The Hausdorff Dimension of the Set of Dissipative Points for a Cantor–Like Model Set for Singly Cusped Parabolic Dynamics

Jörg Schmeling
Bernd O. Stratmann

Vienna, Preprint ESI 1999 (2008) January 25, 2008

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

THE HAUSDORFF DIMENSION OF THE SET OF DISSIPATIVE POINTS FOR A CANTOR-LIKE MODEL SET FOR SINGLY CUSPED PARABOLIC DYNAMICS

JÖRG SCHMELING AND BERND O. STRATMANN

Abstract. In this paper we introduce and study a certain intricate Cantor-like set $\mathcal{C}$ contained in unit interval. Our main result is to show that the set $\mathcal{C}$ itself, as well as the set of dissipative points within $\mathcal{C}$, both have Hausdorff dimension equal to 1. The proof uses the transience of a certain non-symmetric Cauchy-type random walk.

1. Introduction

In this paper we estimate the Hausdorff dimension of the set $\mathcal{C}_\infty$ of dissipative points within a certain Cantor-like subset $\mathcal{C}$ of the unit interval $[0,1) \subset \mathbb{R}$. There are two ways to define these sets. The first is via the interation of a certain interval map Φ, where $\mathcal{C}$ represents the set of points with an infinite forward orbit, and $\mathcal{C}_\infty$ is equal to the basin of attraction of the set of critical points of Φ (see Remark 1.1 at the end of this introduction). The second construction is purely in terms of fractal geometry. Let us remark that our motivation for considering the sets $\mathcal{C}$ and $\mathcal{C}_\infty$ stems from the investigations in [10] of the geometry of limit sets of Kleinian groups with singly cusped parabolic dynamics. For the purposes of this paper this link is irrelevant and therefore we omit the details. However, intuition coming from Kleinian groups has historically played a very important role in in the development of Real and Complex Dynamics, and this paper can be seen as adding to this tradition.

Let us begin with by giving the slightly intricate, but more down-to-earth fractal geometric construction of the sets $\mathcal{C}$ and $\mathcal{C}_\infty$. For this we have to define certain families of fundamental intervals by induction as follows. We start with the unit interval $[0,1)$, and then partition the left half of $[0,1)$ into the infinitely many intervals

$$I_1 := \left[0, \frac{3}{\pi^2}\right), \text{ and } I_{k+1} := \left[\frac{3}{\pi^2}\sum_{l=1}^{k} l^{-2}, \frac{3}{\pi^2}\sum_{l=1}^{k+1} l^{-2}\right), \text{ for } k \in \mathbb{N}.$$

The family of these first-level intervals will be denoted by C_1. Note that the right half $\left[\frac{1}{2}, 1\right)$ of the unit interval, which is clearly not captured by C_1, should be interpreted as 'the hole at the first level'. The second step is to partition each element $I_{k_1} \in C_1$ as follows. By starting from the left

2000 *Mathematics Subject Classification.* Primary: 60D05, 14L35, 51N30.

Key words and phrases. Fractal geometry, Hausdorff dimension, Cauchy random walks, Kleinian groups.

endpoint of I_{k_1}, we partition the left half of I_{k_1} into infinitely many mutually adjacent intervals

$$I_{k_1 k_1+1}, \cdots, I_{k_1 k_1+l}, \cdots,$$

where the diameters of these intervals are given by

$$|I_{k_1 k_1+l}| = \frac{3}{\pi^2} \frac{|I_{k_1}|}{l^2}, \text{ for } l \in \mathbb{N}.$$

Similarly, by starting from the right endpoint of I_{k_1} we insert into the right half of I_{k_1} the (k_1+1) mutually adjacent intervals

$$I_{k_1 k_1}, I_{k_1 k_1-1}, \cdots, I_{k_1 0},$$

with diameters given by

$$|I_{k_1 k_1-l}| = \begin{cases} \frac{3}{\pi^2} \frac{|I_{k_1}|}{(2k_1)^2} & \text{for} \quad l = 0 \\ \frac{3}{\pi^2} \frac{|I_{k_1}|}{l^2} & \text{for} \quad l \in \{1, \ldots, k_1\}. \end{cases}$$

The family of these second-level intervals will be denoted by C_2. Note that in this way we have perforated each $I_{k_1} \in C_1$ such that there is a 'hole' in I_{k_1} with diameter of order $|I_{k_1}|/k_1$.
We then proceed by induction as follows. Suppose that for $n \geq 2$ the n-th level interval $I_{k_1 \ldots k_n}$ has been constructed. The $(n+1)$-th level intervals arising from $I_{k_1 \ldots k_n}$ are then obtained as follows. There are two cases to consider. The first case is that $k_n = 0$, and here the partition only continues in the left half of $I_{k_1 \ldots k_{n-1} 0}$. More precisely, in this case we start from the left endpoint of $I_{k_1 \ldots k_{n-1} 0}$ and partition the left half of $I_{k_1 \ldots k_{n-1} 0}$ into infinitely many mutually adjacent intervals

$$I_{k_1 \cdots k_{n-1} 0 1}, \cdots, I_{k_1 \cdots k_{n-1} 0 l}, \cdots,$$

with diameters given by

$$\left|I_{k_1 \cdots k_{n-1} 0 l}\right| = \frac{3}{\pi^2} \frac{|I_{k_1 \cdots k_n}|}{l^2}, \text{ for } l \in \mathbb{N}.$$

In the second case we have that $k_n \in \mathbb{N}$, and here we start from the left endpoint of $I_{k_1 \ldots k_n}$ and partition the left half of $I_{k_1 \ldots k_n}$ into infinitely many mutually adjacent intervals

$$I_{k_1 \cdots k_n k_n+1}, \cdots, I_{k_1 \cdots k_n k_n+l}, \cdots.$$

The diameters of these intervals are

$$|I_{k_1 \cdots k_n k_n+l}| = \frac{3}{\pi^2} \frac{|I_{k_1 \cdots k_n}|}{l^2}, \text{ for } l \in \mathbb{N}. \tag{1}$$

