ESI **The Erwin Schrödinger International Institute for Mathematical Physics**

Pasteurgasse 6/7
A-1090 Wien, Austria

On the Harmonic Superspace Geometry of (4,4) Supersymmetric Sigma Models with Torsion

Evgenyi A. Ivanov

Vienna, Preprint ESI 196 (1995)

February 13, 1995

Supported by Federal Ministry of Science and Research, Austria
Available via WWW.ESI.AC.AT

Vienna, ESI-...(1995)
Dubna, JINR-E2-95-53

On the harmonic superspace geometry of $(4,4)$ supersymmetric sigma models with torsion

Evgenyi A. Ivanov

Bogoliubov Laboratory of Theoretical Physics, JINR, 141 980 Dubna, Russia

Abstract

Starting with the dual action of $(4,4)$ $2D$ twisted multiplets in the harmonic superspace with two independent sets of $SU(2)$ harmonic variables, we present its generalization which hopefully provides an off-shell description of general $(4,4)$ supersymmetric sigma models with torsion. Like the action of the torsionless $(4,4)$ hyper-Kähler sigma models in the standard harmonic superspace, it is characterized by a number of superfield potentials. They depend on n copies of a triple of analytic harmonic $(4,4)$ superfields. As distinct from the hyper-Kähler case, the potentials prove to be severely constrained by the self-consistency condition which stems from the commutativity of the left and right harmonic derivatives. We show that for $n = 1$ these constraints reduce the general action to that of $(4,4)$ twisted multiplet, while for $n \geq 2$ there exists a wide class of new actions which cannot be written only via twisted multiplets. Their most striking feature is the nonabelian and in general nonlinear gauge invariance which substitutes the abelian gauge symmetry of the dual action of twisted multiplets and ensures the correct number of physical degrees of freedom. We conjecture that these actions describe sigma models with non-commuting left and right complex structures on the bosonic target.

1 Introduction

An interesting and important class of two-dimensional supersymmetric sigma models consists of those with $(4,4)$ worldsheet supersymmetry. The main reason of current interest to them is that they can provide non-trivial backgrounds for $d = 4$ strings (see, e.g., [1]). Relevant bosonic target manifolds in general possess a nontrivial torsion and two sets of covariantly constant complex structures (left and right ones) which in general do not commute [2, 3]. The $(4,4)$ sigma models which can be obtained via a direct dimensional reduction of $N = 2$ $4D$ sigma models constitute merely a subclass in the general variety of $(4,4)$ sigma models; their bosonic target manifolds are hyper-Kähler (or quaternionic-Kähler in the case of local supersymmetry) and so are torsionless and possess only one set of complex structures [4]. A manifestly supersymmetric off-shell description of this latter type of sigma models has been given in [5 - 7] in the harmonic $N = 2$ $4D$ (or

$(4,4)$ $2D$) superspace with one set of harmonic variables parametrizing the $SU(2)$ automorphism group of $N=2$ $4D$ ($(4,4)$ $2D$) supersymmetry [8, 9]. Later on, an analogous formulation with the use of the same type of harmonic superspace has been constructed for sigma models with heterotic worldsheet $(4,0)$ supersymmetry [10] (these models in general possess a torsion).

Since the full automorphism group of $(4,4)$ supersymmetry in two dimensions is $SO(4)_L \times SO(4)_R$, there arises a possibility to consider more general types of harmonic superspaces compared to the one utilized in [8 - 10]. In [11] A. Sutulin and the author have constructed the $(4,4)$ $2D$ harmonic superspace which involves two independent sets of harmonic variables parametrizing two commuting $SU(2)$ automorphism groups in the left and right light-cone sectors, $SU(2)_L$ and $SU(2)_R$ (the automorphism $SU(2)$ group of the conventional $(4,4)$ $2D$ harmonic superspace is a diagonal in the product $SU(2)_L \times SU(2)_R$)[1]. We have shown how to describe in this $SU(2) \times SU(2)$ harmonic superspace the $(4,4)$ twisted supermultiplet [2, 13] and presented the most general off-shell action of the latter as an integral over an analytic subspace of this superspace. The action involves the standard number of auxilary fields (four bosonic ones) and, in accord with reasonings of Refs. [2, 14], corresponds to a general $(4,4)$ supersymmetric sigma model with torsion and mutually commuting sets of left and right complex structures. A new dual form of the action in terms of unconstrained analytic superfields with an infinite number of auxiliary fields has been also given. An interesting peculiarity of the dual action is the abelian gauge invariance which ensures the on-shell equivalence of this action to the original one. We argued that this form of the action is a good starting point to attack the problem (as yet unsolved) of constructing a manifestly $(4,4)$ supersymmetric off-shell description of $(4,4)$ sigma models with non-commuting left and right complex structures. These models *cannot* be described only in terms of $(4,4)$ twisted multiplets [2, 14], so one is led to seek for such generalizations of the dual action which would not allow an equivalent formulation via $(4,4)$ harmonic superfields representing twisted multiplets [2].

In the present paper we generalize the dual action of $(4,4)$ twisted multiplet along these lines. As the main result, we find a wide class of $(4,4)$ sigma model actions with a nonabelian generalization of the abelian gauge invariance of the dual action. They cannot be written through $(4,4)$ twisted superfields only and, for this reason, can be thought of as corresponding to the aforementioned more general type of $(4,4)$ sigma models.

Our consideration is largely based upon an analogy with the description of torsionless $N=2$ $4D$ ($(4,4)$ in two dimensions) hyper-Kähler supersymmetric sigma models in the standard (having one set of harmonic variables) harmonic superspace. So we start in Sect.2 by recapitulating salient features of this description. Then in Sect.3 we recollect the basic facts about the $SU(2) \times SU(2)$ harmonic superspace and off-shell description of the twisted $(4,4)$ multiplet in its framework. In Sect.4 we discuss the dual action of the latter which involves n copies of a triple of unconstrained analytic superfields, and construct its most general extension, proceeding from the analogy with the general hyper-Kähler $(4,4)$ sigma model. This extension includes a few superfield potentials which, as distinct from the unconstrained potentials of the hyper-Kähler $(4,4)$ action, prove to be

[1]See also [12].

[2]For other proposals of how to describe off shell $(4,4)$ and $(2,2)$ sigma models with non-commuting complex structures see Refs. [12, 15, 16].

severely restricted by the integrability condition coming from the commutativity of the left and right harmonic derivatives. In Sect.5 we elaborate the $n=1$ example (with four-dimensional bosonic target) and show that the integrability constraint just mentioned reduces the general $n=1$ action to that of one twisted multiplet. In Sect.6 we come back to considering the generic $n \geq 2$ action. Under some natural simplifying assumptions we partially solve the integrability constraint and find a wide variety of the actions which do not admit a representation through the twisted $(4,4)$ superfields and so presumably describe sigma models with non-commuting left and right complex structures. Besides the inevitable presence of an infinite number of auxiliary fields, one more intriguing feature of these actions is the nonabelian and in general nonlinear gauge invariance which generalizes the abelian gauge symmetry of the dual action of twisted multiplets and restores the correct number of physical degrees of freedom ($4n$ bosonic and $8n$ fermionic ones). We discuss in some detail an interesting subclass of them, direct bi-harmonic analogs of the two-dimensional Yang-Mills action.

2 Sketch of $(4,4)$ sigma models in standard harmonic superspace

To make further consideration more understandable, it is instructive to start with a brief review of the off-shell formulation of $(4,4)$ sigma models in $(4,4)$ $2D$ harmonic superspace obtained by dimensional reduction from the standard $N=2$ $4D$ harmonic superspace [8, 9]. They contain no torsion in the bosonic part of the action; the bosonic target space metric is necessarily hyper-Kähler [4].

The sigma models in question are described in terms of unconstrained analytic harmonic superfields $q^{(+)\,M}(\zeta,u)$ $(M=1,2,...2n)$ defined on the $(2|4)$ dimensional $(4,4)$ $2D$ analytic harmonic superspace (see [5, 6] for details and terminology).

$$(\zeta,u)=(z^{++},z^{--},\theta^{(+)+},\bar{\theta}^{(+)+},\theta^{(+)-},\bar{\theta}^{(+)-},u^{(+)}_i,u^{(-)}_j)\ , \tag{2.1}$$

Here, the harmonic variables $u^{(\pm)i}$,

$$u^{(+)i}u^{(-)}_i=1\ ,$$

parametrize the two-sphere $S^2\sim SU(2)/U(1)$, $SU(2)$ being the diagonal subgroup in the product of two independent $SU(2)$ automorpism groups (the left and right ones) of the $(4,4)$ $2D$ Poincare superalgebra. The indices $\pm$ in the parentheses refer to the harmonic $U(1)$ charge, other $\pm$'s are $2D$ light-cone indices.

The general action of superfields $q^{(+)\,M}$ yields in the bosonic sector a generic sigma model on $4n$ dimensional hyper-Kähler manifold. The action is given by the following integral over the $(4,4)$ $2D$ analytic harmonic superspace

$$S_q=\int \mu^{(-4)}\{L^{(+)}_M(q^{(+)},u)D^{(+2)}q^{(+)\,M}+L^{(+4)}(q^{(+)},u)\}\ . \tag{2.2}$$

The object

$$D^{(+2)}=\partial^{(+2)}+2i(\theta^{(+)+}\bar{\theta}^{(+)+}\partial_{++}+\theta^{(+)-}\bar{\theta}^{(+)-}\partial_{--})\ ,\ \ (\partial^{(+2)}=u^{(+)i}\frac{\partial}{\partial u^{(-)i}})\ , \tag{2.3}$$

is the analyticity-preserving harmonic derivative, $\mu^{(-4)}$ is the analytic superspace integration measure

$$\mu^{(-4)} = d^6\zeta\ [du] = d^2z d^2\theta^{(+)+} d^2\theta^{(+)-}\ [du].$$

Two arbitrary potentials in the superfield Lagrangian in (2.2), $L^{(+)\,M}(q^{(+)\,M}, u)$ and $L^{(+4)}(q^{(+)\,M}, u)$, encode (locally) all the information about the relevant bosonic hyper-Kähler manifold. The fields parametrizing the latter appear as the first components in the harmonic and θ expansions of $q^{(+)\,M}$

$$q^{(+)\,M}(\zeta, u) = q^{iM}(z) u_i^{(+)} + \ldots .$$

The needed number of independent real fields in $q^{iM}(z)$ (just $4n$) comes out as a result of imposing the reality condition on the superfields $q^{(+)\,M}$: $\widetilde{(q^{(+)\,M})} = \Omega_{MN} q^{+\,N}$, where Ω_{MN} is a constant skew-symmetric matrix and the generalized involution "$\sim$" is defined in Refs. [8, 9].

The quantities $L^{(+)}_M$ and $L^{(+4)}$ have a clear geometric meaning: these are the hyper-Kähler potentials, the basic objects of unconstrained formulation of hyper-Kähler geometry given for the first time in [7]. There we started from the standard definition of this geometry as a Riemann geometry with restricted holonomy group (in case of $4n$ dimensional manifold it should belong to $Sp(n)$). We extended the original (arbitrary) hyper-Kähler manifold by a set of harmonic variables which parametrize the $SU(2)$ group rotating complex structures and then solved the constraints on the curvature by passing to a new, analytic basis in such a harmonic extension. The main feature of this extension which is visualized by passing to the analytic basis is the existence of an analytic subspace with twice as few coordinates compared to the manifold one started with (besides the harmonic variables the number of which is the same). The basic geometric objects solving for the hyper-Kähler constraints are just $L^{(+)}_M$ and $L^{(+4)}$ living as unconstrained functions on this analytic subspace.

The fact that the action of most general $N = 2$ $4D$ ($(4, 4)$ upon the reduction to two dimensions) supersymmetric sigma model is expressed via $L^{(+)}_M$ and $L^{(+4)}$, while identifying superfields $q^{(+)\,M}$ with coordinates of the analytic subspace of the harmonic extension of the target hyper-Kähler manifold, and the automorphism $SU(2)$ with the $SU(2)$ group rotating complex structures on this manifold, makes manifest the remarkable one-to-one correspondence between $N = 2$ $4D$ ($(4, 4)$ $2D$) supersymmetry and hyper-Kähler geometry [7]. There exists a clear analogy with $N = 1$ $4D$ ($(2, 2)$ $2D$) sigma models: the most general off-shell superfield Lagrangian of the latter can be interpreted as some Kähler potential, with the involved chiral superfields as the coordinates of the associated Kähler manifold. This makes manifest the one-to-one correspondence between Kähler geometry and $N = 1$ $4D$ ($(2, 2)$ $2D$) supersymmetry [17].

