$; the hypothesis of geometric separation and Proposition 3.5 imply that the sets associated to curves are associated to a pseudo-distance; and, finally, the sets associated to the exponential map are used to prove that all these sets are equivalent.

Ideas of the proof of Proposition 4.1. It is similar to the proofs of Proposition 3.4 (p. 96) and Lemma 3.16 (p. 101) of [CD06b]. Thus we will only give the main articulations.

The first inclusion comes easily from the control of the coefficients of the vector fields L_i in the coordinate system (z_i) in the polydisc (Proposition 3.5). The second one is more complicated.

Let $\exp_p$ be the exponential map based at p relatively to the vector fields $\mathscr{Y}_i$ (real an imaginary parts of the L_i). Let $\Psi^p=\left(\Psi^p_i\right)_{i=2,\ldots,2n}=\left(\exp_p\right)^{-1}$. We establish the following estimate on the derivatives of the functions Ψ^p_i: there exists constants β and K_1, depending on K and the data, such that

$$\text{if } q=\exp_p(u),\ \max\{|u_i|,|u_{i+n}|\}\leq\beta F_i(p,\delta)^{-1/2} \text{ then } \left|\mathscr{Y}_k\Psi^p_j(q)\right|\leq K_1F_k(p,\delta)^{1/2}F_j(p,\delta)^{-1/2}, \tag{4.1}$$

with the notation of Definition 3.4.

To prove this, we estimate the derivatives of the exponential map. Considering, for $u\in\mathbb{R}^n$, the vector field $\mathscr{Y}_u=\sum u_i\mathscr{Y}_i$, the derivatives of $\exp_p$ are estimated via the Campbell-Hausdorff formula. Let $q=q(u)=\exp_p(u)$, $|u|\leq u_0$,

$$\left|d\exp_p\left(\frac{\partial}{\partial u_i}\right)(u)-\mathscr{Y}_i(q)+\sum_{k=2}^{M}\alpha_k\left[\mathscr{Y}_u,[\ldots[\mathscr{Y}_u,\mathscr{Y}_i]\ldots]\right](q)\right|\leq C\left|u\right|^{M+1},$$

where α_k are universal constants corresponding to brackets of length k (see Lemma 1 (p. 97) of [CD06b]). The brackets are then estimated with Proposition 3.5 and (4.1) is easily obtained. The second inclusion of the Proposition is then easily proved.

The equivalence between the sets defined with the exponential map and the curves is a quite simple consequence of (4.1). ☐

Proposition 4.2. *Let Ω be a bounded pseudoconvex domain K-geometrically separated at $p_0\in\partial\Omega$. Then there exists a constant $c_0>0$, depending on K and the data such that, for all $c\leq c_0$, the sets $B(\mathscr{B}(p,\delta),p,\delta)$ are associated to a pseudo-distance in the following sense: there exists a constant C depending on K and the data (but not on c) such that, if $p\in W(p_0)\cap\partial\Omega$ and $\delta\leq\delta_0$, and if $q\in B(\mathscr{B}(p,\delta),p,\delta)\cap\partial\Omega$, then*

$$B(\mathscr{B}(q,\delta),q,\delta)\subset B(\mathscr{B}(p,\delta),p,C\delta),$$

B denoting one of the sets $B^c_{\mathscr{C}}$, $B^c_{\exp}$ or B^c.

Remark. If we define γ, on $W(p_0)\cap\partial\Omega$, by

$$\gamma(p,q)=\inf\{\delta \text{ such that } q\in B(\mathscr{B}(p,\delta),p,\delta)\}., \tag{4.2}$$

then γ is a real pseudo-distance.

Proof.

Lemma. *1. For all $A>0$ there exists B depending on A and K such that*

$$B^{Ac}_{\mathscr{C}}(\mathscr{B}(q,\delta),q\delta)\subset B^c_{\mathscr{C}}(\mathscr{B}(q,B\delta),q,B\delta).$$

2. For all $B>0$ there exists C depending on B such that

$$B^c_{\mathscr{C}}(\mathscr{B}(q,B\delta),q,B\delta)\subset B^{Cc}_{\mathscr{C}}(\mathscr{B}(q,\delta),q,\delta).$$

Proof. Let us denote by $L_i(q,\delta)$ (resp $L_i(q,B\delta)$) the vector fields of $\mathscr{B}(q,\delta)$ (resp. $\mathscr{B}(q,B\delta)$). By the hypothesis on Ω, we have $L_i(q,\delta)=\sum_k\beta_i^kL_k(q,B\delta)$, with β_i^k constants. By extremality,

$$\begin{aligned}\left|\beta_i^k\right| &\leq KF(L_i(q,\delta),q,B\delta)^{1/2}F(L_k(q,B\delta),q,B\delta)^{-1/2}\\ &\leq KB^{-1/M}F(L_i(q,\delta),q,\delta)^{1/2}F(L_k(q,B\delta),q,B\delta)^{-1/2},\end{aligned}$$

which proves the first part of the Lemma with $B=(AK(n-1))^M$. The second part is proved similarly with $C=(BK(n-1))^M$. ☐

To prove the assertion on the pseudo-distance in the Proposition, by Proposition 4.1, it is enough to prove that, there exists a constant K_0such that if $q,q'\in B^c_{\mathscr{C}}(\mathscr{B}(p,\delta),p,\delta)$ then $z\in B^{K_0c}_{\mathscr{C}}(\mathscr{B}(q,\delta),q,\delta)$. But there exists φ, $\mathscr{C}^1$ piecewise smooth, such that $\varphi(0)=q$, $\varphi(1)=z$ and, almost everywhere, $\varphi'(t)=\sum_{i=1}^{2n}a_i(t)\mathscr{Y}_i(\varphi(t))$, with $\max\{|a_i(t)|,|a_{i+n}(t)|\}\leq 2cF(L_i(p,\delta),p,\delta)\leq 4cF(L_i(p,\delta),q,\delta)$, if we choose c small enough (Proposition 3.5). Now, as in the Lemma, writing $L_i(p,\delta)=\sum\alpha_i^kL_k(q,\delta)$ (with α_i^k constants), using extremality we easily conclude $q'\in B^{K_0c}_{\mathscr{C}}(\mathscr{B}_1^{q,\delta},q,\delta)$. ☐

Let us now define "pseudo-balls" centered at points of $\Omega\cap W(p_0)$, denoted ${}^\pi B^c(q,\delta)$ (resp. ${}^\pi B^c_{\mathscr{C}}(q,\delta)$ ${}^\pi B^c_{\exp}(q,\delta)$) by

$${}^\pi B^c(q,\delta)=\left\{q'\in V(p_0)\text{ such that }\pi(q')\in B^c\left(\mathscr{B}(\pi(q),\delta),\pi(q),\delta\right)\text{ and }\rho(q')\in[\rho(q)-c\delta,\rho(q)+c\delta]\right\}.$$

Then:

Theorem 4.1. *Let Ω be a pseudo-convex domain geometrically separated at $p_0\in\partial\Omega$. There exists a constant $c_0>0$, depending on K and the data, such that, for all $c\leq c_0$, the sets $B^c(q,\delta)$ define a structure of "homogeneous space" on $W(p_0)\cap\overline{\Omega}$ in the following sense: there exists a constant C, depending only on K and the data (not on c) such that, if $q_1\in W(p_0)\cap\overline{\Omega}$, $\delta<\delta_0$, and $q_2\in B(q_1,\delta)$, we have*

$$B(q_2,\delta)\subset B(q_1,C\delta)$$

and

$$\mathrm{Vol}\left(B(q,2\delta)\right)\leq C\mathrm{Vol}\left(q,\delta)\right),$$

B denoting one of the sets $B^c_{\mathscr{C}}$, $B^c_{\exp}$ or B^c.

Proof. The first assertion follows immediately the Proposition. To prove the second, we use that both $B^c_{\mathscr{C}}\left(\mathscr{B}(p,\delta),p,\delta\right)$ and $B^c_{\exp}\left(\mathscr{B}(p,\delta),p,\delta\right)$ are equivalent to $B^c\left(\mathscr{B}(p,\delta),p,\delta\right)$, the fact that the coordinate system associated to the extremal basis have a Jacobian uniformly bounded from above and below and the preceding Lemma. ☐

Remark 4.2. (1) For $p\in\partial\Omega$, the sets ${}^\pi B^c(q,\delta)\cap\partial\Omega$ (for each definition) are the pseudo-balls of a structure of homogeneous space on $\partial\Omega\cap W(p_0)$.
(2) On $\partial\Omega$, as in [NRSW89], we could define equivalent pseudo-balls using complex tangent curves.

4.3. **Localization.** Suppose that Ω is K-geometrically separated at $p_0\in\partial\Omega$, and consider the domain D constructed in Section 3.5.1 near that point. Then D is K-geometrically separated at each point of $\partial\Omega\cap\partial D$, and, by strict pseudo-convexity, the same is true on $\partial D\setminus\overline{\partial\Omega\cap\partial D}$.

Suppose that P is a point of the boundary of $\partial\Omega\cap\partial D$, and let p be a point of $V(P)\cap\partial D$ and δ small enough (with the notations of the previous Section). Let us denote by $\mathscr{B}(p,\delta)=\left\{L_1^{p,\delta},\ldots,L_{n-1}^{p,\delta}\right\}$ the extremal basis given by Proposition 3.13 and by $\mathscr{B}^{0,\Omega}=\left\{L_1^{0,\Omega},\ldots,L_{n-1}^{0,\Omega}\right\}$ the basis denoted B^0 in Definition 4.1. Then, by the construction of $\mathscr{B}(p,\delta)$ made in the previous Section, we have $L_i^{p,\delta}=L_i^\rho\circ\pi-\beta(L_i^\rho)N^\rho\circ\pi$ with $L\mapsto\beta(L)$ linear. Thus, if we define $\mathscr{B}^{0,D}=\left\{L_1^{0,D},\ldots,L_{n-1}^{0,D}\right\}$ by $L_i^{0,D}=L_i^{0,\Omega}\circ\pi-\beta(L_i^{0,\Omega})N^\rho\circ\pi$, we see that the vector fields of $\mathscr{B}(p,\delta)$ are linear combinations (with constant coefficients) of the vector fields of $\mathscr{B}^{0,D}$. Thus, we have proved the following result:

Theorem 4.2. *If Ω is K-geometrically separated at $p_0\in\partial\Omega$, then the domain D defined in Definition 3.6 is K'-geometrically separated (at every point of its boundary) for a constant K' depending only on K and the data.*

Remark. Recall that every point of $\partial\Omega$ is of finite 1-type.