Similarly, by starting from the right endpoint of $I_{k_1 \ldots k_n}$ we insert into the right half of $I_{k_1 \ldots k_n}$ the (k_n+1) mutually adjacent intervals

$$I_{k_1 \cdots k_n k_n}, I_{k_1 \cdots k_n k_n-1}, \cdots, I_{k_1 \cdots k_n 0},$$

with diameters given by

$$|I_{k_1 \cdots k_n k_n-l}| = \begin{cases} \frac{3}{\pi^2} \frac{|I_{k_1 \cdots k_n}|}{(2k_n)^2} & \text{for} \quad l = 0 \\ \frac{3}{\pi^2} \frac{|I_{k_1 \cdots k_n}|}{l^2} & \text{for} \quad l \in \{1, \ldots, k_n\}. \end{cases} \tag{2}$$

The so obtained set of intervals of the $(n+1)$-th level will be denoted by C_{n+1}. That is,

$$C_n = \{I_{k_1\ldots k_n} : k_1 \in \mathbb{N}, k_{i+1} \in \mathbb{N}_0 \text{ for } i \in \mathbb{N}, \text{ and if } k_i = 0 \text{ then } k_{i+1} \neq 0\}.$$

Again, note that by this we have perforated $I_{k_1\ldots k_n}$ such that in the first case 'the hole' is precisely the right half of $I_{k_1\ldots k_n}$, whereas in the second case the diameter of the hole is of order $|I_{k_1\ldots k_n}|/k_n$. Also, let us emphasize that by construction, the state 0 necessarily has to renew itself. That is, the generation following the interval $I_{k_1\cdots k_{n-1}0}$ is given by $\{I_{k_1\cdots k_{n-1}0k_{n+1}} : k_{n+1} \in \mathbb{N}\}$. Moreover, note that the system can only be stationary at states $k_n \in \mathbb{N}$, which means that if $I_{k_1\ldots k_n}$ is a given interval of some level n then $I_{k_1\ldots k_n k_n}$ exists if and only if $k_n \neq 0$. Finally, note that we always assume that the intervals $I_{k_1\ldots k_n}$ are half open, namely closed to the left and open to the right.
With this inductive construction of the generating intervals $I_{k_1\ldots k_n}$ at hand, the Cantor-like set $\mathcal{C}$ is defined by

$$\mathcal{C} := \bigcap_{n\in\mathbb{N}} \bigcup_{I\in C_n} I.$$

Next, we define the set of dissipative points in $\mathcal{C}$. For this we require the following canonical coding of the elements in $\mathcal{C}$. A finite or infinite sequence $(k_1, k_2, \ldots)$ is called admissable if $I_{k_1\cdots k_n} \in C_n$, for all $n \in \mathbb{N}$. Clearly, the diameter of $I_{k_1\cdots k_n}$ tends to zero as n tends to ∞, for every fixed infinite admissable sequence $(k_n)_{n\in\mathbb{N}}$, and therefore,

$$\bigcap_{n=1}^{\infty} I_{k_1\cdots k_n} \text{ is a singleton.}$$

In particular, each $x \in \mathcal{C}$ is coded uniquely by an infinite admissable sequence, and this gives rise to the bijection

$$\rho : \Sigma \to \mathcal{C}, (k_1, k_2, \ldots) \mapsto \bigcap_{n=1}^{\infty} I_{k_1\cdots k_n},$$

where Σ refers to the set of all admissable sequences. Using this coding, the set $\mathcal{C}_\infty \subset \mathcal{C}$ of dissipative points is then given by

$$\mathcal{C}_\infty := \left\{ x \in \mathcal{C} \,:\, x = \rho(k_1, k_2, \ldots) \text{ and } \lim_{n\to\infty} k_n = \infty \right\}.$$

The following theorem gives the main result of this paper. Here, $\dim_H$ refers to the Hausdorff dimension.

Main Theorem. *For $\mathcal{C}$ and $\mathcal{C}_\infty$ as defined above, we have*

$$\dim_H(\mathcal{C}_\infty) = \dim_H(\mathcal{C}) = 1.$$

Remark 1.1. As already mentioned at the beginning of the introduction, the sets $\mathcal{C}$ and $\mathcal{C}_\infty$ can alternatively be defined in terms of a certain interval map

$$\Phi : \bigcup_{I_k\in C_1} I_k \to [0,1).$$

Namely, the map Φ is given piecewise by $\Phi|_{I_k} := \phi_k$, for each $k \in \mathbb{N}$, where the maps $\phi_k : \bigcup_{I_{kl} \in C_2} I_{kl} \to I_k$ are piecewise linear in the following sense. For each $k \in \mathbb{N}$ and $l \in \mathbb{N}_0$, the map $\phi_k|_{I_{kl}}$ is a linear and

$$\phi_k|_{I_{kl}}(I_{kl}) = \begin{cases} \bigcup_{m=l-1}^{\infty} I_{km} & \text{for} \quad l \neq 0 \\ \\ I_k \setminus I_{k0} & \text{for} \quad l = 0. \end{cases}$$

One then immediately verifies that the set $\mathcal{C}$ is equal to the set of points which have an infinite forward orbit under Φ. Moreover, the centre c_k of I_k is the critical point of ϕ_k, for each $k \in \mathbb{N}$, and thus $Crit(\Phi) := \{c_k : k \in \mathbb{N}\}$ represents the countable set of critical points of Φ. With $\omega_\Phi(x)$ referring to the ω-limit set of an element $x \in [0, 1/2)$ with respect to Φ (that is, $\omega_\Phi(x)$ denotes the set of accumulation points of $\{\Phi^n(x) : n \in \mathbb{N}\}$), and with $\mathcal{B}(Crit(\Phi)) := \{x : \omega_\Phi(x) \subset Crit(\Phi)\}$ denoting the basin of attraction of $Crit(\Phi)$ under Φ, we then have

$$\mathcal{C}_\infty = \mathcal{B}(Crit(\Phi)).$$

Similar types of interval maps have been studied for instance in [4] and [11] in connection with 'wild Cantor sets' and the search for Julia sets of positive Lebesgue measure. It seems worthwhile to point out that the 'Martingale Argument' of Keller (see [4], Section 4.1), which gives a criterion for the basin of attraction of a critical point to be of positive Lebesgue measure, is not applicable to the map Φ and hence does not allow to draw any conclusion for the Lebesgue measure of $\mathcal{C}_\infty$. Nevertheless, recent studies in the theory of Kleinian groups (cf. [1] [3] [5], and also [6]) have confirmed the Ahlfors Conjecture, and applying these to our situation here strongly suggests that $\mathcal{C}$ and $\mathcal{C}_\infty$ are both of 1-dimensional Lebesgue measure equal to 0. However, currently it is still a conjecture that the Lebesgue measure of $\mathcal{C}$ and $\mathcal{C}_\infty$ is equal to zero, and it would be desirable to have an elementary proof of this conjecture.