It is important to point out that the superfield action (2.2) has been written and interpreted as the most general $N = 2$ $4D$ supersymmetric sigma model action [5, 6] *prior* to recognizing the potentials $L^{(+)}_M$ and $L^{(+4)}$ as the basic objects of hyper-Kähler geometry and deducing them ¿from the primary principles of the latter in [7]. Many characteristic features of the analytic space formulation of this geometry can be read off by inspecting the action (2.2). For instance, it is invariant under arbitrary analytic reparametrizations of $q^{(+)\,M}$

$$\delta q^{(+)\,M} = \Lambda^{(+)\,M}(q^{(+)}, u) \;, \tag{2.4}$$

provided $L^{(+)}_M$ and $L^{(+4)}$ transform as

$$\delta L^{(+)}_M = -L^{(+)}_N \frac{\partial \Lambda^{(+)\,N}}{\partial q^{(+)\,M}} \;, \;\; \delta L^{(+4)} = -L^{(+)}_N \partial^{(+2)} \Lambda^{(+)\,N} \;, \tag{2.5}$$

as well as under the following transformations called in [6] the hyper-Kähler ones (because these are a direct analog of Kähler transformations $K(x, \bar{x}) \Rightarrow K(x, \bar{x}) + \Lambda(x) + \bar{\Lambda}(\bar{x})$)

$$\delta q^{(+)\,M} = 0 \;, \; \delta L^{(+)}_M = \frac{\partial \Lambda^{(+2)}}{\partial q^{(+)\,M}} \;, \; \delta L^{(+4)} = \partial^{(+2)} \Lambda^{(+2)} \;, \;\; \Lambda^{(+2)} = \Lambda^{(+2)}(q^{(+)}, u) \;. \tag{2.6}$$

Here $\partial^{(+2)}$ acts only on the explicit harmonics in the arguments of $\Lambda^{(+)\,M}$, $\Lambda^{(+2)}$. The geometric origin of these transformations have been fully understood later on [7] within the analytic space formulation of hyper-Kähler manifolds. Note that these invariances can be used to gauge $L^{(+)}_M$ into its "flat" part $q^{(+)\,M}$

$$L^{(+)}_M = -\Omega_{MN} q^{(+)\,N} \;, \tag{2.7}$$

thus demonstrating that the only essential hyper-Kähler potential is $L^{(+4)}$ (the sign "$-$" in (2.7) ensures the correct sign of the kinetic term of physical bosonic fields in the component action).

The equation of motion for $q^{(+)\,M}$ following from (2.2)

$$\begin{aligned} D^{(+2)} q^{(+)\,M} &= -H^{MN} \left(\frac{\partial L^{(+4)}}{\partial q^{(+)\,N}} - \partial^{(+2)} L^{(+)}_N \right) \;, \\ H^{MN} H_{NT} &= \delta^M_T \;, \quad H_{NT} = \frac{\partial L^{(+)}_N}{\partial q^{(+)\,T}} - \frac{\partial L^{(+)}_T}{\partial q^{(+)\,N}} \end{aligned} \tag{2.8}$$

also has a nice geometric interpretation. Defining the target space harmonic derivative $\mathcal{D}^{(+2)}$ which acts in the target analytic subspace spanned by the coordinates $q^{(+)\,M}, u^{(\pm)i}$

$$\mathcal{D}^{(+2)} = \partial^{(+2)} + D^{(+2)} q^{(+)\,M} \frac{\partial}{\partial q^{(+)\,M}} \equiv \partial^{(+2)} + E^{(+3)\,M} \frac{\partial}{\partial q^{(+)\,M}} \;, \tag{2.9}$$

one observes that eq. (2.8) is none other than the expression of the target space analytic vielbein $E^{(+3)\,M}$ in terms of the hyper-Kähler potentials [7]. Moreover, just eq. (2.8) tells us that $D^{(+2)} q^{(+)\,M} \equiv E^{(+3)\,M}$ is analytic and, hence, that $\mathcal{D}^{(+2)}$ (2.9) preserves the target space harmonic analyticity.

In what follows we will refer to a slightly different representation of the general action (2.2). Let us split the target space world index M as $M = (i\alpha)$, $i = 1, 2;$ $\alpha = 1, 2, ...n$ and, using the completeness property of harmonics

$$u^{(+)i} u^{(-)k} - u^{(+)k} u^{(-)i} = \epsilon^{ki} \;, \;\; (\epsilon^{12} = -\epsilon^{12} = -1) \;,$$

equivalently re-express $q^{(+)\,M} = q^{(+)\,i\alpha}$ through the pair of analytic superfields $\omega^\alpha(\zeta, u)$, $l^{(+2)\,\alpha}(\zeta, u)$

$$
\begin{aligned}
q^{(+)\,i\alpha} &= u^{(+)i}\omega^\alpha - u^{(-)i} l^{(+2)\,\alpha} \, , \\
\omega^\alpha &= u^{(-)}_i q^{(+)\,i\alpha} \, , \;\; l^{(+2)\,\alpha} \; = \; u^{(+)}_i q^{(+)\,i\alpha} \, .
\end{aligned} \tag{2.10}
$$

In terms of these superfields the action (2.2) can be rewritten as

$$
S_{\omega,l} = \int \mu^{(-4)} \{ L^{(+2)}_\alpha(\omega, l, u) D^{(+2)} \omega^\alpha + L_\alpha(\omega, l, u) D^{(+2)} l^{(+2)\,\alpha} + \tilde{L}^{+4}(\omega, l, u) \} \, . \tag{2.11}
$$

To know the precise form of the relation between the potentials in (2.11) and the previous ones $L^{(+)}_M$, $L^{(+4)}$, as well as the ω, l realization of groups (2.4), (2.6), is of no need for our further purposes. We only note that the potentials $L^{(+2)}_\alpha$, L_α are also pure gauge. They can be gauged into their flat limits

$$
L^{(+2)}_\alpha = l^{(+2)\,\alpha} \, , \;\; L_\alpha = -\omega^\alpha \, , \tag{2.12}
$$

where, without loss of generality, we have chosen $\Omega_{MN} = \epsilon_{ij}\delta_{\alpha\beta}$.

Note that in this gauge and with $\tilde{L}^{(+4)}$ displaying no dependence on ω^α, the general action reduces to

$$
S_l = \int \mu^{(-4)} \{ -2\omega^\alpha D^{(+2)} l^{(+2)\,\alpha} + \tilde{L}^{+4}(l, u) \} \, . \tag{2.13}
$$

This reduced action is the general action of the so called "$N = 2$ tensor multiplets". Indeed, varying (2.13) with respect to ω^α we arrive at the action which contains only the $\tilde{L}^{+4}(l, u)$ part,

$$
S_l = \int \mu^{(-4)} \tilde{L}^{+4}(l, u) \, , \tag{2.14}
$$

with the superfield $l^{(+)\,\alpha}$ subjected to the constraint

$$
D^{(+2)} l^{(+2)\,\alpha} = 0 \, . \tag{2.15}
$$

This is just the harmonic superspace action and constraint of $N = 2 \;\; 4D$ $((4,4) \;\; 2D)$ tensor multiplet [5]. Alternatively, one could vary (2.13) with respect to $l^{(+2)\,\alpha}$ and, expressing $l^{(+2)\,\alpha}$ from the resulting algebraic equation as a function of $D^{(+2)}\omega^\alpha$, rewrite (2.13) through the unconstrained analytic superfields ω^α. This kind of $N = 2 \;\; 4D$ $((4,4) \;\; 2D)$ duality relates to each other two different off-shell descriptions of the same scalar supermultiplet (4+4 components on shell): with a finite number of auxiliary fields (l representation of the action) and with an infinite number of auxiliary fields (ω representation of the action). Note that the passing to the ω form is possible for the general action (2.11) as well, because for the superfield $l^{(+2)\,\alpha}$ the equation of motion is always algebraic,

$$
l^{(+)\,\alpha} \sim D^{(+2)} \omega^\alpha + \ldots \, , \tag{2.16}
$$

and by means of this equation $l^{(+2)\,\alpha}$ can be expressed in terms of ω^α. Actually, the l, ω and ω actions are the first and second order forms of the same general $(4,4)$ supersymmetric hyper-Kähler sigma model action.

3 (4,4) $SU(2)\times SU(2)$ harmonic superspace

In fixing our further notation we will basically follow Ref. [11] with minor deviations. We start with some definitions.

The standard $(4,4)$ $2D$ superspace is defined as

$$\mathbf{S}^{(1,1|4,4)} = (x^{++}, x^{--}, \theta^{+\, i\underline{k}}, \theta^{-\, a\underline{b}}).$$

Here $+, -$ are light-cone indices and $i, \underline{k}, a, \underline{b}$ are doublet indices of four commuting $SU(2)$ groups which constitute the full automorphism group $SO(4)_L \times SO(4)_R$ of $(4,4)$ $2D$ Poincare superalgebra. The harmonic $(4,4)$ superspace constructed in [11] is an extension of $\mathbf{S}^{(1,1|4,4)}$ by two independent sets of harmonic variables $u_i^{\pm 1}$, $v_a^{\pm 1}$, each parametrizing one of the $SU(2)$ factors of $SO(4)_L$ and $SO(4)_R$, respectively (we denote them by $SU(2)_L$ and $SU(2)_R$):

$$\mathbf{HS}^{(1+2,1+2|4,4)} = \mathbf{S}^{(1,1|4,4)} \otimes (u_i^{\pm 1}, v_a^{\pm 1})\,, \quad u^{1\,i} u_i^{-1} = 1, \;\; v^{1\,a} v_a^{-1} = 1\,.$$

The harmonics u and v carry two independent $U(1)$ charges which are assumed to be strictly conserved (like in the standard $N=2$ $4D$ harmonic superspace [8, 9]). This requirement actually implies u and v to parametrize the 2-spheres $S_L^2 \sim SU(2)_L/U(1)_L$ and $S_R^2 \sim SU(2)_R/U(1)_R$. All superfields given on $\mathbf{HS}^{(1+2,1+2|4,4)}$ possess two definite $U(1)$ charges and, correspondingly, are assumed to be decomposable in the double harmonic series on the above 2-spheres.

Like in the $N=2$ $4D$ case, the main merit of passing to the $(4,4)$ harmonic superspace in question is the existence of an analytic subspace in it which is closed under the transformations of $(4,4)$ supersymmetry and has twice as few odd coordinates

$$\mathbf{AS}^{(1+2,1+2|2,2)} = (z^{++}, z^{--}, \theta^{1,0\,\underline{i}}, \theta^{0,1\,\underline{a}}, u_i^{\pm 1},\; v_a^{\pm 1}) \equiv (\zeta^{\mu}, u_i^{\pm 1},\; v_a^{\pm 1})\,, \tag{3.1}$$

where

$$\theta^{1,0\,\underline{i}} = \theta^{+\,i\underline{i}}\, u_i^1\;, \theta^{0,1\,\underline{a}} = \theta^{-\,a\underline{a}}\, v_a^1\;,$$

and the relation between $z^{\pm\pm}$ and $x^{\pm\pm}$ can be found in [11]. Superfields given on the superspace (3.1), $\Phi^{p,q}(\zeta, u, v)$ (p and q are values of the left and right harmonic $U(1)$ charges), are called analytic $(4,4)$ superfields.

The analytic superspace (3.1) is real with respect to the generalized involution "$\sim$" which is the product of ordinary complex conjugation and an antipodal map of the 2-spheres $SU(2)_L/U(1)_L$ and $SU(2)_R/U(1)_R$

$$(\widetilde{\theta^{1,0\,\underline{i}}}) = \theta^{1,0}_{\underline{i}}\;, \;\; (\widetilde{u^{\pm 1\,i}}) = -u_i^{\pm 1}\;, \tag{3.2}$$

(and similarly for $\theta^{1\,\underline{a}}, v_a^{\pm 1}$). The analytic superfields $\Psi^{p,q}$ can be chosen real with respect to this involution, provided $|p+q| = 2n$

$$(\widetilde{\Psi^{p,q}}) = \Psi^{p,q}\;, |p+q| = 2n\;. \tag{3.3}$$

In what follows we will need the fact of existence of two mutually commuting sets of derivatives with respect to harmonics $u^{\pm 1\, i}$ and $v^{\pm 1\, a}$, each forming an $SU(2)$ algebra

$$\begin{aligned} \partial^{\pm 2,0} &= u^{\pm 1\, i}\frac{\partial}{\partial u^{\mp 1\, i}}\;, \quad \partial^0_u \;=\; u^{1\, i}\frac{\partial}{\partial u^{1\, i}} - u^{-1\, i}\frac{\partial}{\partial u^{-1\, i}} \\ \partial^{0,\pm 2} &= v^{\pm 1\, a}\frac{\partial}{\partial v^{\mp 1\, a}}\;, \quad \partial^0_v \;=\; v^{1\, a}\frac{\partial}{\partial v^{1\, a}} - v^{-1\, a}\frac{\partial}{\partial v^{-1\, a}}\;. \end{aligned} \tag{3.4}$$

The full analyticity preserving harmonic derivatives $D^{2,0}$, D^0_u, $D^{0,2}$, D^0_v, when applied on analytic superfields, are given by the expressions

$$\begin{aligned} D^{2,0} &= \partial^{2,0} + i\theta^{1,0}\theta^{1,0}\partial_{++}\;, \quad D^{0,2} \;=\; \partial^{0,2} + i\theta^{0,1}\theta^{0,1}\partial_{--} \\ D^0_u &= \partial^0_u + \theta^{1,0\, \underline{i}}\frac{\partial}{\partial\theta^{1,0\, \underline{i}}}\;, \quad D^0_v \;=\; \partial^0_v + \theta^{0,1\, \underline{a}}\frac{\partial}{\partial\theta^{0,1\, \underline{a}}}\;. \end{aligned} \tag{3.5}$$

The operators D^0_u, D^0_v count the $U(1)$ charges of analytic $(4,4)$ superfields

$$D^0_u\Phi^{p,q}(\zeta,u,v) = p\Phi^{p,q}(\zeta,u,v)\;, \quad D^0_v\Phi^{p,q}(\zeta,u,v) = q\Phi^{p,q}(\zeta,u,v)\;. \tag{3.6}$$

The last topic of this Section will be the harmonic superspace off-shell description of $(4,4)$ twisted chiral multiplet.

