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On Nurowski's Conformal Structure Associated to a Generic Rank Two Distribution in Dimension Five

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Abstract

For a generic distribution of rank two on a manifold M of dimension five, we introduce the notion of a generalized contact form. To such a form we associate a generalized Reeb field and a partial connection. From these data, we explicitly constructed a pseudo–Riemannian metric on M of split signature. We prove that a change of the generalized contact form only leads to a conformal rescaling of this metric, so the corresponding conformal class is intrinsic to the distribution.

In the second part of the article, we relate this conformal class to the canonical Cartan connection associated to the distribution. This is used to prove that it coincides with the conformal class constructed by Nurowski.

1 Introduction

The study of generic rank two subbundles in the tangent bundles of five–dimensional manifolds goes back to Elie Cartan's famous "five variables paper" [5] from 1910. This paper is remarkable in several respects. First, by constructing a canonical Cartan connection associated to such distributions, Cartan showed that they have non–trivial local invariants. Second, for the simplest instance of such a distribution, Cartan showed that the infinitesimal symmetries form an exceptional simple Lie algebra of type G_2. This was the first instance of an exceptional simple Lie algebra showing up "in real life".

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In modern terminology, the homogeneous model of generic rank two distributions in dimension five is the quotient of the split real form of an exceptional Lie group G of type G_2 by a maximal parabolic subgroup $P \subset G$. An explicit description of this homogeneous space and its generic rank two distribution can be found in [10]. Cartan's construction associates to an arbitrary generic rank two distribution on a five–manifold M a Cartan geometry of type (G, P).

In pioneering work culminating in [12], N. Tanaka showed that Cartan geometries modelled on the quotient of a semisimple group by a parabolic subgroup are (under small restrictions, which have later been eliminated) equivalent to simpler underlying geometric structures. These geometric structures have been intensively studied during the last years under the name "parabolic geometries". The description of the structures equivalent to parabolic geometries is best phrased in terms of filtered manifolds, an up to date version can be found in [3]. In particular, for many geometries this underlying structure is only a filtration of the tangent bundle of a manifold with certain properties, see [9]. In the special case of the split real form of G_2 with the appropriate parabolic subgroup, this recovers Cartan's result.

For generic rank two distributions in dimension five, progress was made recently by P. Nurowski in [6]. Using the canonical Cartan connection associated to a distribution $\mathcal{H}$ on M, Nurowski constructed a canonical conformal structure of signature $(2, 3)$ on M. Rather than a rank two distribution, Nurowski's starting point was an underdetermined system of ODEs of certain type, which can be equivalently described by a generic distribution. The system of ODEs is determined by a single smooth function, and in his article Nurowski gives an impressive (and frightening) formula for a metric in the conformal class in terms of this function.

It was soon realized, see [3], that Nurowski's construction can be interpreted as an analog of the Fefferman construction, which to a non–degenerate CR–manifold of hypersurface type associates a canonical conformal structure on the total space of a circle bundle. The main point in this analogy is not that a conformal structure occurs in both cases, but the interpretation of the construction in terms of Cartan connections. In terms of this analogy, Nurowski's construction corresponds to the version of the Fefferman construction based on Cartan connections, which was developed in [1].

The aim of this paper is to present a description of Nurowski's conformal class which is analogous to J. Lee's description of the Fefferman construction, see [7,8]. In section 2, we introduce the notion of a generalized contact form for a generic rank two distribution $\mathcal{H}$ on a five–manifold M. Starting from such a form α we explicitly construct a pseudo–Riemannian metric g_α of signature $(2, 3)$ on M. Then we prove that another choice of generalized contact form leads to a conformally related metric, so the conformal class $[g_\alpha]$ is intrinsic to the distribution $\mathcal{H}$. The formula for g_α in terms of data derived from α is rather simple and explicit, and requires no knowledge of the canonical Cartan connection.

In section 3 we show that our conformal structure coincides with the one constructed by Nurowski. This is based on the theory of Weyl–structures for parabolic geometries, which interprets the canonical Cartan connection in terms of underlying data. This also explains how the formula for the metric g_α was actually found. The developments in section 3 are also interesting from the point of view of the general theory of parabolic geometries, since this is the first time that essential parts of a Weyl structure are explicitly computed for one of the more involved parabolic geometries. This article is based on results obtained during the work on the second author's PhD thesis, which will contain a complete description of this Weyl structure.

2 A canonical conformal structure

In this section, we first introduce the notion of a generalized contact form for a generic rank two distribution $\mathcal{H}$ on a smooth manifold M of dimension five. Given a generalized contact form α, we construct a pseudo–Riemannian metric g_α on M. Then we show that the metrics associated to different generalized contact forms are always conformal to each other. Hence the conformal class $[g_\alpha]$ on M depends only on the distribution $\mathcal{H}$.

2.1 Generic rank 2 distributions in dimension 5

We start by collecting some facts on such distributions and fixing some notation. Recall that a rank 2 distribution $\mathcal{H}$ on a 5–manifold M is called *generic* if the values of linear combinations of iterated Lie brackets of at most three sections of $\mathcal{H}$ in each $x \in M$ span the tangent space T_xM. In particular, sections of $\mathcal{H}$ and their Lie brackets have to span a rank 3 subbundle $[\mathcal{H},\mathcal{H}]$. Defining $T^{-1}M = \mathcal{H}$ and $T^{-2}M = [\mathcal{H},\mathcal{H}]$, we obtain a filtration

$$T^{-1}M \subset T^{-2}M \subset TM$$

of the tangent bundle by smooth subbundles. We use the convention that $T^iM = 0$ for $i \geq 0$ and $T^iM = TM$ for $i \leq -3$. Then the filtration is compatible with the Lie bracket of vector fields in the sense that for $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$ we get $[\xi,\eta] \in \Gamma(T^{i+j}M)$. For $i = -1,-2,-3$ we define $\mathrm{gr}_i(M) = T^iM/T^{i+1}M$ and then $\mathrm{gr}(M) = \oplus_{i=-3}^{-1}\mathrm{gr}_i(TM)$ is the associated graded vector bundle to the tangent bundle. For $i = -2,-3$, we denote by $q_i : T^iM \to \mathrm{gr}_i(TM)$ the natural quotient map.

The Lie bracket of vector fields induces a skew symmetric bilinear bundle map $\{\ ,\ \} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$, called the Levi bracket, which is homogeneous of degree zero. Note that the two nontrivial components

$$\{\ ,\ \} : \Lambda^2 T^{-1}M \to \mathrm{gr}_{-2}(TM); \quad \{\xi(x), \eta(x)\} := q_{-2}([\xi, \eta](x))$$

and

$$\{\ ,\ \} : T^{-1}M \otimes \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_{-3}(TM); \quad \{\xi(x), q_{-2}(\zeta(x))\} := q_{-3}([\xi, \zeta](x))$$

are both isomorphisms of vector bundles.

The filtration of TM dualizes to filtration of the cotangent bundle T^*M and we consider the associated graded bundle

$$\mathrm{gr}(T^*M) = \oplus_{i=1}^{3} \mathrm{gr}_i(T^*M).$$

By construction, $\mathrm{gr}_i(T^*M) = (\mathrm{gr}_{-i}(TM))^*$.

2.2 Generalized contact forms and Reeb fields

Definition 1 *Let $\mathcal{H} \subset TM$ be a generic rank two distribution on a 5–manifold M. A generalized contact form for $\mathcal{H}$ is a smooth section α of the bundle $(T^{-2}M)^*$ such that for each $x \in M$ the kernel of the linear map $\alpha(x) : T_x^{-2}M \to \mathbb{R}$ is $T_x^{-1}M = \mathcal{H}_x$.*

By definition, a generalized contact form is a partially defined one form. We will see below, that α can be canonically extended to a true one form on M. Note the the condition on the kernel implies that α is nowhere vanishing.

Given a generalized contact form α for $\mathcal{H}$, we next want to introduce an analog of the Reeb vector field. Consider a local section r of $T^{-2}M$ which is transversal to $T^{-1}M$, i.e. such that $\varphi := q_{-2}(r) \in \Gamma(\mathrm{gr}_{-2}(TM))$ is nowhere vanishing. Then for $\xi, \eta \in \Gamma(T^{-1}M)$, there is a unique smooth section $\nabla_\xi \eta \in \Gamma(T^{-1}M)$ such that

$$\{\nabla_\xi \eta, \varphi\} = q_{-3}([\xi, [\eta, r]]). \tag{1}$$

By construction, the operator $\nabla : \Gamma(T^{-1}M) \times \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$ is bilinear over $\mathbb{R}$. Using that $q_{-3}(\xi) = 0$ and $q_{-3}([\eta, r]) = \{\eta, \varphi\}$ one immediately concludes from the defining equation that $\nabla_{f\xi}\eta = f\nabla_\xi \eta$ and $\nabla_\xi f\eta = (\xi \cdot f)\eta + f\nabla_\xi \eta$ for $f \in C^\infty(M, \mathbb{R})$. This means that ∇ defines a *partial connection* on $T^{-1}M$.