Acknowledgement: We like to thank the *EPSRC* for supporting a one week visit of the first author to the School of Mathematics at the University of St. Andrews. During this visit we began with the work towards this paper. Also, we like to thank the *Erwin Schrdinger Institute, Vienna* for supporting the workshop *Ergodic Theory - Limit Theorems and Dimensions* (Vienna, 17 - 21 December 2007) during which we made the finishing touches to this paper. Finally, we are grateful to Tatyana Turova for very helpful discussions concerning the random walk argument which we employ here.

2. **Proof of the Main Theorem**

Since $\mathcal{C}_\infty \subset \mathcal{C} \subset [0, 1)$, we have

$$\dim_H(\mathcal{C}_\infty) \leq \dim_H(\mathcal{C}) \leq 1.$$

Therefore, our strategy will be to construct a family of probability measures μ_α on $\mathcal{C}_\infty$, for $1/2 < \alpha < 1$, such that the Hausdorff dimension $\dim_H(\mu_\alpha)$ of the measure μ_α tends to 1 for α tending to 1. Clearly, this will then be sufficient for the proof of the Main Theorem.

2.1. **The family of measures μ_α.**
Let $1/2 < \alpha \le 1$ be fixed. We then define a set function μ_α on the intervals $I_{k_1\cdots k_n}$ by induction in the following way. Define $I_0 := [0,1)$ and set $I_{0k} := I_k$ for all $k \in \mathbb{N}$. Then let

$$\mu_\alpha(I_0) := 1,$$

and define $\mu_\alpha\left(I_{k_1\cdots k_n k_{n+1}}\right)$ for each finite admissable sequence $(k_1, \cdots, k_{n+1})$ as follows. With $\zeta(s) := \sum_{m=1}^\infty m^{-s}$ referring to the Riemann zeta function, we define for $k_n \neq k_{n+1}$,

$$\mu_\alpha\left(I_{k_1\cdots k_n k_{n+1}}\right) := \begin{cases} \frac{\mu_\alpha\left(I_{k_1\cdots k_n}\right)}{2\zeta(2\alpha)}\left(\frac{1}{|k_{n+1}-k_n|^{2\alpha}} + \frac{1}{(k_{n+1}+k_n)^{2\alpha}}\right) & \text{for } k_{n+1} \neq 0 \\ \frac{\mu_\alpha\left(I_{k_1\cdots k_n}\right)}{2\zeta(2\alpha)}\frac{1}{k_n^{2\alpha}} & \text{for } k_{n+1} = 0. \end{cases}$$

Also, for $k_n = k_{n+1}$ let

$$\mu_\alpha\left(I_{k_1\cdots k_n k_n}\right) := \frac{\mu_\alpha\left(I_{k_1\cdots k_n}\right)}{2\zeta(2\alpha)}\frac{1}{(2k_n)^{2\alpha}}.$$

On the first sight, this definition of the set function μ_α might appear to be slightly artificial. However, in the next section we will see that this definition reflects the transition probabilities of a certain (transient) random walk on $\mathbb{N}_0$, and therefore is rather canonical. Before we come to this, let us first state the following consistency property for μ_α. This property can also be deduced using the random walk of Section 2.2. Nevertheless, the following gives an elementary proof of this consistency property.

Lemma 2.1. *For each finite admissable sequence $(k_1, \cdots, k_n)$, we have*

$$\mu_\alpha\left(I_{k_1\cdots k_n}\right) = \sum_{k_{n+1}\ge 0} \mu_\alpha\left(I_{k_1\cdots k_n k_{n+1}}\right).$$

Proof. For $k_n = 0$, we have

$$\sum_{\substack{k_{n+1}\ge 0\\ k_{n+1}\neq 0}} \mu_\alpha\left(I_{k_1\cdots k_{n-1}0k_{n+1}}\right) = \frac{\mu_\alpha\left(I_{k_1\cdots k_{n-1}0}\right)}{2\zeta(2\alpha)}\sum_{l=1}^\infty \frac{2}{l^{2\alpha}} = \mu_\alpha\left(I_{k_1\cdots k_{n-1}0}\right).$$

If $k_n \neq 0$, then we compute

$$\begin{aligned}
&\sum_{k_{n+1}\geq 0} \mu_\alpha\left(I_{k_1\cdots k_n k_{n+1}}\right) \\
&= \sum_{\substack{k_{n+1}>0\\ k_{n+1}\neq k_n}} \frac{1}{2\zeta(2\alpha)}\left(\frac{1}{|k_{n+1}-k_n|^{2\alpha}}+\frac{1}{(k_{n+1}+k_n)^{2\alpha}}\right)\mu_\alpha\left(I_{k_1\cdots k_n}\right)+ \\
&\qquad + \frac{1}{2\zeta(2\alpha)k_n^{2\alpha}}\mu_\alpha\left(I_{k_1\cdots k_n}\right)+\frac{1}{2\zeta(2\alpha)(2k_n)^{2\alpha}}\mu_\alpha\left(I_{k_1\cdots k_n}\right) \\
&= \frac{\mu_\alpha\left(I_{k_1\cdots k_n}\right)}{2\zeta(2\alpha)}\left(\sum_{\substack{k_{n+1}>0\\ k_{n+1}\neq k_n}}\left(\frac{1}{|k_{n+1}-k_n|^{2\alpha}}+\frac{1}{(k_{n+1}+k_n)^{2\alpha}}\right)+\right. \\
&\qquad \left. + \frac{1}{k_n^{2\alpha}}+\frac{1}{(2k_n)^{2\alpha}}\right) \\
&= \frac{\mu_\alpha\left(I_{k_1\cdots k_n}\right)}{2\zeta(2\alpha)}\left(\sum_{k_{n+1}>k_n}\frac{1}{(k_{n+1}-k_n)^{2\alpha}}+\sum_{0<k_{n+1}<k_n}\frac{1}{(k_n-k_{n+1})^{2\alpha}}+\right. \\
&\qquad \left. + \sum_{0<k_{n+1}<k_n}\frac{1}{(k_{n+1}+k_n)^{2\alpha}}+\sum_{k_{n+1}>k_n}\frac{1}{(k_{n+1}+k_n)^{2\alpha}}+\frac{1}{k_n^{2\alpha}}+\frac{1}{(2k_n)^{2\alpha}}\right) \\
&= \frac{\mu_\alpha\left(I_{k_1\cdots k_n}\right)}{2\zeta(2\alpha)}\cdot 2\sum_{l=1}^{\infty}\frac{1}{l^{2\alpha}} = \mu_\alpha\left(I_{k_1\cdots k_n}\right).
\end{aligned}$$