This multiplet is represented by an analytic $(4,4)$ superfield $q^{1,1}(\zeta,u,v)$ obeying the harmonic constraints

$$D^{2,0}q^{1,1} = D^{0,2}q^{1,1} = 0\;. \tag{3.7}$$

They leave in $q^{1,1}$ $8+8$ independent components [11], just the off-shell field content of $(4,4)$ twisted multiplet [2, 13]. Notice a formal similarity of the constraints (3.7) to the constraint defining $N=2$ tensor multiplet in the harmonic $N=2$ $4D$ superspace (2.15). The crucial difference between either constraints is that (2.15) implies a differential condition for a vector component of the relevant superfield, requiring it to be divergenceless, while this is not the case for the constraints (3.7). These constraints are purely algebraic and express the higher dimension components of $q^{1,1}$ through z-derivatives of the physical dimension ones (they leave as independent also four auxiliary fields which enter the θ expansion of $q^{1,1}$ with the monomial $\theta^{1,0\underline{i}}\theta^{0,1\underline{a}}$).

To understand the origin of the difference between two types of constraints, let us perform the reduction of the $(4,4)$ $SU(2)_L\times SU(2)_R$ harmonic superspace to the standard $(4,4)$ $SU(2)$ one. It is accomplished by identifying harmonic variables $u^{\pm 1\, i} = v^{\pm 1\, a}$ and, correspondingly, both harmonic $U(1)$ charges. The harmonic derivative $D^{(+2)}$ (2.3) is recognized as the sum of the left and right ones

$$D^{(+2)} = D^{2,0} + D^{0,2}\;.$$

¿From this consideration it is already clear that there is no smooth transition between the constraints (3.7) and (2.15). The field content of $q^{1,1}$ also changes. While before identifying harmonics u and v the matrix of physical bosons $q^{ia}(z)$ ($q^{1,1} = q^{ia}u^1_iv^1_a + ...$) comprises 4 independent fields, after the identification this number is reduced to 3 (only the symmetric part of q^{ia} survives). As a result of imposing the constraint (2.15) on the reduced superfield, the lost fourth scalar field reappears as a solution to the divergencelessness condition for the $2D$ vector field components multiplying the θ monomials $(\theta^{(+)+})^2$, $(\theta^{(+)-})^2$. Note that the smooth transition between the two superfield systems becomes possible in the dual action of $q^{1,1}$ (see below).

Despite the essential difference between the constraints (3.7) and (2.15), invariant actions of $q^{1,1}$ look similar to those of $l^{(+2)}$ (2.14). The general off-shell action of n superfields $q^{1,1\,M}$ $(M = 1, 2, ...n)$ is given by

$$S_q = \int \mu^{-2,-2}\ L^{2,2}(q^{1,1\,M}(\zeta, u, v),\ u,\ v)\ . \tag{3.8}$$

Here

$$\mu^{-2,-2} = d^6\zeta\ [du\ dv] = d^2z\ d^2\theta^{1,0}\ d^2\theta^{0,1}\ [du\ dv]$$

is the analytic superspace integration measure. The dimensionless analytic superfield Lagrangian $L^{2,2}(q^{1,1\,M}, u_i^{\pm 1}, v_a^{\pm 1})$ bears in general an arbitrary dependence on its arguments, the only restriction being a compatibility with the external $U(1)$ charges $2, 2$. The free action of $q^{1,1\,M}$ is given by

$$S_q^{free} \sim \int \mu^{-2,-2}\ q^{1,1\,M}\ q^{1,1\,M}\ , \tag{3.9}$$

so for consistency we are led to assume

$$\det\left(\frac{\partial^2 L^{2,2}}{\partial q^{1,1\,M}\partial q^{1,1\,N}}\right)\ |_{q^{1,1}=0} \neq 0\ . \tag{3.10}$$

For completeness, we also add the constraints on $q^{1,1\,M}(\zeta, u, v)$

$$D^{2,0}q^{1,1\,M} = D^{0,2}q^{1,1\,M} = 0\ . \tag{3.11}$$

The passing to the component form of the action is straightforward [11]. The bosonic sigma model action consists of two parts related to each other by $(4,4)$ supersymmetry: the metric part and the part including the torsion potential.

As an important particular example of $q^{1,1}$ action we give the action of $(4,4)$ extension of the group manifold $SU(2) \times U(1)$ WZNW sigma model

$$S_{wzw} = -\frac{1}{4\kappa^2}\int \mu^{-2,-2}\ \hat{q}^{1,1}\hat{q}^{(1,1)}\left(\frac{1}{(1+X)X} - \frac{\ln(1+X)}{X^2}\right)\ . \tag{3.12}$$

Here

$$\hat{q}^{1,1} = q^{1,1} - c^{1,1}\ ,\ X = c^{-1,-1}\hat{q}^{1,1}\ ,\ c^{\pm 1,\pm 1} = c^{ia}u_i^{\pm 1}v_a^{\pm 1}\ ,\ c^{ia}c_{ia} = 2\ . \tag{3.13}$$

Despite the presence of an extra quartet constant c^{ia} in the analytic superfield Lagrangian, the action (3.12) actually does not depend on c^{ia} [11] as it is invariant under arbitrary rescalings and $SU(2) \times SU(2)$ rotations of this constant.

4 Dual form of the $q^{1,1}$ action and its generalization

By adding the constraints (3.11) with Lagrange multipliers to the general $q^{1,1}$ action (3.8) one puts the latter in the form analogous to the tensor supermultiplet master action (2.13)

$$S_{q,\omega} = \int \mu^{-2,-2}\{\omega^{-1,1\,M} D^{2,0} q^{1,1\,M} + \omega^{1,-1\,M} D^{0,2} q^{1,1\,M} + L^{2,2}(q^{1,1}, u, v)\}\ . \qquad (4.1)$$

The analytic superfields $q^{1,1\,M}$, $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$ are now unconstrained and one can vary them to get the superfield equations of motion. Varying $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$ yields the constraints (3.11) and we recover the original action (3.8). Alternatively, one can vary (4.1) with respect to $q^{1,1\,M}$, which gives rise to the equation

$$\frac{\partial L^{2,2}}{\partial q^{1,1\,M}} = D^{2,0}\omega^{-1,1\,M} + D^{0,2}\omega^{1,-1\,M} \equiv A^{1,1\,M}\ . \qquad (4.2)$$

This algebraic equation is a kind of Legendre transformation expressing $q^{1,1\,M}$ as a function of $A^{1,1\,M}$

$$(4.2) \Rightarrow q^{1,1\,M} = q^{1,1\,M}(A^{1,1}, u, v)\ . \qquad (4.3)$$

Substituting this expression back into (4.1), one arrives at the dual form of the $q^{(1,1)}$ action

$$\begin{aligned} S_\omega &= \int \mu^{-2,-2} L^{2,2}_\omega(A^{1,1}, u, v)\ , \\ L^{2,2}_\omega(A^{1,1}, u, v) &\equiv L^{2,2}(q^{1,1\,M}(A, u, v), u, v) - q^{1,1\,M}(A, u, v) A^{1,1\,M}\ . \end{aligned} \qquad (4.4)$$

The dual action (4.4) provides a new off-shell formulation of $(4,4)$ sigma models with commuting left and right complex structures via *unconstrained* analytic $(4,4)$ superfields. The most characteristic feature of such formulations is the presence of infinite number of auxiliary fields [8, 9]. Thus, in the case at hand the physical component action for $4n$ bosons and $8n$ fermions is restored only after eliminating an infinite tower of auxiliary fields coming from the double harmonic expansion of superfields $\omega^{1,-1\,N}(\zeta, u, v)$, $\omega^{-1,1\,N}(\zeta, u, v)$.

To see in more detail how this occurs, let us focus on the bosonic degrees of freedom. The action (4.1) originally involves three independent superfields $q^{1,1\,N}$, $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$, each including $4n$ real bosonic fields in the first term of its double harmonic expansion (higher rank bosonic fields finally prove to be auxiliary and we should not care about them). Varying $q^{1,1\,N}$ yields an algebraic equation (4.2) by which $q^{1,1\,N}$ is eliminated in terms of the remaining two superfields

$$q^{1,1\,N} \sim D^{2,0}\omega^{-1,1\,N} + D^{0,2}\omega^{1,-1\,N} + ... \qquad (4.5)$$

(cf. eq. (2.16)). Thereby, the number of physical dimension bosonic fields is reduced from $12n$ to $8n$. However, the number of such fields carried by two ω superfields is still twice the number of those carried by $q^{1,1}$ in the original formulation. So one may wonder how the on-shell equivalence of these two off-shell formulations is achieved. The answer is that the equivalence is guaranteed due to the invariance of the action (4.1) and its ω version (4.4) under the abelian gauge transformations

$$\delta\,\omega^{1,-1\,M} = D^{2,0}\sigma^{-1,-1\,M}\;,\;\delta\,\omega^{-1,1\,M} = -D^{0,2}\sigma^{-1,-1\,M}\;, \tag{4.6}$$

with $\sigma^{-1,-1\,M} = \sigma^{-1,-1\,M}(\zeta,u,v)$ being arbitrary analytic functions. This gauge freedom takes away just half of the lowest superisospin multiplets in the superfields $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$, thus restoring the correct physical field content of the theory. For instance, the first components in the θ expansion of these superfields are transformed as

$$\delta\,\omega_0^{1,-1\,M}(z) = \partial^{2,0}\sigma^{-1,-1\,M}(z)\;,\;\delta\,\omega_0^{-1,1\,M}(z) = -\partial^{0,2}\sigma^{-1,-1\,M}(z)\;, \tag{4.7}$$

and one may fix the gauge so as to entirely eliminate one set of these fields (other gauge choices are also possible). Thus, in contrast to the $q^{1,1}$ superfield formulation, where the necessary set of the physical fields is ensured by imposing the harmonic constraints on $q^{1,1}$, the same goal in the dual formulation is achieved thanks to the gauge freedom (4.6) (and after eliminating an infinite set of auxiliary fields). This gauge invariance is the main novel feature of the dual formulation of the $q^{1,1}$ action compared to an analogous formulation of the $l^{(+2)}$ action in the conventional harmonic superspace. It is a necessary ingredient of the free action of the triple $q^{1,1\,N}$, $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$ (corresponding to the choice $L^{2,2} = q^{1,1\,N}q^{1,1\,N}$ in (4.1)) and one can expect that any reasonable generalization to the case with interaction should enjoy this important symmetry. Below we will see that this is indeed so, the abelian gauge invariance getting nonabelian in general.

For what follows it will be important to realize that the gauge freedom in question reflects the commutativity of the left and right harmonic derivatives $D^{2,0}$ and $D^{0,2}$. Indeed, the equations of motion which follow by varying Lagrange multipliers $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$, viz. the constraints (3.11), are not entirely independent: due to the above commutativity they obey the evident integrability condition

$$D^{2,0}(D^{0,2}q^{1,1\,M}) - D^{0,2}(D^{2,0}q^{1,1\,M}) = 0\;. \tag{4.8}$$

In the simplest case we are considering, this condition is identically satisfied (since $L^{2,2}$ does not depend on $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$). However, in more general cases it puts non-trivial restrictions on the structure of the action. Below we will see that in all examples in which the condition (4.8) is satisfied the relevant actions respect gauge symmetry (4.6) or a nonabelian extension of it.

It is to the point here to mention a clarifying analogy with the abelian gauge theory in two dimensions. The harmonic derivatives $D^{2,0}, D^{0,2}$ are analogous to the x derivatives ∂_μ, $\mu = 1,2$, two Lagrange multipliers $\omega^{1,-1\,N}$ and $-\omega^{-1,1\,N}$ are analogs of the two-dimensional $U(1)$ gauge connection A_μ (actually, of N independent copies of it), the quantity $A^{1,1\,N}$ in (4.2) is an analog of the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \equiv \epsilon_{\mu\nu}F$. Then the dual action (4.1) is analogous to the first order form of the Maxwell action of A_μ[3], while the constraints (3.7) are the precise analog of the sourceless Maxwell equation

$$\partial^\mu F_{\mu\nu} = \partial^\mu \epsilon_{\mu\nu}F = 0\;. \tag{4.9}$$

[3]Just as the dual action of $(4,4)$ supersymmetric hyper-Kähler sigma model (2.11) is analogous to the first order form of a scalar field action.

The self-consistency condition (4.8) is a counterpart of the "kinematical" conservation law

$$\partial^{\nu}(\partial^{\mu}F_{\mu\nu}) = 0 \ . \tag{4.10}$$

The conservation law (4.10) ceases to be trivial after inserting a matter current into the r.h.s. of (4.9): in this case it requires the current to be conserved as a consequence of the equations of motion, which imposes severe restrictions on the structure of this current and implies the gauge symmetry of the free action to extend to the whole action. Quite similarly, after allowing for a $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$ dependence in $L^{2,2}$ there will appears a non-zero "current" in the r.h.s. of eqs. (3.7) and the condition (4.8) will become the harmonic conservation law for this current, severely restricting the structure of the latter and, hence, of $L^{2,2}$. In the sequel we will often resort to this analogy.

The last comment concerning transformations (4.6) is that they define a *genuine* symmetry of the actions (4.1), (4.4), contrary, e.g., to the transformations (2.4), (2.5), (2.6) which are a kind of equivalence redefinitions of the involved superfields and potentials. These latter transformations leave the relevant actions form-invariant but change the precise structure of the potentials in them.