5. ADAPTED PLURI-SUBHARMONIC FUNCTION FOR GEOMETRICALLY SEPARATED DOMAINS

5.1. Definition and examples.

Definition 5.1. Let Ω be geometrically separated at p_0. Let E be the vector space generated by $\mathscr{B}^0\cup\{N\}$, and, if $L=\sum_{i=1}^{n-1}b_iL_i^0+b_nN=L_\tau+b_nN\in E$ denotes, for $\delta\leq\delta_0$, $F(L,q,\delta)=F(L_\tau,q,\delta)+\frac{|b_n|^2}{\delta^2}$. A pluri-subharmonic function H_δ is said to be β-adapted to $\mathscr{B}^0$ at p_0 if there exists a constant β such that the following properties hold:

(1) $|H_\delta|\leq 1$ on Ω;

(2) For $q \in W(p_0) \cap \Omega \cap \{\rho \geq -2\delta\}$ and for all vector field $L \in E$,

$$\left\langle \partial\overline{\partial} H_\delta; L, \overline{L} \right\rangle (q) \geq \frac{1}{\beta} F(L, q, \delta);$$

(3) For $q \in W(p_0) \cap \Omega \cap \{\rho \geq -2\delta\}$ and for all list $\mathscr{L} \in \mathscr{L}_3(E)$,

$$|\mathscr{L} H_\delta|(q) \leq \beta \prod_{L \in \mathscr{L}} F(L, q, \delta)^{1/2}.$$

Remark 5.1. Note that (3) implies in particular that, for all $\mathscr{L} \in \mathscr{L}_3(\mathscr{B}(\pi(q), \delta) \cup \{N\})$,

$$|\mathscr{L} H_\delta|(q) \lesssim F(\mathscr{B}(\pi(q), \delta), q, \delta)^{\mathscr{L}/2}.$$

Definition 5.2. A bounded pseudo-convex domain Ω is called "K-completely geometrically separated" at p_0 if it is K-geometrically separated and, there exists $\delta_0 > 0$ such that, for all $0 < \delta \leq \delta_0$, there exists a pluri-subharmonic function K-adapted to $\mathscr{B}^0$ at p_0.

Example 5.1.

1. If the boundary of Ω is locally convex near p_0 (a point of finite type), it is proved in [McN94, MN02] that it is completely geometrically separated at p_0.
2. In [Cho02b, Cho02a, Cho03], it is proved that, at a point of finite type, if the eigenvalues of the Levi form are comparable at p_0 then it is also completely geometrically separated at p_0.
3. In the next Section, we prove that geometrically separated domains whose extremal basis are strongly extremal with a sufficiently small α are completely geometrically separated, and, moreover that, for those domains, the local domain defined in Section 3.5 is completely geometrically separated at every point of its boundary. In particular, this applies when the Levi form is locally diagonalizable at p_0.

5.2. **The case of geometrically separated domains with strongly extremal basis.** In this Section we prove the two following Theorems:

Theorem 5.1. *Suppose Ω is K-geometrically separated at $p_0 \in \partial\Omega$. Then there exists a constant α_0, depending on K and the data, such that, if for all $p \in W(p_0) \cap \partial\Omega$ and $\delta \leq \delta_0$, the basis $\mathscr{B}(p, \delta)$ are (K, α, p, δ)-strongly extremal (c.f. Definition 3.5) with $\alpha \leq \alpha_0$ then it is completely geometrically separated at p_0.*

The second deals with the local domain D constructed in Section 3.5.1, and, in fact contains the first one. We state the two theorems separately because the proof of Theorem 5.1 is used to prove the second:

Theorem 5.2. *Suppose that Ω is K-geometrically separated at $p_0 \in \partial\Omega$. There exists a constant α_1, depending on K and the data such that, if for all $p \in W(p_0) \cap \partial\Omega$ and $\delta \leq \delta_0$, the basis $\mathscr{B}(p, \delta)$ are (K, α, p, δ)-strongly extremal with $\alpha \leq \alpha_1$, then the local domain constructed in Section 3.5.1 is K'-completely geometrically separated at every point of its boundary for a constant K' depending only on K and Ω.*

5.2.1. *Proof of Theorem 5.1.* Here we suppose that the basis $\mathscr{B}(p, \delta)$, $p \in W(p) \cap \partial\Omega$, $\delta \leq \delta_0$, are (K, α, p, δ)-strongly extremal for a constant α not yet fixed. During the proof, we will impose successive conditions on α (depending on K, M and n) to be able to construct the good pluri-subharmonic function. The existence of α will be clear at the end of the proof but we will not give an explicit value. Now, we fix $\delta > 0$.

The ideas of construction are comparable to those developed in [CD06b] (following ideas of [Cat87]) but the technical proofs are slightly different. On one hand the basis are local instead to be global and we have to construct local "almost pluri-subharmonic" functions and then add them using the structure of homogeneous space instead to construct directly a global function. On the other hand, the control of lists following our hypothesis are weaker than those following the local diagonalizability of the Levi form. Thus, for reader's convenience, we will write the proof with enough details.

Theorem 5.1 is proved using a local construction. We need to introduce new notations.

Let us fix δ and denote $Q^c(p, \delta)$ the points q in $V(p_0)$ such that $\pi(q)$ belongs to $B^c(p, \delta)$, the polydisc associated to the extremal basis $\mathscr{B}(p, \delta)$ (see Definition 3.3). Let L be a vector field in E (the vector space generated by $\mathscr{B}^0$ ans N). We write $L = L_\tau + a_n N$, where L_τ is tangent to ρ. Because Ω is geometrically separated we can write $L_\tau = \sum_{i=1}^{n-1} a_i^p L_i^{p,\delta}$ ($a_i^p \in \mathbb{C}$), and finally, c_{ii}^p will denote the coefficient of the Levi form associated to the vector field $L_i^{p,\delta} \in \mathscr{B}(p, \delta)$.

Now we state the local result and show how it leads to Theorem 5.1. For the proof we need only estimates in the strip $\Omega_{3\delta} = \{-3\delta \leq \rho \leq 0\}$, but in Section 5.2.3we need corresponding results in a larger domain, and thus we state the local result for the sets $Q^c(p, \delta)$:

Proposition 5.1. *For all constant $C > 1$ there exists constants α_0 (depending only on K, c, C and the data), β and γ_1 such that if the basis $\mathscr{B}(p, \delta)$ are (K, α, p, δ)-extremal with $\alpha \leq \alpha_0$, then for all $\delta \leq \delta(\alpha_0)$ (depending on α_0, K and the data) and all point $p \in W(p_0) \cap \partial\Omega$, there exists a function $H_{p,\delta} = H$ with support in $Q^c(p, \delta)$ satisfying, for every vector field L, the following conditions:*

(1) $|H| \leq 1$*;*

(2) *$\left\langle\partial\bar\partial H;L,\bar L\right\rangle(q)\geq\beta F(L_\tau,q,\delta)-\gamma_1\left(\sum_{i=1}^{n-1}\left|a_i^p\right|^2\frac{c_{ii}}{\delta}+\frac{|a_n|^2}{\delta^2}+\frac{\partial_\Omega(q)}{\delta}\right)(q)$, for $q\in Q^{c/2}(p,\delta)$;*

(3) *For all q,$\left\langle\partial\bar\partial H;L,\bar L\right\rangle(q)\geq-\frac{\beta}{C}F(L_\tau^q,q,\delta)-\gamma_1\left(\sum_{i=1}^{n-1}\left|a_i^{p,q}\right|^2\frac{c_{ii}}{\delta}+\frac{|a_n|^2}{\delta^2}+\frac{\partial_\Omega(q)}{\delta}\right)(q)$.*

(4) *For $\mathscr{L}\in\mathscr{L}_3\left(\mathscr{B}(p,\delta)\cup\{N\}\right)$, $|\mathscr{L}H|(q)\leq\gamma_2\prod_{L\in\mathscr{L}}F(L,q,\delta)^{1/2}+\left(\frac{\delta_\Omega(q)}{\delta}\right)^{|\mathscr{L}|/2}$, where is the distance to $\partial\Omega$.*

We will prove this Proposition in the next Section. Now we show how the proof of Theorem 5.1 follows this Proposition:

Proof of Theorem 5.1. We cover $\partial\Omega\cap W(p_0)$ with a minimal system of pseudo-balls ${}^\pi B^{c/2}(p_k,\delta)\cap\partial\Omega$, $p_k\in\partial\Omega$. As the pseudo-balls are associated to a structure of homogeneous space, there exists an integer S, independent of δ, such that each point of $W(p_0)$ belongs to at most S sets $Q^c(p_j,\delta)$. We apply Proposition 5.1 with $C=2SC_1$ to get the function $H_{p_k,\delta}$.

For all point $q\in V(P_0)$ there exists j_0 such that $q\in Q^{c/2}(p_{j_0},\delta)$ and thus (denoting c_{ii}^k the coefficient of the Levi form in the direction $L_i^{p_k}$ and $a_i^k=a_i^{p_k,q}$), by Proposition 5.1,

$$\left\langle\partial\bar\partial\sum_k H_{p_k,\delta};L,\bar L\right\rangle(q)\geq\frac{\beta}{2}F(L_\tau,q,\delta)-\gamma_1\sum_{k\text{ s.t. }q\in Q^c(p_k,\delta)}\left(\sum_{i=1}^{n-1}\left|a_i^k\right|^2\frac{\left|c_{ii}^k(q)\right|}{\delta}+\frac{|a_n|^2}{\delta^2}+\frac{\delta_\Omega(q)}{\delta}\right).\tag{5.1}$$

We now suppose that we are in the strip $\Omega_{3\delta}$. Let us consider the function

$$H=\sum_k H_{p_k,\delta}+Ae^{-\rho/\delta}+B|Z|^2,$$

for suitable constant A and B and α small enough:

Claim. There exists constants A, B, γ and α'_0 depending only on K and the data such that if $\alpha\leq\alpha'_0$,

(1) H is uniformly bounded, independently of $\delta\leq\delta_0$, on $\Omega_{3\delta}$;

(2) For any vector field L, for every k, for $q\in Q^{c/2}(p_k,\delta)\cap\Omega_{3\delta}\cap W(p_0)$, $\left\langle\partial\bar\partial H;L,\bar L\right\rangle(q)\geq\frac{\beta}{2}F(L_\tau^q,q,\delta)+\frac{|a_n|^2}{\delta^2}$;

(3) For $q\in\Omega_{3\delta}\cap W(p_0)$ and all list $\mathscr{L}\in\mathscr{L}_3(E)$, $|\mathscr{L}H|(q)\leq\gamma_2\prod_{L\in\mathscr{L}}F(L,q,\delta)^{1/2}$.