Via the isomorphism $\Lambda^2 T^{-1}M \cong \mathrm{gr}_{-2}(TM)$, we obtain an induced partial connection on the line bundle $\mathrm{gr}_{-2}(TM)$. This is an operator $\nabla : \Gamma(T^{-1}M) \times \Gamma(\mathrm{gr}_{-2}(TM)) \to \Gamma(\mathrm{gr}_{-2}(TM))$, which is linear over smooth functions in the first variable and satisfies a Leibniz rule in the second variable. The induced connection is characterized by

$$\nabla_\gamma\{\xi,\eta\} = \{\nabla_\gamma\xi,\eta\} + \{\xi,\nabla_\gamma\eta\}, \tag{2}$$

for $\xi,\eta,\gamma \in \Gamma(T^{-1}M)$.

This immediately singles out a preferred class of fields r as above, namely those, for which the nowhere vanishing section $\varphi = q_{-2}(r)$ is parallel for the induced partial connection. We can completely describe all such fields:

Proposition 2 *Let φ be a local non–vanishing smooth section of* $\mathrm{gr}_{-2}(TM)$*. Then there is a unique smooth section $r \in \Gamma(T^{-2}M)$ such that $q_{-2}(r) = \varphi$ and such that φ is parallel for the partial connection determined by r.*

PROOF. Since $\mathrm{gr}_{-2}(TM)$ is a quotient bundle of $T^{-2}M$ we find a local section $r_0 \in \Gamma(T^{-2}M)$ such that $q_{-2}(r_0) = \varphi$. Any other section with this property is of the form $r_0 + \delta$ for some $\delta \in \Gamma(T^{-1}M)$. We have to compute how the choice of δ influences the partial connection. We will denote partial connections associated to r_0 by ∇ and those associated to $r_0+\delta$ by ∇^δ. We first compute what happens on $T^{-1}M$. From the defining equation (1), we immediately get

$$\{\nabla^\delta_\gamma\xi,\varphi\} = \{\nabla_\gamma\xi,\varphi\} + \{\gamma,\{\xi,\delta\}\},$$

for all $\xi,\gamma \in \Gamma(T^{-1}M)$. Now we can define a skew symmetric bilinear map $a : T^{-1}M \times_M T^{-1}M \to \mathbb{R}$ by $\{\xi,\gamma\} = a(\xi,\gamma)\varphi$ for all $\xi,\gamma \in \Gamma(T^{-1}M)$. Then the above equation shows that $\nabla^\delta_\gamma\xi = \nabla_\gamma\xi + a(\xi,\delta)\gamma$. Now choose local smooth sections $\xi,\eta \in \Gamma(T^{-1}M)$ such that $\varphi = \{\xi,\eta\}$. Using the defining equation (2) we immediately conclude that

$$\nabla^\delta_\gamma\{\xi,\eta\} = \nabla_\gamma\{\xi,\eta\} + a(\xi,\delta)\{\gamma,\eta\} + a(\eta,\delta)\{\xi,\gamma\}.$$

For fixed ξ, η, and δ, the last two terms in the right hand side define a bundle map $T^{-1}M \to \mathrm{gr}_{-2}(TM)$. For $\gamma = \xi$, we obtain $a(\xi,\delta)\{\xi,\eta\} = a(\xi,\delta)\varphi = \{\xi,\delta\}$. Likewise, inserting $\gamma = \eta$, we obtain $\{\eta,\delta\}$. Since $\{\xi,\eta\} = \varphi$ is nowhere vanishing, the two fields form a local frame for $T^{-1}M$. Hence the bundle map reduces to $\{\gamma,\delta\}$ for any γ, and locally $\nabla^\delta_\gamma\varphi = \nabla_\gamma\varphi + \{\gamma,\delta\}$, which then has to hold globally. But $\gamma \mapsto -\nabla_\gamma\varphi$ is a bundle map $T^{-1}M \to \mathrm{gr}_{-2}(TM)$, so there is a unique section $\delta \in \Gamma(T^{-1}M)$ such that $-\nabla_\gamma\varphi = \{\gamma,\delta\}$ for all $\gamma \in \Gamma(T^{-1}M)$. $\square$

The proposition immediately implies that for a generalized contact form α for $\mathcal{H}$, there is a unique section $r \in \Gamma(T^{-2}M)$ such that $\alpha(r) = 1$ and such that $q_{-2}(r)$ is parallel for the partial connection determined by r. This section is called the *generalized Reeb field* associated to α.

Notice that a generalized contact form α can be equivalently viewed as a section of the bundle $\mathrm{gr}_2(T^*M) = (\mathrm{gr}_{-2}(TM))^*$. Any partial connection on $\mathrm{gr}_{-2}(TM)$ induces a partial connection on the dual bundle. Then a section $r \in \Gamma(T^{-2}M)$ such that $\alpha(r) = 1$ is the generalized Reeb field if and only if $\alpha \in \Gamma(\mathrm{gr}_2(T^*M))$ is parallel for the partial connection induced by r.

2.3 The canonical extension of a generalized contact form

Using the generalized Reeb field we can next construct a canonical extension of a generalized contact form to a true one–form on M.

Proposition 3 *Let α be a generalized contact form with generalized Reeb field r. Then there is a unique one form $\tilde{\alpha} \in \Omega^1(M)$ extending α such that $i_r d\tilde{\alpha}|_{T^{-1}M} = 0$.*

PROOF. For a vector field $\zeta \in \mathfrak{X}(M)$, we can write $q_{-3}(\zeta) \in \Gamma(\mathrm{gr}_{-3}(TM))$ as $\{r, \zeta_1\} = q_{-3}([r, \zeta_1])$ for a unique $\zeta_1 \in \Gamma(T^{-1}M)$. But this exactly means that $\zeta - [r, \zeta_1] \in \Gamma(T^{-2}M)$. Hence for any extension $\tilde{\alpha}$ of α, we obtain

$$\tilde{\alpha}(\zeta) = \tilde{\alpha}([r, \zeta_1]) + \alpha(\zeta - [r, \zeta_1]) = -i_r d\tilde{\alpha}(\zeta_1) + \alpha(\zeta - [r, \zeta_1]). \qquad \square$$

In the sequel, we will use the same symbol α to denote a generalized contact form and its canonical extension to a one form on M.

2.4 The projection associated to a generalized contact form

The objects we have associated to a generalized contact form α so far amount to a partial splitting of the filtration of the tangent bundle. Denoting by r the generalized Reeb field associated to α, we obtain a projection from $T^{-2}M$ onto the subbundle $T^{-1}M$ by $\zeta \mapsto \zeta - \alpha(\zeta)r$. Likewise, the canonical extension $\alpha \in \Omega^1(M)$ of the generalized contact form allows one to project an arbitrary tangent vector onto a multiple of r. To construct a metric, we need a complete splitting of the filtration. Hence we need a projection $\pi_{-1} : TM \to T^{-1}M$, which extends the one on $T^{-2}M$ from above.

As we have noted in the proof of Proposition 3, given a vector field $\zeta \in \mathfrak{X}(M)$, we find a unique $\zeta_1 \in \Gamma(T^{-1}M)$ such that $\zeta - [r, \zeta_1] \in \Gamma(T^{-2}M)$. To decompose in a slightly finer way, observe that this implies that $\zeta_2 := \zeta - [r, \zeta_1] - \alpha(\zeta)r \in \Gamma(T^{-1}M)$, so we can write

$$\zeta = [r, \zeta_1] + \alpha(\zeta)r + \zeta_2$$

for uniquely determined $\zeta_1, \zeta_2 \in \Gamma(T^{-1}M)$. In particular, for any projection π_{-1} as above, we get $\pi_{-1}(\zeta) = \pi_{-1}([r, \zeta_1]) + \zeta_2$. Such a projection is therefore equivalent to the operator $A : \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$ defined by $A(\xi) := \pi_{-1}([r, \xi])$. If we want π_1 to be linear over smooth functions, we have to require that $A(f\xi) = fA(\xi) + (r\cdot f)\xi$ for all $f \in C^\infty(M, \mathbb{R})$.

There are two candidates for such an operator. First, define $\Phi : \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$ by

$$\nabla_\xi \nabla_\eta \gamma - \nabla_\eta \nabla_\xi \gamma - \nabla_{[\xi,\eta]-\alpha([\xi,\eta])r}\gamma = \alpha([\xi, \eta])\Phi(\gamma), \tag{3}$$

for $\xi, \eta, \gamma \in \Gamma(T^{-1}M)$. This makes sense since the left hand side of (3) is alternating in ξ and η and hence depends only on $\alpha([\xi, \eta])$. Note further that $[\xi, \eta] - \alpha([\xi, \eta])r$ is the projection of $[\xi, \eta] \in \Gamma(T^{-2}M)$ to a section of $T^{-1}M$. Extending ∇ to a linear connection on TM and denoting by R the curvature of this extension, the left hand side of (3) (which is evidently independent of the extension) can be written as $R(\xi, \eta)(\gamma) + \alpha([\xi, \eta])\nabla_r\gamma$. This shows that $\Phi(f\gamma) = f\Phi(\gamma) + (r\cdot f)\gamma$.