Let us turn to generalizations of the action (4.1). As was argued in [2, 14], with making use of the $(4,4)$ twisted supermultiplet alone one may construct only the $(4,4)$ sigma models with mutually commuting left and right complex structures. Then a natural way to approach the problem of constructing off-shell $(4,4)$ superfield actions with non-commuting structures is to seek for such generalizations of the action (4.1) which *do not admit the passing to a pure* $q^{1,1}$ *form.* The rest of the paper is devoted to deducing such generalizations.

The action (4.1) is an analog of the dual $l^{(+2)}$ action (2.13), the triple of superfields $q^{(1,1)}$, $\omega^{1,-1}$, $\omega^{1,-1}$ being an analog of the pair $l^{(+2)}, \omega$. So one may write the most general action of this triple, making in (4.1) the substitutions like those which lead from (2.13) to the general l, ω action (2.11). In this way one obtains

$$\begin{aligned} S_{q,\omega} &= \int \mu^{-2,-2}\{H^{2,2} + H^{-1,1\,M}D^{2,0}q^{1,1\,M} + H^{1,-1\,M}D^{0,2}q^{1,1\,M} + H^{1,1\,M}D^{2,0}\omega^{1,-1\,M} \\ &\quad + \tilde{H}^{1,1\,M}D^{0,2}\omega^{-1,1\,M} + H^{-1,3\,M}D^{2,0}\omega^{1,-1\,M} + H^{3,-1\,M}D^{0,2}\omega^{-1,1\,M}\} \\ &\equiv \int \mu^{-2,-2}\mathcal{L}^{2,2}_{q,\omega}(q,\omega,u,v) \ , \end{aligned} \tag{4.11}$$

where *a priori* all the potentials are arbitrary functions of the superfields $q^{1,1\,M}$, $\omega^{1,-1\,M}$, $\omega^{-1,1\,M}$ and harmonics u, v. For the time being we leave aside the important question of implementing the gauge freedom (4.6) in this action and try to use the set of invariances of the type (2.4), (2.6) to reduce the number of independent potentials as much as possible.

One type of such invariances of the action (4.11) is related to reparametrizations of the involved superfields

$$\begin{aligned} \delta q^{1,1\,M} &= \Lambda^{1,1\,M}(q,\omega,u,v) \ , \ \ \delta\omega^{1,-1\,M} = \Lambda^{1,-1\,M}(q,\omega,u,v) \ , \\ \delta\omega^{-1,1\,M} &= \Lambda^{-1,1\,M}(q,\omega,u,v) \ . \end{aligned} \tag{4.12}$$

It is straightforward to find the transformations of the potentials such that the action is form-invariant. Their explicit structure is not too enlightening.

Another type of invariance is similar to the hyper-Kähler one (2.6) and is related to the freedom of adding full harmonic deriavtives to the superfield Lagrangian in (4.11)

$$\mathcal{L}^{2,2}_{q,\omega} \Rightarrow \mathcal{L}^{2,2}_{q,\omega} + D^{2,0}\Lambda^{0,2} + D^{0,2}\Lambda^{2,0} \,, \tag{4.13}$$

$$\Lambda^{2,0} = \Lambda^{2,0}(q,\omega,u,v) \,, \quad \Lambda^{0,2} = \Lambda^{0,2}(q,\omega,u,v) \,.$$

Once again, it is easy to indicate how the potentials should transform to generate the shifts (4.13). It will be important for our consideration that, assuming the existence of the flat limit (given by the action (4.1) with $L^{2,2}(q,u,v) = q^{1,1\,N}q^{1,1\,N}$), the full gauge freedom (4.12), (4.13) can be fixed so that

$$\begin{aligned} H^{-1,1\,N} &= \alpha\omega^{-1,1\,N} \,, \quad H^{1,-1\,N} = \beta\omega^{1,-1\,N} \,, \\ H^{1,1\,N} &= (1+\beta)q^{1,1\,N} \,, \quad \tilde{H}^{1,1\,N} = (1+\alpha)q^{1,1\,N} + \hat{H}^{1,1\,N} \,, \end{aligned} \tag{4.14}$$

α, β being arbitrary parameters. In this gauge (which is an analog of the gauges (2.7), (2.12)) the action still contains four independent potentials, $H^{2,2}$, $H^{-1,3\,N}$, $H^{3,-1\,N}$ and $\hat{H}^{1,1\,N}$,

$$\begin{aligned} S_{q,\omega} = \int \mu^{-2,-2}\{ & q^{1,1\,M}D^{0,2}\omega^{1,-1\,M} + (q^{1,1\,N} + \hat{H}^{1,1\,N})D^{2,0}\omega^{-1,1\,M} \\ & + H^{-1,3\,M}D^{2,0}\omega^{1,-1\,M} + H^{3,-1\,M}D^{0,2}\omega^{-1,1\,M} + H^{2,2}\} \,, \end{aligned} \tag{4.15}$$

and is invariant under the following transformations which are a mixture of (4.12) and (4.13) (the parameters $\Lambda^{2,0}, \Lambda^{0,2}$ below are not the same as those in eq. (4.13), but are related to them)

$$\begin{aligned} \delta\hat{H}^{1,1\,M} &= -\Lambda^{1,1\,M} + \frac{\partial\Lambda^{0,2}}{\partial\omega^{-1,1\,M}} + \Lambda^{1,-1\,N}\frac{\partial H^{-1,3\,N}}{\partial\omega^{-1,1\,M}} + \Lambda^{-1,1\,N}\frac{\partial\hat{H}^{1,1\,N}}{\partial\omega^{-1,1\,M}} \\ \delta H^{-1,3\,M} &= \frac{\partial\Lambda^{0,2}}{\partial\omega^{1,-1\,M}} + \Lambda^{1,-1\,N}\frac{\partial H^{-1,3\,N}}{\partial\omega^{1,-1\,M}} + \Lambda^{-1,1\,N}\frac{\partial\hat{H}^{1,1\,N}}{\partial\omega^{1,-1\,M}} \\ \delta H^{3,-1\,M} &= \frac{\partial\Lambda^{2,0}}{\partial\omega^{-1,1\,M}} + \Lambda^{-1,1\,N}\frac{\partial H^{3,-1\,N}}{\partial\omega^{-1,1\,M}} \\ \delta H^{2,2} &= \partial^{2,0}\Lambda^{0,2} + \partial^{0,2}\Lambda^{2,0} + \Lambda^{1,-1\,N}\partial^{2,0}H^{-1,3\,N} \\ & \quad + \Lambda^{-1,1\,N}(\partial^{2,0}\hat{H}^{1,1\,N} + \partial^{0,2}H^{3,-1\,N}) \end{aligned} \tag{4.16}$$

with

$$\begin{aligned} \Lambda^{1,1\,M} &= \frac{\partial\Lambda^{2,0}}{\partial\omega^{1,-1\,M}} - (B^{-1})^{FN}\frac{\partial H^{3,-1\,N}}{\partial\omega^{1,-1\,M}}\left\{\frac{\partial\Lambda^{0,2}}{\partial q^{1,1\,F}} - \frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,T}}\frac{\partial H^{-1,3\,T}}{\partial q^{1,1\,F}}\right\} \\ \Lambda^{1,-1\,M} &= -\frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,M}} + (B^{-1})^{FN}\frac{\partial H^{3,-1\,N}}{\partial q^{1,1\,M}}\left\{\frac{\partial\Lambda^{0,2}}{\partial q^{1,1\,F}} - \frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,T}}\frac{\partial H^{-1,3\,T}}{\partial q^{1,1\,F}}\right\} \\ \Lambda^{-1,1\,M} &= -(B^{-1})^{NM}\left\{\frac{\partial\Lambda^{0,2}}{\partial q^{1,1\,N}} - \frac{\partial\Lambda^{2,0}}{\partial q^{1,1\,T}}\frac{\partial H^{-1,3\,T}}{\partial q^{1,1\,N}}\right\} \\ B^{MN} &= \delta^{MN} + \frac{\partial\hat{H}^{1,1\,M}}{\partial q^{1,1\,N}} - \frac{\partial H^{3,-1\,M}}{\partial q^{1,1\,F}}\frac{\partial H^{-1,3\,F}}{\partial q^{1,1\,N}} \,, \; B^{MN}(B^{-1})^{NL} = \delta^{ML} \end{aligned} \tag{4.17}$$

(one should add, of course, the coordinate transformations (4.12) with the parameters (4.17)). Note that in the case of general manifold ($M = 1, 2...n, n > 1$) it is impossible to gauge away any of the surviving potentials with the help of this remaining gauge freedom, though one can still put them in the form similar to the normal gauge of the hyper-Kähler potential $L^{(+4)}$ [7]. The fact that there remain three more potentials besides $H^{2,2}$ (which is a direct analog of $L^{(+4)}$) is the essential difference of the considered case with torsion from the torsionless hyper-Kähler case. It is worth mentioning that upon the reduction to the $(4,4)$ $SU(2)$ harmonic superspace the superfields $\omega^{1,-1\,N}$ and $\omega^{-1,1\,N}$ in (4.11) are identified with each other and recognized as the superfield ω^N, $q^{1,1\,N} \Rightarrow l^{(+2)\,N}$, $H^{2,2} \Rightarrow L^{(+4)}$, and the potentials $\hat{H}^{1,1\,N}$, $H^{-1,3\,N}$, $H^{3,-1\,N}$ are combined into a shift of $l^{(+2)\,N}$. This shift can be absorbed in an equivalence redefinition of $l^{(+2)\,N}$, after which one recovers the ω, l action (2.11) of the general $(4,4)$ hyper-Kähler sigma model in the "flat" gauge (2.12).

As was noticed in Sect.2, the $q^{(+)}$ equation of motion (2.8) following from the general $q^{(+)}$ action (2.2) has a transparent interpretation within the analytic target space geometry: it expresses the vielbein $E^{(+3)\,M} \equiv D^{(+2)}q^{(+)\,M}$ of the analytic target space harmonic derivative via the unconstrained hyper-Kähler potential $L^{(+4)}$. At present we have no clear understanding which kind of the central basis geometry underlies the general $(4,4)$ action (4.11) (in view of the presence of nontrivial torsion terms it certainly cannot be Riemannian). The direct study of the geometry of the relevant analytic space (spanned by the coordinates $q^{1,1\,N}$, $\omega^{1,-1\,N}$, $\omega^{-1,1\,N}$ and harmonics u, v) starting, by analogy with the hyper-Kähler case, from the action (4.11), (4.15) could help to clarify this point.

We will deal with the gauge-fixed action (4.15). Let us introduce, like in the hyper-Kähler case (eq. (2.9)), the target space harmonic derivatives $\mathcal{D}^{2,0}$, $\mathcal{D}^{0,2}$. When acting on the analytic space coordinates u, v, $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$, they are given by

$$\begin{aligned}
\mathcal{D}^{2,0} &= \partial^{2,0} + E^{3,1\,M}\frac{\partial}{\partial q^{1,1\,M}} + E^{1,1\,M}\frac{\partial}{\partial \omega^{-1,1\,M}} + E^{3,-1\,M}\frac{\partial}{\partial \omega^{1,-1\,M}}\,, \\
\mathcal{D}^{0,2} &= \partial^{0,2} + E^{1,3\,M}\frac{\partial}{\partial q^{1,1\,M}} + E^{-1,3\,M}\frac{\partial}{\partial \omega^{-1,1\,M}} + \tilde{E}^{1,1\,M}\frac{\partial}{\partial \omega^{1,-1\,M}}\,,
\end{aligned} \qquad (4.18)$$

$$\begin{aligned}
E^{3,1\,M} &\equiv D^{2,0}q^{1,1\,M}\,, \quad E^{1,1\,M} \equiv D^{2,0}\omega^{-1,1\,M}\,, \quad E^{3,-1\,M} \equiv D^{2,0}\omega^{1,-1\,M} \\
E^{1,3\,M} &\equiv D^{0,2}q^{1,1\,M}\,, \quad E^{-1,3\,M} \equiv D^{0,2}\omega^{-1,1\,M}\,, \quad \tilde{E}^{1,1\,M} \equiv D^{0,2}\omega^{1,-1\,M}\,.
\end{aligned} \qquad (4.19)$$

The complete target space geometry is as yet unclear to us, so there could exist additional target space coordinates and $\mathcal{D}^{2,0}$, $\mathcal{D}^{0,2}$ could contain extra partial derivatives appearing with the related vielbeins (e.g., one may expect that the full harmonic traget space in the analytic basis involves, in addition to the triple of the analytic subspace coordinates $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$, one more coordinate $l^{-1,-1\,N}$ which is represented by a general harmonic superfield). In what follows we will not specify the full structure of $\mathcal{D}^{2,0}$, $\mathcal{D}^{0,2}$ and simply assume that they have the proper action on all the objects depending on harmonics u and v. In particular, when acting on an arbitrary analytic harmonic $(4,4)$ superfield (it can be, e.g., a local function of superfields $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$), they always coincide with $D^{2,0}$ and $D^{0,2}$.