□

The following is an immediate consequence of the previous lemma.

Corollary 2.2. *The measure μ_α is a probability measure on $\mathcal{C}$.*

2.2. **The associated random walk.**

In this section we show that the measure μ_α can be interpreted in terms of a certain random walk. In particular, this will give that μ_α has the Markov property. For this, let the random variables $\tilde{X}_n^\alpha$ be defined by the probability (with respect to μ_α) being in the interval $I_{k_1\cdots k_n k_{n+1}}$ given that in the previous step the process has been in the interval $I_{k_1\cdots k_n}$. That is, the random variables $\tilde{X}_n^\alpha$ is given as follows.

- For $k_{n+1}>0$ such that $k_{n+1}\neq k_n$, let

$$\begin{aligned}
&\mathbb{P}\left(\tilde{X}_{n+1}^\alpha = I_{k_1\cdots k_n k_{n+1}} \mid \tilde{X}_n^\alpha = I_{k_1\cdots k_n}\right) := \frac{\mu_\alpha\left(I_{k_1\cdots k_n k_{n+1}}\right)}{\mu_\alpha\left(I_{k_1\cdots k_n}\right)} \\
&= \frac{1}{2\zeta(2\alpha)}\left(\frac{1}{|k_{n+1}-k_n|^{2\alpha}}+\frac{1}{(k_{n+1}+k_n)^{2\alpha}}\right).
\end{aligned}$$

- For $k_{n+1}>0$ such that $k_{n+1}=k_n$, let

$$\mathbb{P}\left(\tilde{X}_{n+1}^\alpha = I_{k_1\cdots k_n k_n} \mid \tilde{X}_n^\alpha = I_{k_1\cdots k_n}\right) := \frac{\mu_\alpha\left(I_{k_1\cdots k_n k_n}\right)}{\mu_\alpha\left(I_{k_1\cdots k_n}\right)} = \frac{1}{2\zeta(2\alpha)}\frac{1}{(2k_n)^{2\alpha}}.$$

- If $k_{n+1}=0$, then

$$\mathbb{P}\left(\tilde{X}_{n+1}^\alpha = I_{k_1\cdots k_n 0} \mid \tilde{X}_n^\alpha = I_{k_1\cdots k_n}\right) := \frac{\mu_\alpha\left(I_{k_1\cdots k_n 0}\right)}{\mu_\alpha\left(I_{k_1\cdots k_n}\right)} = \frac{1}{2\zeta(2\alpha)}\frac{1}{k_n^{2\alpha}}.$$

Clearly, these conditional probabilities do not depend on $k_1, \cdots, k_{n-1}$. Hence, we can define an associated random walk X_n^α on $\mathbb{N}_0$ by the following transition probabilities.

- For $l, m \in \mathbb{N}_0$, let

$$\mathbb{P}(X_{n+1}^\alpha = l \,|\, X_n^\alpha = m) := \begin{cases} \frac{1}{2\zeta(2\alpha)} \left(\frac{1}{|m-l|^{2\alpha}} + \frac{1}{(m+l)^{2\alpha}} \right) & \text{for} \quad l \neq 0, l \neq m \\ \frac{1}{2\zeta(2\alpha)} \frac{1}{(2m)^{2\alpha}} & \text{for} \quad l \neq 0, l = m \\ \frac{1}{2\zeta(2\alpha)} \frac{1}{m^{2\alpha}} & \text{for} \quad l = 0, m \neq 0 \\ 0 & \text{for} \quad l = m = 0. \end{cases}$$

The random walk X_n^α is very closely connected to our original geometric setting, since it allows to recover the measure μ_α as follows.

$$\begin{aligned} &\mathbb{P}(X_1^\alpha = k_1, \cdots, X_n^\alpha = k_n) \\ &= \mathbb{P}(X_n^\alpha = k_n \,|\, X_{n-1}^\alpha = k_{n-1}) \cdot \mathbb{P}(X_{n-1}^\alpha = k_{n-1} \,|\, X_{n-2}^\alpha = k_{n-2}) \cdot \\ &\qquad \ldots \cdot \mathbb{P}(X_1^\alpha = k_1 \,|\, X_0^\alpha = 0) \\ &= \frac{\mu_\alpha\left(I_{k_1 \cdots k_n}\right)}{\mu_\alpha\left(I_{k_1 \cdots k_{n-1}}\right)} \frac{\mu_\alpha\left(I_{k_1 \cdots k_{n-1}}\right)}{\mu_\alpha\left(I_{k_1 \cdots k_{n-2}}\right)} \cdots \frac{\mu_\alpha\left(I_{k_1}\right)}{\mu_\alpha\left([0,1)\right)} = \mu_\alpha\left(I_{k_1 \cdots k_n}\right). \end{aligned}$$

The aim now is to show that the random walk X_n^α is transient. This will then allow us to deduce that μ_α is non-trivial on $\mathcal{C}_\infty$.