Second, $\gamma \mapsto q_{-3}([r, [\gamma, r]])$ defines an operator $\Gamma(T^{-1}M) \to \Gamma(T^{-3}M)$. Since $q_{-2}(r)$ is nowhere vanishing, there is a unique operator $\Psi : \Gamma(T^{-1}M) \to \Gamma(T^{-1}M)$ such that

$$\{\Psi(\gamma), q_{-2}(r)\} = \tfrac{1}{2}q_{-3}([r, [\gamma, r]]). \tag{4}$$

From this definition, $\Psi(f\gamma) = f\Psi(\gamma) + (r\cdot f)\gamma$ follows easily.

Any convex combination of the operators Φ and Ψ also has the right behavior under multiplication by smooth functions. To define our projection $\pi_{-1} : TM \to T^{-1}M$ we use the combination $\frac{-2}{5}\Phi + \frac{7}{5}\Psi$, so we define

$$\pi_{-1}(\zeta) := \tfrac{-2}{5}\Phi(\zeta_1) + \tfrac{7}{5}\Psi(\zeta_1) + \zeta_2, \tag{5}$$

where ζ_1 is the unique vector field in $\Gamma(T^{-1}M)$ such that $q_{-3}(\zeta) = q_{-3}([r, \zeta_1])$ and $\zeta_2 = \zeta - [r, \zeta_1] - \alpha(\zeta)r$. The motivation for the choice of factors will become clear from the following computations and from section 3.

We can now define the pseudo–Riemannian metric associated to α. Let ζ, ζ' be tangent vectors on M. Using the components ζ_1 and ζ_2 as above and likewise for ζ', we define

$$g_\alpha(\zeta, \zeta') := d\alpha(\zeta_1, \pi_{-1}(\zeta')) - \tfrac{4}{3}\alpha(\zeta)\alpha(\zeta') + d\alpha(\zeta'_1, \pi_{-1}(\zeta)). \tag{6}$$

Proposition 4 *For any generalized contact form α, the map g_α defined above is a pseudo–Riemannian metric of signature $(2, 3)$ on M.*

PROOF. Evidently, g_α is a smooth, symmetric bilinear bundle map. For

$\zeta \in T^{-1}M$ we have $\zeta_1 = \alpha(\zeta) = 0$ and $\pi_{-1}(\zeta) = \zeta$. This shows that the rank two subbundle $T^{-1}M \subset TM$ is isotropic for g_α. Likewise, $\ker(\pi_{-1}) \cap \ker(\alpha)$ is a rank two subbundle of TM, which is transversal to $T^{-1}M$ and isotropic for g_α. Taking ζ' in $\ker(\pi_{-1}) \cap \ker(\alpha)$ and $\zeta \in T^{-1}M$, we by definition get $g_\alpha(\zeta, \zeta') = d\alpha(\zeta'_1, \zeta)$. This vanishes for all ζ if and only if $\zeta'_1 = 0$ and hence $\zeta' \in T^{-2}M$. But then $\alpha(\zeta') = 0$ implies $\zeta' \in T^{-1}M$ and $\pi_{-1}(\zeta') = 0$ shows $\zeta' = 0$. Hence g_α induces a non–degenerate pairing between the two isotropic subbundles. Since r spans a rank one subbundle transversal to the two isotropic subbundles on which g_α is negative definite, the result follows. $\square$

2.5 The dependence on the generalized contact form

To analyze how g_α depends on α, we have to study the dependence of the ingredients used in the construction. Given a generalized contact form α, any other generalized contact form is obtained by multiplying α by a nowhere vanishing smooth function. We will following the convention that we denote the changed generalized contact form by $\hat{\alpha}$ and indicate all quantities referring to the new form by a hat.

Let us first check what happens if we replace $\alpha \in \Gamma((T^{-2}M)^*)$ by $\hat{\alpha} := -\alpha$. If in the defining equation (1) we replace r by $-r$ (and hence φ by $-\varphi$) we obtain the same partial connection ∇. This shows that $\hat{r} = -r$ and $\hat{\nabla} = \nabla$. Then it follows that $\hat{\alpha} = -\alpha$ also holds for the extensions to one forms on M. Decomposing $\zeta \in \mathfrak{X}(M)$ as introduced in 2.4, we obtain $\hat{\zeta}_1 = -\zeta_1$ and $\hat{\zeta}_2 = \zeta_2$. In the defining equation (3), the left hand side remains unchanged while in the right hand side α has to be replaced by $-\alpha$, so $\hat{\Phi} = -\Phi$. Similarly, the definition in (4) shows that $\hat{\Psi} = -\Psi$. Hence we obtain $\hat{\pi}_{-1} = \pi_{-1}$. Putting all these results together, we conclude that $g_{-\alpha} = g_\alpha$ from the definition of the metric.

Hence it suffices to analyze the behavior of g_α under rescaling α by a positive smooth function, which we write as e^f for $f \in C^\infty(M, \mathbb{R})$.

Lemma 5 *Let $\alpha \in \Gamma((T^{-2}M)^*)$ be a generalized contact form, consider a smooth function $f \in C^\infty(M, \mathbb{R})$, and the generalized contact form $\hat{\alpha} = e^f\alpha \in \Gamma((T^{-2}M)^*)$. Then we have:*

(i) The Reeb vector field $\hat{r}$ associated to $\hat{\alpha}$ is given by $\hat{r} = e^{-f}r + \delta$, where $\delta \in \Gamma(T^{-1}M)$ is the unique vector field such that $\{\gamma, \delta\} = 4e^{-f}df(\gamma)q_{-2}(r)$ for all $\gamma \in \Gamma(T^{-1}M)$.

(ii) The canonical extensions of the two generalized contact forms to one forms on M are related by $\hat{\alpha}(\zeta) = e^f\alpha(\zeta) + 3df(\zeta_1)$, where $\zeta_1 \in \Gamma(T^{-1}M)$ is characterized by $q_{-3}(\zeta) = \{\varphi, \zeta_1\}$.

PROOF. (i) Put $r_0 = e^{-f}r$. Then $\hat{\alpha}(r_0) = 1$, and we can compute $\hat{r}$ following the proof of Proposition 2. We first have to compute the partial connection $\tilde{\nabla}$ induced by r_0. Putting $\varphi = q_{-2}(r)$ and using that for $\gamma \in \Gamma(T^{-1}M)$ we have $\gamma{\cdot}e^{-f} = -e^{-f}df(\gamma)$ one easily computes that

$$q_{-3}([\gamma,[\xi,r_0]]) = e^{-f}\Big(q_{-3}([\gamma,[\xi,r]]) - df(\gamma)\{\xi,\varphi\} - df(\xi)\{\gamma,\varphi\}\Big).$$

Via the defining equation (1) in 2.2 and $q_{-2}(r_0) = e^{-f}\varphi$, this shows that the partial connection $\tilde{\nabla}$ on $T^{-1}M$ determined by r_0 is given by

$$\tilde{\nabla}_\gamma\xi = \nabla_\gamma\xi - df(\gamma)\xi - df(\xi)\gamma.$$

For the induced connection on $\mathrm{gr}_{-2}(TM)$ we therefore get

$$\tilde{\nabla}_\gamma\{\xi,\eta\} = \nabla_\gamma\{\xi,\eta\} - 2df(\gamma)\{\xi,\eta\} - df(\xi)\{\gamma,\eta\} - df(\eta)\{\xi,\gamma\}.$$

Fixing ξ and η, the last two terms in the right hand side define a bundle map $T^{-1}M \to \mathrm{gr}_{-2}(TM)$. For $\gamma = \xi$ and $\gamma = \eta$, this bundle map coincides with $-df(\gamma)\{\xi,\eta\}$. To have $\{\xi,\eta\} \neq 0$, the two sections have to form a frame of $T^{-1}M$, so we conclude that the induced connection on $\mathrm{gr}_{-2}(TM)$ is characterized by $\tilde{\nabla}_\gamma\varphi = \nabla_\gamma\varphi - 3df(\gamma)\varphi$. Using $\nabla_\gamma\varphi = 0$, this implies

$$\tilde{\nabla}_\gamma e^{-f}\varphi = -4df(\gamma)e^{-f}\varphi,$$

and (i) follows from the proof of Proposition 2.

(ii) From (i) we know that $\hat{r} = e^{-f}r + \delta$. For $\gamma \in \Gamma(T^{-1}M)$, we thus have

$$[\hat{r},\gamma] = e^{-f}[r,\gamma] + e^{-f}df(\gamma)r + [\delta,\gamma]. \tag{7}$$

Since $\alpha([r,\gamma]) = 0$, the one form $e^f\alpha \in \Omega^1(M)$ maps this to $df(\gamma) - e^f\alpha([\gamma,\delta])$. By definition, $\{\gamma,\delta\} = 4e^{-f}df(\gamma)\varphi$, and hence $\alpha([\gamma,\delta]) = 4e^{-f}df(\gamma)$. Thus we obtain

$$e^f\alpha([\hat{r},\gamma]) = -3df(\gamma).$$

But this exactly means that the claimed formula for $\hat{\alpha}$ defines a form which annihilates each field of the form $[\hat{r},\gamma]$ for $\gamma \in \Gamma(T^{-1}M)$. Since it obviously is an extension of $\hat{\alpha} \in \Gamma((T^{-2}M)^*)$,this completes the proof. □

Remark 6 *Note that the transformation laws in the lemma both depend only on $df|_{T^{-1}M}$, i.e. on the class of df in $\Gamma(\mathrm{gr}_1(T^*M))$.*

It remains to analyze the dependence of π_{-1} on the generalized contact form.