Using the definition (4.19), the equations of motion following from (4.15) can be written in a geometric way as the relations expressing some of the harmonic vielbeins via the potentials $H^{2,2}$, $H^{-1,3\,N}$, $H^{3,-1\,N}$, $\hat{H}^{1,1\,N}$

$$
\begin{aligned}
&\tilde{E}^{1,1\,M} + E^{1,1\,N}\left(\delta^{NM} + \frac{\partial \hat{H}^{1,1\,N}}{\partial q^{1,1\,M}}\right) - E^{-1,3\,N}\frac{\partial H^{3,-1\,N}}{\partial q^{1,1\,M}} \\
&+E^{3,-1\,N}\frac{\partial H^{-1,3\,N}}{\partial q^{1,1\,M}} = -\frac{\partial H^{2,2}}{\partial q^{1,1\,M}}\,, \\
&E^{1,3\,M} + E^{3,-1\,N}\left(\frac{\partial H^{-1,3\,M}}{\partial \omega^{1,-1\,N}} - \frac{\partial H^{-1,3\,N}}{\partial \omega^{1,-1\,M}}\right) + E^{1,1\,N}\left(\frac{\partial H^{-1,3\,M}}{\partial \omega^{-1,1\,N}} - \frac{\partial \hat{H}^{1,1\,N}}{\partial \omega^{1,-1\,M}}\right) \\
&-E^{-1,3\,N}\frac{\partial H^{3,-1\,N}}{\partial \omega^{1,-1\,M}} + E^{3,1\,N}\frac{\partial H^{-1,3\,M}}{\partial q^{1,1\,N}} = \frac{\partial H^{2,2}}{\partial \omega^{1,-1\,M}} - \partial^{2,0}H^{-1,3\,M}\,, \\
&E^{3,1\,N}\left(\delta^{MN} + \frac{\partial \hat{H}^{1,1\,M}}{\partial q^{1,1\,N}}\right) + E^{1,3\,N}\frac{\partial H^{3,-1\,M}}{\partial q^{1,1\,N}} + \tilde{E}^{1,1\,N}\frac{\partial H^{3,-1\,M}}{\partial \omega^{1,-1\,N}} \\
&+E^{1,1\,N}\left(\frac{\partial \hat{H}^{1,1\,M}}{\partial \omega^{-1,1\,N}} - \frac{\partial \hat{H}^{1,1\,N}}{\partial \omega^{-1,1\,M}}\right) + E^{3,-1\,N}\left(\frac{\partial \hat{H}^{1,1\,M}}{\partial \omega^{1,-1\,N}} - \frac{\partial H^{-1,3\,N}}{\partial \omega^{-1,1\,M}}\right) \\
&+E^{-1,3\,N}\left(\frac{\partial H^{3,-1\,M}}{\partial \omega^{-1,1\,N}} - \frac{\partial H^{3,-1\,N}}{\partial \omega^{-1,1\,M}}\right) = \frac{\partial H^{2,2}}{\partial \omega^{-1,1\,M}} - \partial^{2,0}\hat{H}^{1,1\,M} - \partial^{0,2}H^{3,-1\,M}\,. \quad (4.20)
\end{aligned}
$$

Looking at these relations one immediately realizes what is the main difference ¿from the hyper-Kähler relation (2.8). Only three harmonic vielbeins, namely, $E^{1,3\,N}$, $E^{3,1\,N}$ and some linear combination of $\tilde{E}^{1,1\,N}$, $E^{1,1\,N}$, are really eliminated. Three remaining ones are not constrained by these equations and so are to be treated as some independent quantities. One cannot even conclude that they are local functions of the analytic target space coordinates u, v, $q^{1,1\,N}$, $\omega^{-1,1\,N}$, $\omega^{1,-1\,N}$. Nonetheless, two of them, $E^{-1,3\,N}$ and $E^{3,-1\,N}$, can be eliminated by the kinematic relations following from the natural requirement that full target space harmonic derivatives $\mathcal{D}^{2,0}$, $\mathcal{D}^{0,2}$ commute with each other

$$[\mathcal{D}^{2,0}, \mathcal{D}^{0,2}] = 0 \Rightarrow \quad (4.21)$$

$$
\begin{aligned}
\mathcal{D}^{0,2}E^{3,1\,N} - \mathcal{D}^{2,0}E^{1,3\,N} &= 0 &(4.22)\\
\mathcal{D}^{2,0}E^{-1,3\,N} - \mathcal{D}^{0,2}E^{1,1\,N} &= 0 &(4.23)\\
\mathcal{D}^{0,2}E^{3,-1\,N} - \mathcal{D}^{2,0}\tilde{E}^{1,1\,N} &= 0\,. &(4.24)
\end{aligned}
$$

Of course, these relations are identically satisfied with the definition (4.19). For the time being let us forget about the latter (from the point of view of the target space geometry these expressions mean that the harmonic vielbeins are induced as a result of passing to the analytic basis of the target space from some central basis where the harmonic derivatives are short, $\mathcal{D}^{2,0} = \partial^{2,0}$, $\mathcal{D}^{0,2} = \partial^{0,2}$) and treat (4.22) - (4.24) as the consistency conditions for the harmonic vielbeins. Then (4.23), (4.24) are differential equations defined on the product of two 2-spheres $S^2_L \times S^2_R$. Such equations can be solved for $E^{-1,3\,N}$, $E^{3,-1\,N}$ (taking into account that $E^{1,1}$ and $\tilde{E}^{1,1}$ are harmonic derivatives of analytic superfields). As a result, these vielbeins prove to be completely specified by $E^{1,1\,N}$, $\tilde{E}^{1,1\,N}$ and so are in a sense auxiliary.

Thus, at this step we are left with four basic harmonic vielbeins $E^{1,3\,N}$, $E^{3,1\,N}$, $E^{1,1\,N}$, $\tilde{E}^{1,1\,N}$ three of which are expected to be related by eqs. (4.20) to the potentials present in the action (4.15), while the fourth one (a combination of $E^{1,1\,N}$, $\tilde{E}^{1,1\,N}$ which is "orthogonal" to the one entering first of eqs. (4.20)) remains arbitrary. We will come back to this general case after studying, in the next Section, the $n = 1$ example.

The last (but not least) problem is to satisfy the consistency constraint (4.22) which is none other than the integrability condition (4.8). The natural requirement is that this condition is obeyed as a consequence of the initial equations of motion (4.20) and does not produce additional dynamical restrictions on the involved superfields (let us recall the aforementioned analogy with two-dimensional gauge theory). Thus one should express $D^{0,2}q^{1,1} \equiv E^{1,3\,N}$, $D^{2,0}q^{1,1} \equiv E^{3,1\,N}$ ¿from eqs. (4.20), substitute these expressions into (4.22), evaluate the result of action of the harmonic derivatives on them and use once more eqs. (4.20) to eliminate three harmonic derivatives: $D^{2,0}q^{1,1}$, $D^{0,2}q^{1,1}$ and, say, $D^{2,0}\omega^{-1,1}$. Finally, one should equate the coefficients before the same harmonic derivatives in both sides of the resulting equality, as well as the terms containing no such derivatives. In this way one deduces a set of constraints on the potentials $\hat{H}^{1,1\,N}$, $H^{-1,3\,N}$, $H^{3,-1\,N}$, $H^{2,2}$. Thus, in contradistinction to the hyper-Kähler case, the potentials of the general sigma model action in the considered case with torsion prove to be *necessarily constrained*. At present we do not know how to solve these constraints in general. Moreover, they look rather ugly, for which reason we do not present them here. However, the study of a simple $n = 1$ example in the next Section shows that these constraints are very restrictive. Now we turn to considering this particular case.

5 Digression: $n = 1$ example

In this case we deal with one triple of analytic superfields $q^{1,1}$, $\omega^{1,-1}$, $\omega^{-1,1}$ and four-dimensional manifold of physical bosons (provided the gauge freedom (4.6) or some its generalization hold). The action (4.15) can be further simplified because the potentials $H^{-1,3}$, $H^{3,-1}$ become pure gauge

$$H^{-1,3} = H^{3,-1} = 0 \tag{5.1}$$

$$\Rightarrow\; S^{(1)}_{q,\omega} = \int \mu^{-2,-2}\{q^{1,1}D^{0,2}\omega^{1,-1} + (q^{1,1} + \hat{H}^{1,1})D^{2,0}\omega^{-1,1} + H^{2,2}\}\;. \tag{5.2}$$

Thus, the general action of the triple $q^{1,1}$, $\omega^{1,-1}$, $\omega^{-1,1}$ is characterized by two potentials $H^{2,2} = H^{2,2}(q,\omega,u,v)$ and $\hat{H}^{1,1} = \hat{H}^{1,1}(q,\omega,u,v)$ which, before enforcing the integrability conditions (4.22) - (4.24), are arbitrary functions of their arguments. The action is still invariant under the restricted class of reparametrizations preserving the gauge (5.1)

$$\delta\hat{H}^{1,1} = -\Lambda^{1,1} + \frac{\partial\Lambda^{0,2}}{\partial\omega^{-1,1}} + \Lambda^{-1,1}\frac{\partial\hat{H}^{1,1}}{\partial\omega^{-1,1}}\;,\; \delta H^{2,2} = \partial^{2,0}\Lambda^{0,2} + \partial^{0,2}\Lambda^{2,0} + \Lambda^{-1,1}\partial^{2,0}\hat{H}^{1,1} \tag{5.3}$$

$$\begin{aligned} \delta q^{1,1} &\equiv \Lambda^{1,1} = \frac{\partial\Lambda^{2,0}}{\partial\omega^{1,-1}}\;,\; \delta\omega^{1,-1} \equiv \Lambda^{1,-1} = -\frac{\partial\Lambda^{2,0}}{\partial q^{1,1}}\;, \\ \delta\omega^{-1,1} &\equiv \Lambda^{-1,1} = -\left(1 + \frac{\partial\hat{H}^{1,1}}{\partial q^{1,1}}\right)^{-1}\frac{\partial\Lambda^{0,2}}{\partial q^{1,1}} \equiv -B^{-1}\frac{\partial\Lambda^{0,2}}{\partial q^{1,1}} \end{aligned} \tag{5.4}$$

$$\frac{\partial \Lambda^{2,0}}{\partial \omega^{-1,1}} = 0 \Rightarrow \Lambda^{2,0} = \Lambda^{2,0}(q^{1,1}, \omega^{1,-1}, u, v)\;, \quad \frac{\partial \Lambda^{0,2}}{\partial \omega^{1,-1}} - B^{-1}\frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}}\frac{\partial \Lambda^{0,2}}{\partial q^{1,1}} = 0\;. \tag{5.5}$$

The set of equations (4.20) is also essentially simplified

$$\begin{aligned} \tilde{E}^{1,1} + E^{1,1}B &= -\frac{\partial H^{2,2}}{\partial q^{1,1}}\;, \quad E^{1,3} - E^{1,1}\frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}} = \frac{\partial H^{2,2}}{\partial \omega^{1,-1}}\;, \\ E^{3,1}B + E^{3,-1}\frac{\partial \hat{H}^{1,1}}{\partial \omega^{1,-1}} &= \frac{\partial H^{2,2}}{\partial \omega^{-1,1}} - \partial^{2,0}\hat{H}^{1,1}\;. \end{aligned} \tag{5.6}$$

However, even for this simple set of equations the analysis of the restrictions imposed by the integrability condition (4.22), (4.8) proves to be a rather difficult task. We postpone it to the end of this Section and first consider the reduced case with

$$\hat{H}^{1,1} = 0\;. \tag{5.7}$$

The action reduced in this way is still invariant under the transformations (5.3), (5.4) with

$$\Lambda^{2,0} = \lambda^{2,0}(q,u,v) + \omega^{1,-1}\lambda^{1,1}(q,u,v)\;, \quad \Lambda^{0,2} = \lambda^{0,2}(q,u,v) + \omega^{-1,1}\lambda^{1,1}(q,u,v)\;. \tag{5.8}$$

The set of equations (5.6) is reduced to

$$\tilde{E}^{1,1} + E^{1,1} = -\frac{\partial H^{2,2}}{\partial q^{1,1}}\;, \quad E^{1,3} = \frac{\partial H^{2,2}}{\partial \omega^{1,-1}}\;, E^{3,1} = \frac{\partial H^{2,2}}{\partial \omega^{-1,1}}\;. \tag{5.9}$$

To extract the consequences of the integrability condition (4.22), we act on the r.h.s. of the second and third equations in (5.9) by $D^{2,0}$ and $D^{0,2}$, use once again (5.9) to eliminate $E^{1,3}$, $E^{3,1}$ and one of $E^{1,1}$, $\tilde{E}^{1,1}$, say $E^{1,1}$, and finally equate the obtained expressions. Equating, in both sides of the resulting equality, the coefficients before independent objects $E^{-1,3}$, $E^{3,-1}$ and $\tilde{E}^{1,1}$ as well as the terms without derivatives, we get four conditions on the potential $H^{2,2}$

$$\frac{\partial^2 H^{2,2}}{\partial \omega^{-1,1}\partial \omega^{-1,1}} = \frac{\partial^2 H^{2,2}}{\partial \omega^{1,-1}\partial \omega^{1,-1}} = \frac{\partial^2 H^{2,2}}{\partial \omega^{1,-1}\partial \omega^{-1,1}} = 0\;, \tag{5.10}$$

$$\left(\partial^{2,0} + \frac{\partial H^{2,2}}{\partial \omega^{-1,1}}\frac{\partial}{\partial q^{1,1}}\right)\frac{\partial H^{2,2}}{\partial \omega^{1,-1}} - \left(\partial^{0,2} + \frac{\partial H^{2,2}}{\partial \omega^{1,-1}}\frac{\partial}{\partial q^{1,1}}\right)\frac{\partial H^{2,2}}{\partial \omega^{-1,1}} = 0\;. \tag{5.11}$$

¿From eqs. (5.10) we find

$$H^{2,2}(u,v,q,\omega) = h^{2,2}(u,v,q) + \omega^{1,-1}h^{1,3}(u,v,q) + \omega^{-1,1}h^{3,1}(u,v,q)\;, \tag{5.12}$$

after which eq. (5.11) can be rewritten as

$$\left(\partial^{2,0} + h^{3,1}\frac{\partial}{\partial q^{1,1}}\right)h^{1,3} - \left(\partial^{0,2} + h^{1,3}\frac{\partial}{\partial q^{1,1}}\right)h^{3,1} \equiv \nabla^{2,0}h^{1,3} - \nabla^{0,2}h^{3,1} = 0\;. \tag{5.13}$$

The action and constraints are covariant under the transformations (5.3), (5.4) with the restricted parameters (5.8)

$$\delta h^{2,2} = \nabla^{2,0}\lambda^{0,2} + \nabla^{0,2}\lambda^{2,0}\;, \quad \delta h^{1,3} = \nabla^{0,2}\lambda^{1,1}\;, \quad \delta h^{3,1} = \nabla^{2,0}\lambda^{1,1}\;. \tag{5.14}$$

It is easy to see that the action, with taking account of the constraint (5.13), is invariant under the following generalization of gauge transformations (4.6)

$$\delta\omega^{1,-1} = \left(D^{2,0} + \frac{\partial h^{3,1}}{\partial q^{1,1}}\right)\sigma^{-1,-1}\ , \quad \delta\omega^{-1,1} = -\left(D^{0,2} + \frac{\partial h^{1,3}}{\partial q^{1,1}}\right)\sigma^{-1,-1}\ , \tag{5.15}$$

and so propagates just 4 bosonic fields like the action (4.1).