Theorem 2.3. *For each $1/2 < \alpha < 1$, the random walk X_n^α on $\mathbb{N}_0$ is transient. That is, we have $\mathbb{P}$-almost surely,*

$$\lim_{n \to \infty} X_n^\alpha = \infty.$$

Proof. Let $1 < \beta < 2$ be fixed, and consider the Cauchy-type random walk Y_n^β on $\mathbb{Z}$, given by the transition probabilities

$$\mathbb{P}(Y_{n+1}^\beta = m + l \,|\, Y_n^\beta = m) := \frac{1}{2\zeta(\beta)} \frac{1}{|l|^\beta} \text{ for } n \in \mathbb{N}, m \in \mathbb{Z} \text{ and } l \in \mathbb{Z} \setminus \{0\}.$$

It is well known that Y_n^β is symmetric, and that Y_n^β is transient if and only if $\beta < 2$. Let $\tilde{Y}_n^\beta$ denote the random walk which arises from Y_n^β in the following way. Let $l_1, \cdots, l_n$ be the sequence of jumps of Y_n^β after the first n steps. With r_1 referring to the first time at which Y_k^β crosses 0, we set $\tilde{Y}_{r_1}^\beta := |Y_{r_1}^\beta|$. Subsequently, we apply the jumps $|l_{r_1+1}|$ up to $|l_{r_2}|$, where r_2 refers to the next time Y_n^β crosses 0. After that, we apply the jumps $|l_{r_2+1}|$ up to $|l_{r_3}|$, where r_3 denotes the next time at which the process Y_k^β first crosses 0 again. More precisely, with r_i referring to the i-th time the process Y_k^β crosses 0, we let $\tilde{Y}_{r_i}^\beta := |Y_{r_i}^\beta|$, and between each two consecutive crossings r_i and r_{i+1} we define the jumps of $\tilde{Y}_{r_i}^\beta$ to be $|l_{r_i+1}|, \ldots, |l_{r_{i+1}}|$. Note that the random walk we have just described is equal to the random walk $|Y_n^\beta|$. Also, note that since Y_n^β is a symmetric random walk, the above modification of the sample path does not alter its probability. Therefore, it immediately follows that the transience of Y_n^β implies that $\tilde{Y}_n^\beta$ is transient. Using the fact that by symmetry of Y_n^β we have $\mathbb{P}(Y_n^\beta = m) = \mathbb{P}(Y_n^\beta = -m)$, we now compute the transition probabilities of the random walk $\tilde{Y}_n^\beta$ on $\mathbb{N}_0$ as follows. For $l, m \in \mathbb{N}_0$ such that $l \neq 0$, and $m \neq l$, we have

$$\begin{aligned}
&\mathbb{P}(\tilde{Y}_{n+1}^\beta = l \,|\, \tilde{Y}_n^\beta = m) = \mathbb{P}(|Y_{n+1}^\beta| = l \,|\, |Y_n^\beta| = m) \\
&= \frac{\mathbb{P}(|Y_{n+1}^\beta| = l,\, Y_n^\beta = m \text{ or } Y_n^\beta = -m)}{\mathbb{P}(Y_n^\beta = m \text{ or } Y_n^\beta = -m)} \\
&= \frac{\mathbb{P}(|Y_{n+1}^\beta| = l,\, Y_n^\beta = m)}{2\mathbb{P}(Y_n^\beta = m)} + \frac{\mathbb{P}(|Y_{n+1}^\beta| = l,\, Y_n^\beta = -m)}{2\mathbb{P}(Y_n^\beta = -m)} \\
&= \frac{\mathbb{P}(|Y_{n+1}^\beta| = l,\, Y_n^\beta = m)}{\mathbb{P}(Y_n^\beta = m)} = \frac{1}{2\zeta(\beta)}\left(\frac{1}{|m-l|^\beta} + \frac{1}{(m+l)^\beta}\right).
\end{aligned}$$

Similarly, we obtain for $l = m \neq 0$,

$$\mathbb{P}(\tilde{Y}_{n+1}^\beta = m \,|\, \tilde{Y}_n^\beta = m) = \frac{1}{2\zeta(\beta)}\frac{1}{(2m)^\beta},$$

and for $l = 0$ and $m \neq l$,

$$\mathbb{P}(\tilde{Y}_{n+1}^\beta = 0 \,|\, \tilde{Y}_n^\beta = m) = \frac{1}{2\zeta(\beta)}\frac{1}{m^\beta}.$$

Finally, note that we immediately have

$$\mathbb{P}(\tilde{Y}_{n+1}^\beta = 0 \,|\, \tilde{Y}_n^\beta = 0) = 0.$$

This shows that the transition probabilities of $\tilde{Y}_n^\beta$ coincide with the ones of $X_n^{\beta/2}$. Therefore, since $\tilde{Y}_n^\beta$ is transient, it follows that $X_n^{\beta/2}$ is transient. This finishes the proof of the theorem. □

As already announced before, Theorem 2.3 has the following important implication.

Corollary 2.4. *For every* $1/2 < \alpha < 1$*, we have*

$$\mu_\alpha(\mathcal{C}_\infty) = 1.$$

Remark 2.5. Note that the proof of Theorem 2.3 relies heavily on the fact that $1/2 < \alpha < 1$. Namely, for instance for $\alpha = 1$ the associated random walk is recurrent, and consequently the measure μ_α vanishes on $\mathcal{C}_\infty$.