Lemma 7 *Let $\alpha \in \Gamma((T^{-2}M)^*)$ be a generalized contact form and consider a rescaling $\hat{\alpha} = e^f\alpha$ for $f \in C^\infty(M,\mathbb{R})$. Let r be the generalized Reeb field for α*

and put $\varphi = q_{-2}(r)$. Then the projection $\hat{\pi}_1$ determined by $\hat{\alpha}$ is given by

$$\hat{\pi}_{-1}(\zeta) = \pi_{-1}(\zeta) - \tfrac{3}{2}df(r)\zeta_1 - \tfrac{3}{2}e^f df(\zeta_1)\delta - e^f\alpha(\zeta)\delta,$$

where $\zeta_1 \in \Gamma(T^{-1}M)$ is the unique vector field such that $q_{-3}(\zeta) = \{\varphi, \zeta_1\}$, and $\delta \in \Gamma(T^{-1}M)$ is characterized by $\{\gamma, \delta\} = 4e^{-f}df(\gamma)q_{-2}(r)$ for all $\gamma \in \Gamma(T^{-1}M)$.

PROOF. Let us first compare the decompositions of $\zeta \in \mathfrak{X}(M)$ from 2.4 with respect to the two generalized contact forms. For α, this reads as

$$\zeta = [r, \zeta_1] + \alpha(\zeta)r + \zeta_2,$$

where $\zeta_1 \in \Gamma(T^{-1}M)$ is characterized by $q_{-3}(\zeta) = \{\varphi, \zeta_1\}$, and then $\zeta_2 \in T^{-1}M$ is defined by the equation. Since $\hat{\varphi} = e^{-f}\varphi$ we see that $\widehat{\zeta_1} = e^f\zeta_1$. From Lemma 5, we know that $\hat{r} = e^{-f}r + \delta$. Note that by definition of δ, we get $df(\delta) = 0$. Using this and formula (7) from the proof of Lemma 5, we obtain

$$[\hat{r}, \hat{\zeta}_1] = [r, \zeta_1] + df(r)\zeta_1 + df(\zeta_1)r + e^f[\delta, \zeta_1]. \tag{8}$$

Further, the formulae from Lemma 5 show that

$$\hat{\alpha}(\zeta)\hat{r} = (\alpha(\zeta) + 3df(\zeta_1))(r + e^f\delta).$$

Putting the results obtained so far together, we get

$$\widehat{\zeta_2} - \zeta_2 = -e^f\alpha(\zeta)\delta - df(r)\zeta_1 - 3e^f df(\zeta_1)\delta - 4df(\zeta_1)r - e^f[\delta, \zeta_1]. \tag{9}$$

Using (8), we next compute

$$\begin{aligned} q_{-3}([\hat{r}, [\hat{\zeta}_1, \hat{r}]]) =& e^{-f}q_{-3}([r, [\zeta_1, r]]) + 2q_{-3}([\delta, [\zeta_1, r]]) - q_{-3}([\zeta_1, [\delta, r]]) \\ & - e^f\{\delta, \{\delta, \zeta_1\}\} - e^{-f}df(r)\{\varphi, \zeta_1\} - df(\zeta_1)\{\delta, \varphi\}. \end{aligned}$$

The second and third term can be rewritten in terms of the partial connection ∇ associated to r using the definition in formula (1) in 2.2. On the other hand, $\{\delta, \zeta_1\} = -4e^{-f}df(\zeta_1)\varphi$ by definition. In view of equation (4) from 2.4 this shows that

$$\hat{\Psi}(\hat{\zeta}_1) - \Psi(\zeta_1) = e^f\nabla_\delta\zeta_1 - \tfrac{1}{2}e^f\nabla_{\zeta_1}\delta + \tfrac{3}{2}e^f df(\zeta_1)\delta + \tfrac{1}{2}df(r)\zeta_1. \tag{10}$$

To compute $\hat{\Phi}(\hat{\zeta}_1)$ we have to analyze the relation between the partial connections associated to α and $\hat{\alpha}$. Using (7) from the proof of Lemma 5, one computes $q_{-3}([\xi, [\eta, \hat{r}]])$, and via the defining equation (1) from 2.2 this shows that

$$\hat{\nabla}_\xi\eta = \nabla_\xi\eta - df(\xi)\eta + 3df(\eta)\xi.$$

Using this, one easily verifies directly that for $\xi, \eta, \gamma \in \Gamma(T^{-1}M)$, the difference $\hat{\nabla}_\xi \hat{\nabla}_\eta \gamma - \nabla_\xi \nabla_\eta \gamma$ can, up to terms symmetric in ξ and η, be expressed as

$$\begin{aligned} 3df(\nabla_\eta \gamma)\xi - (\xi \cdot df(\eta))\gamma + 3(\xi \cdot df(\gamma))\eta \\ + 3df(\gamma)\nabla_\xi \eta + 9df(\gamma)df(\eta)\xi \end{aligned} \tag{11}$$

Now $[\xi, \eta] \in \Gamma(T^{-2}M)$, which implies that

$$\hat{\alpha}([\xi, \eta])\hat{r} = \alpha([\xi, \eta])(r + e^f \delta).$$

Using this and $df(\delta) = 0$, one shows that $\hat{\nabla}_{[\xi,\eta]-\hat{\alpha}([\xi,\eta])\hat{r}}\gamma - \nabla_{[\xi,\eta]-\alpha([\xi,\eta])r}\gamma$ is given by

$$\begin{aligned} &- df([\xi, \eta] - \alpha([\xi, \eta])r)\gamma + 3df(\gamma)([\xi, \eta] - \alpha([\xi, \eta])r) \\ &- \hat{\alpha}([\xi, \eta]) \left(\nabla_\delta \gamma + 3df(\gamma)\delta\right). \end{aligned} \tag{12}$$

Now we need a few identities. First, expanding $0 = ddf(\xi, \eta)$, we obtain

$$\xi \cdot df(\eta) - \eta \cdot df(\xi) - df([\xi, \eta] - \alpha([\xi, \eta])r) = \alpha([\xi, \eta])df(r). \tag{13}$$

For $\gamma_1, \gamma_2 \in \Gamma(T^{-1}M)$, the Jacobi identity implies

$$q_{-3}([\gamma_1, [\gamma_2, r]]) - q_{-3}([\gamma_2, [\gamma_1, r]]) = q_{-3}([[\gamma_1, \gamma_2], r]).$$

The right hand side remains unchanged if we replace $[\gamma_1, \gamma_2]$ by $[\gamma_1, \gamma_2] - \alpha([\gamma_1, \gamma_2])r \in \Gamma(T^{-1}M)$ and then gives $\{[\gamma_1, \gamma_2] - \alpha([\gamma_1, \gamma_2])r, \varphi\}$. Hence

$$\nabla_{\gamma_1}\gamma_2 - \nabla_{\gamma_2}\gamma_1 = [\gamma_1, \gamma_2] - \alpha([\gamma_1, \gamma_2])r, \tag{14}$$

which is the analog of torsion freeness for ∇. To obtain a formula for $\hat{\Phi}(\gamma) - \Phi(\gamma)$, we have to take (11), then subtract the analogous terms with ξ and η exchanged and further subtract (12). Using (13) and (14), we obtain

$$\begin{aligned} 3df(\nabla_\eta \gamma)\xi - 3df(\nabla_\xi \gamma)\eta + 3(\xi \cdot df(\gamma))\eta - 3(\eta \cdot df(\gamma))\xi \\ -\alpha([\xi, \eta])df(r)\gamma + 9df(\gamma)(df(\eta)\xi - df(\xi)\eta) \\ +\hat{\alpha}([\xi, \eta])(\nabla_\delta \gamma + 3df(\gamma)\delta). \end{aligned} \tag{15}$$

Inserting (14) into (13), we get

$$df(\nabla_\eta \gamma) - \eta \cdot df(\gamma) = df(\nabla_\gamma \eta) - \gamma \cdot df(\eta) - \alpha([\eta, \gamma])df(r).$$

Using this and the analogous formula for ξ and γ, we see that the first line of (15) can be rewritten as

$$\begin{aligned} 3(\gamma \cdot df(\xi))\eta - 3df(\nabla_\gamma \xi)\eta - 3(\gamma \cdot df(\eta))\xi + 3df(\nabla_\gamma \eta)\xi \\ +3df(r)(\alpha([\xi, \gamma])\eta - \alpha([\eta, \gamma])\xi). \end{aligned} \tag{16}$$

Now $\gamma \mapsto \alpha([\xi,\gamma])\eta - \alpha([\eta,\gamma])\xi$ defines a bundle map $T^{-1}M \to T^{-1}M$. This map coincides with $\alpha([\xi,\eta])\gamma$ for $\gamma = \xi$ and $\gamma = \eta$, and since $\{\xi,\eta\}$ is nowhere vanishing by assumption, this holds for all γ.

