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EXPLICIT DOUBLY-HERMITIAN METRICS

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ABSTRACT. We construct explicit examples of 4-dimensional Riemannian metrics which admit precisely two independent hermitian structures with the same orientation. It is shown that metrics of this type exist on four-dimensional tori.

1. INTRODUCTION

If M is an oriented Riemannian 4-manifold then it is well known that the *(local)* existence of a hermitian structure I compatible with the orientation imposes constraints on the selfdual part W_+ of the Weyl tensor of M (see for example [Sal91]). More precisely, such hermitian structures must necessarily be roots of W_+ thought of as a real quartic polynomial (W_+ lives in the spin bundle S^4V_+ while almost hermitian structures are sections of the twistor space $\mathbb{P}V_+$). In particular, if M admits three independent hermitian structures with the same orientation then it is half conformally flat, that is, $W_+ = 0$. The same happens if there are two anticommuting hermitian structures on M since their composition is again hermitian. But $W_+ = 0$ implies that the twistor space $\mathbb{P}V_+$ is integrable [AHS78] so locally M admits a whole family of hermitian structures parametrised by the germs of holomorphic functions in two variables (hermitian structures on M are parametrised by complex submanifolds of the twistor space, transversal to the fibres). If, on the other hand, W_+ is nontrivial then there are at most two (modulo sign) candidates for hermitian structures compatible with the orientation on M. A natural question arises: does the existence of two compatible hermitian structures imply that $W_+ = 0$? In other words, we are asking whether there exist 4-dimensional metrics which admit locally *precisely* two independent compatible hermitian structures. The aim of this note is to present a simple proof of existence of such metrics. We shall exhibit metrics of this type on four-dimensional tori.

Conventions: For brevity, we shall say that two almost hermitian structures I, J are independent if $I_x \neq \pm J_x$ for all $x \in M$; an almost hermitian structure is said to be compatible if it has the same orientation as M. Hermitian means 'almost hermitian and integrable'. Riemannian 4-manifolds which admit precisely two compatible hermitian structures will be called doubly-hermitian. Finally, we emphasize again that questions dealt with in this paper are of *local* nature.

2. DEFORMING THE FLAT METRIC

Let us consider the complex manifold $M = \mathbb{C}^2$ with complex coordinates (z_1, z_2), orientation $-dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2$ and with the metric

$$g_f = dz_1 \odot d\bar{z}_1 + f dz_2 \odot d\bar{z}_2 \qquad (1)$$

where $f : \mathbb{C}^2 \to \mathbb{R}_{>0}$ is a smooth strictly positive function.

We let I denote the standard complex structure on M. If J is a compatible almost hermitian structure such that I and J are independent then there is a unique smooth complex valued function $u : \mathbb{C}^2 \to \mathbb{C}$ such that the space of $(1,0)$-forms with respect to J is spanned by

$$\alpha_1 = \bar{u} dz_1 - d\bar{z}_2 \qquad \text{and} \qquad \alpha_2 = \bar{u} f dz_2 + d\bar{z}_1. \tag{2}$$

(The complex conjugate over u is introduced purely for convenience.) The reader familiar with the twistor language will guess that $\bar{u}$ is in fact a section of the twistor space $M \times \mathbb{CP}^1$: the complex structure I corresponds to the section $M \times \{\infty\}$ (see Appendix).

There is no need, however, to resort to twistor spaces here: in order to convince oneself that equations (2) do define an almost hermitian structure note that the annihilator of the 1-forms $\bar{\alpha}_1 = u d\bar{z}_1 - dz_2$ and $\bar{\alpha}_2 = u f d\bar{z}_2 + dz_1$ is spanned by the vectors

$$Z_1 = -uf\partial_1 + \bar{\partial}_2, \qquad Z_2 = u\partial_2 + \bar{\partial}_1. \tag{3}$$

where $\partial_k = \frac{\partial}{\partial z_k}$ and $\bar{\partial}_k = \frac{\partial}{\partial \bar{z}_k}$. Since $g_f(Z_i, Z_j) = 0$ for $i, j \in \{1, 2\}$, the 2-dimensional distribution $\langle Z_1, Z_2 \rangle_{\mathbb{C}}$ (the complex linear span of Z_1 and Z_2) is isotropic in $T^{\mathbb{C}}M$. Then

$$T^{\mathbb{C}}M = \langle Z_1, Z_2 \rangle_{\mathbb{C}} \oplus \langle \bar{Z}_1, \bar{Z}_2 \rangle_{\mathbb{C}}$$

and J is determined by the property that the above is a decomposition into J-eigenspaces with eigenvalues $+i$ and $-i$. Moreover, I and J have the same orientation since J is a smooth deformation of $-I$. This can be also verified by a direct calculation:

$$\alpha_1 \wedge \bar{\alpha}_1 \wedge \alpha_2 \wedge \bar{\alpha}_2 = (1 + fu\bar{u})^2 dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