2.3. **Approximating the essential support of μ_α.**
In order to prepare our estimate of the lower pointwise dimension of μ_α, we need a further approximation of the essential support of this measure. We will see that μ_α-almost surely the diameters of the coding intervals of an element of $\mathcal{C}_\infty$ do not shrink too fast. For this we define, for $\gamma \in \mathbb{R}$,

$$\mathcal{C}_\infty^\gamma := \left\{ x \in \mathcal{C}_\infty \,:\, x = \rho(k_1, k_2, \cdots) \text{ such that } \limsup_{n\to\infty} \frac{|k_{n+1} - k_n|}{n^\gamma} \leq 1 \right\}.$$

Lemma 2.6. *For each* $1/2 < \alpha < 1$ *and* $\gamma > 1/(2\alpha - 1)$*, we have*

$$\mu_\alpha(\mathcal{C}_\infty^\gamma) = 1.$$

Proof. The proof is an easy consequence of the Borel-Cantelli lemma. Indeed, first note that for $\beta = 2\alpha$ and $k \in \mathbb{N}$ we have

$$\mathbb{P}(|Y_n^\beta - Y_{n-1}^\beta| \geq k) \geq \mathbb{P}(|\tilde{Y}_n^\beta - \tilde{Y}_{n-1}^\beta| \geq k).$$

The latter is an immediate consequence of the fact that the random walk Y_n^β has the same distribution as $\tilde{Y}_n^\beta$, but without reflections at 0. Hence, it suffices to prove the lemma for the symmetric random walk Y_n^β. For this we define

$$p_n^\gamma := \mathbb{P}(|Y_n^\beta - Y_{n-1}^\beta| \geq n^\gamma).$$

We then have, for each $n \in \mathbb{N}$ and with $c(\beta) > 0$ referring to some universal constant,

$$p_n^\gamma = 2 \sum_{k=n^\gamma}^{\infty} \frac{1}{k^\beta} \leq c(\beta) \frac{1}{n^{\gamma(\beta-1)}}.$$

Since the series $\sum_{n=1}^\infty p_n^\gamma$ converges for $\gamma > 1/(\beta - 1)$, the Borel-Cantelli Lemma implies that $\mathbb{P}$-almost surely there are at most finitely many n which satisfy the inequality

$$|Y_n^\beta - Y_{n-1}^\beta| \geq n^\gamma.$$

This shows that for μ_α-almost every $x = \rho(k_1, k_2, \ldots) \in \mathcal{C}$ we have

$$\limsup_{n\to\infty} \frac{|k_{n+1} - k_n|}{n^\gamma} \leq 1.$$

□

2.4. **The lower pointwise dimension on fundamental intervals.**
The main result of this section will be the following estimate for the lower pointwise dimension of the measure μ_α restricted to the fundamental intervals $I_{k_1 \ldots k_n}$.

Proposition 2.7. *For each $\epsilon > 0$ there exists $1/2 < \alpha < 1$ and $\gamma > 1/(2\alpha - 1)$ such that for every $x = \rho(k_1, k_2, \ldots) \in \mathcal{C}_\infty^\gamma$,*

$$\liminf_{n\to\infty} \frac{\log \mu_\alpha(I_{k_1 \cdots k_n})}{\log |I_{k_1 \cdots k_n}|} \geq \alpha - \epsilon.$$

Furthermore, in here we have that α tends to 1 for ϵ tending to 0.

Proof. Since we are interested in the asymptotic behaviour of $I_{k_1 \ldots k_n}$ for points $x = \rho(k_1, k_2, \ldots) \in \mathcal{C}_\infty^\gamma$, Corollary 2.4 and Lemma 2.6 imply that we can assume without loss of generality that $k_n > 0$ and $|k_{n+1} - k_n| \leq n^\gamma$, for all $n \in \mathbb{N}$. Furthermore, for ease of exposition we only consider sequences which do not contain repetitions. That is, we assume that $k_n \neq k_{n+1}$, for all $n \in \mathbb{N}$. The case with repetitions can be dealt with in similar way and is left to the reader. Using the definition of μ_α, we then have

$$\begin{aligned}
&\frac{\log \mu_\alpha(I_{k_1\cdots k_n k_{n+1}})}{\log |I_{k_1\cdots k_n k_{n+1}}|} \\
&= \frac{-\log(2\zeta(2\alpha))}{\log |I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log\left[\left(\frac{1}{|k_{n+1}-k_n|^{2\alpha}} + \frac{1}{(k_n+k_{n+1})^{2\alpha}}\right)\mu_\alpha(I_{k_1\cdots k_n})\right]}{\log |I_{k_1\cdots k_n k_{n+1}}|} \\
&\geq \frac{-\log(2\zeta(2\alpha))}{\log |I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log\left[\frac{2}{|k_{n+1}-k_n|^{2\alpha}}\mu_\alpha(I_{k_1\cdots k_n})\right]}{\log |I_{k_1\cdots k_n k_{n+1}}|} \\
&= \frac{-\log(2\zeta(2\alpha))}{\log |I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log\left[\frac{2\cdot 2^\alpha \zeta(2)^\alpha |I_{k_1\cdots k_{n+1}}|^\alpha}{|I_{k_1\cdots k_n}|^{2\alpha}}\mu_\alpha(I_{k_1\cdots k_n})\right]}{\log |I_{k_1\cdots k_n k_{n+1}}|} \\
&= \frac{\log \frac{2^\alpha \zeta(2)^\alpha}{\zeta(2\alpha)}}{\log |I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log(|I_{k_1\cdots k_{n+1}}|^\alpha)}{\log(I_{k_1\cdots k_{n+1}})} + \frac{\log \frac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log |I_{k_1\cdots k_n k_{n+1}}|} \\
&= \alpha + \frac{\log \frac{2^\alpha \zeta(2)^\alpha}{\zeta(2\alpha)}}{\log |I_{k_1\cdots k_n k_{n+1}}|} + \frac{\log \frac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log |I_{k_1\cdots k_n k_{n+1}}|}.
\end{aligned}$$

Since

$$|I_{k_1\cdots k_{n+1}}| < \left(\frac{1}{2\zeta(2)}\right)^n, \tag{3}$$

it follows for each $\kappa > 0$ and for all n sufficiently large,

$$\frac{\log \frac{2^\alpha \zeta(2)^\alpha}{\zeta(2\alpha)}}{\log |I_{k_1\cdots k_n k_{n+1}}|} > -\kappa. \tag{4}$$

Clearly, we even have that the limit of the latter expression is equal to 0. This settles the second term in the final line in the above calculation. The third term is more subtle, and for this we proceed as follows. Using (1) and (2), we derive with the convention $k_0 \equiv 0$,

$$|I_{k_1\cdots k_n}|^\alpha = \prod_{i=0}^{n-1}\left(\frac{1}{2^\alpha \zeta(2)^\alpha}\frac{1}{|k_{i+1}-k_i|^{2\alpha}}\right).$$