Proof of the Claim. As we are in $\Omega_{3\delta}$, (5.1) implies

$$\left\langle\partial\bar\partial\sum_k H_{p_k,\delta};L,\bar L\right\rangle(q)\geq\frac{\beta}{2}F(L_\tau,q,\delta)-\gamma_1\sum_{k\text{ s.t. }q\in Q^c(p_k,\delta)}\left(\sum_{i=1}^{n-1}\left|a_i^k\right|^2\frac{\left|c_{ii}^k(q)\right|}{\delta}+\frac{|a_n|^2}{\delta^2}\right)(q).$$

Moreover, for every k such that $q\in Q^c(p_k,\delta)$,

$$\left\langle\partial\bar\partial e^{\rho/\delta};L,\bar L\right\rangle(q)=e^{\rho/\delta}\left[\frac{1}{\delta}\left(\frac{1}{2}\sum_{i,j=1}^{n-1}a_i^k\overline{a_j^k}c_{ij}^k+2\Re\mathrm{e}\left(\sum_{i=1}^{n-1}a_i^k\overline{a_n^k}\left\langle\partial\bar\partial\rho;L_i^{p_k},\bar N\right\rangle\right)+|a_n|^2\left\langle\partial\bar\partial\rho;N,\bar N\right\rangle\right)+\frac{|a_n|^2}{\delta^2}\right].$$

Then, we use the hypothesis of strong extremality and Taylor's formula to estimate $\left|c_{ij}^k\right|$, $i\neq j$, in the set $Q^c(p_k,\delta)\cap\Omega_{3\delta}$. Using the fact that $c_{ii}=|c_{ii}|+\mathrm{O}(\delta)$ (recall Ω is pseudo-convex), this gives a constant K_0 depending on K and the data such that

$$\left\langle\partial\bar\partial e^{\rho/\delta};L,\bar L\right\rangle(q)\geq-K_0+\frac{e^{-3}}{2\delta^2}|a_n|^2+\frac{e^{-3}}{2\delta}\sum_{i=1}^{n-1}\left|a_i^k\right|^2\left|c_{ii}^k(q)\right|-4n\alpha F(L_\tau^q,q,\delta),$$

because, by definition of c, in the sets $Q^c(p_k,\delta)$, $F(L_\tau^{p_k},q)\leq 3F(L_\tau^{p_k},p_k)$ (see Proposition 3.5).

Now we choose $A=2Se^3\gamma_1+1$, then $B=K_0A$, and $\alpha'_0<\frac{\beta}{16}nA$, which proves the two first properties of the Claim.

The third property follows (4) of Proposition 5.1, the extremality of every basis $\mathscr{B}(p,\delta)$, Proposition 3.5, and the fact that, in the considered domain, $\delta_\Omega(q)=O(\delta)$. □

To finish the proof of Theorem 5.1, we cut H to adapt it to good neighborhoods $V(p_0)$ and $W(p_0)$ and the required properties in the strip $\{\delta_\Omega(p)<2\delta\}$, and we add $D|z|^2$ for a large constant D. More precisely, the cutting functions are defined as follows:

Let $\vartheta=\vartheta_1\vartheta_2$ where $\vartheta_1(q)=\chi_1\left(\frac{1}{2}\frac{|q-p_0|}{r}\right)$, with χ_1 a $\mathscr{C}^\infty$ increasing function equal to 0 on $]-\infty,0]$, 1 on $[1/4,+\infty[$ and $\chi_1(t)=t^4$ on $[0,1/8]$, and $\vartheta_2(q)=\chi_\delta(\rho(q))$ with $\chi_\delta(t)=\chi(t/\delta)$, χ being even, increasing on $]-\infty,0[$, equal to 0 on $]-\infty,-4]$, to 1 on $]-2,0[$ and to $\frac{(t+4)^4}{16}$ for $t\in[-4,-8/3]$.

Then the final calculus is made as in [CD06b]. □

5.2.2. *Proof of Proposition 5.1.* The proof uses essentially the ideas developed in Section 4.1 of [CD06b], except that we have to work locally around the point p. Thus the technique is more complicated (it needs to use the structure of homogeneous space) and we will give it with some details.

For $p \in W(p_0) \cap \partial\Omega$ and $\delta \leq \delta_0$ fixed, let $\mathscr{B}(p,\delta) = \{L_i^{p,\delta} = L_i, 1 \leq i \leq n-1\}$ be the (K,α,p,δ)-strongly extremal basis associated, and $\Phi = \Phi_p^\delta$ the adapted change of coordinates at (p,δ).

For $i = 1,\dots,n-1$ and $l = 3,\dots,M$, let us define

$$\mathscr{E}_l^i = \{\Re\mathrm{e}(\mathscr{L}(\partial\rho), \Im\mathrm{m}(\mathscr{L}(\partial\rho), |\mathscr{L}| = l-1, \mathscr{L} = \{L^1,\dots,L^{l-1}\}, L^k \in \{L_i,\overline{L_i}\}\},$$

$$\mathscr{E}^i = \bigcup_l \mathscr{E}_l^i.$$

If $\varphi \in \mathscr{E}_l^i$, we denote $l(\varphi) = l$.

Note that $F_i(.,\delta) = F(L_i,.,\delta) \simeq \frac{|c_{ii}|}{\delta} + \sum_{\varphi\in\mathscr{E}^i} \left|\frac{L_i\varphi}{\delta}\right|^{2/l(\varphi)}$. The functions $\frac{|c_{ii}|}{\delta}$ and $\left|\frac{L_i\varphi}{\delta}\right|^{2/l(\varphi)}$ are called the *components* of F_i and are denoted generically f_i. We also define $l(c_{ii}) = 2$, and, for the other functions f_i, $l(f_i) = l(\varphi)$. In the following proof, these components cannot be considered individually. Thus, we introduce the terminology of "$(n-1)$-uplet" of components: $f = (f_1,\dots,f_{n-1})$, where f_i are component of F_i, is called a $(n-1)$-uplet of components of the weights F_i. The set of all such $(n-1)$-uplet is denoted by $\mathscr{H}$. $\mathscr{H}$ is ordered by the lexicographic order.

First we define a cutoff function with support the point p and in the set where a component is "dominant". More precisely, if B is a positive number and $f = (f_i)$ a $(n-1)$-uplet of components of F_i, we define, for fixed $c \leq c_0$,

$$\chi_{f,B} = \prod_i \chi_B\left(\frac{f_i\circ\pi}{F_i(p,\delta)}\right)\chi_0 = \chi'_{f,B}\chi_0,$$

where $\chi_B(t) = \chi(Bt)$, $\chi : [0,+\infty[\mapsto [0,1]$, being a $\mathscr{C}^\infty$ function equal to 0 on $[0,1/2]$ and 1 on $[1,+\infty[$, and $\chi_0(q) = \chi_1\left(\left(\frac{F_i(p,\delta)^{1/2}}{c}\Phi_p(\pi(q))_i\right)_i\right)$, with χ_1 a $\mathscr{C}^\infty$ function identically 1 on $B(0,1/2)$ and with compact support in $B(0,1)$.

We say that f is B dominant if $\chi'_{f,B} = 1$.

Then, to each component of F_i of type $f_i = \left|\frac{L_i^p\varphi}{\delta}\right|^{2/l(\varphi)}$, we associate, for $\lambda > 1$ the function

$$H_i(f,\lambda,B) = \lambda^{-3/2}e^{\lambda\psi_i}\chi_{f,B},$$

where $\psi_i(q) = \frac{\varphi_i(\pi(q))}{\delta}F_i(p,\delta)^{\frac{1-l(\varphi_i)}{2}}$.

Lemma 5.1. *For each constant $B > 0$, there exists a constant K_0 depending only on B, c, K and the data such that, for each i, if $q \in Q^c(p,\delta)$, for each $L = \sum_{i=1}^n a_iL_i$, $\sum\left|a_j\right|^2 = 1$, we have the following estimates:*

(1) $|L\psi_i(q)| \leq K_0\left(F(L_\tau,q,\delta)^{1/2} + \frac{|a_n|}{\delta} + \left(\frac{\delta_\Omega(q)}{\delta}\right)^{1/2}\right)$, *and* $|\overline{L}L(\psi_i)(q)| \leq K_0\left(F(L_\tau,q,\delta) + \frac{|a_n|^2}{\delta^2} + \frac{\delta_\Omega(q)}{\delta}\right)$;

(2) $\left|L\chi_{f,B}(q)\right| \leq K_0\left(F(L_\tau,q,\delta)^{1/2} + \frac{|a_n|}{\delta} + \left(\frac{\delta_\Omega(q)}{\delta}\right)^{1/2}\right)$, *and* $\left|\overline{L}L\chi_{f,B}\right| \leq K_0\left(F(L_\tau,q,\delta) + \frac{|a_n|^2}{\delta^2} + \frac{\delta_\Omega(q)}{\delta}\right)$;

(3) $|[L,\overline{L}]\left(\partial(H_i(f,\lambda,B))\right| \leq K_0\lambda^{-1/2}e^{\lambda\psi_i}\left(F(L_\tau,q,\delta)^{1/2} + \frac{|a_n|}{\delta} + \left(\frac{\delta_\Omega(q)}{\delta}\right)^{1/2}\right)$.