Similarly, we see that the bundle map $T^{-1}M \to \mathrm{gr}_{-2}(TM)$ given by $\gamma \mapsto df(\xi)\{\gamma,\eta\} - df(\eta)\{\gamma,\xi\}$ coincides with $df(\gamma)\{\xi,\eta\}$. By definition of δ, this implies

$$4(df(\xi)\eta - df(\eta)\xi) = \alpha([\xi,\eta])e^f\delta. \tag{17}$$

It follows that

$$\begin{gathered}4(df(\nabla_\gamma\xi)\eta - df(\eta)\nabla_\gamma\xi + df(\xi)\nabla_\gamma\eta - df(\nabla_\gamma\eta)\xi) = \\ \alpha([\nabla_\gamma\xi,\eta])e^f\delta - \alpha([\nabla_\gamma\eta,\xi])e^f\delta.\end{gathered}$$

Next we use the characterisation (2) of the induced connection on $\mathrm{gr}_{-2}(TM)$, the fact that $\nabla_\gamma\varphi = 0$ and $\{\gamma_1,\gamma_2\} = \alpha([\gamma_1,\gamma_2])\varphi$ for all $\gamma_1,\gamma_2 \in \Gamma(T^{-1}M)$ to get

$$\alpha([\nabla_\gamma\xi,\eta])e^f\delta - \alpha([\nabla_\gamma\eta,\xi])e^f\delta = (\gamma\cdot\alpha([\xi,\eta]))e^f\delta.$$

Now we apply ∇_γ to (17) and simplify using the last two equations to obtain

$$\begin{aligned}df(\nabla_\gamma\eta)\xi - (\gamma\cdot df(\eta))\xi - df(\nabla_\gamma\xi)\eta + (\gamma\cdot df(\xi))\eta \\ = \tfrac{1}{4}\hat\alpha([\xi,\eta])(df(\gamma)\delta + \nabla_\gamma\delta).\end{aligned} \tag{18}$$

Collecting our results and using the defining equation (3) from 2.4, we see that

$$\hat\Phi(\gamma) = \Phi(\gamma) + 2e^{-f}df(r)\gamma + \tfrac{3}{2}df(\gamma)\delta + \nabla_\delta\gamma + \tfrac{3}{4}\nabla_\gamma\delta.$$

Inserting $\gamma = \hat\zeta_1 = e^f\zeta_1$, and using $df(\delta) = 0$ we get

$$\hat\Phi(\hat\zeta_1) - \Phi(\zeta_1) = 3df(r)\zeta_1 + \tfrac{3}{2}e^f df(\zeta_1)\delta + e^f\nabla_\delta\zeta_1 + \tfrac{3}{4}e^f\nabla_{\zeta_1}\delta \tag{19}$$

Using appropriate multiples of this, (10), and (9) we obtain the following expression for $\hat\pi_1(\zeta) - \pi_1(\zeta)$:

$$e^f(\nabla_\delta\zeta_1 - \nabla_{\zeta_1}\delta - [\delta,\zeta_1]) - 4df(\zeta_1)r - \tfrac{3}{2}df(r)\zeta_1 - \tfrac{3}{2}e^f df(\zeta_1)\delta - e^f\alpha(\zeta)\delta. \tag{20}$$

Using (14), the first four terms simplify to

$$-(e^f\alpha([\delta,\zeta_1]) + 4df(\zeta_1))r,$$

and we have seen in the proof of Lemma 5 that this vanishes. □

2.6 The main result

Having the technical results at hand, it is now easy to prove that a change of generalized contact form just leads to a conformal rescaling of the associated pseudo–Riemannian metric.

Theorem 8 *Replacing a generalized contact form $\alpha \in \Gamma(T^{-2}M)$ by $\hat{\alpha} = e^f \alpha$, the pseudo–Riemannian metrics associated to α and $\hat{\alpha}$ as in (6) in 2.4 are related by $g_{\hat{\alpha}} = e^{2f} g_\alpha$. In particular, the conformal class of g_α depends only on the generic distribution $\mathcal{H}$.*

PROOF. In the definition of g_α, the exterior derivative $d\alpha$ is only applied to two elements of $T^{-1}M$, whence passing to $\hat{\alpha}$ this only rescales by e^f. Consequently, the Lemma shows that $d\hat{\alpha}(\hat{\zeta}_1, \hat{\pi}_1(\zeta')) - e^{2f} d\alpha(\zeta_1, \pi_1(\zeta'))$ is given by

$$-e^{2f}\Big(\tfrac{3}{2}df(r)d\alpha(\zeta_1, \zeta_1') + e^f(\alpha(\zeta') + \tfrac{3}{2}df(\zeta_1'))d\alpha(\zeta_1, \delta)\Big).$$

The first summand in the bracket is skew symmetric in ζ and ζ' and hence will not contribute to the final result. In the end of the proof of the Lemma, we have seen that

$$e^f d\alpha(\zeta_1, \delta) = -e^f \alpha([\zeta_1, \delta]) = -4df(\zeta_1).$$

Inserting this and using part (ii) of Lemma 5, the result follows by a simple direct computation. □

3 The relation to Nurowski's construction

In this section, we will describe the relation of the conformal class constructed in section 2 to the canonical Cartan connection associated to a generic rank two distribution on a five–manifold. In particular, this will show that our conformal class coincides with the one constructed by P. Nurowski in [6]. Moreover, this will put our construction in a broader context of general tools for parabolic geometries which have been developed during the last years. We will start by describing the canonical Cartan connection associated to a generic rank two distribution in dimension five.

3.1 On G_2

By Cartan's classical result [5], generic rank two distributions in dimension five admit a canonical Cartan connection on a certain principal bundle. The structure group of this bundle is a subgroup in a Lie group whose Lie algebra is the split real form of the exceptional Lie algebra of type G_2. We next discuss the necessary background on this Lie algebra put Cartan's result into the perspective of the general theory of parabolic geometries.

A Lie group G with this Lie algebra can be realized as the automorphism group of the algebra $\mathbb{O}_s$ of split octonions, see [11]. $\mathbb{O}_s$ is an eight dimensional non–associative unital real algebra with a multiplicative inner product of split signature $(4,4)$. Any automorphism of $\mathbb{O}_s$ preserves the inner product and the unit element, so the automorphism group naturally acts on the orthocomplement of the unit element. This is the seven dimensional space $\mathrm{im}(\mathbb{O}_s)$ of purely imaginary split octonions, which carries an invariant inner product of signature $(3,4)$. Hence G can be naturally viewed as a closed subgroup of $SO(3,4)$. The Lie algebra $\mathfrak{g}$ of G is the algebra of derivations of $\mathbb{O}_s$. Since any derivation vanishes on the unit element, also $\mathfrak{g}$ is naturally represented on $\mathrm{im}(\mathbb{O}_s)$ and $\mathfrak{g} \subset \mathfrak{so}(3,4)$.

To obtain an explicit description of $\mathfrak{g}$, we first fix the inner product of signature $(3,4)$. In terms of coordinates $x_0, \dots, x_6$ on $\mathbb{R}^7$ consider the quadratic form $x_0x_6 + x_1x_4 + x_2x_5 - (x_3)^2$, which is evidently induced by an inner product of signature $(3,4)$. The explicit form of $\mathfrak{g}$ for this inner product can be essentially read off from [9]. In the notation of that article, one has to use the ordered basis $\{X_1, X_6, X_7, X_4, X_2, X_3, X_5\}$ to obtain

$$\mathfrak{g} = \left\{ \begin{pmatrix} \mathrm{tr}(A) & Z & s & W & 0 \\ X & A & \sqrt{2}\mathbb{J}Z^t & \frac{s}{\sqrt{2}}\mathbb{J} & -W^t \\ r & -\sqrt{2}X^t\mathbb{J} & 0 & -\sqrt{2}Z\mathbb{J} & s \\ Y & -\frac{r}{\sqrt{2}}\mathbb{J} & \sqrt{2}\mathbb{J}X & -A^t & -Z^t \\ 0 & -Y^t & r & -X^t & -\mathrm{tr}(A) \end{pmatrix} \right\}$$

with $A \in \mathfrak{gl}(2,\mathbb{R})$, $X, Y, Z^t, W^t \in \mathbb{R}^2$, $r, s \in \mathbb{R}$ and $\mathbb{J} := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Indeed, one may easily verify directly that this forms a Lie subalgebra of $\mathfrak{so}(3,4)$, the diagonal matrices contained in $\mathfrak{g}$ act diagonalizably under the adjoint action, and the resulting root decomposition of $\mathfrak{g}$ has a root system of type G_2.

Let us decompose $\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ as in

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 & \mathfrak{g}_3 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & 0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ 0 & \mathfrak{g}_{-3} & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix}.$$

Then this is immediately seen to define a grading on $\mathfrak{g}$, i.e. $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_\ell = \{0\}$ for $|\ell| > 3$. In particular, the Lie bracket defines a representation of the subalgebra $\mathfrak{g}_0$ on each $\mathfrak{g}_i$, which is compatible with the Lie brackets. It is easy to see that the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1} \cong \mathbb{R}^2$ is faithful, so we can use this representation to identify $\mathfrak{g}_0$ with $\mathfrak{gl}(2,\mathbb{R})$. The Lie bracket induces isomorphisms $\Lambda^2\mathfrak{g}_{-1} \to \mathfrak{g}_{-2}$ and $\mathfrak{g}_{-1} \otimes \mathfrak{g}_{-2} \to \mathfrak{g}_{-3}$ of $\mathfrak{g}_0$–modules, which already indicates the relation to generic rank two distributions. On the other hand, $\mathfrak{g}_i$ is dual to $\mathfrak{g}_{-i}$ as a $\mathfrak{g}_0$–module for $i = 1, 2, 3$. A convenient way to express these dualities is via mapping two matrices to $\frac{1}{6}$ times the trace of their product. Let us denote all these pairings by B. In the notation introduced above, this is explicitly given by $(X, Z) \mapsto ZX$, $(r, s) \mapsto \frac{1}{2}rs$, and $(Y, W) \mapsto \frac{1}{3}WY$.