Similarly, using the recursive definition of μ_α, we obtain

$$\mu_\alpha(I_{k_1\cdots k_n}) = \prod_{i=0}^{n-1}\left(\frac{1}{2\zeta(2\alpha)}\left[\frac{1}{|k_{i+1}-k_i|^{2\alpha}} + \frac{1}{(k_{i+1}+k_i)^{2\alpha}}\right]\right).$$

Hence,

$$\frac{\log \frac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log|I_{k_1\cdots k_n k_{n+1}}|} = \frac{\log \frac{\prod_{i=0}^{n-1}\left(\frac{1}{2\zeta(2\alpha)}\left[\frac{1}{|k_{i+1}-k_i|^{2\alpha}}+\frac{1}{(k_{i+1}+k_i)^{2\alpha}}\right]\right)}{\prod_{i=0}^{n-1}\left(\frac{1}{2^\alpha\zeta(2)^\alpha}\frac{1}{|k_{i+1}-k_i|^{2\alpha}}\right)}}{\log\left(\prod_{i=0}^{n}\left(\frac{1}{2^\alpha\zeta(2)^\alpha}\frac{1}{|k_{i+1}-k_i|^{2\alpha}}\right)\right)}$$

$$= \frac{\log\left(\left[\frac{2^\alpha\zeta(2)^\alpha}{2\zeta(2\alpha)}\right]^n \prod_{i=0}^{n-1}\left[1+\left(\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right)^{2\alpha}\right]\right)}{\log\left(\left[\frac{1}{2^\alpha\zeta(2)^\alpha}\right]^{n+1}\prod_{i=0}^{n}\frac{1}{|k_{i+1}-k_i|^{2\alpha}}\right)}$$

$$= \frac{n\alpha\log(2\zeta(2)) - n\log(2\zeta(2\alpha)) + \sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{-(n+1)\log(2\zeta(2)) + \sum_{i=0}^{n}\log\frac{1}{|k_{i+1}-k_i|^{2\alpha}}}$$

$$= \frac{\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) - \frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{\frac{n+1}{n}\log(2\zeta(2)) + \frac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}}.$$

Let $\kappa > 0$ be fixed. We then distinguish the following two cases.
First, if for some $n \in \mathbb{N}$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right) < \kappa,$$

then we obtain for α sufficiently close to 1,

$$\frac{\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) - \frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{\frac{n+1}{n}\log(2\zeta(2)) + \frac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}}$$
$$\geq -\left(\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2))\right) - \frac{\kappa}{\log(2\zeta(2)) + \frac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}}$$
$$\geq -\left(\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2))\right) - \kappa \geq -2\kappa.$$

Here we made use of the fact that

$$\lim_{\alpha\to 1}\left(\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2))\right) = 0,$$

which implies $\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) < \kappa$, for all α sufficiently close to 1. Also, note that in here the lower bound on α depends only on κ and not on n. In particular, we also have that α tends to 1 as κ tends to 0.
Before we start with the discussion of the second case, first note that since $0 < \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \leq 1$, we clearly always have

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right) \leq \log 2. \tag{5}$$

Furthermore, since k_n tends to infinity, there exists $j(\kappa) \in \mathbb{N}$ such that

$$\log\frac{\kappa^2 k_i^2}{4} > \frac{2\log 2}{\kappa^2}, \quad \text{for all } i \geq j(\kappa). \tag{6}$$

Let us now come to the second case. That is, we now assume that for some $n \in \mathbb{N}$ we have

$$\frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)\geq\kappa.$$

Since $x > \log(1+x)$ for all $x > 0$, we then have

$$\frac{1}{n}\sum_{i=0}^{n-1}\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\geq\kappa.$$

Let us make the following two observations. Firstly, using the fact that $0 < \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \leq 1$, we can apply Chebyshev's Inequality, which gives that for n suffciently large,

$$\operatorname{card}\mathcal{I}_n > \kappa n,$$

where

$$\mathcal{I}_n := \left\{ i \in [j(\kappa), n] : \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \geq \frac{\kappa}{2}\right\}.$$

Secondly, note that for $1/2 < \alpha < 1$ the following implication holds.

$$\text{If } \left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha} \geq \frac{\kappa}{2}, \quad \text{then} \quad |k_{i+1}-k_i| > \frac{\kappa}{2}k_i.$$

Combining these two observations with (6), we then compute

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha} &\geq \frac{\alpha}{n}\sum_{i\in\mathcal{I}_n}\log|k_{i+1}-k_i|^2 \geq \frac{\alpha}{n}\sum_{i\in\mathcal{I}_n}\log\frac{\kappa^2k_i^2}{4}\\ &\geq \frac{\alpha}{n}\sum_{i\in\mathcal{I}_n}\frac{2\log 2}{\kappa^2} > \frac{\log 2}{\kappa^2 n}\operatorname{card}\mathcal{I}_n > \frac{\log 2}{\kappa}.\end{aligned}$$

Inserting this into our estimate above and using (5), it follows

$$\begin{aligned}&\frac{\log(2\zeta(2\alpha)) - \alpha\log(2\zeta(2)) - \frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\left[\frac{|k_{i+1}-k_i|}{k_{i+1}+k_i}\right]^{2\alpha}\right)}{\frac{n+1}{n}\log(2\zeta(2)) + \frac{1}{n}\sum_{i=0}^{n}\log|k_{i+1}-k_i|^{2\alpha}}\\ &\geq \frac{-\log 2}{\log(2\zeta(2)) + \frac{\log 2}{\kappa}} = -\kappa\frac{\log 2}{\kappa\log(2\zeta(2)) + \log 2}\\ &\geq -\kappa.\end{aligned}$$

This finishes the second case.