Despite the appearance of nonlinearities, these transformations are abelian like (4.6) and this property already suggests that the action (5.2) with the additional condition (5.7) is actually a reparametrization of the dual form of the $q^{1,1}$ action (4.1). This is indeed so. It is easy to show (starting with a linearized level) that the general solution to the constraint (5.13) is given by

$$h^{1,3} = \nabla^{0,2}\Sigma^{1,1}(u,v,q)\ , \quad h^{3,1} = \nabla^{2,0}\Sigma^{1,1}(u,v,q)\ , \tag{5.16}$$

with $\Sigma^{1,1}(u,v,q)$ being an arbitrary function (the covariant derivatives $\nabla^{2,0}$, $\nabla^{0,2}$ commute as a consequence of (5.13)). Then we can make use of the invariance (5.14) to entirely gauge away $h^{1,3}$ and $h^{3,1}$.

Let us return to the generic $n = 1$ action (5.2). In order to simplify the analysis of the relevant set of equations of motion (5.6) we will stick to a natural assumption which will be also used in an analogous analysis of the general set (4.20) in the next Section. Let us recall that the object $E^{3,-1} \equiv D^{2,0}\omega^{1,-1}$ entering third of eqs. (5.6) can be expressed through $\tilde{E}^{1,1} \equiv D^{0,2}\omega^{1,-1}$ from the harmonic differential equation (4.24) and so can be treated as an auxiliary quantity. Besides, it does not appear at all in the equations corresponding to the free action. So it seems natural to postulate that *it drops out as well from the equations of motion with interaction*. In the considered case this postulate amounts to the constraint

$$\frac{\partial \hat{H}^{1,1}}{\partial\omega^{1,-1}} = 0 \Rightarrow \hat{H}^{1,1} = \hat{H}^{1,1}(q^{1,1},\omega^{-1,1},u,v)\ . \tag{5.17}$$

It is covariant under the transformations (5.3) - (5.5) with the following additional restrictions on the parameters

$$\begin{aligned}
\frac{\partial^2\Lambda^{2,0}}{\partial\omega^{1,-1}\partial\omega^{1,-1}} &= 0 \Rightarrow \Lambda^{2,0} = \lambda^{2,0}(q,u,v) + \omega^{1,-1}\lambda^{1,1}(q,u,v)\ , \\
\frac{\partial\Lambda^{0,2}}{\partial\omega^{1,-1}} &= 0 \Rightarrow \Lambda^{0,2} = \Lambda^{0,2}(q^{1,1},\omega^{-1,1},u,v)\ .
\end{aligned} \tag{5.18}$$

Keeping in mind that $\hat{H}^{1,1}$ is independent of $\omega^{1,-1}$, this remaining gauge freedom is sufficient to entirely gauge away $\hat{H}^{1,1}$, and we recover the $\hat{H}^{1,1} = 0$ case already considered.

As a final step in our analysis of the $n = 1$ case we briefly discuss how to solve the integrability constraint (4.22) for the general set (5.6) without any *ad hoc* assumptions. We will reproduce the condition (5.17) within this setting and thereby justify our dynamical postulate.

In short, applying the same procedure as in the $\hat{H}^{1,1} = 0$ case (though it becomes much more involved) we arrive at several constraints. Most essential of them proves to be the following one

$$\frac{\partial G^{0,2}}{\partial\omega^{-1,1}} = 0\ , \quad \left(G^{0,2} \equiv B^{-1}\frac{\partial\hat{H}^{1,1}}{\partial\omega^{1,-1}}\right)\ . \tag{5.19}$$

It is straightforward to check that this condition is covariant under the *whole* target space gauge group (5.3) - (5.5). Further, $G^{0,2}$ transforms as

$$\delta G^{0,2} = -\frac{\partial^2 \Lambda^{2,0}}{\partial\omega^{1,-1}\partial\omega^{1,-1}} + \ldots \,, \tag{5.20}$$

where dots stand for field-dependent terms. Taking into account that both $G^{0,2}$ and $\Lambda^{2,0}$ do not depend on $\omega^{-1,1}$ and eq. (5.19) is covariant, we observe from (5.20) that $G^{0,2}$ can be gauged away, thus effectively giving rise to the condition (5.17):

$$G^{0,2} = 0 \quad \Rightarrow \quad \frac{\partial \hat{H}^{1,1}}{\partial\omega^{1,-1}} = 0 \,. \tag{5.21}$$

Other corollaries of the integrability condition (4.22) are reduced to those already explored in the $\hat{H}^{1,1} = 0$ case.

Thus, under the natural assumption (just proved) that the harmonic derivative $E^{3,-1} = D^{2,0}\omega^{1,-1}$ never appears in the equations of motion, the general $n = 1$ action (5.2) coincides, modulo a field redefinition, with the general dual action (4.1) of one self-interacting twisted $(4,4)$ multiplet. So the relevant $(4,4)$ sigma models always admit a formulation in terms of single twisted superfield $q^{1,1}$ (constrained by (3.7)) and, in accord with the arguments of Refs. [2, 14], correspond to the case of mutually commuting left and right complex structures on the target. In the next Section we will see that, beginning with $n = 2$, this equivalence to the action (4.1) ceases to hold in general.

6 Back to the general case

In order to simplify the general set of equations (4.20) we accept the same dynamical postulate as in the $n = 1$ case. Namely, we require that eqs. (4.20) *pose no any restrictions on the quantities* $E^{3,-1\,N} \equiv D^{2,0}\omega^{1,-1\,N}$ *and* $E^{-1,3\,N} \equiv D^{0,2}\omega^{-1,1\,N}$. This requirement gives rise to the following constraints on the involved potentials (we equate to zero the coefficients of $E^{3,-1\,N}$, $E^{-1,3\,N}$ in eqs. (4.20))

$$\frac{\partial H^{3,-1\,N}}{\partial\omega^{1,-1\,M}} = \frac{\partial H^{3,-1\,N}}{\partial q^{1,1\,M}} = \frac{\partial H^{3,-1\,N}}{\partial\omega^{-1,1\,M}} - \frac{\partial H^{3,-1\,M}}{\partial\omega^{-1,1\,N}} = 0 \tag{6.1}$$

$$\frac{\partial H^{-1,3\,N}}{\partial q^{1,1\,M}} = \frac{\partial H^{-1,3\,N}}{\partial\omega^{-1,1\,M}} - \frac{\partial \hat{H}^{1,1\,M}}{\partial\omega^{1,-1\,N}} = \frac{\partial H^{-1,3\,N}}{\partial\omega^{1,-1\,M}} - \frac{\partial H^{-1,3\,M}}{\partial\omega^{1,-1\,N}} = 0 \,. \tag{6.2}$$

We believe that, like in the $n = 1$ case, these relations can be rigorously derived by applying the integrability condition (4.22) to the set (4.20) without any prerequisite assumptions. We postpone the check of this natural hypothesis to the future.

The set of constraints (6.1), (6.2) is closed under the action of the group (4.16), (4.17) with the parameters $\Lambda^{2,0}$, $\Lambda^{0,2}$ restricted by

$$\frac{\partial^2 \Lambda^{2,0}}{\partial q^{1,1\,N}\partial\omega^{-1,1\,M}} = \frac{\partial^2 \Lambda^{2,0}}{\partial\omega^{1,-1\,N}\partial\omega^{-1,1\,M}} = \frac{\partial^2 \Lambda^{2,0}}{\partial\omega^{1,-1\,M}\partial\omega^{1,-1\,N}} = 0 \Rightarrow$$

$$\Lambda^{2,0} = \lambda^{2,0}(q,u,v) + \omega^{1,-1\,N}\lambda^{1,1\,N}(q,u,v) + \omega^{-1,1\,N}\lambda^{3,-1\,N}(\omega^{-1,1},u,v) \tag{6.3}$$

$$\frac{\partial^2 \Lambda^{0,2}}{\partial q^{1,1\,N}\partial\omega^{1,-1\,M}} = 0 \Rightarrow \Lambda^{0,2} = \lambda^{0,2}(q^{1,1},\omega^{-1,1},u,v) + \omega^{1,-1\,N}\lambda^{-1,3\,N}(\omega^{1,-1},\omega^{-1,1},u,v)\,. \tag{6.4}$$

It is easy to see that this restricted gauge freedom combined with the constraints (6.1), (6.2) allows one to entirely gauge away the potentials $H^{3,-1\,N}$, $H^{-1,3\,N}$

$$H^{3,-1\,N} = H^{-1,3\,N} = 0\,. \tag{6.5}$$

Taking into account that in this gauge the whole set of eqs. (6.1), (6.2) is reduced to the single constraint

$$\frac{\partial \hat{H}^{1,1\,M}}{\partial\omega^{1,-1\,N}} = 0\,, \tag{6.6}$$

it is easy to find the relevant remaining gauge freedom

$$\begin{aligned}
\frac{\partial\Lambda^{2,0}}{\partial\omega^{-1,1\,M}} &= 0 \;\Rightarrow\; \Lambda^{2,0} \;=\; \lambda^{2,0}(q,u,v) + \omega^{1,-1\,N}\lambda^{1,1\,N}(q,u,v) \\
\frac{\partial\Lambda^{0,2}}{\partial\omega^{1,-1\,M}} &= 0 \;\Rightarrow\; \Lambda^{0,2} = \lambda^{0,2}(q^{1,1},\omega^{-1,1},u,v)\,.
\end{aligned} \tag{6.7}$$

Since $\hat{H}^{1,1\,M}$, in virtue of (6.6), depends only on $q^{1,1\,N},\omega^{-1,1\,N}$, u, v, while the gauge function $\lambda^{0,2}$ still involves an arbitrary dependence on these variables, the transformation law of $\hat{H}^{1,1}$ in (4.16) shows that in the coefficients of the expansion of $\hat{H}^{1,1\,M}$ in powers of $\omega^{-1,1\,N}$ one may gauge away all the pieces totally symmetric in indices $M,N,...$ (in the case of $n=1$ this corresponds to entirely killing $\hat{H}^{1,1}$). The relevant gauge condition is as follows

$$\omega^{-1,1\,N}\hat{H}^{1,1\,N}_0 \;=\; 0 \;\Rightarrow \tag{6.8}$$

$$\begin{aligned}
\hat{H}^{1,1\,N}_0(q^{1,1},\omega^{-1,1},u,v) &= \int_0^1 d\alpha\,\omega^{-1,1\,M}\left(\frac{\partial\hat{H}^{1,1\,N}(q^{1,1},\alpha\omega^{-1,1},u,v)}{\partial\omega^{-1,1\,M}} - (M\leftrightarrow N)\right) \\
&\equiv \omega^{-1,1\,M}H^{2,0\,[M,N]}(q^{1,1},\omega^{-1,1},u,v)\,,
\end{aligned} \tag{6.9}$$

with α being a real parameter.

Thus, under the dynamical assumption (6.1), (6.2) and before examining consequences of the integrability condition (4.8), the general analytic superspace action of n superfield triples $q^{1,1\,N},\omega^{-1,1\,N},\omega^{1,-1\,N}$ can be written as

$$\begin{aligned}
S_{q,\omega} &= \int \mu^{-2,-2}\{q^{1,1\,M}D^{0,2}\omega^{1,-1\,M} + q^{1,1\,M}D^{2,0}\omega^{-1,1\,M} \\
&\quad +\omega^{-1,1\,M}H^{2,0\,[M,N]}D^{2,0}\omega^{-1,1\,N} + H^{2,2}\}\,.
\end{aligned} \tag{6.10}$$

Here $H^{2,0\,[M,N]}$ is defined in eq. (6.9) and $H^{2,2}$ for the moment is an arbitrary function of the involved superfields and harmonics.