Proof. If $q \in \partial\Omega$, the inequality $|L\psi_i(q)| \leq K_0\left(F(L_\tau,q,\delta)^{1/2} + \frac{|a_n|}{\delta}\right)$ follows immediately Proposition 3.5 and the extremality of the basis (L_i) at (p,δ). The general case for (1) follows.

(2) is obtained using the fact that, if (z) is the change of coordinates associated to Φ and $L_i = \sum a_i^j\frac{\partial}{\partial z_j}$, then $\left|a_i^j\right| \lesssim F_i^{1/2}(p,\delta)F_j^{-1/2}(p,\delta)$ for $q \in Q^c(p,\delta) \cap \partial\Omega$, and similar techniques as for (1).

(3) is proved similarly, using the estimates of the coefficients of the brackets $[L_i\overline{L_j}]$ in $Q^c(p,\delta) \cap \partial\Omega$ (Proposition 3.5). $\square$

For $f = (f_1,\dots,f_{n-1})$, a $(n-1)$-uplet of components of the weights F_i, let us denote by I the set of indices i such that $f_i = \left|\frac{L_i^p\varphi_i}{\delta}\right|^{2/l(\varphi_i)}$. Then we consider the function

$$H(f,\lambda,B) = \sum_{i\in I} H_if,\lambda,B).$$

The next Lemma gives some properties of the function $H(f,\lambda,B)$. To state it we need to introduce the following set:

For f a $(n-1)$-uplet of components of the weights F_i and B' a positive number, we denote

$$U_{B',f} = \left\{q \in Q^c(p,\delta) \text{ for which there exists } f' < f \text{ such that } f'(q) \text{ is } B' \text{ dominant}\right\}.$$

Lemma 5.2. *Let f be a $(n-1)$-uplet of component, A, B and ε three positive fixed real numbers. Then there exists constants α_0, λ, A', B', $A' > A$, $B' > B$, ε' and K_1, depending only on A, B, ε, K and the data, such that, if the constant α of strong extremality is $\leq \alpha_0$, then the function $H(f,A,B,\varepsilon) = H(f,\lambda,B) = H$ satisfies:*

(1) *$|H|\leq K_1$;*

(2) *If $L=\sum_{i=1}^n a_iL_i^p=L_\tau+a_nN$, $a_i\in\mathbb{C}$, $\sum|a_i|^2=1$, then$\left|\left\langle\partial\bar{\partial}H;L,\bar{L}\right\rangle\right|(q)\leq A'\left(F(L_\tau,q,\delta)+\frac{|a_n|^2}{\delta^2}+\frac{\delta_\Omega(q)}{\delta}\right)$;*

(3) *if $q\notin U_{B'}$, $\chi'_{f,B}(q)=1$, $\chi_0(q)\geq\varepsilon$, for the same L,*

$$\left\langle\partial\bar{\partial}H;L,\bar{L}\right\rangle(q)\geq AF(L_\tau,q,\delta)-K_2\left(\sum_{i=1}^{n-1}|a_i|^2\frac{|c_{ii}(q)|}{\delta}+\frac{|a_n|^2}{\delta^2}+\frac{\delta_\Omega(q)}{\delta}\right);$$

(4) *$\left\langle\partial\bar{\partial}H;L,\bar{L}\right\rangle(q)\leq-\left(F(L_\tau,q,\delta)+\frac{|a_n|^2}{\delta^2}+\frac{\delta_\Omega(q)}{\delta}\right)$ implies $q\in U_{B'}$ and $\chi_0(q)\geq\varepsilon'$.*

(5) *For all list $\mathscr{L}\in\mathscr{L}_3(\mathscr{B}(p,\delta)\cup\{N\})$, $|\mathscr{L}H(q)|\leq K_2\left(\prod_{L\in\mathscr{L}}F(L,q,\delta)^{1/2}+\left(\frac{\delta_\Omega(q)}{\delta}\right)^{|\mathscr{L}|/2}\right)$.*

Proof. Recall that $H=\sum_{i\in I}H_i$, thus the properties are trivially satisfied if $I=\emptyset$ and we suppose $I\neq\emptyset$. The functions $|\psi_i|$ being bounded by 2 (see Proposition 3.5), (1) is satisfied with a constant K_1 depending only on λ and n.

Let $i\in I$. Then $\left\langle\partial\bar{\partial}H_i;L\bar{L}\right\rangle=\bar{L}LH_i+[L,\bar{L}](\partial H_i)$, and as

$$\bar{L}LH_i=\lambda^{-3/2}e^{\lambda\psi_i}\left[\left(\lambda^2|L\psi_i|^2+\lambda\bar{L}L\psi_i\right)\chi_{f,B}+\lambda\left(L\psi_i\bar{L}\chi_{f,B}+\bar{L}\psi_iL\chi_{f,B}\right)+\bar{L}L\chi_{f,B}\right],$$

Lemma 5.1 implies $\left\langle\partial\bar{\partial}H_i;L\bar{L}\right\rangle(q)\geq\lambda^{-3/2}e^{\lambda\psi_i}\left(\lambda^2|L\psi_i|^2\chi_{f,B}-K'_0\lambda F(L_\tau,q,\delta)+\frac{|a_n|^2}{\delta^2}+\frac{\delta_\Omega(q)}{\delta}\right)$ and thus the existence of a constant A', depending only on the choice of λ, B, c, K and the data, satisfying (2).

Now, if for all $i\in I$, $|\lambda\psi_i|\leq 1$, then, for λ large enough, we have $\left\langle\partial\bar{\partial}H;L,\bar{L}\right\rangle\geq-F(L)$. Thus we suppose that there exists an $i\in I$ such that $|\lambda\psi_i(q)|=\frac{\lambda|\varphi_i(\pi(q))|}{\delta}F_i(p,\delta)^{(1-l(\varphi_i))/2}\geq 1$. Thus there exists a constant $B'>B$, depending on λ, such that $\left|\frac{\varphi_i(\pi(q))}{\delta}\right|^{2/(l(\varphi_i)-1)}>\frac{4}{B'}F_i(p,\delta)$, and this implies that there exists a $(n-1)$-uplet $f'<f$ which is B'-dominant at the point q. In other words, to each choice of λ we can associate B' such that the first conclusion in (4) is true. Moreover, λ, B and c being fixed, χ_1 being $\mathscr{C}^\infty$, there exists ε', depending on λ, B, c and χ_1, such that the hypothesis of (4) implies the second conclusion.

Let us now show that we can choose λ (thus A', B', K_1 and ε' will be fixed) such that (3) is satisfied if α is small enough. Suppose then $\chi'_{f,B}(q)=1$and $\chi_0(q)>\varepsilon$. The hypothesis of strong extremality and the invariance of the $F_i(q)$ and a_{ij}^k in $B^c(p,\delta)$ (Propositions 3.5 and 3.9) gives, if $\delta\leq\delta(\alpha)$,

$$|L\psi_i(q)|^2\geq\frac{1}{4}\left|\sum_{j\leq i}a_j(L_j\psi_i)(q)\right|^2-4nC(K)\left(2\alpha\sum_{n-1\geq j>i}|a_j|^2F_j(p)+\frac{|a_n|^2}{\delta^2}\right),$$

and then, by extremality at p,

$$|L\psi_i(q)|^2\geq\frac{1}{4}\left|\sum_{j\leq i}a_jL_j\psi_i(q)\right|^2-C_1(K)\left(\alpha F(L,q,\delta)+\frac{|a_n|^2}{\delta^2}+\frac{\delta_\Omega(q)}{\delta}\right).$$

Now we make use of the following Lemma:

Lemma. *Let β_i^j be complex numbers, $i=1,2,\cdots,n-1$, $j\leq i$, verifying $\left|\beta_i^i\right|\geq c\alpha_i$ and $\left|\beta_i^j\right|\leq C\alpha_j$ for $j<i$. Then there exists a constant $W=W(c,C,n)$ such that $\sum_{i=1}^{n-1}\left|\sum_{j=1}^i\beta_i^j\right|^2\geq W\sum_{i=1}^{n-1}(\alpha_i)^2$.*

It implies, using the invariance of $F_i(q)$ and $F(L,q)$ in the ball and the extremality of the basis at p, that there exists constants W, K_3 and K_4, depending on B, M, K and the data, such that:

$$\sum_{i\in I}|L\psi_i(q)|^2+\sum_{i\notin I}\frac{|c_{ii}(q)|}{\delta}\geq\frac{W}{2K}F(L_\tau,q,\delta)-\alpha K_3\left(F(L_\tau,q,\delta)+\frac{\delta_\Omega(q)}{\delta}\right)-K_4\frac{|a_n|^2}{\delta^2},$$

and thus, for $\alpha_0=W/4KK_3$ (depending only on the data M, K, B, c and n),

$$\sum_{i\in I}|L\psi_i(q)|^2+\sum_{i\notin I}\frac{|c_{ii}(q)|}{\delta}\geq W'F(L_\tau,q,\delta)-K_4\left(\frac{|a_n|^2}{\delta^2}+\frac{\delta_\Omega(q)}{\delta}\right).$$

This finishes the proof of the Lemma for a choice of λ depending on A, ε, B, M, K and c, c depending itself only on M, K and the data, the property (5) being trivial. □

Proof of Proposition 5.1. First, note that there exists a constant D, depending on M and n, such that, for $p\in W(p_0)$ and $\delta\geq\frac{1}{3}|\rho(p)|$, there exists a component f_i of $F_i(p,\delta)$ verifying $f_i(q)\geq\frac{1}{D}F_i(p,\delta)$ for all points $q\in B^c(p,\delta)$, $c\leq c_0$ (Proposition 3.5) .