With the vectors Z_1 and Z_2 at hand it is easy to establish when J is hermitian: this happens precisely when the distribution $\langle Z_1, Z_2 \rangle_{\mathbb{C}}$ is integrable, that is,

$$[Z_1, Z_2] \in \langle Z_1, Z_2 \rangle_{\mathbb{C}}\,. \tag{4}$$

We have:

$$\begin{aligned} [Z_1, Z_2] &= \big[-uf\partial_1 + \bar{\partial}_2, u\partial_2 + \bar{\partial}_1\big] \\ &= (-uf\partial_1 u + \bar{\partial}_2 u)\partial_2 + \big(u\partial_2(uf) + \bar{\partial}_1(uf)\big)\partial_1\,. \end{aligned}$$

The above vector field has no components in $\bar{\partial}_1$ and $\bar{\partial}_2$ so it must vanish, and the condition (4) is equivalent to the following system of equations:

$$\begin{cases} uf\partial_1 u - \bar{\partial}_2 u = 0, \\ u\partial_2(uf) + \bar{\partial}_1(uf) = 0. \end{cases} \tag{5}$$

Obviously, $u = 0$ is a solution of (5) for any f. This corresponds to the complex structure $-I$. The equations 'do not see' I since it comes from the section at infinity. On the other hand if u (corresponding to J) is a solution of the equations then so is $-\frac{1}{f\bar{u}}$ as it gives rise to $-J$. According to what was said in the introduction, we have:

Lemma 1. *The metric g_f defined in* (1) *is doubly-hermitian if and only if (for fixed f) the equation* (5) *has precisely two nonzero solutions in u.*

3. Examples of doubly-hermitian metrics

Since we are only interested in explicit examples, we will solve the system (5) with the condition $u = c = \text{const}$, $c \neq 0$. In this case it reduces to a single equation

$$(c\partial_2 + \bar{\partial}_1)f = 0$$

which has solutions of the form

$$f(z_1, z_2) = h(z_2 - c\bar{z}_1)$$

where $h : \mathbb{C} \to \mathbb{R}_{>0}$ is an arbitrary smooth function. To simplify things further, set $c = -1$ (there is no loss of generality as one can get rid of nonzero c by redefining the coordinates z_i and the function f. In this case

$$\partial_2 f = \bar{\partial}_1 f,$$

and

$$f(z_1, z_2) = h(z_2 + \bar{z}_1). \tag{6}$$

Let us find now how many solutions system (5) can have. To do this note that it can be rewritten simply as $[Z_1, Z_2] = 0$, or

$$\begin{cases} Z_1 u = 0, \\ Z_2(uf) = 0. \end{cases} \tag{7}$$

We will get more equations by differentiating the above:

$$Z_1 Z_2(uf) = 0.$$

Since $[Z_1, Z_2] = 0$, this is the same as

$$Z_2 Z_1(uf) = 0.$$

Now expand this, remembering that $Z_1 u = 0$:

$$0 = Z_2 Z_1(uf) = Z_2(uZ_1 f) = Z_2 u Z_1 f + uZ_2 Z_1 f. \tag{8}$$

The idea is to get rid of the partial derivatives of u. From (7) we have $Z_2 u = -\frac{1}{f} u Z_2 f$, so (8) gives

$$u(fZ_2 Z_1 f - Z_2 f Z_1 f) = 0. \tag{9}$$

According to formula (3)

$$Z_1 f = -uff_1 + f_{\bar{2}}, \qquad Z_2 f = uf_2 + f_{\bar{1}},$$

and, since $Z_2(uf) = 0$,

$$\begin{aligned} Z_2 Z_1 f &= Z_2(-uff_1 + f_{\bar{2}}) = -ufZ_2 f_1 + Z_2 f_{\bar{2}} \\ &= -u^2 ff_{12} - uff_{1\bar{1}} + uf_{2\bar{2}} + f_{\bar{1}\bar{2}}. \end{aligned}$$