Let us now pass to analysing the integrability condition (4.8). It is still rather difficult to analyse it for the whole action (6.10), so we postpone this task to the future study and in the rest of the paper consider the simplified case with

$$H^{2,0\,[M,N]} = 0 \Rightarrow \hat{H}^{1,1\,N}_0 = 0\,. \tag{6.11}$$

The relevant action respects the reparametrization freedom (4.16) with $H^{3,-1\,N} = H^{-1,3\,N} = \hat{H}^{1,1\,N} = 0$ and

$$
\begin{aligned}
\Lambda^{2,0} &= \lambda^{2,0}(q,u,v) + \omega^{1,-1\,N}\lambda^{1,1\,N}(q,u,v)\,, \\
\Lambda^{0,2} &= \lambda^{0,2}(q,u,v) + \omega^{-1,1\,N}\lambda^{1,1\,N}(q,u,v)\,.
\end{aligned} \tag{6.12}
$$

The equations of motion are

$$
\begin{aligned}
D^{2,0}\omega^{-1,1\,M} + D^{0,2}\omega^{1,-1\,M} &= -\frac{\partial H^{2,2}(q,\omega,u,v)}{\partial q^{1,1\,M}}\,, \\
D^{2,0}q^{1,1\,M} = \frac{\partial H^{2,2}(q,\omega,u,v)}{\partial \omega^{-1,1\,M}}\,,\quad D^{0,2}q^{1,1\,M} &= \frac{\partial H^{2,2}(q,\omega,u,v)}{\partial \omega^{1,-1\,M}}\,.
\end{aligned} \tag{6.13}
$$

To find out the restrictions imposed by the integrability condition (4.8) on the structure of $H^{2,2}$, we proceed in the same way as in the $n=1$ example. The $n \geq 2$ generalization of the conditions (5.10), (5.11) proves to be

$$
\frac{\partial^2 H^{2,2}}{\partial\omega^{-1,1\,N}\partial\omega^{-1,1\,M}} = \frac{\partial^2 H^{2,2}}{\partial\omega^{1,-1\,N}\partial\omega^{1,-1\,M}} = \frac{\partial^2 H^{2,2}}{\partial\omega^{1,-1\,(N}\partial\omega^{-1,1\,M)}} = 0\,, \tag{6.14}
$$

$$
\begin{aligned}
&\left(\partial^{2,0} + \frac{\partial H^{2,2}}{\partial\omega^{-1,1\,N}}\,\frac{\partial}{\partial q^{1,1\,N}} - \frac{1}{2}\,\frac{\partial H^{2,2}}{\partial q^{1,1\,N}}\,\frac{\partial}{\partial\omega^{-1,1\,N}}\right)\frac{\partial H^{2,2}}{\partial\omega^{1,-1\,M}} \\
&-\left(\partial^{0,2} + \frac{\partial H^{2,2}}{\partial\omega^{1,-1\,N}}\,\frac{\partial}{\partial q^{1,1\,N}} - \frac{1}{2}\,\frac{\partial H^{2,2}}{\partial q^{1,1\,N}}\,\frac{\partial}{\partial\omega^{1,-1\,N}}\right)\frac{\partial H^{2,2}}{\partial\omega^{-1,1\,M}} = 0
\end{aligned} \tag{6.15}
$$

Eqs. (6.14) imply

$$
\begin{aligned}
H^{2,2} = \;& h^{2,2}(q,u,v) + \omega^{1,-1\,N}h^{1,3\,N}(q,u,v) + \omega^{-1,1\,N}h^{3,1\,N}(q,u,v) \\
&+ \omega^{-1,1\,N}\omega^{1,-1\,M}h^{2,2\,[N,M]}(q,u,v)\,.
\end{aligned} \tag{6.16}
$$

Notice the presence of the term bilinear in ω's in the general case. Plugging this expression into the constraint (6.15), we finally deduce four independent constraints on the potentials $h^{2,2}$, $h^{1,3\,N}$, $h^{3,1\,N}$ and $h^{2,2\,[N,M]}$

$$
\nabla^{2,0}h^{1,3\,N} - \nabla^{0,2}h^{3,1\,N} + h^{2,2\,[N,M]}\,\frac{\partial h^{2,2}}{\partial q^{1,1\,M}} = 0 \tag{6.17}
$$

$$
\nabla^{2,0}h^{2,2\,[N,M]} - \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,M]} + \frac{\partial h^{3,1\,M}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,N]} = 0 \tag{6.18}
$$

$$
\nabla^{0,2}h^{2,2\,[N,M]} - \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,M]} + \frac{\partial h^{1,3\,M}}{\partial q^{1,1\,T}}\,h^{2,2\,[T,N]} = 0 \tag{6.19}
$$

$$
h^{2,2\,[N,T]}\,\frac{\partial h^{2,2\,[M,L]}}{\partial q^{1,1\,T}} + h^{2,2\,[L,T]}\,\frac{\partial h^{2,2\,[N,M]}}{\partial q^{1,1\,T}} + h^{2,2\,[M,T]}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,T}} = 0 \tag{6.20}
$$

where

$$
\nabla^{2,0} = \partial^{2,0} + h^{3,1\,N}\frac{\partial}{\partial q^{1,1\,N}}\,,\quad \nabla^{0,2} = \partial^{0,2} + h^{1,3\,N}\frac{\partial}{\partial q^{1,1\,N}}\,. \tag{6.21}
$$

It is convenient to rewrite the action and the equations of motion through the newly introduced potentials

$$
\begin{aligned}
S_{q,\omega} &= \int \mu^{-2,-2}\{\, q^{1,1\,M} D^{0,2}\omega^{1,-1\,M} + q^{1,1\,M} D^{2,0}\omega^{-1,1\,M} + \omega^{1,-1\,M} h^{1,3\,M} \\
&\quad + \omega^{-1,1\,M} h^{3,1\,M} + \omega^{-1,1\,M}\omega^{1,-1\,N}\, h^{2,2\,[M,N]} + h^{2,2}\,\}
\end{aligned}
\tag{6.22}
$$

$$
\begin{aligned}
&\left(D^{2,0}\delta^{MN} + \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,M}}\right)\omega^{-1,1\,N} + \left(D^{0,2}\delta^{MN} + \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,M}}\right)\omega^{1,-1\,N} \\
&+\omega^{-1,1\,S}\omega^{1,-1\,T}\,\frac{\partial h^{2,2\,[S,T]}}{\partial q^{1,1\,M}} = -\frac{\partial h^{2,2}}{\partial q^{1,1\,M}}\,, \\
&D^{2,0}q^{1,1\,M} - h^{3,1\,M} + \omega^{1,-1\,N}h^{2,2\,[N,M]} = 0 \\
&D^{0,2}q^{1,1\,M} - h^{1,3\,M} - \omega^{-1,1\,N}h^{2,2\,[N,M]} = 0
\end{aligned}
\tag{6.23}
$$

These action and equations enjoy a rich set of invariances.

One of them is the form-invariance under the restricted reparametrizations (4.16), (6.12). These are realized on the superfields and potentials in the following way

$$
\begin{aligned}
\delta q^{1,1\,N} &= \lambda^{1,1\,N}\,, \quad \delta\omega^{-1,1\,N} = -\frac{\partial\lambda^{0,2}}{\partial q^{1,1\,N}} - \frac{\partial\lambda^{1,1\,M}}{\partial q^{1,1\,N}}\,\omega^{-1,1\,M}\,, \\
\delta\omega^{1,-1\,N} &= -\frac{\partial\lambda^{2,0}}{\partial q^{1,1\,N}} - \frac{\partial\lambda^{1,1\,M}}{\partial q^{1,1\,N}}\,\omega^{1,-1\,M}\,, \\
\delta h^{2,2} &= \nabla^{2,0}\lambda^{0,2} + \nabla^{0,2}\lambda^{2,0}\,, \\
\delta h^{3,1\,M} &= \nabla^{2,0}\lambda^{1,1\,M} + h^{2,2\,[M,N]}\,\frac{\partial\lambda^{2,0}}{\partial q^{1,1\,N}} \\
\delta h^{1,3\,M} &= \nabla^{0,2}\lambda^{1,1\,M} - h^{2,2\,[M,N]}\,\frac{\partial\lambda^{0,2}}{\partial q^{1,1\,N}} \\
\delta h^{2,2\,[N,M]} &= \frac{\partial\lambda^{1,1\,N}}{\partial q^{1,1\,L}}\,h^{2,2\,[L,M]} - \frac{\partial\lambda^{1,1\,M}}{\partial q^{1,1\,L}}\,h^{2,2\,[L,N]}\,.
\end{aligned}
\tag{6.24}
$$

It is a simple exercise to check also the covariance of constraints (6.17) - (6.20) under these reparametrizations.

More interesting is the gauge invariance inherent to the action (6.22). It is a highly nontrivial nonabelian (and in general nonlinear) generalization of the gauge invariance (4.6)

$$
\begin{aligned}
\delta\omega^{1,-1\,M} &= \left(D^{2,0}\delta^{MN} + \frac{\partial h^{3,1\,N}}{\partial q^{1,1\,M}}\right)\sigma^{-1,-1\,N} - \omega^{1,-1\,L}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,M}}\,\sigma^{-1,-1\,N}\,, \\
\delta\omega^{-1,1\,M} &= -\left(D^{0,2}\delta^{MN} + \frac{\partial h^{1,3\,N}}{\partial q^{1,1\,M}}\right)\sigma^{-1,-1\,N} - \omega^{-1,1\,L}\,\frac{\partial h^{2,2\,[L,N]}}{\partial q^{1,1\,M}}\,\sigma^{-1,-1\,N}\,, \\
\delta q^{1,1\,M} &= \sigma^{-1,-1\,N}h^{2,2\,[N,M]}\,.
\end{aligned}
\tag{6.25}
$$

As expected, the action is invariant only provided the integrability conditions (6.17) - (6.20) are obeyed. In general, these gauge transformations close with a field-dependent

Lie bracket parameter. Indeed, commuting two such transformations, say, on $q^{1,1\,N}$, and using the cyclic constraint (6.20), we find

$$\delta_{br} q^{1,1\,M} = \sigma_{br}^{-1,-1\,N} h^{2,2\,[N,M]}\,, \quad \sigma_{br}^{-1,-1\,N} = -\sigma_1^{-1,-1\,L}\sigma_2^{-1,-1\,T}\frac{\partial h^{2,2\,[L,T]}}{\partial q^{1,1\,N}}\,. \tag{6.26}$$

We see that eq. (6.20) guarantees the nonlinear closure of the algebra of gauge transformations (6.25) and so it is a group condition similar to the Jacobi identities. It is curious that the gauge transformations (6.25) with the relation (6.20) are precise biharmonic counterparts of the basic relations of a two-dimensional version of the recently proposed nonlinear extension of Yang-Mills theory [18] (with the evident correspondence $D^{2,0}, D^{0,2} \leftrightarrow \partial_\mu;\ \omega^{1,-1\,M}, -\omega^{-1,1\,M} \leftrightarrow A^M_\mu$).

We point out that it is the presence of the antisymmetric potential $h^{2,2\,[N,M]}$ that makes the considered case nontrivial and, in particular, the gauge invariance (6.25) nonabelian. If $h^{2,2\,[N,M]}$ is vanishing, the invariance gets abelian and the constraints (6.17) - (6.20) except for (6.17) are identically satisfied, while (6.17) can be solved on the pattern of the $n = 1$ case, eqs. (5.16). As a result, the potentials $h^{1,3\,N}$, $h^{3,1\,N}$ can be gauged away using the $\lambda^{1,1\,N}$ freedom (6.24), and we return to the general twisted multiplet action (4.1). On the contrary, with nonvanishing $h^{2,2\,[N,M]}$ eq. (6.17) does not imply $h^{1,3\,N}$, $h^{3,1\,N}$ to be pure gauge. We cannot remove the ω dependence from second and third of eqs. (6.23) by any local field redefinition with preserving harmonic analyticity. Moreover, in contradistinction to the constraints (3.7), these equations are compatible only with using the first equation. So, the obtained system definitely does not admit in general an equivalent description in terms of twisted $(4,4)$ analytic superfields. Hence, the left and right complex structures on the target space can be non-commuting. On the other hand, $q^{1,1\,N}$ can be expressed by first of eqs. (6.23) (at least, iteratively) via ω superfields to yield ω representation of the action similar to (4.4). The main distinguishing feature of this general ω action is the nonlinear and nonabelian nature of the underlying gauge symmetry.

It remains to solve the constraints (6.17) - (6.20). They have a nice geometric form and certainly encode a nontrivial geometry. For the time being we are not aware of their general solution and are able to present only a particular one. Nonetheless, it is very remarkable on its own and seems to share most of characteristic features of the general case.