To define completely our function H, we have to define, for each $(n-1)$-uplet of component $f\in\mathscr{H}$ (the set of $(n-1)$-uplets of components of the weights $F_i(p,\delta)$), the constants A_f, B_f and ε_f from which $\lambda(f)$ is constructed. Let f^0 be the largest element of $\mathscr{H}$ for the lexicographic order. Define $A_{f_0}=C4^{Mn+1}$, $B_{f_0}=D$ and $\varepsilon_{f_0}=1$. Suppose we have constructed the constants A_f, B_f and ε_f for $f \geq f^1$. Consider the constants A'_{f^1}, B'_{f^1} and ε'_{f^1} obtained applying Lemma 5.2 for the constants A_{f^1}, B_{f^1} and ε_{f^1}, and define, for f^2 preceding f^1, $A_{f^2} = 3C\sum_{f>f^2} A'_f$, $B_{f^2} = B'_{f^1}$ and $\varepsilon_{f^2} = \varepsilon_{f^1}$. Thus $H = \sum_{f \in \mathscr{H}} H(f, A_f, B_f \varepsilon_f)$ is well defined.

For $q \in Q^c(p,\delta)$ define the following subsets of $\mathscr{H}$:

$$\begin{aligned} E_1(q) &= \left\{ f \in \mathscr{H} \text{ such that there exists } f' < f, \text{ such that } f'(q) \text{ is } B'_f\text{-dominant and } \chi_0(q) \geq \varepsilon'_f \right\}, \\ E_3(q) &= \left\{ f \in \mathscr{H} \text{ such that } \chi'_{f,B_f}(q) = 1 \text{ and } \chi_0(q) \geq \varepsilon_f \right\}, \\ E_2(q) &= \mathscr{H} \setminus \{E_1(q) \cup E_3(q)\}. \end{aligned}$$

Note that if $E_1(q)$ is not empty, and if f is it's smallest element, then there exists $f' < f$ such that $f'(q)$ is B'_f dominant, that is $\chi'_{f',B_{f'}}(q) = 1$, and, as $\varepsilon_{f'} \leq \varepsilon'_f$, we also have $\chi_0(q) \geq \varepsilon_{f'}$ which means $f' \in E_3(q)$, f being the smallest element of $E_1(q)$.

Now suppose first that $q \in Q^{c/2}(p,\delta)$. Then, by definition of D, $E_3(q)$ is not empty, and, if $E_1(q)$is also not empty there exists in $E_3(q)$ some strict minorant of $E_1(q)$. Then, by Lemma 5.2

$$\left\langle \partial\bar{\partial} H; L, \bar{L} \right\rangle (q) \geq \left(\sum_{f \in E_3(q)} A_f - \sum_{f \in E_1(q)} A'_f - \#E_2(q) \right) F(L_\tau, q, \delta) - \sum K_2(A_f, B_f, A'_f) \left(\sum_{i=1}^{n-1} |a_i|^2 \frac{c_{ii}(q)}{\delta} + \frac{|a_n|^2}{\delta^2} + \frac{\delta_\Omega(q)}{\delta} \right),$$

for α small enough, depending only on M, K and n ($\#E_2(f)$ denoting the number of elements of $E_2(f)$). Then, by the preceding remark and the fact that $\#E_2(q) \leq 4^{Mn} \leq \frac{1}{4C} A_{f^0}$ imply

$$\left\langle \partial\bar{\partial} H; L, \bar{L} \right\rangle (q) \geq C4^{Mn} F(L_\tau, q, \delta) - \gamma_1 \left(\sum_{i=1}^{n-1} |a_i|^2 \frac{|c_{ii}(q)|}{\delta} + \frac{|a_n|^2}{\delta^2} + \frac{\delta_\Omega(q)}{\delta} \right).$$

Finally, if q is any point in $B^c(p,\delta)$ then $E_3(q)$ may be empty, but then $E_1(q)$ is also empty, and thus

$$\left\langle \partial\bar{\partial} H; L, \bar{L} \right\rangle (q) \geq -4^{Mn} F(L_\tau, q, \delta) - \gamma_1 \left(\sum_{i=1}^{n-1} |a_i|^2 \frac{|c_{ii}(q)|}{\delta} + \frac{|a_n|^2}{\delta^2} + \frac{\delta_\Omega(q)}{\delta} \right).$$

This finishes the proof of Proposition 5.1, property (4) being trivial. ☐

5.2.3. *Proof of Theorem 5.2.* If P is a point of the boundary of D, by the definition of D and Theorem 5.1, to prove that there exists a pluri-subharmonic function adapted to the structure of geometrically separated domain near P, we have only to consider the case where P is in the boundary of $\partial\Omega \cap \partial D$. Thus, with the notations introduced just before, we prove the following reformulation of Theorem 5.2:

Proposition 5.2. *Let P be a point of the boundary of $\partial\Omega \cap \partial D$, and $V(P)$ the neighborhood considered in the previous Section. For all $K > 0$, there exists constants α_1 and δ_1 depending on K and the data such that if Ω is K-geometrically separated at $p_0 \in \partial\Omega$ and if the extremal basis of Ω are (K, α, p, δ)-strongly extremal with $\alpha \leq \alpha_1$, then, for $0 < \delta \leq \delta_1$, there exists a pluri-subharmonic function H_δ on the local domain D which is (δ, K')-adapted to $\mathscr{B}^{0,D}$.*

Proof. We fix δ small enough and then will omit the subscript δ in the notations of the vector fields. Consider, as in Section 5.2.1 the covering of $\partial\Omega \cap V(P)$ by the pseudo-balls $B^{c/2}(p_k,\delta) \cap \partial\Omega$ (note that here P plays the role of p_0 in the previous Sections).

Let L' be a vector field in E^D the vector space generated by the vector fields in the basis $\mathscr{B}^{0,D}$ on D and the normal N^D. To evaluate the hessian $\left\langle \partial\bar{\partial} .; L', \overline{L'} \right\rangle (q)$, only the value of L' at the point q is relevant and then we ca associate to L' a vector field in E^Ω (the vector space generated by $\mathscr{B}^{0,\Omega}$ and N^Ω) such that $L(q) = L'(q)$. Thus for the estimation of the hessian, we will assume that the vector field belongs to E^Ω.

Let us denote (L_i^{Ω,p_k}) the extremal basis for the domain Ω at (p_k,δ), and, for $L \in E^\Omega$, let us write $L = L_\tau^\Omega + a_n N^\Omega = \sum_{i=1}^{n-1} a_i^k L_i^{\Omega,p_k} + a_n N^\Omega$, with $a_i^k \in \mathbb{C}$.

With these notations, (5.1) say that the function $H_\delta^\Omega = \sum_k H_{p_k,\delta}$ satisfies

$$\left\langle \partial\bar{\partial} H_\delta^\Omega; L, \bar{L} \right\rangle (q) \geq \frac{\beta}{2} F\left(L_\tau^\Omega, q, \delta \right) - \gamma_1 \sum_{k \text{ s. t. } q \in Q^c(p_k,\delta)} \left(\sum_{i=1}^{n-1} \left| a_i^k \right|^2 \frac{\left| c_{ii}^k(q) \right|}{\delta} + \frac{|a_n|}{\delta} + \frac{\delta_\Omega(q)}{\delta} \right). \tag{5.2}$$

Let us consider now the function $\mathscr{H} = H_\delta^\Omega + Ae^{r/\delta} + B|z|^2$. First we prove the following Lemma:

Lemma. *There exists constants A, B and α_0, depending only on K and the data, such that, on $V(P) \cap \{0 > r > -3\delta\}$ the function $\mathscr{H}$ satisfy, for $\alpha \leq \alpha_0$ and $L = \sum a_i L_i^{0,D}$ of norm 1,*

$$\left\langle \partial\bar{\partial} \mathscr{H}; L, \bar{L} \right\rangle (q) \geq \frac{\beta_0}{4} \left(F(L_\tau^\Omega, q, \delta) + \frac{|b_n|^2}{\delta^2} + \frac{\varphi'\left(|q|^2\right)}{\delta} + \frac{\varphi''\left(|q|^2\right)}{\delta} \left| \left\langle L_\tau^\Omega, z \right\rangle \right|^2 \right)$$

with the notation $L = L_\tau^D + b_n N^D$.

Proof of the Lemma. First we estimate the hessian of $e^{r/\delta}$:

$$\left\langle \partial\overline{\partial} e^{r/\delta}; L, \overline{L} \right\rangle = e^{r/\delta} \left(\frac{2\Re\mathrm{e}\left(b_n \left\langle \partial\overline{\partial} r; L_\tau^D, \overline{N} \right\rangle \right)}{\delta} + \frac{|b_n|^2}{\delta^2} \right)(q) + \left\langle \partial\overline{\partial} e^{r/\delta}; L_\tau^D, \overline{L_\tau^D} \right\rangle (q). \tag{5.3}$$

For $q \in \{r \geq -3\delta\}$, the first term of (5.3) is $\geq \frac{1}{2e^3} \frac{|b_n|^2}{\delta^2} - K_0$. Let us look at the second term of (5.3).

$$\left\langle \partial\overline{\partial} e^{r/\delta}; L_\tau^D, \overline{L_\tau^D} \right\rangle = \frac{e^{r/\delta}}{\delta} \left(\left\langle \partial\overline{\partial}\rho; L_\tau^D, \overline{L_\tau^D} \right\rangle + \left\| L_\tau^D \right\|^2 \varphi'\left(|q|^2\right) + \left| \left\langle L_\tau^D, q \right\rangle \right|^2 \varphi''\left(|q|^2\right) \right).$$

Note that $\left\langle \partial\overline{\partial}\rho; L_\tau^D, \overline{L_\tau^D} \right\rangle = \left\langle \partial\overline{\partial}\rho; L, \overline{L} \right\rangle + O(b_n)$, thus we estimate $\left\langle \partial\overline{\partial}\rho; L, \overline{L} \right\rangle$. Using the vector fields $L^{\Omega, p_k, q}$, we have

$$\left\langle \partial\overline{\partial}\rho; L, \overline{L} \right\rangle (q) = \left(\sum_{i=1}^{n-1} a_i^k \overline{a_j^k} c_{ij}^k(q) + 2\Re\mathrm{e} \left(\overline{a_n} \sum_{j<n} a_j^k \left\langle \partial\overline{\partial}\rho; L_j^{\Omega, p_k}, N^\Omega \right\rangle \right) + |a_n|^2 \right),$$

and, as in the proof of Theorem 5.1, we get

$$\left\langle \partial\overline{\partial}\rho; L, \overline{L} \right\rangle \geq \sum_{i=1}^{n-1} \left| a_i^k \right|^2 \left| c_{ii}^k(q) \right| - 4n\alpha C(K) \delta F\left(L_\tau^\Omega, q, \delta\right) + O(\delta_\Omega(q)) + O(a_n).$$