Combining the latter results with (4), and putting $\epsilon := 3\kappa$, we have now shown that

$$\liminf_{n\to\infty}\frac{\log\frac{\mu_\alpha(I_{k_1\cdots k_n})}{|I_{k_1\cdots k_n}|^\alpha}}{\log|I_{k_1\cdots k_n k_{n+1}}|} \geq -\epsilon,$$

and hence,

$$\liminf_{n\to\infty}\frac{\log\mu_\alpha(I_{k_1\cdots k_n})}{\log|I_{k_1\cdots k_n}|} \geq \alpha - \epsilon,$$

Since α tends to 1 for ϵ tending to 0, the proof is complete

□

2.5. **The proof of the Main Theorem.**
Recall that the lower pointwise dimension $\underline{d}_\nu(x)$ of a Borel measure ν on $\mathbb{R}$ at a point $x \in \mathbb{R}$ is given by

$$\underline{d}_\nu(x) := \liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r},$$

where $B(x,r)$ refers to the interval centred at x with diameter equal to $2r$. The idea is to apply the well-known Mass Distribution Principle of Frostman [8] and Billingsley [2] (see also e.g. [7]).
In order to be able to apply the Mass Distribution Principle, we still require the following straight forward generalization of Furstenberg's Lemma [9].

Lemma 2.8. *Let ν be a Borel measure on $\mathbb{R}$, and let (r_n) be a sequence of positive numbers for which $\lim_{n\to\infty} r_n = 0$ and $\lim_{n\to\infty}(\log r_{n+1}/\log r_n) = 1$. We then have for every $x \in \mathbb{R}$,*

$$\underline{d}_\nu(x) = \liminf_{n\to\infty} \frac{\log \nu(B(x,r_n))}{\log r_n}.$$

Proof. For $r > 0$ we define $n = n(r) := \max\{k \in \mathbb{N} : r_k \geq r\}$. The assertion of the lemma is then an immediate consequence of the following simple calculation.

$$\frac{\log \nu(B(x,r))}{\log r} \geq \frac{\log \nu(B(x,r_n))}{\log r_{n+1}} = \frac{\log r_n}{\log r_{n+1}} \frac{\log \nu(B(x,r_n))}{\log r_n}.$$

□

Proof of the Main Theorem. Let $\epsilon > 0$ be given, and then fix $1/2 < \alpha < 1$ and $\gamma > 1/(2\alpha - 1)$ as in Proposition 2.7. By Lemma 2.6 we have that in order to find a lower bound for $\dim_H(\mu_\alpha)$ it is sufficient to give an estimate for $\underline{d}_{\mu_\alpha}(x)$ from below, for each $x = \rho(k_1, k_2, \ldots) \in \mathcal{C}_\infty^\gamma$. For this note that Proposition 2.7 implies

$$\liminf_{n\to\infty} \frac{\log \mu_\alpha(I_{k_1\cdots k_n})}{\log |I_{k_1\cdots k_n}|} \geq \alpha - \epsilon.$$

In order to deduce the desired lower bound for $\underline{d}_{\mu_\alpha}(x)$, we then use (3) and the definition of $\mathcal{C}_\infty^\gamma$, which gives for $r_n := |I_{k_1\cdots k_n}|$,

$$\begin{aligned}\lim_{n\to\infty} \frac{\log r_{n+1}}{\log r_n} &= 1 + \lim_{n\to\infty} \frac{\log \frac{|I_{k_1\cdots k_n k_{n+1}}|}{|I_{k_1\cdots k_n}|}}{\log |I_{k_1\cdots k_n}|} \\ &\leq 1 + \lim_{n\to\infty} \frac{\gamma \log n}{n \log(2\zeta(2))} = 1.\end{aligned}$$

Therefore, Lemma 2.8 implies that for each $x \in \mathcal{C}_\infty^\gamma$,

$$\underline{d}_{\mu_\alpha}(x) \geq \alpha - \epsilon.$$

Combining this with Corollary 2.2, Corollary 2.4 and Lemma 2.6, we have by the Mass Distribution Principle,

$$\dim_H(\mathcal{C}) \geq \dim_H(\mathcal{C}_\infty) \geq \dim_H(\mathcal{C}_\infty^\gamma) \geq \dim_H(\mu_\alpha) \geq \alpha - \epsilon.$$

Finally, note that by Proposition 2.7 we have that α tends to 1 for ϵ tending to 0. This completes the proof of the theorem.

□

References

[1] I. Agol, 'Tameness of hyperbolic 3-manifolds', *preprint* arXiv:math.GT/0405568.
[2] P. Billingsley, *Ergodic theory and information*, Wiley, New York, 1979.
[3] J. F. Brock, R. D. Canary, and Y. N. Minsky, 'Classification of kleinian surface groups ii: the ending lamination conjecture', *preprint*, arXiv:math.GT/0412006.
[4] H. Bruin, G. Keller, T. Nowicki, S. van Strien, 'Wild Cantor attractors exist', *Ann. of Math.* (2) **143** (1996) 97 - 130.
[5] D. Calegari, D. Gabai. 'Shrinkwrapping and the taming of hyperbolic 3-manifolds', *preprint*, arXiv:math.GT/0407161.
[6] S. Choi, 'Drilling cores of hyperbolic 3-manifolds to prove tameness', *preprint*, arXiv:math.GT/0410381.
[7] K.J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications*, J. Wiley, 1990.
[8] O. Frostman, 'Potential d'equilibre et capacité des ensembles avec quelques applications à la théorie des fonctions', *Meddel. Lunds Univ. Math. Sem.* **3** (1935) 1-118.
[9] H. Furstenberg, 'Intersection of Cantor sets and transversality of semigroups I', in *Problems in analysis*, Princeton Univ. Press (1970) 41-59.
[10] J.R. Parker, B.O. Stratmann, 'Kleinian groups with singly cusped parabolic fixed points', *Kodai Journ. of Math.* **24** (2001) 169 - 206.
[11] B.O. Stratmann, R. Vogt, 'Fractal dimensions for dissipative sets', *Nonlinearity* **10** (1997) 565-577.

Jörg Schmeling: Department of Mathematics, LTH, University of Lund, P.O. Box 118, SE-221 00 LUND, Sweden
E-mail address: joerg@maths.lth.se

Bernd O. Stratmann, Mathematical Institute, University of St. Andrews, North Haugh, St. Andrews KY16 9SS, Scotland
E-mail address: bos@st-and.ac.uk