For $i = -3, \dots, 3$, we define $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_3$. This makes $\mathfrak{g}$ into a filtered Lie algebra, i.e. $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$, where we agree that $\mathfrak{g}^\ell = \mathfrak{g}$ for $\ell < -3$ and $\mathfrak{g}^\ell = \{0\}$ for $\ell > 3$. In particular, $\mathfrak{p} := \mathfrak{g}^0$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{p}_+ := \mathfrak{g}^1$ is a nilpotent ideal in $\mathfrak{p}$.

3.2 The canonical Cartan connection

As we have seen above, we may view $G = \mathrm{Aut}(\mathbb{O}_s)$ as a closed subgroup of $SO(3, 4)$. Define $P \subset G$ to be the intersection of G with the stabilizer of the isotropic line spanned by the first basis vector. By construction, P corresponds to the Lie subalgebra $\mathfrak{p} \subset \mathfrak{g}$. For $g \in P$, the adjoint action $\mathrm{Ad}(g)$ preserves the filtration on $\mathfrak{g}$, i.e. $\mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i$ for all i. Define $G_0 \subset P$ as the subgroup of those g for which $\mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i$ for all i. Then the Lie algebra of G_0 is $\mathfrak{g}_0$. On the other hand, it turns out that the exponential map defines a diffeomorphism from $\mathfrak{p}_+$ onto a closed normal subgroup $P_+ \subset P$, and P/P_+ is naturally isomorphic to G_0.

It is well known (and easy to see from the explicit form of $\mathfrak{g}$ above) that the 7–dimensional representation of $\mathfrak{g}$ defined by the above matrix form is irreducible. By Schur's lemma, this implies that the center of G consists only of multiples of the identity matrix, so since $G \subset SO(3, 4)$, we see that G has trivial center. Thus the adjoint representation maps G injectively into the group of automorphisms of $\mathfrak{g}$ and thus G_0 is mapped injectively into the group $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g})$ of automorphisms of the *graded* Lie algebra $\mathfrak{g}$. From [9] we see that the latter group coincides with $\mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-) \cong GL(\mathfrak{g}_{-1})$.

On the other hand, one easily verifies directly that any invertible linear map on $\mathfrak{g}_{-1}$ can be obtained from the adjoint action of an element $g \in G_0$. Hence we conclude that $G_0 \cong GL(\mathfrak{g}_{-1}) \cong \mathrm{Aut}_{\mathrm{gr}}(\mathfrak{g}_-)$. As explained in [9], this shows that parabolic geometries of type (G, P) are equivalent to filtrations of the tangent bundle such that the bundle of symbol algebras is locally trivial and modelled on $\mathfrak{g}_-$, and hence to generic rank two distributions in dimension five.

This can be easily made more explicit. Suppose that M is a five dimensional smooth manifold endowed with a generic distribution $\mathcal{H} \subset TM$ of rank two, and consider the corresponding filtration $\{T^iM\}$ of the tangent bundle as introduced in 2.1. Let $p_0 : \mathcal{G}_0 \to M$ be the linear frame bundle of $\mathcal{H} = T^{-1}M$. This has structure group $GL(2, \mathbb{R})$ which we view as G_0. Since all the components $\mathrm{gr}_i(TM)$ of the associated graded can be constructed from $T^{-1}M$, they are naturally associated to $\mathcal{G}_0$. This can be expressed via G_0–equivariant partially defined one–forms with values in $\mathfrak{g}_-$, which define a *regular infinitesimal flag structure* in the sense of [4, 2.6].

The prolongation procedures for parabolic geometries show that $\mathcal{G}_0$ naturally extends to a principal P–bundle $p : \mathcal{G} \to M$, which can be endowed with a canonical normal Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. The bundle $\mathcal{G}_0$ can be recovered from $\mathcal{G}$ as $\mathcal{G}/P_+$. Further, the generic distribution $\mathcal{H}$ and, more generally, the filtration of TM can be recovered from $(p : \mathcal{G} \to M, \omega)$. For a tangent vector $\xi \in T_xM$ choose any $u \in \mathcal{G}$ and $\tilde{\xi} \in T_u\mathcal{G}$ such that $T_up{\cdot}\tilde{\xi} = \xi$. Then $\xi \in T^i_xM$ for $i = -1, -2$ if and only if $\omega(\tilde{\xi}) \in \mathfrak{g}^i$. This is independent of the choices by the defining properties of a Cartan connection an the fact that each $\mathfrak{g}^i$ is a P–invariant subspace of $\mathfrak{g}$.

Normality is a condition on the curvature of the Cartan connection ω. The detailed form of the condition is not important for our purposes. The curvature of ω is most easily viewed as the two form $\mathcal{K} \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by

$$\mathcal{K}(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

The only fact about the normality condition we will need in the sequel is a restriction on the homogeneity of $\mathcal{K}$. Namely, if $Tp{\cdot}\xi \in T^iM$ and $Tp{\cdot}\eta \in T^jM$ for some $i, j = -3, \dots, -1$, then $\mathcal{K}(\xi, \eta) \in \mathfrak{g}^{i+j+4}$ (and for the sequel even $i + j + 3$ would be sufficient).

3.3 Nurowski's conformal structure

We have realized the group G as a subgroup of $\tilde{G} := SO(3, 4)$. Denoting by $\tilde{P} \subset \tilde{G}$ the stabilizer of the isotropic line generated by the first basis vector, we see that $P = G \cap \tilde{P}$. Now it is well known that $\tilde{G}/\tilde{P}$ is the Möbius space $S^{2,3}$ with $\tilde{G}$ acting as the group of all conformal isometries of the canonical (locally conformally flat) conformal structure. Denoting by $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}}$ the Lie algebras, this conformal structure is induced by a conformal class of inner products on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ which is invariant under the natural action of $\tilde{P}$.

The inclusion $G \hookrightarrow \tilde{G}$ induces a smooth injection $G/P \to \tilde{G}/\tilde{P}$. Since both spaces have the same dimension, this must be an open embedding. It is well known that quotients of semisimple Lie groups by parabolic subgroups are always compact, whence $G/P \cong \tilde{G}/\tilde{P}$. The derivative at the base point eP

of this map is a linear isomorphism $\mathfrak{g}/\mathfrak{p} \to \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$, which by construction is equivariant over the inclusion $P \hookrightarrow \tilde{P}$. Hence the conformal class of inner products on $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$ from above pulls back to a P–invariant conformal class of inner products on $\mathfrak{g}/\mathfrak{p}$. For any Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) we get $TM \cong \mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$ via ω, and hence an induced conformal structure on M.

Nurowski's original construction in [6] is obtained by using a degenerate inner product on $\mathfrak{g}$ to induce an inner product from the conformal class on $\mathfrak{g}/\mathfrak{p}$. Basically, this amounts to applying the given inner product on $\mathbb{R}^7$ to the first columns of matrices. Via the Cartan connection, this is carried over to a degenerate metric on $\mathcal{G}$, which is shown to induce a well defined conformal class on M.

3.4 Weyl structures

We next explain how to obtain the representative metric in Nurowski's conformal class as described in section 2. The basic tool is provided by Weyl structures as introduced in [4], see also [2] for an alternative approach. For a five manifold M and a generic rank two distribution $\mathcal{H} \subset TM$, let $p_0 : \mathcal{G}_0 \to M$ be the frame bundle of $\mathcal{H}$ and let $(p : \mathcal{G} \to M, \omega)$ be the canonical Cartan geometry. A (local) Weyl structure then is a G_0–equivariant (local) smooth section σ of the natural projection $\pi : \mathcal{G} \to \mathcal{G}/P_+ = \mathcal{G}_0$. There always exist global Weyl structures, but local ones suffice for our purposes.

Given a Weyl structure σ, one may pull back the Cartan connection ω to obtain $\sigma^*\omega \in \Omega^1(\mathcal{G}_0, \mathfrak{g})$. By construction, for each $i = -3, \dots, 3$ the component $\sigma^*\omega_i \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_i)$ is G_0–equivariant. It is better to decompose the pullback as $\sigma^*\omega = \sigma^*\omega_- + \sigma^*\omega_0 + \sigma^*\omega_+$ according to the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. Then the equivariant form $\sigma^*\omega_-$ descends to an element of $\Omega^1(M, \mathrm{gr}(TM))$. Its value in each point $x \in M$ induces a linear isomorphism $T_xM \to \mathrm{gr}(T_xM)$ which splits the filtration, i.e. the restriction of the $\mathrm{gr}_i(TM)$–component to T^iM coincides with the canonical projection. Second, the component $\sigma^*\omega_0$ defines a principal connection on $\mathcal{G}_0$, called the *Weyl connection* associated to σ. This induces a linear connection on any vector bundle associated to $\mathcal{G}_0$. Finally, $\sigma^*\omega_+$ descends to a one–form $P \in \Omega^1(M, \mathrm{gr}(T^*M))$, called the *Rho tensor* associated to σ.