As $L(q) = L_\tau^\Omega + a_n N^\Omega = L_\tau^D + b_n N^D$, we have $a_n = b_n + O\left(\varphi'\left(|q|^2\right)|q|\right)$, and then, for $q \in V(O) \cap \{-3\delta < r < 0\}$ and all k,

$$\frac{1}{\delta} \left\langle \partial\overline{\partial}\rho; L, \overline{L} \right\rangle (q) \geq \frac{1}{e^3\delta} \sum_{i=1}^{n-1} \left| a_i^k \right|^2 \left| c_{ii}^k(q) \right| - 4n\alpha C(K) F\left(L_\tau^\Omega, q, \delta\right) + \frac{1}{\delta} O\left(b_n + |q| \varphi'\left(|q|^2\right) + \left| \varphi\left(|q|^2\right) \right| + \delta \right).$$

Now, shrinking $V(P)$ if necessary, note that $\varphi\left(|q|^2\right) \ll \varphi'\left(|q|^2\right)$ and $|q| \ll 1$, and, for δ small, separating the cases $\left\| L_\tau^D \right\| \geq 1/2$ and $\left\| L_\tau^D \right\| \leq 1/2$ which implies $|b_n|$large, we obtain

$$\begin{aligned} \left\langle \partial\overline{\partial} e^{r/\delta}; L, \overline{L} \right\rangle \geq \frac{1}{e^3\delta} \sum_{i=1}^{n-1} \left| a_i^k \right|^2 \left| c_{ii}^k(q) \right| - 4n\alpha C(K) F\left(L_\tau^\Omega, q, \delta\right) + \frac{1}{4e^3} \frac{|b_n|^2}{\delta^2} - K_1 \\ + \frac{1}{8e^3\delta} \left(\varphi'\left(|q|^2\right) + \left| \left\langle L_\tau^D, q \right\rangle \right|^2 \varphi''\left(|q|^2\right) \right). \end{aligned}$$

Now, taking $A = 2Se^3 \max(2, \gamma_1) + 1$ and $B = K_1 A$ (constants depending only on K, and the data), and noting that the term $-\gamma_1 \frac{|a_n|}{\delta}$ coming from $\partial\partial H_\delta^\Omega$ is absorbed (for δ small) by the terms $\frac{|b_n|^2}{\delta^2}$ and $\frac{\varphi'\left(|q|^2\right)}{\delta}$ we finally get

$$\begin{aligned} \left\langle \partial\overline{\partial} \left(H_\delta^\Omega + Ae^{r/\delta} + B|z|^2 \right); L, \overline{L} \right\rangle (q) \geq \frac{\beta}{2} F\left(L^\Omega, q, \delta\right) + \sum_{i=1}^{n-1} \left| a_i^k \right|^2 \left| c_{ii}^k(q) \right| + \frac{|b_n|^2}{\delta} \\ - 2n\alpha C(K) A \sum_{k \text{ s. t. } q \in Q^c(p_k, \delta)} F\left(L_\tau^\Omega, q, \delta\right) + \frac{\varphi'\left(|q|^2\right)}{\delta} + \frac{\left| \left\langle L_\tau^D, q \right\rangle \right|^2 \varphi''\left(|q|^2\right)}{\delta}, \end{aligned}$$

which proves the Lemma for α_1 small enough, depending only on K, and the data, because $\left\langle L_\tau^\Omega, q \right\rangle = \left\langle L_\tau^D, q \right\rangle + O(\varphi')$. □

To control derivatives (of order less than three) of the function H relatively to vector fields belonging to E^D, i.e. of the form $L = \sum_{i=1}^{n-1} a_i L_i^{0,D} + a_n N^D$. By definition of the basis $\left(L_i^{0,D}\right)$, for a point belonging to ∂D, we have $L_i^{0,D} = L_i^{0,\Omega} \circ \pi + \beta_i N^\Omega \circ \pi$, and it is easy to remark that $L = \sum_{i=1}^{n-1} L_i^{0,\Omega} + a_n N^\Omega + W$ where W has a uniform norm (resp. $\mathscr{C}^1$ norm, $\mathscr{C}^2$ norm) controlled by $O_K\left(\varphi'\left(|q|^2\right)\right)$ (resp. $O_K\left(\varphi''\left(|q|^2\right)\right)$, $O_K\left(\varphi'''\left(|q|^2\right)\right)$). Then the expected controls of derivatives are obtained by the same methods than in the study of the hessian.

Now, the proof of Proposition 5.2 is finished using arguments similar to the one used at the end of the proof of Theorem 5.1. □

6. APPLICATIONS TO COMPLEX ANALYSIS

6.1. **Statements of the results for geometrically separated domains.** In [CD06b] and [CD06a] we proved that the methods introduced, for the study of the Bergman and Szegö projection, by A. Nagel, J. P. Rosay, E. M. Stein and S. Wainger in $\mathbb{C}^2$ ([NRSW89]) and by J. McNeal and E. M. Stein for convex domains ([MS94, MS97]) can be adapted to pseudo-convex domains having an "adapted geometry". The study made in the previous Sections show that it is the case for completely geometrically separated domains and thus we have the following sharp estimates:

Theorem 6.1. *Suppose Ω is completely geometrically separated at $p_0 \in \partial\Omega$. Let $K_B(z,w)$ be the Bergman kernel of Ω. There exists a neighborhood $W(p_0)$ of p_0 such that:*

1. *For $p \in W(p_0) \cap \Omega$, $K_B(p,p) \simeq \Pi_{i=1}^{n} F(L_i^{p,\delta(p)}, p, \delta_\Omega(p))$, where $\delta_\Omega(p)$ is the distance from p to $\partial\Omega$.*
2. *For $p_1, p_2 \in W(p_0) \cap \Omega$, for all integer N, there exists a constant C_N depending on Ω and N, such that for all list $\mathscr{L}_{Z_1} = \{L_1^1, \dots, L_1^k\}$ (resp $\mathscr{L}_{Z_2} = \{L_2^1, \dots L_2^{k'}\}$) of length $k \leq N$ (resp. $k' \leq N$) with $L_1^j \in \mathscr{B}(\pi(p_1),\tau) \cup \{N\}$ (resp. $L_2^j \in \overline{\mathscr{B}(\pi(p_1),\tau) \cup \{N\}}$), we have*

   $$\left|\mathscr{L}_{Z_1}\overline{\mathscr{L}_{Z_2}} K_B(Z_1,Z_2)(p_1,p_2)\right| \leq C_N \prod_{i=1}^{n} F(L_i^{\pi(p_1),\tau}, \pi(p_1), \tau)^{1+l_i/2},$$

   where $\tau = \delta_{\partial\Omega}(p_1) + \delta_{\partial\Omega}(p_2) + \gamma(\pi(p_1), \pi(p_2))$, $\gamma(\pi(p_1), \pi(p_2))$ is the pseudo-distance from $\pi(p_1)$ to $\pi(p_2)$ associated to the structure of homogeneous space and l_i is the number of times the vector fields $L_i^{\pi(p_1),\tau}$ or $\overline{L_i^{\pi(p_1),\tau}}$ appear in the union of the lists $\mathscr{L}_{Z_1}$ and $\mathscr{L}_{Z_2}$.

Corollary. *Suppose Ω satisfies the hypothesis of Theorem 5.2. Let D be the local domain considered in Theorem 5.2. Then the Bergman kernel $K_D(z,w)$ of D satisfy all the estimates stated in the Theorem at any point of its boundary.*

Using the methods of Section 5 of [CD92] the following result on invariant metrics is easily proved:

Theorem 6.2. *Suppose Ω is completely geometrically separated at $p_0 \in \partial\Omega$. Let us denote by $B_\Omega(z,L)$ (resp. $C_\Omega(z,L)$, resp. $K_\Omega(z,L)$) the Bergman (resp. Caratheodory, resp. Kobayashi) metric of Ω at the point $z \in \Omega$. Then there exists a neighborhood $V(p_0)$ such that, for all vector field $L \in E$, $L = L_\tau + a_n N$, we have, for $q \in V(p_0) \cap \Omega$,*

$$B_\Omega(q,L) \simeq C_\Omega(q,L) \simeq K_\Omega(q,L) \simeq F(L_\tau, q, \delta(q)) + \frac{|a_n|}{\delta(q)},$$

where $\delta(q)$ is the distance of q to the boundary of Ω and the constants in the equivalences depend only on the constant of geometric separation and the data.

Theorem 6.3. *Suppose Ω is completely geometrically separated at every point of its boundary. Then the following results hold:*

1. *Let P_B be the Bergman projection of Ω. Then:*
   1. *for $1 < p < +\infty$ and $s \geq 0$, P_B maps continuously the Sobolev space $L_s^p(\Omega)$into itself;*
   2. *for $0 < \alpha < +\infty$, P_B maps continuously the Lipschitz space $\Lambda_\alpha(\Omega)$ into itself;*
   3. *for $0 < \alpha < 1/M$, P_B maps continuously the Lipschitz space $\Lambda_\alpha(\Omega)$ into the non-isotropic Lipschitz space $\Gamma_\alpha(\Omega)$.*
2. *Let P_S be the Szegö projection of Ω. Then:*
   1. *for $1 < p < +\infty$ and $s \in \mathbb{N}$, P_S maps continuously the Sobolev space $L_s^p(\partial\Omega)$into itself;*
   2. *for $0 < \alpha < +\infty$, P_S maps continuously the Lipschitz space $\Lambda_\alpha(\partial\Omega)$ into itself;*
   3. *for $0 < \alpha < 1/M$, P_S maps continuously the Lipschitz space $\Lambda_\alpha(\partial\Omega)$ into the non-isotropic Lipschitz space $\Gamma_\alpha(\partial\Omega)$.*

Note. (1) (c) and (2) (c) can be extended to all $\alpha > 0$ with convenient definitions of the spaces $\Gamma_\alpha(\Omega)$ and $\Gamma_\alpha(\partial\Omega)$.