The solution is given by the following ansatz

$$\begin{aligned} h^{1,3\,N} &= h^{3,1\,N} = 0\,;\ h^{2,2} = h^{2,2}(t,u,v)\,,\ t^{2,2} = q^{1,1\,M}q^{1,1\,M}\,; \\ h^{2,2\,[N,M]} &= b^{1,1} f^{NML} q^{1,1\,L}\,,\ b^{1,1} = b^{ia}u^1_i v^1_a\,,\ b^{ia} = \text{const}\,, \end{aligned} \tag{6.27}$$

where the constants f^{NML} are totally antisymmetric. The constraints (6.17) - (6.19) are identically satisfied with this ansatz, while (6.20) is now none other than the Jacobi identity which tells us that the constants f^{NML} should be the structure constants of some real semi-simple Lie algebra (the minimal possibility is $n = 3$, the corresponding algebra being $so(3)$). Thus the $(4,4)$ sigma models associated with the above solution can be interpreted as a kind of Yang-Mills theories in the harmonic superspace. They provide a natural nonabelian generalization of the twisted multiplet sigma models with the action

(4.1) which, as was mentioned in Sect.4, are analogs of two-dimensional abelian gauge theory. The action (6.22), equations of motion (6.23) and the gauge transformation laws (6.25) specialized to the case (6.27) are as follows

$$\begin{aligned} S^{YM}_{q,\omega} &= \int \mu^{-2,-2}\{ \, q^{1,1\,M}(\, D^{0,2}\omega^{1,-1\,M} + D^{2,0}\omega^{-1,1\,M} + b^{1,1} \, \omega^{-1,1\,L}\omega^{1,-1\,N} f^{LNM} \,) \\ &\quad + h^{2,2}(q,u,v)\} \end{aligned} \tag{6.28}$$

$$\begin{aligned} &D^{2,0}\omega^{-1,1\,N} + D^{0,2}\omega^{1,-1\,N} + b^{1,1} \, \omega^{-1,1\,S}\omega^{1,-1\,T} f^{STN} \equiv B^{1,1\,N} = -\frac{\partial h^{2,2}}{\partial q^{1,1\,N}} \, , \\ &D^{2,0}q^{1,1\,M} + b^{1,1} \, \omega^{1,-1\,N} f^{NML} q^{1,1\,L} \equiv \Delta^{2,0}q^{1,1\,M} = 0 \\ &D^{0,2}q^{1,1\,M} - b^{1,1} \, \omega^{-1,1\,N} f^{NML} q^{1,1\,L} \equiv \Delta^{0,2}q^{1,1\,M} = 0 \end{aligned} \tag{6.29}$$

$$\begin{aligned} &\delta\omega^{1,-1\,M} = \Delta^{2,0}\sigma^{-1,-1\,M} \, , \; \delta\omega^{-1,1\,M} = -\Delta^{0,2}\sigma^{-1,-1\,M} \, , \\ &\delta q^{1,1\,M} = b^{1,1} \, \sigma^{-1,-1\,N} f^{NML} q^{1,1\,L} \, . \end{aligned} \tag{6.30}$$

These formulas make the analogy with two-dimensional nonabelian gauge theory almost literal, especially for $h^{2,2} = q^{1,1\,M}q^{1,1\,M}$. Under this choice

$$q^{1,1\,N} = -\frac{1}{2} \, B^{1,1\,N}$$

by first of eqs. (6.29), then two remaining equations are direct analogs of two-dimensional Yang-Mills equations

$$\Delta^{2,0}B^{1,1\,N} = \Delta^{0,2}B^{1,1\,N} = 0 \, , \tag{6.31}$$

and we recognize (6.28) and (6.29) as the harmonic counterpart of the first order formalism of two-dimensional Yang-Mills theory. In the general case $q^{1,1\,M}$ is a nonlinear function of $B^{1,1\,N}$, however for $B^{1,1\,N}$ one still has the same equations (6.31).

It is instructive to see how the fundamental integrability condition (4.8) is satisfied with the ansatz (6.27):

$$[\Delta^{2,0}, \Delta^{0,2}] \, q^{1,1\,M} = -b^{1,1} \, B^{1,1\,N} f^{NML} q^{1,1\,L} = 2b^{1,1} \, \frac{\partial h^{2,2}}{\partial t^{2,2}} \, q^{1,1\,N} f^{NML} q^{1,1\,L} \equiv 0 \, .$$

We stress once more that in checking this condition in the nonabelian case one necessarily needs first of eqs. (6.29), while in the abelian, twisted multiplet case (4.1) the integrability condition is satisfied without any help from eq. (4.2). As was already mentioned, this property reflects the fact that the class of $(4,4)$ sigma models we have found cannot be described only in terms of twisted $(4,4)$ multiplets (of course, in general the above gauge group has the structure of a direct product which can include abelian factors; the relevant $q^{1,1}$'s satisfy the linear twisted multiplet constraints (3.11)).

An interesting specific feature of this "harmonic Yang-Mills theory" is the presence of the doubly charged "coupling constant" $b^{1,1}$ in all formulas, which is necessary for the correct balance of the harmonic $U(1)$ charges. Since $b^{1,1} = b^{ia}u^1_i v^1_a$, we conclude that in the geometry of the considered class of $(4,4)$ sigma models a very essential role is played by the quartet constant b^{ia}. When $b^{ia} \rightarrow 0$, the nonabelian structure contracts into the abelian one and we reproduce the twisted multiplet action (4.1). If the action (6.28) indeed corresponds to non-commuting left and right complex structures on the target space (this is still to be checked), then the complex structures become commuting in this limit and b^{ia} can be interpreted as a measure of non-commutativity of complex structures.

In forthcoming publications we will present more detailed study of all these issues, including those related to the target space geometry and complex structures, at the component level.

Finally, to avoid a confusing, we point out that the analogy with two-dimensional gauge theories is somewhat formal because there is no any genuine propagating gauge fields among the components of the superfields ω. The only role of the gauge freedom (6.30) seems to consist in ensuring the correct number of physical degrees of freedom in the action (6.28) (after elimination of $q^{1,1\,N}$). It is also unclear, in what sense the transformations (4.6), (6.25), (6.30) could be interpreted as gauging of some rigid ones. Indeed, in the present case the naive definition of the rigid group via imposing the conditions $D^{2,0}\sigma^{-1,-1\,M} = D^{0,2}\sigma^{-1,-1\,M} = 0$ leads to the trivial result $\sigma^{-1,-1\,M} = 0$. The group-theoretical and geometric meaning of this gauge invariance remains to be understood.

7 Summary and outlook

For reader's convenience, we summarize here the basic steps and results of our analysis.

We have proceeded from the dual action (4.1) of $(4,4)$ twisted multiplet in the analytic harmonic $SU(2) \times SU(2)$ superspace and written down its most general conceivable extension (4.11) involving n copies of the triple of analytic harmonic superfields $q^{1,1\,M}, \omega^{1,-1\,M}, \omega^{-1,1\,M}$ $(M = 1, ...n)$. Then, using a freedom with respect to the redefinitions (4.12) and (4.13), we reduced it to the form (4.15). In order to simplify the relevant set of equations of motion (4.20), we assumed that the harmonic derivatives $E^{3,-1\,N} \equiv D^{2,0}\omega^{1,-1\,N}$, $E^{-1,3\,N} \equiv D^{0,2}\omega^{-1,1\,N}$ which do not appear in the free equations of motion are not present in the equations with interaction as well. For $n = 1$ this assumption follows from the basic integrability condition (4.8), while for $n \geq 2$ we took it as a natural postulate. It gives rise to the constraints (6.1), (6.2) which eventually simplify the $q^{1,1}, \omega^{1,-1}, \omega^{-1,1}$ action to the form (6.10). After enforcing further simplifying constraint (6.11) we studied the restrictions imposed on the structure of the reduced action by the integrability condition (4.8). The latter entirely fixes the ω dependence of the superfield Lagrangian, bringing the action to the form (6.22) with the potentials $h^{2,2}$, $h^{1,3\,N}$, $h^{3,1\,N}$ and $h^{2,2\,[N,M]}$ constrained by eqs. (6.17) - (6.20). The action (6.22) reveals new features compared to the twisted multiplet action (4.1) only provided the potential $h^{2,2\,[N,M]}$ is non-vanishing; otherwise (6.22) can be reduced to (4.1) by a field redefinition. For $n = 1$ the potential $h^{2,2\,[N,M]}$ identically vanishes, so the novel class of $(4,4)$ sigma model actions with non-zero $h^{2,2\,[N,M]}$ exists beginning with $n = 2$. Its main novelty is the nonabelian and in general nonlinear gauge invariance (6.25) which substitutes the abelian gauge invariance (4.6) of the twisted multiplets action. These new actions involve an infinite number of auxiliary fields and do not admit a formulation in terms of the twisted $(4,4)$ superfields only. For the latter reason they are good candidates for off-shell description of $(4,4)$ sigma models with non-commuting left and right triplets of complex structures.

There remains a lot of things to be done and questions to be answered. Besides a general problem of inquiring the intrinsic geometric aspects of the action (6.22) and constraints (6.17) - (6.20) as well as revealing their links with the full target space geometry, there are a few more specific (and urgent) ones two of which we will outline here.

An interesting problem is to find out whether the constraints (6.17) - (6.20) admit solutions corresponding to $(4,4)$ supersymmetric WZNW sigma models on the group manifolds from the list given in [19]. Only for the simplest manifolds from this list, namely $[U(1)]^4$ and $SU(2) \times U(1)$, the left and right complex structures commute [14] and only for the related WZNW models there exists a description via twisted multiplets (in the $q^{1,1}$ language, these models are described by the free action (3.9) and the action (3.12), respectively). On higher-dimensional manifolds which are not reduced to products of these two, the left and right structures do not commute. We conjecture that the associated $(4,4)$ WZNW sigma models are described off shell by the actions (6.22) with proper potentials $h^{2,2\,[N,M]}$. The minimal dimension of the superfield triples at which $h^{2,2\,[N,M]}$ exists, $n=2$, amounts to the dimension 8 of the bosonic target. This precisely matches with the dimension of the first nontrivial manifold from the aforementioned list, that of the group $SU(3)$.

One more problem is to prove that the general action of the triples $q^{1,1}, \omega^{1,-1}, \omega^{-1,1}$ in the analytic $SU(2) \times SU(2)$ harmonic superspace indeed yields a most general $(4,4)$ supersymmetric sigma model. Our starting point in this paper was the analytic superfield $q^{1,1}$ which represents a $(4,4)$ twisted multiplet. But this is merely one type of $(4,4)$ twisted multiplet. There exist other types which reveal the same irreducible $(8+8)$ off-shell content, but differ in the $SU(2)_L \times SU(2)_R$ assignment of component fields (see, e.g., [20, 21]). For the time being it is unclear how to simultaneously decribe all these types within the same $SU(2) \times SU(2)$ harmonic superspace. Perhaps, they can be related to each other by a duality transformation (just as all $N=2$ $4D$ matter multiplets are related to the ultimate analytic $q^{(+)}$ multiplet [5]). Alternatively, it may happen that for their consistent description one will need to harmonize the whole $(4,4)$ supersymmetry automorphism group $SO(4)_L \times SO(4)_R$, i.e. to introduce two extra sets of $SU(2)$ harmonic variables, and to consider appropriate analytic superfields in this maximally extended $(4,4)$ harmonic superspace. The relevant actions will be certainly more general than those discussed in this paper. Anyway, in order to distinguish between all these possibilities, one should, before all, understand in full the geometry of the target space and various harmonic extensions of the latter for general $(4,4)$ sigma models, like this has been done for their hyper-Kähler subclass in [7].

Acknowledgements

The major part of this work has been accomplished during the author's visit to The Erwin Schrödinger International Institute for Mathematical Physics in Vienna. The author thanks Professors D. Alekseevsky and P. Michor for their kind hospitality in ESI. He is grateful to Emery Sokatchev for stimulating discussions of the geometry of $(4,4)$ models. A partial support from the Russian Foundation of Fundamental Research, grant 93-02-03821, and the International Science Foundation, grant M9T000, is also acknowledged.

References

[1] E. Kiristis, C. Kounnas and D. Lüst, Int. J. Mod. Phys. **A9** (1994) 1361

[2] S. J. Gates Jr., C. Hull and M. Roček, Nucl. Phys. **B 248** (1984) 157

[3] P.S. Howe and G. Papadopoulos, Class. Quantum Grav. **5** (1988) 1647

[4] L. Alvarez-Gaumé and D.Z. Freedman, Commun. Math. Phys. **80** (1981) 443;
J. Bagger and E. Witten, Nucl. Phys. **B 222** (1983) 1

[5] A. Galperin, E. Ivanov and V. Ogievetsky, Nucl. Phys. **B282** (1987) 74

[6] J.A. Bagger, A.S. Galperin, E.A. Ivanov and V.I. Ogievetsky, Nucl. Phys **B303** (1988) 522

[7] A.S. Galperin, E.A. Ivanov, V.I. Ogievetsky and E. Sokatchev, Ann. Phys. **185** (1988) 22

[8] A. Galperin, E. Ivanov, V. Ogievetsky and E. Sokatchev, Jetp. Lett. **40** (1984) 912

[9] A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1** (1984) 469

[10] F. Delduc, S. Kalitzin and E. Sokatchev, Class. Quantum Grav. **7** (1990) 1567

[11] E. Ivanov and A. Sutulin, Nucl. Phys. **B 432** (1994) 246

[12] T. Buscher, U. Lindström and M. Roŏcek, Phys. Lett. **B 202** (1988) 202

[13] E. A. Ivanov and S. O. Krivonos, J. Phys. A: Math. and Gen. **17** (1984) L671

[14] M. Roček, K. Schoutens and A. Sevrin, Phys. Lett. **B 265** (1991) 303

[15] U. Lindström, I.T. Ivanov and M. Roček, Phys. Lett. **B 328** (1994) 49

[16] F. Delduc and E. Sokatchev, Int. J. Mod. Phys. **B 8** (1994) 3725

[17] B. Zumino. Phys. Lett. **B 87** (1979) 203

[18] N. Ikeda, Ann. Phys. **235** (1994) 435

[19] Ph. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, Phys. Lett. **B 206** (1988) 71; Nucl. Phys. **B 308** (1988) 662

[20] O. Gorovoy and E. Ivanov, Nucl. Phys. **B 381** (1992) 394

[21] S. James Gates Jr., Phys. Lett. **B 338** (1994) 31