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Abstract

Some new results on higher order analogs $\mathbf{dR}_\sigma$, σ being a sequence of positive integers, of the standard de Rham complex $\mathbf{dR} \equiv \mathbf{dR}_{(1,\ldots,1,..)}$ are presented. First we sketch an algebraic machinery which allows us to prove an analog of the infinitesimal Stoke's formula (also called the Cartan homotopy formula) for $\mathbf{dR}_\sigma$'s with non-decreasing σ's. Second, we outline the key points of the proof that the cohomology of $\mathbf{dR}_\sigma$ does not depend on σ, under some smoothness assumptions.

1 Definitions

We recall briefly the necessary definitions from [1] and [2] (see also [3]). Let K be a commutative ring with unity and A a commutative, associative unitary K-algebra. $A-\mathbf{Mod}$ will designate the category of A-modules and, if $\mathbf{C}$ is a full subcategory of $A-\mathbf{Mod}$, $[\mathbf{C},\mathbf{C}]$ the category of functors $\mathbf{C} \longrightarrow \mathbf{C}$. A functor $T : \mathbf{C} \longrightarrow A-\mathbf{Mod}$ will be said *strictly representable* in $\mathbf{C}$ if it is representable in $\mathbf{C}$, say by τ, and if moreover there is a functorial isomorphism $T \simeq Hom_A(\tau,\cdot)$ in $[\mathbf{C}, A-\mathbf{Mod}]$.

If P and Q are A-modules and $a \in A$ we define:

$$\begin{gathered}\delta_a : Hom_K(P,Q) \longrightarrow Hom_K(P,Q) \\ \Phi \longmapsto \{\delta_a\Phi : p \longmapsto \Phi(ap) - a\Phi(p)\} \quad , \; p \in P\end{gathered}$$

(where, as always if no confusion may arise, we use mere juxtaposition to indicate both A-module multiplications in P and Q). For each $a \in A$, δ_a is a morphism of K-modules,

$$\delta_{a_1} \circ \delta_{a_2} = \delta_{a_2} \circ \delta_{a_1} \quad , \forall a_1, a_2 \in A.$$

Definition 1.1 *A ($K-$) differential operator of order $\leq s$ from the A-module P to the A-module Q , is an element $\Delta \in Hom_K(P,Q)$ such that:*

$$[\delta_{a_0} \circ \delta_{a_1} \circ ... \circ \delta_{a_s}] (\Delta) = 0, \ \forall \{a_0, a_1, ..., a_s\} \subset A.$$

The *set* $\mathcal{D}iff_k(P,Q)$ of differential operators of order $\leq k$ from P to Q comes naturally endowed with two different A-module structures:

(i) $(\mathcal{D}iff_k(P,Q), \tau) \doteq Diff_k(P,Q)$ (left),

$\tau : A \times \mathcal{D}iff_k(P,Q) \longrightarrow \mathcal{D}iff_k(P,Q) : (a, \Delta) \longmapsto \tau(a,\Delta) : p \longmapsto a\Delta(p)$

(ii) $(\mathcal{D}iff_k(P,Q), \tau^+) \doteq Diff_k^+(P,Q)$ (right),

$\tau^+ : A \times \mathcal{D}iff_k(P,Q) \longrightarrow \mathcal{D}iff_k(P,Q) : (a, \Delta) \longmapsto \tau^+(a,\Delta) : p \longmapsto \Delta(ap)$.

We will often write, to be coincise, $\tau(a,\Delta) \equiv a\Delta$ and $\tau^+(a,\Delta) \equiv a^+\Delta$. As is easily seen, $(\mathcal{D}iff_k(P,Q), (\tau, \tau^+)) \doteq Diff_k^{(+)}(P,Q)$ turns out to be an (A,A)-bimodule.

Remark 1.1 *Since*

$$\delta_{a_0}(\Delta) \equiv 0 \Leftrightarrow \Delta(a_0 p) = a_0 \Delta p$$
$$\forall \ a_0 \in A, \ \forall p \in P$$

we have $\mathcal{D}iff_0(P,Q) \underset{\mathbf{Ens}}{\equiv} Hom_A(P,Q)$ and also $Hom_A(P,Q) \simeq Diff_0(P,Q) \simeq Diff_0^+(P,Q)$ as A-modules.

The obvious inclusions (of sets):

$$\mathcal{D}iff_k(P,Q) \hookrightarrow \mathcal{D}iff_l(P,Q), \ k \leq l$$

induce the monomorphisms of (A,A)-bimodules:

$$Diff_k^{(+)}(P,Q) \hookrightarrow Diff_l^{(+)}(P,Q), \ k \leq l;$$

thus giving rise to a directed system (over $\mathbf{N}$) in $(A,A) - Bi\mathbf{Mod}$ (the category of (A,A)-bimodules):

$$Diff_0^{(+)}(P,Q) \hookrightarrow Diff_1^{(+)}(P,Q) \hookrightarrow ... \hookrightarrow Diff_n^{(+)}(P,Q) \hookrightarrow ...$$

whose direct limit is the (A,A)-bimodule:

$$dir \lim_{n \geq 0} \ Diff_n^{(+)}(P,Q) \equiv \bigcup_{n \geq 0} Diff_n^{(+)}(P,Q) \doteq Diff^{(+)}(P,Q)$$

filtered by $\left\{Diff_n^{(+)}(P,Q)\right\}_{n \geq 0}$.