Corollary. *Suppose Ω satisfy the hypothesis of Theorem 5.2. Let D be the local domain considered in Theorem 5.2. Then all the results stated for Ω in the previous Theorem are valid for D.*

Using an idea of M. Machedon [Mac88] we deduce local estimates for the Szegö projection:

Theorem 6.4. *Suppose Ω satisfies the hypothesis of Theorem 5.2. Let P_S be the Szegö projection of Ω. Then if f is a $L^2(\partial\Omega)$ function which is locally near $p_0$in the Sobolev space L_s^p, $1 < p < +\infty$ and $s \in \mathbb{N}$, (resp. in the Lipschitz space Λ_α, $0 < \alpha < 1/M$) then its projection $P_S(f)$ is locally near $p_0$in L_s^p (resp. in the non-isotropic Lipschitz space Γ_α). In particular this applies if the Levi form of Ω is locally diagonalizable at p_0.*

Proof. if $f \in L^2(\Omega)$ and if $\chi \in \mathscr{C}^\infty(\partial\Omega)$ has compact support in a sufficiently small neighborhood of p_0 and $\chi = 1$ in a neighborhood of p_0, then the subelliptic estimates for $\Box_b$ and Kohn's theory ([Koh85, KN65]) implies $P_S((1-\chi)f)$ is $\mathscr{C}^\infty$ near p_0, and, denoting P_S^D the Szegö projection of D, $(P_S - P_S^D)(\chi f)$ is $\mathscr{C}^\infty$ in a neighborhood of p_0 (see also [Kan90]); the result follows thus the previous Corollary. ☐

6.2. **A guide of the proofs of the results of Section 6.1.** Let U be a neighborhood of $\partial\Omega$ where we can define a projection π onto $\partial\Omega$ using the integral curve of the real normal to ρ. We will always suppose that $V(p_0) \subset U$.

The two notions of "weak homogeneous space" and "adapted pluri-subharmonic function" plays a crucial role in [CD06b, CD06a]:

Definition 6.1. We say that the domain Ω satisfy the hypothesis of "weak homogeneous space" at a boundary point p_0 of finite type τif there exists two neighborhoods $V(p_0)$ and $W(p_0) \Subset V(p_0)$ and a constant K such that:

1. There exists $\delta_0 > 0$ such that, for every $p \in W(p_0)$, $\forall \delta \in [-\frac{1}{3}\rho(p), \delta_0]$, there exists a basis of vector fields tangent to ρ in $V(p_0)$, $\mathscr{B}(p,\delta)$, for which there exists a K-adapted coordinate system
2. There exists two constants C and c_0, depending on K and τ, such that, for $c \le c_0$, the sets $B^c(\mathscr{B}(p,\delta),p,\delta)$ (associated to the coordinate system), $B^c_{\mathscr{C}}(\mathscr{B}(p,\delta),p,\delta)$ and $B^c_{\exp}(\mathscr{B}(p,\delta),p,\delta)$ satisfy the following conditions, for all $p \in W(p_0) \cap \overline{\Omega}$ and all $\delta \in [-\frac{1}{3}\rho(p), \delta_0]$:
   (a) for $q \in B^c_0(p,\delta)$, $B^c_0(\mathscr{B}(q,\delta),q,\delta) \subset B^c_1(\mathscr{B}(p,\delta),p,C\delta)$, where B^c_0 and B^c_1 denotes one of the sets B^c, $B^c_{\mathscr{C}}$ or $B^c_{\exp}$.
   (b) $\mathrm{Vol}\left(B^c_0(\mathscr{B}(p,2\delta),p,2\delta)\right) \le C\mathrm{Vol}\left(B^c_0(\mathscr{B}(p,\delta),p,\delta)\right)$.

Note that, in this Definition the weights F_i are defined with $M = M'(\tau)$.

Definition 6.2. Let $\mathscr{B} = \{L_1,\dots,L_{n-1}\}$ be a basis of vector fields tangent to ρ in a neighborhood $V(p_0)$ of a boundary point p_0 and $0 < \delta \le \delta_0$. We say that a pluri-subharmonic function $H \in \mathrm{PSH}(\Omega)$ is (p_0,K,c,δ)-*adapted to this basis* $\mathscr{B}$ if the following properties are satisfied:

$|H| \le 1$ in Ω, and, for all point $p \in W(p_0) \cap \overline{\Omega}$, $\rho(p) \ge -3\delta$, the two following inequalities are verified for points $q \in B^c_{\mathscr{C}}(\mathscr{B},p,\delta) \cap \Omega$:

1. For all $L = \sum_{i=1}^n a_i L_i$, $a_i \in \mathbb{C}$,
$$\left\langle \partial\overline{\partial} H, L, \overline{L} \right\rangle \ge \frac{1}{K}\sum_{i=1}^{n} |a_i|^2 F(L_i,p,\delta).$$
2. For $\mathscr{L} \in \mathscr{L}_3(\mathscr{B} \cup \{N\})$,
$$|\mathscr{L}H| \le K \prod_{L \in \mathscr{L}} F(L,p,\delta)^{1/2}.$$

Note that this Definition depends on the values of the vector fields L_i^p at points q in Ω. But, in the situation of the applications below (i.e. with a finite type hypothesis) it can be shown that it depends only (up to uniform constants) on the restriction of the basis on $\partial\Omega$.

The following Proposition follows the work in [CD06b, CD06a]:

Proposition. *Let Ω be a bounded pseudo-convex domain and p_0 be a boundary point of finite type (resp. a bounded pseudo-convex domain of finite type). Then, if Ω satisfies the hypothesis of "weak homogeneous space" at p_0 (resp. at every point of its boundary) and if there exists a pluri-subharmonic function $\mathscr{H}_\delta$ adapted to $\mathscr{B}(p,\delta)$ for all $p \in W(p_0) \cap \overline{\Omega}$ and all $\delta \in [-\frac{1}{3}, \delta_0]$ (resp. if this property holds at every point p_0 of $\partial\Omega$) then the conclusions of Theorem 6.1 (resp. Theorem 6.3)are satisfied.*

To prove Theorems 6.1 and 6.3it suffices then to use the properties of extremal basis and to note the two following facts:

1. The existence of extremal basis and adapted coordinate systems for points of $\partial\Omega \cap W(p_0)$ allows us to define basis and coordinate systems for points inside Ω (see Remark 4.1) and,
2. if $p_1 \in W(p_0) \cap \Omega$, $p = \pi(p_1)$, the sets $\widetilde{B^c_0}(\mathscr{B}(p,\delta),p_1,\delta)$, $-\frac{1}{3}\rho(p_1) < \delta \le \delta_0$, defined by $q \in \widetilde{B^c_0}(\mathscr{B}(p,\delta),p_1,\delta)$ if $\pi(q) \in B^c_0(\mathscr{B}(p,\delta),p,\delta)$ and $|\rho(q) - \rho(p_1)| < c\delta$ induce a structure of "weak homogeneous space".

6.3. **Main articulations of the proof of the Proposition.** In the Section 2 of [CD06b] we showed that if the Levi form is locally diagonalizable then the local hypothesis of the Proposition is satisfied, and in [CD06a, CD06b], even if the statements are given in the case of a locally diagonalizable Levi form, the proofs of the estimates on the Bergman and Szegö projections are made only using the hypothesis of the Proposition. We just give here the main articulations of the proofs:

- The Bergman kernel estimates on the diagonal is done using Theorem 6.1 of [Cat89] and the change of coordinates Φ_p adapted to the basis $\mathscr{B}(p,\delta(p))$.
- The estimates on the derivatives of the Bergman kernel outside the diagonal follow the methods developed by A. Nagel, J. P. Rosay, E. M. Stein and S. Wainger [NRSW89] and J. Mc Neal [McN89] for the pseudoconvex domains of finite type in $\mathbb{C}^2$, and used for some generalizations (see the introduction) in particular by J. Mc Neal [McN94] in the case of convex domains. It consists to obtain uniform local estimates for the Neumann operator $\mathscr{N}$ and then to apply the ideas developed by N. Kerzman [Ker72] in the study of strictly pseudoconvex case. This requires scaling.
  The starting point is to write the Bergman kernel K_B^Ω using the Bergman projection. More precisely, if ψ_ζ is a radial function centered at ζ with compact support in Ω and of integral 1, and P_B^Ω is the Bergman projection of Ω, then $D^\mu \overline{D}^\nu K_B^\Omega(w,\zeta) = D_w^\mu P_B^\Omega(\overline{D}_\zeta^\nu \psi_\zeta)(w)$. Then, P_B^Ω being related to the $\overline{\partial}$-Neumann problem by the formula $P_B^\Omega = \mathrm{Id} - \vartheta\mathscr{N}\overline{\partial}$, where ϑ is the formal adjoint to $\overline{\partial}$ and $\mathscr{N}$ the inverse operator of $\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$, the estimates on P_B^Ω are obtained via estimates on $\mathscr{N}$. To obtain these estimates, we use the theory developed by J. J. Kohn and L. Nirenberg [KN65] which gives local Sobolev estimates for $\mathscr{N}$ if there exists a local sub-elliptic estimates for the $\overline{\partial}$-Neumann problem and the famous work of D. Catlin ([Cat87]), where it is proved that the existence of an adapted pluri-subharmonic function implies the existence of sub-elliptic estimates for the $\overline{\partial}$-Neumann operator.
  The study of the Bergman kernel is not directly done in Ω but in $\Phi_p(\Omega)$, where Φ_p is a coordinate system adapted to the basis $\mathscr{B}(p, \delta_{\partial\Omega}(p) + \delta_{\partial\Omega}(q) + \gamma(\pi(p),\pi(q)))$, where γ is the pseudo-distance on $\partial\Omega$. One difficulty is to

see that all the constants appearing in the estimates and all the domains where the estimates are done are uniformly controlled.
- The estimates for the Bergman and Szegö projectors are obtained adapting the methods developed by J. Mc Neal and E. M. Stein in [MS94, MS97] (and also [NRSW89]), related, in particular, to the theory of non isotropic smoothing operators, to non convex domains.