The above can be used to expand (9); in order to make the result more compact, let p_{ij} denote the following determinant:

$$p_{ij} := \det \begin{vmatrix} f & f_i \\ f_j & f_{ij} \end{vmatrix}$$

where $i, j \in \{1, 2, \bar{1}, \bar{2}\}$. Then (9) can be written as

$$u(-u^2 fp_{12} + u(p_{2\bar{2}} - fp_{1\bar{1}}) + p_{\bar{1}\bar{2}}) = 0. \tag{10}$$

Let us consider functions f which satisfy (6). In this case $f_2 = f_{\bar{1}} = h_z$ and $f_{\bar{2}} = f_1 = h_{\bar{z}}$, so all p_{ij} which appear in (10) are equal to the determinant $hh_{z\bar{z}} - h_z h_{\bar{z}}$. This can be written simply as $h^2 \partial\bar{\partial} \ln h$. As a result (10) gives

$$u(1-uf)(1+u)\,\partial\bar{\partial}\ln h = 0. \tag{11}$$

It is clear that, unless $\partial\bar{\partial}\ln h = 0$, the system (5) can only have solutions $u = 0$, $u = -1$, or $u = \frac{1}{f}$. We know that these actually are solutions of (5) with f given by (6) so, from Lemma 1, we get the following simple

Proposition 2. *Let $h : \mathbb{C} \to \mathbb{R}_{>0}$ be such that it* does not *satisfy the equation*

$$\partial\bar{\partial}\ln h = 0, \tag{12}$$

and let $f(z_1, z_2) = h(z_2 + \bar{z}_1)$. Then the metric g_f defined by (1) is doubly-hermitian.

Note that $\partial\bar{\partial}\ln h = 0$ (i.e. $\ln h$ is a harmonic function) iff $W_+ = 0$; for the metrics of type (1) this means that also $W_- = 0$ so the solutions of (12) give conformally flat metrics.

Example 3. Consider $h(z) = 2 + \cos(z + \bar{z})$. Then (12) evaluates to $-2\cos(z+\bar{z}) - 1$ so the corresponding metric g_f is doubly-hermitian. Moreover, both g_f and the complex structures $u = 0$ and $u = -1$ are periodic so they define a doubly-hermitian structure on the 4-dimensional torus $\mathbb{C}^2/2\pi\mathbb{Z}^4$.

Remark 4. Rather than looking for the obstructions to integrability of (5) one could calculate the W_+ part of the Weyl tensor directly from the metric g_f. We chose a different approach in order to avoid tedious computations. In fact it is known that the obstruction to integrability of an almost hermitian structure on a 4-manifold does give W_+ (see [Nur93, Ch.3]), so we did compute W_+ after all. (The polynomial (10) is cubic rather than quartic since we chose the coordinates on the twistor space in such a way that one of the roots of W_+ lies at infinity.)

Remark 5. If W_+ has double roots then there is at most one (modulo sign) hermitian structure compatible with the orientation. One can get explicit examples of such 'one-hermitian' metrics by choosing the function f so that $p_{12} = 0$ and the polynomial (10) does not vanish. For example set $f = z_1 + \bar{z}_1$. This gives a metric which admits one independent compatible hermitian structure in each orientation.

Another class of examples can be obtained from the following well-known result (cf. [Der83, Prop.2] and the references therein): For a Kählerian 4-manifold the tensor W_+ (thought of as an endomorphism of $\Lambda^2_+ T^*M$) has multiple eigenvalues (this means that $W_+ \in S^4 V_+$ has multiple roots). For example consider $\mathbb{CP}^2$ with the Fubini-Study metric and the orientation compatible with the standard Kähler structure. It follows that the standard complex structure on $\mathbb{CP}^2$ is the unique (up to sign) hermitian structure compatible with the Fubini-Study metric. (Of course the Fubini-Study metric, being antiselfdual, admits locally many hermitian structures with opposite orientation.)

For recent results concerning existence of complex structures on symmetric spaces see [BGMR93] and [Gau94].