Now suppose that we have two tangent vectors $\xi, \eta \in T_xM$. To compute the values of the metrics in the conformal class on these two vectors, we have to choose $u \in \mathcal{G}$ with $p(u) = x$ and lifts $\tilde{\xi}, \tilde{\eta} \in T_u\mathcal{G}$. Then we evaluate the elements in the preferred class of inner products on $\mathfrak{g}/\mathfrak{p}$ on $\omega(\tilde{\xi}) + \mathfrak{p}$ and $\omega(\tilde{\eta}) + \mathfrak{p}$. Now we may linearly identify $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{g}_-$. Denoting elements of

$\mathfrak{g}_-$ as triples (X, r, Y) as suggested by the presentation of matrices in 3.1, the conformal class of inner products consists of all multiples of

$$((X, r, Y), (X', r', Y')) \mapsto X^t Y' - rr' + Y^t X'. \tag{21}$$

Fix a (local) Weyl structure σ, choose a point $u_0 \in \mathcal{G}_0$ and lifts $\hat{\xi}, \hat{\eta} \in T_{u_0}\mathcal{G}_0$ of the two tangent vectors. Then put $u := \sigma(u_0)$, $\tilde{\xi} = T_{u_0}\sigma \cdot \hat{\xi}$ and likewise for $\tilde{\eta}$. Then the $\mathfrak{g}_-$–components of $\omega(\tilde{\xi})$ represent the components of the image of ξ in $\operatorname{gr}(TM)$ under the isomorphism $TM \cong \operatorname{gr}(TM)$ determined by σ.

To interpret the individual terms in the right hand side of (21), take an element $s \in \mathfrak{g}_2$. Then from the matrix presentation and the definition of the duality B in 3.1 one immediately computes that

$$B([s, (X, 0, 0)], [s, (0, 0, Y')]) = 2s^2 X^t Y' \tag{22}$$

$$B(s, (0, r, 0))B(s, (0, r', 0)) = \frac{s^2}{4} rr'. \tag{23}$$

Passing to associated bundles, brackets and B correspond to geometric operations. Using these, one can translate (21) into a geometrically meaningful formula for a metric.

Hence it remains to compute the isomorphism $TM \to \operatorname{gr}(TM)$ induced by some Weyl structure. To pin down one Weyl structure we use an analog of the scales used in [4]. The associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_2$ is the line bundle $(\operatorname{gr}_{-2}(TM))^*$. For any Weyl structure σ, the Weyl connection $\sigma^*\omega_0$ induces a linear connection on $(\operatorname{gr}_{-2}(TM))^*$. Since the grading element acts non–trivially on $\mathfrak{g}_2$, the proof of Theorem 3.8 of [4] shows that mapping Weyl structures to induced linear connections on $(\operatorname{gr}_{-2}(TM))^*$ is bijective. In particular, given a local nowhere vanishing section α of $(\operatorname{gr}_{-2}(TM))^*$, there is a unique local Weyl structure such that α is covariantly constant for the induced connection. In the language of 2.2 this means that any generalized contact form for $\mathcal{H}$ determines a Weyl structure. The main point about the method is that the isomorphism $TM \to \operatorname{gr}(TM)$ can be computed without knowing the canonical Cartan connection. In fact, one only has to go through the first steps in the prolongation/normalization procedure.

We next describe how to encode the individual parts of a Weyl structure. Since $\mathcal{G}_0$ is the full frame bundle of $\mathcal{H} = T^{-1}M$ a principal connection on $\mathcal{G}_0$ is equivalent to a linear connection ∇ on $T^{-1}M$. Concerning the isomorphism $TM \to \operatorname{gr}(TM)$, the component in $\operatorname{gr}_{-3}(TM)$ is just given by the canonical projection q_{-3}, so this contains no information. Suppose that we have given a (local) generalized contact form $\alpha \in \Gamma((\operatorname{gr}_{-2}(TM))^*)$. Since this is nowhere vanishing, there is a unique section $\varphi \in \Gamma(\operatorname{gr}_{-2}(TM))$ such that $\alpha(\varphi) = 1$. Viewing α as a section of $L(T^{-2}M, \mathbb{R})$, the canonical projection $T^{-2}M \to \operatorname{gr}_{-2}(TM)$ is then given by $\xi \mapsto \alpha(\xi)\varphi$. Hence we can describe the component in $\mathrm{gr}_{-2}(TM)$ of the isomorphism $TM \to \mathrm{gr}(TM)$ equivalently by an extension of α to a one–form on M, which we will again denote by the same symbol. Finally, the component in $\mathrm{gr}_{-1}(TM)$ of the isomorphism can be viewed as a projection π_{-1} from TM onto the subbundle $T^{-1}M$. Restricting this projection to $T^{-2}M$, the kernel is a line subbundle and q_{-2} identifies this line subbundle with $\mathrm{gr}_{-2}(TM)$. In particular, there is a unique section $r \in \Gamma(T^{-2}M)$ such that $\alpha(r) = 1$ and $\pi_{-1}(r) = 0$.

3.5 The Weyl structure associated to a generalized contact form

Let $\alpha \in \Gamma((\mathrm{gr}_{-2}(TM))^*)$ be a (local) generalized contact form. As we have seen above, a choice of Weyl structure gives us a linear connection ∇ on $T^{-1}M$, an extension of α to a one–form on M, a section $r \in \Gamma(T^{-2}M)$, and a projection π_{-1} from TM onto the subbundle $T^{-1}M$. We want to prove that for the unique Weyl structure such that α is parallel for the induced linear connection, these specialize to the objects obtained in 2.2–2.4 (where we used only a part of the connection). We denote all linear connections induced by our Weyl connection by ∇.

The key for verifying this comes from the fact that the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is normal. We have noted in 3.2 that this implies restrictions on the homogeneity of its curvature $\mathcal{K}$. For any Weyl structure σ, this implies that the form

$$W(\xi, \eta) := d\sigma^*\omega(\xi, \eta) + [\sigma^*\omega(\xi), \sigma^*\omega(\eta)] \tag{24}$$

maps tangent vectors ξ such that $Tp_0{\cdot}\xi \in T^iM$ and $Tp_0{\cdot}\eta \in T^jM$ to $\mathfrak{g}^{i+j+3}$. Now we can split the right hand side of (24) into components, which admit a direct interpretation in terms of the Rho–tensor and a curvature/torsion quantity K associated to the components of $\sigma^*\omega$, see section 4 of [4].

For the first step, we will only need components of K with values in $\mathrm{gr}(TM)$, for which there is an explicit formula in Proposition 4.2 of [4]. For $\zeta \in \mathfrak{X}(M)$ let us denote components in $\mathrm{gr}(TM)$ under the isomorphism provided by a Weyl structure by ζ_i for $i = -3, -2, -1$. Then for $\ell < 0$, the formula for the $\mathfrak{g}_\ell$–component of $K(\zeta, \zeta')$ reads as

$$K_\ell(\zeta, \zeta') = \nabla_\zeta \zeta'_\ell - \nabla_{\zeta'}\zeta_\ell - [\zeta, \zeta']_\ell + \sum_{i,j<0, i+j=\ell} \{\zeta_i, \zeta'_j\}. \tag{25}$$

The analysis is best done homogeneity by homogeneity. From Proposition 4.3 of [4] we see that the homogeneous component of degree one of K coincides with the one of W and hence has to vanish.

Claim 9 *Vanishing of the homogeneous component of degree one of K implies* *that r is the Reeb field associated to α as in 2.2, the extension of α to a one–form coincides with the one from Proposition 3, and ∇ restricts to the partial connection associated to r as in formula* (1) *in 2.2.*

PROOF. The curvature K automatically has positive homogeneity. Hence vanishing of the homogeneous component of degree one implies vanishing of K_i on $T^{-1}M \times T^iM$ for $i = -3, -2, -1$, and for $\xi \in \Gamma(T^{-1}M)$ and $\zeta \in \Gamma(T^iM)$ the value of $K_i(\xi, \zeta)$ depends only on $q_i(\zeta)$. In particular, for $i = 2$ it suffices to compute $K_{-2}(\xi, r)$. Since $\alpha \in \Gamma(\mathrm{gr}_{-2}(T^*M))$ is parallel for the induced connection, then so is the dual section $\varphi = r_{-2}$. Using that $r_{-1} = 0$, (25) simplifies to give

$$K_{-2}(\xi, r) = \alpha([\xi, r])\varphi = [\xi, r]_{-2}. \tag{26}$$

Next, for $\xi \in \Gamma(T^{-1}M)$ and $\zeta \in \mathfrak{X}(M)$, we obtain from (25)

$$K_{-3}(\xi, \zeta) = \nabla_\xi \zeta_{-3} - [\xi, \zeta]_{-3} + \{\xi, \zeta_{-2}\}. \tag{27}$$