Remark. The results on the Szegö projection are thus obtained adapting the theory of NIS operators to our settings. The Λ_α estimates, for example, for the domains considered by M. Derridj in [Der99] can also be obtained using the estimate for $\Box_b$ of Derridj's paper, the estimate on the Bergman projection derived from the fact that these domains are completely geometrically separated and the results on the comparison of the Bergman and Szegö projection obtained by K. D. Koenig in [Koe07].

7. APPENDIX

The following Lemma is an improvement of Lemma 3.9 of [CD06b]:

Lemma 7.1. *Let B_j be the unit ball in $\mathbb{C}^j$. Let K_1be a positive real number, M and n two positive integers. There exists a constant $C(K_1)$ depending on K_1, M and n such that, for $j=1,\dots,n-1$, if g is a non negative function of class $\mathscr{C}^M$on B_j satisfying $\sup_{B_j}\{\left|D^{\alpha\beta}g(w)\right|, |\alpha+\beta|\leq M\}\leq K_1$, where $D^{\alpha\beta}=\frac{\partial^{|\alpha+\beta|}}{\partial w^\alpha\partial\overline{w}^\beta}$, then, for all $(\alpha^0,\beta^0)\in\left(\mathbb{N}^j\right)^2$, $\left|\alpha^0+\beta^0\right|<M$, there exists $a\in\mathbb{N}^j$, $2\,|a|\leq\left|\alpha^0+\beta^0\right|$ such that*

$$\left(\prod_{i=1}^{j}\Delta_i^{a_i}\right)g(0)\geq\frac{1}{C(K_1)}\left|D^{\alpha^0\beta^0}g(0)\right|^{2^{\left|\alpha^0+\beta^0\right|}},$$

where $\Delta_i=\frac{\partial^2}{\partial z_i\partial\overline{z_i}}$ denotes the Laplacian in the z_i coordinate.

Note that there is no absolute value in the left hand side of the inequality.

Proof. We only indicate how the proof of Lemma 3.9 of [CD06b] has to be modified.

Without loss of generality, we can suppose $\left|D^{\alpha_0\beta_0}g(0)\right|=\max_{|\alpha+\beta|=\left|\alpha^0+\beta^0\right|}\left|D^{\alpha\beta}g(0)\right|$. By induction, it is enough to prove that there exists two constants c and C, depending on M and n, such that one of the following two cases holds:

First case there exists $a\in\mathbb{N}^j$, $2\,|a|=\left|\alpha^0+\beta^0\right|$ such that $\left(\prod_{i=1}^{j}\Delta_i^{a_i}\right)g(0)\geq c\left|D^{\alpha^0\beta^0}g(0)\right|$;

Second case there exists $(\tilde\alpha,\tilde\beta)\in\in\left(\mathbb{N}^j\right)^2$, $\left|\tilde\alpha+\tilde\beta\right|<\left|\alpha^0+\beta^0\right|$ such that

$$\left|D^{\tilde\alpha\tilde\beta}g(0)\right|\geq\frac{1}{C}\left|D^{\alpha^0\beta^0}g(0)\right|^{-\left|\tilde\alpha+\tilde\beta\right|+\left|\alpha^0+\beta^0\right|+1}.$$

Let $p=\left|\alpha^0+\beta^0\right|$, $\xi=\mu\varepsilon$, $\mu\in]0,1[$, $\varepsilon=(\varepsilon_i)$, $|\varepsilon_i|\leq1$, and, as in the proof of Lemma 3.9 of [CD06b] let us write Taylor formula:

$$\begin{aligned}g(\xi) &= \sum_{k=0}^{p-1}\mu^k\sum_{|\alpha+\beta|=k}*D^{\alpha\beta}g(0)\varepsilon^\alpha\overline{\varepsilon}^\beta+\mu^p\sum_{|\alpha+\beta|=p}*D^{\alpha\nu}g(0)\varepsilon^\alpha\overline{\varepsilon}^\beta+\mu^{p+1}R(\varepsilon,\mu)\\ &= A_1(\xi)+\mu^pA_2(\xi)+\mu^{p+1}R(\varepsilon,\mu),\end{aligned}$$

where $*$ are multinomial coefficients and $|R|\leq K_1K_2$, K_2 depending only on M and n.

Remark now that, g being non negative,

(*) $\left\{\begin{array}{l}\text{If there exists } \mu\simeq\left|D^{\alpha^0\beta^0}g(0)\right| \text{ such that } A_2(\xi)+\mu R(\varepsilon,\mu)<-c_1\left|D^{\alpha^0\beta^0}g(0)\right|,\ c_1>0, \text{ then the \textit{Second}}\\ \text{\textit{case} hold.}\end{array}\right.$

In the proof of Lemma 3.9 of [CD06b] we introduced a multiindex c ($|c|=p$), depending on g, and complex numbers ε_i ($\forall i$, $|\varepsilon_i|\geq c(M,n)$), depending on g and $K(M,n)$, such that

$$\sum_{\substack{|\alpha+\beta|=p\\ \alpha+\beta\neq c}}\left|*D^{\alpha\beta}g(0)\varepsilon^\alpha\overline{\varepsilon}^\beta\right|\leq\frac{\left|D^{\alpha^0\beta^0}g(0)\right|}{K}\tag{7.1}$$

and

$$\left|\sum_{\alpha+\beta=c}*D^{\alpha\beta}g(0)\varepsilon^\alpha\overline{\varepsilon}^\beta\right|\geq4\frac{\left|D^{\alpha^0\beta^0}g(0)\right|}{K}.\tag{7.2}$$

To finish the proof, we show now that, either we can find ε and μ satisfying the hypothesis of (*), or we are in the *First case*.

We take $\mu=\frac{\left|D^{\alpha^0\beta^0}g(0)\right|}{KK_1K_2}$. Then $|A_2(\xi)+\mu R(\xi)|\geq\frac{\left|D^{\alpha^0\beta^0}g(0)\right|}{K}$ and $A_2(\xi)+\mu R(\xi)$ has the sign of $\sum_{\alpha+\beta=c}*D^{\alpha\beta}g(0)\varepsilon^\alpha\overline{\varepsilon}^\beta$.

If $\sum_{\alpha+\beta=c} {*D^{\alpha\beta}} g(0)\varepsilon^{\alpha}\overline{\varepsilon}^{\beta} < 0$, (*) is then satisfied, thus consider the case where $\sum_{\alpha+\beta=c} {*D^{\alpha\beta}} g(0)\varepsilon^{\alpha}\overline{\varepsilon}^{\beta} > 0$.

If there exists an index i such that c_i is odd, taking ε' defined by $\varepsilon'_j = \varepsilon_j$ if $j \neq i$ and $\varepsilon'_i = -\varepsilon_i$, then

$$\sum_{\alpha+\beta=c} {*D^{\alpha\beta}} g(0)\varepsilon'^{\alpha}\overline{\varepsilon'}^{\beta} \leq -\frac{4}{K}\left|D^{\alpha^0\beta^0}g(0)\right|,$$

and, by (7.1), (*) is verified.

So we suppose that for all i, $c_i = 2c'_i$, and we write

$$\sum_{\alpha+\beta=c} {*D^{\alpha\beta}} g(0)\varepsilon'^{\alpha}\overline{\varepsilon'}^{\beta} = \sum_{k=0}^{c_1} \varepsilon'^{k}_{1}\overline{\varepsilon'}^{c_1-k}_{1} A^1_k(\varepsilon'_2,\dots,\varepsilon'_n),$$

with $|\varepsilon'_i| = |\varepsilon_i|$, and we choose $c \ll 4/K$. We separate two cases.

First suppose that $A^1_{c'_1}(\varepsilon_2,\dots,\varepsilon_n) \leq c\left|D^{\alpha^0\beta^0}g(0)\right|$. If $c_1 = 0$ then (7.2) implies

$$\sum_{\alpha+\beta=c} {*D^{\alpha\beta}} g(0)\varepsilon^{\alpha}\overline{\varepsilon}^{\beta} \leq -c'\left|D^{\alpha^0\beta^0}g(0)\right|$$

which gives (*). Thus suppose $c_1 \neq 0$. Let

$$\mathscr{E}_0 = \{\varepsilon',\ \text{such that } \varepsilon'_i = \varepsilon_i,\ i > 1,\ \varepsilon'_1 = \vartheta\varepsilon_1,\ \text{with } \vartheta^{c_1} = 1\}.$$

Thus

$$\sum_{\varepsilon'\in\mathscr{E}_0}\sum_{\alpha+\beta=c} {*D^{\alpha\beta}} g(0)\varepsilon'^{\alpha}\overline{\varepsilon'}^{\beta} = c_1 A^1_{c'_1}\left|\varepsilon_1\right|^{c_1}.$$

Then, by (7.2), there exists $\varepsilon' \in \mathscr{E}_0$ such that

$$\sum_{\alpha+\beta=c} {*D^{\alpha\beta}} g(0)\varepsilon'^{\alpha}\overline{\varepsilon'}^{\beta} \leq -c''\left|D^{\alpha^0\beta^0}g(0)\right|,$$

(recall $|\varepsilon_1| > c(M,n)$) and (*) is verified as before.

Suppose now $A^1_{c'_1}(\varepsilon_2,\dots,\varepsilon_n) > c'\left|D^{\alpha^0\beta^0}g(0)\right|$. Write

$$A^1_{c'_1} = \sum_{k=0}^{c_2} \varepsilon_2^k\overline{\varepsilon_2}^{c_2-k}A^2_k(\varepsilon_3,\dots\varepsilon_n).$$

As before, if $c_2 = 0$ or if $A^2_{c'_2}(\varepsilon_3,\dots\varepsilon_n) \leq c'''\left|D^{\alpha^0\beta^0}g(0)\right|$ we can change ε_2 such that we obtain

$$A^1_{c'_1}(\varepsilon'_2,\dots\varepsilon'_n) \leq -c''''\left|D^{\alpha^0\beta^0}g(0)\right|,$$

and we conclude that (*) is satisfied. If $A^2_{c'2}(\varepsilon_3,\dots\varepsilon_n) \geq c'''\left|D^{\alpha^0\beta^0}g(0)\right|$, we do an other time the same thing, on the third variable. Then, by induction, if the process does not stop, the last step shows that if (*) is not satisfied, then the inequality on $D^{c'c'}g(0)$ implies that we are in the *First case*. □

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