The condition $\#\,\mathrm{spec}_{\Lambda_+} \leq 2$ (i.e. W_+ has multiple roots) was extensively studied in [Der83]. For example it is shown that if M is Einstein (or, more generally, the divergence δW_+ vanishes) then the condition is equivalent to M being locally conformally Kählerian. Interestingly, for Einstein manifolds the existence of a compatible *hermitian* structure already implies that W_+ has multiple roots, see [PB83] and [Nur93, Ch.3].

4. Appendix: parametrizing hermitian structures

Consider the vector space $W = \mathbb{R}^4$ with the standard metric ${dx_1}^2 + \ldots + {dx_4}^2$ and with orientation $dx_1 \wedge \ldots \wedge dx_4$ and denote by $Z(W)$ the set of all hermitian structures on W compatible with the orientation. There are many equivalent descriptions of $Z(W)$, for example one can identify W with quaternions $\mathbb{H}$ and then $Z(W)$ is the 2-sphere of unit quaternions. Another well known way is to use the spinor formalism. In what follows we use the notation from [Sal91]. Let us write $(W^*)^{\mathbb{C}} = V_+ \otimes V_-$ where $V_\pm$ are spin bundles with real symplectic forms $\eta_\pm$ and quaternionic structures $j_\pm$ (so $j_\pm$ are anti-$\mathbb{C}$-linear, and $(j_\pm)^2 = -1$). The metric g can be expressed in terms of $\eta_\pm$ while the complex conjugation on $(W^*)^{\mathbb{C}}$ is equal to $j_+ \otimes j_-$. Choosing a hermitian structure I on W is equivalent to a choice of a spinor $v \in V_+$ as this gives the decomposition

$$(W^*)^{\mathbb{C}} = \big(\langle v\rangle_{\mathbb{C}} \otimes V_-\big) \oplus \big(\langle \tilde{v}\rangle_{\mathbb{C}} \otimes V_-\big) = \Lambda_v^{(1,0)} \oplus \Lambda_v^{(0,1)}$$

into $(1,0)$ and $(0,1)$-forms. Here $\langle v\rangle_{\mathbb{C}}$ denotes the complex vector space spanned by v and $\tilde{v}$ stands for $j_+(v)$ (similarly, for $w \in V_-$ we will write $\tilde{w}$ to denote $j_-(w)$). To be more explicit, we can write (with a choice of sections v, w such that $\eta_+(v,\tilde{v}) = \eta_-(w,\tilde{w}) = 1$):

$$\Lambda_v^{(1,0)} = \langle dz_1, dz_2\rangle_{\mathbb{C}}, \qquad \Lambda_v^{(0,1)} = \langle d\bar{z}_1, d\bar{z}_2\rangle_{\mathbb{C}}$$

with

$$\begin{aligned} dz_1 &= v \otimes w, & dz_2 &= v \otimes \tilde{w}, \\ d\bar{z}_1 &= \tilde{v} \otimes \tilde{w}, & d\bar{z}_2 &= -\tilde{v} \otimes w. \end{aligned}$$

Since $Z(W) = \mathbb{P}(V_+)$ we can introduce an affine coordinate c on $Z(W)$ in such a way that I is the point at infinity. Then every hermitian structure $J \neq I$ compatible with the orientation corresponds to the spinor

$$v' = [c:1] = cv + \tilde{v}$$

with unique $c \in \mathbb{C}$. Note that $c = 0$ corresponds to the complex structure $-I$. In general, $-J$ corresponds to $j_+(v')$ so it is given by the complex number $-\frac{1}{\bar{c}}$. Finally, the $(1,0)$-forms for J can be written as

$$\begin{aligned} \zeta_1 &= (cv + \tilde{v}) \otimes w = c\,dz_1 - d\bar{z}_2, \\ \zeta_2 &= (cv + \tilde{v}) \otimes \tilde{w} = c\,dz_2 + d\bar{z}_1. \end{aligned}$$

The above can be applied to the metric (1), we just have to normalize dz_2 and $d\bar{z}_2$ — this involves rescaling them by $f^{\frac{1}{2}}$. We see that the $(1,0)$ forms for J are spanned by

$$\begin{aligned} f^{-\frac{1}{2}}\zeta_1 &= cf^{-\frac{1}{2}}dz_1 - d\bar{z}_2, \\ \zeta_2 &= cf^{\frac{1}{2}}dz_2 + d\bar{z}_1. \end{aligned}$$

and one gets (2) simply by changing the notation (take $\bar{u} = cf^{-\frac{1}{2}}$).

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