Now put $\zeta := [r, \eta]$ for $\eta \in \Gamma(T^{-1}M)$. Then $\zeta_{-3} = q_{-3}(\zeta) = \{\varphi, \eta\}$, and since ∇ is compatible with $\{\ ,\ \}$ and $\nabla\varphi = 0$ we see that

$$K_{-3}(\xi, [r, \eta]) = \{\varphi, \nabla_\xi \eta\} - q_{-3}([\xi, [r, \eta]]) - \{\xi, K_{-2}(\eta, r)\}. \tag{28}$$

Vanishing of (27) and (26) thus implies that ∇ is the connection determined by r. But then the fact that φ is parallel implies that r is the Reeb field associated to α as in 2.2. Given this, vanishing of (26) says that we get the right extension of α to a one–form. $\square$

Remark 10 *It can be actually shown that the opposite implication holds as well. If we use r, the extension of α and ∇ as the data associated to the Weyl form and α is parallel for the induced connection, then the facts that r is the Reeb field, we have the right extension of α, and ∇ restricts to the partial connection determined by r imply that the homogeneous component of degree one of the curvature K vanishes.*

It remains to show that our Weyl structure produces the right projection π_{-1}. For this we have to analyze the homogeneous components of degree two of W and K. According to Proposition 4.3 of [4], the difference between these two components is determined the homogeneous component of degree 2 of the Rho–tensor. Since we will not need any other part of the Rho–tensor, we simply denote this component by P. It can be either interpreted as a partially (on $T^{-1}M$) defined one–form with values in $(T^{-1}M)^*$ or as a bilinear form on $T^{-1}M$. Further, we will also need components of W and K in degree zero. These are sections of the bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_0$, so in particular such a section induces an endomorphism of $\mathrm{gr}_i(TM)$ for $i = -3, -2, -1$. This action is induced by the components $\mathfrak{g}_0 \times \mathfrak{g}_i \to \mathfrak{g}_i$ of the Lie bracket on $\mathfrak{g}$.

We will also need some of the other tensorial maps induced by the Lie brackets on $\mathfrak{g}$, and we will denote them all by $\{\ ,\ \}$. In particular, these define bilinear bundle maps $\mathrm{gr}_{-i}(TM) \times \mathrm{gr}_i(T^*M) \to \mathcal{G}_0 \times_{G_0} \mathfrak{g}_0$ for all i, as well as

$$\mathrm{gr}_{-j}(TM) \times \mathrm{gr}_i(T^*M) \to \mathrm{gr}_{i-j}(TM)$$

for $i < j$.

For K, the component K_0 is the curvature of the Weyl connection, see Proposition 4.2 of [4]. Hence the induced endomorphism on $\mathrm{gr}_i(TM)$ is simply the curvature R of the corresponding linear connection. Since we need the component of degree 2, we are interested in K_0 and W_0 as two–forms acting on $T^{-1}M \times T^{-1}M$, and there the difference between W and K is given by

$$(\xi, \eta) \mapsto \{P(\xi), \eta\} - \{P(\eta), \xi\}.$$

Now we first observe that the bundle $\mathrm{gr}_{-2}(TM)$ admits the nonzero parallel section φ, so R has to act trivially on $\mathrm{gr}_{-2}(TM)$. Hence vanishing of the restriction of W_0 to $T^{-1}M \times T^{-1}M$ implies that also $\{P(\xi), \eta\} - \{P(\eta), \xi\}$ acts trivially on $\mathrm{gr}_{-2}(TM)$. But one immediately verifies that for $Z \in \mathfrak{g}_1$ and $X \in \mathfrak{g}_{-1}$ the action of $[Z, X] \in \mathfrak{g}_0$ on $\mathfrak{g}_{-2}$ is by multiplication by a nonzero multiple of $B(Z, X) = ZX$. Hence we conclude that, viewed as a bilinear form on $T^{-1}M$, P is symmetric.

Further, one verifies directly that for $Z \in \mathfrak{g}_1$ and $X_1, X_2 \in \mathfrak{g}_{-1}$, one has

$$\begin{aligned}
[[Z, X_1], X_2] &= B(Z, X_1)X_2 - 3B(Z, X_2)X_1 \\
[Z, [X_1, X_2]] &= 4(B(Z, X_1)X_2 - B(Z, X_2)X_1).
\end{aligned}$$

Using these two identities and the symmetry of P one immediately verifies that

$$\{\{P(\xi), \eta\} - \{P(\eta), \xi\}, \xi'\} = -\tfrac{3}{4}\{P(\xi'), \{\xi, \eta\}\}$$

for all $\xi, \eta, \xi' \in T^{-1}M$.

Now let us assume that $\{\xi, \eta\} = \varphi$. Then by definitions of the curvature R and of Φ in (3) in 2.4 we get $R(\xi, \eta)(\xi') = \Phi(\xi') - \nabla_r \xi'$. Hence we conclude that vanishing of $W_0(\xi, \eta)$ implies (renaming ξ' to ξ) that

$$\Phi(\xi) - \nabla_r \xi + \tfrac{3}{4}\{\varphi, P(\xi)\} = 0 \tag{29}$$

for all $\xi \in \Gamma(T^{-1}M)$.

For the remaining components, we can use formula (25) from 3.5 to compute K, and the correction to W is given by those P–terms which involve entries from $T^{-1}M$. Vanishing of $W_{-1}(r, \xi)$ for $\xi \in \Gamma(T^{-1}M)$ implies

$$\nabla_r \xi - \pi_{-1}([r, \xi]) + \{\varphi, P(\xi)\} = 0. \tag{30}$$

Finally, vanishing of $W_{-3}(r, \zeta)$ for $\zeta \in \mathfrak{X}(M)$ gives

$$\nabla_r q_{-3}(\zeta) - q_{-3}([r, \zeta]) + \{\varphi, \pi_{-1}(\zeta)\} = 0.$$

Inserting $\zeta = [r, \xi]$ and using equation (4) from 2.4 we see that we can pull off $\{\varphi, \ \}$ to conclude that

$$\nabla_r \xi - 2\Psi(\xi) + \pi_{-1}([r, \xi]) = 0. \tag{31}$$

Using (30) to compute $\{\varphi, P(\xi)\}$ and (31) to compute $\nabla_r \xi$ and inserting both into (29), we obtain

$$\pi_{-1}([r, \xi]) - \tfrac{2}{5}\Phi(\xi) + \tfrac{7}{5}\Psi(\xi) = 0,$$

which exactly means that we get the right projection π_{-1}.

Remark 11 *The formula for $[Z, [X_1, X_2]]$ from above shows that the bracket $\{\ ,\ \} : \mathrm{gr}_1(T^*M) \times \mathrm{gr}_{-2}(TM) \to \mathrm{gr}_{-1} TM$ is explicitly given by $\{\psi, \{\xi, \eta\}\} = 4(\psi(\xi)\eta - \psi(\eta)\xi)$. The other components of $\{\ ,\ \}$ can be computed similarly. Using these formulae, one easily verifies that the transformation laws in Lemmas 5 and 7 are the specializations of Proposition 3.4. of [4], which gives a general formula for the change of the isomorphism $TM \to \mathrm{gr}(TM)$ caused by a change of Weyl structure.*

3.6 Computing the metric

With the description of the isomorphism $TM \to \mathrm{gr}(TM)$ at hand, we can now verify the formula for the metric. We only have to interpret the expressions (22) and (23) from 3.4 in geometric terms. In these formulae, $s \in \mathfrak{g}_2$ corresponds to the generalized contact form α. Let us further suppose that $\zeta, \zeta' \in \mathfrak{X}(M)$, are vector fields. Then in (23), the element $r \in \mathfrak{g}_{-2}$ corresponds to $\zeta_{-2} = \alpha(\zeta)\varphi$ and likewise for r'. Thus the geometric interpretation of (23) simply is $\alpha(\zeta)\alpha(\zeta')$.

In (22), the element $X \in \mathfrak{g}_{-1}$ corresponds to $\pi_{-1}(\zeta)$ and Y' corresponds to $q_{-3}(\zeta') = \{\varphi, \zeta'_1\}$. Hence what we actually have to do is interpreting (again in the notation of 3.4)

$$B([s, (X, 0, 0)], [s, [r_0, (X', 0, 0)]]),$$

where $r_0 \in \mathfrak{g}_{-2}$ is characterized by $B(r_0, s) = 1$. Now one easily computes that in the Lie algebra $\mathfrak{g}$, one has $[s, [r_0, X']] = 3X'$ and using this, one verifies that $B([s, X], [s, [r_0, X']])r_0 = 6[X, X']$. Using that

$$\{\xi, \eta\} = \alpha([\xi, \eta])\varphi = -d\alpha(\xi, \eta)\varphi,$$

we see that the geometric interpretation of (22) is $-6d\alpha(\pi_{-1}(\zeta), \zeta'_1)$. But then formula (21) from 3.4 shows that Nurowski's conformal class contains the metric

$$(\zeta, \zeta') \mapsto -3d\alpha(\pi_{-1}(\zeta), \zeta'_1) - 3d\alpha(\pi_{-1}(\zeta'), \zeta_1) - 4\alpha(\zeta)\alpha(\zeta'),$$

which proves

Theorem 12 *The metric g_α defined in formula* (6) *in 2.4 is contained in Nurowski's conformal class.*

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