$\mathbf{Pr} : (A, A) - Bi\mathbf{Mod} \longrightarrow A - \mathbf{Mod} : (P, P^+) \longmapsto P$

$\mathbf{Pr}^+ : (A, A) - Bi\mathbf{Mod} \longrightarrow A - \mathbf{Mod} : (P, P^+) \longmapsto P^+$

we get the two filtered A-modules ($\mathbf{Pr}$ and $\mathbf{Pr}^+$ commutes with direct limits):

$$\mathbf{Pr}\left(Diff^{(+)}(P, Q)\right) = Diff(P, Q) \doteq dir\lim_{n\geq o} Diff_n(P, Q) \equiv \bigcup_{n\geq 0} Diff_n(P, Q)$$
$$\mathbf{Pr}^+\left(Diff^{(+)}(P, Q)\right) = Diff^+(P, Q) \doteq dir\lim_{n\geq o} Diff_n^+(P, Q) \equiv \bigcup_{n\geq 0} Diff_n^+(P, Q)$$

(where direct limits are to be understood, now, in $A - \mathbf{Mod}$).

Putting $Diff_k(A, Q) \doteq Diff_k Q$ and $Diff_k^+(A, Q) \doteq Diff_k^+ Q$, we obtain the functors[1]:

$$Diff_k : Q \longmapsto Diff_k Q,$$
$$Diff_k^+ : Q \longmapsto Diff_k^+ Q$$

from $A - \mathbf{Mod}$ to itself. Remark 1.1 implies $Diff_0^+ = Diff_0 = Id_{A-\mathbf{Mod}}$.

Defining $D_{(k)}(Q) \doteq \{\Delta \in Diff_k Q \mid \Delta(1) = 0\}$, which is an A-submodule of $Diff_k Q$ but not of $Diff_k^+ Q$, we get a functor $D_{(k)} : A - \mathbf{Mod} \to A - \mathbf{Mod}$, together with the short exact sequence:

$$0 \to D_{(k)} \xrightarrow{i_k} Diff_k \xrightarrow{p_k} Id_{A-\mathbf{Mod}} \to 0 \tag{1}$$

in $[A - \mathbf{Mod}, A - \mathbf{Mod}]$, where i_k is the obvious functorial inclusion and p_k is defined by:

$$p_k(Q) : Diff_k Q \to Q : \Delta \mapsto \Delta(1) \quad , \Delta \in Diff_k Q$$

for any A-module Q. The functorial monomorphism $Id_{A-\mathbf{Mod}} \equiv Diff_0 \hookrightarrow Diff_k$ splits (1), so that $Diff_k = D_{(k)} \bigoplus Id_{A-\mathbf{Mod}}$. $D_{(1)}(Q)$ is nothing but the A-module of all Q-valued K-linear derivations on A, denoted in literature, mostly, by $Der_{A/K}(Q)$ (see [5], for example).

Let P and P^+ be the left and right A-modules corresponding to an (A, A)-bimodule $P^{(+)} \equiv (P, P^+)$ (P and P^+ coincide as K-modules, hence as sets). Let's denote by $Diff_k^\bullet(P^+)$ (resp. $D_{(k)}^\bullet(P^+)$) the A-module which coincides with $Diff_k(P^+)$ (resp. $D_{(k)}(P^+)$) as a K-module and whose A-module structure is inherited by that of P (and not of P^+)[2]. For an A-submodule $S \subset P$ we define submodules:

$$Diff_k^\bullet(S \subset P^+) \doteq \{\Delta \in Diff_k^\bullet(P^+) \mid \Delta(A) \subset S\} \underset{A-\mathbf{Mod}}{\subset} Diff_k^\bullet(P^+)$$
$$D_{(k)}^\bullet(S \subset P^+) \doteq \left\{\Delta \in D_{(k)}^\bullet(P^+) \mid \Delta(A) \subset S\right\} \underset{A-\mathbf{Mod}}{\subset} D_{(k)}^\bullet(P^+)$$

[1] We omit the obvious rules on morphisms.

[2] These A-module structures are well defined due to the fact that $(P, P^+) \equiv P^{(+)}$ is a bimodule.

$(\sigma_1, \ldots, \sigma_n)$, *we define by induction on* n, *the functors* $D_{\sigma(n)} : A - \mathbf{Mod} \to A - \mathbf{Mod}$, *by:*

$$D_{\sigma(1)} \doteq D_{(\sigma_1)}$$
$$D_{\sigma(n)} : P \longmapsto D^{\bullet}_{(\sigma_1)} \left(D_{(\sigma_2,\ldots,\sigma_n)}(P) \subset Diff^{+}_{\sigma_2,\ldots,\sigma_n}(P) \right)$$

where, to simplify the notation, we have put $Diff^{+}_{\sigma_2,\ldots,\sigma_n}$ *in place of* $Diff^{+}_{\sigma_2} \circ \ldots \circ Diff^{+}_{\sigma_n}$.

For each $\sigma \in \mathbf{N}^{\infty}_{+}$ and each $n \in \mathbf{N}_+$, we have an exact sequence in $[A - \mathbf{Mod}, A - \mathbf{Mod}]$:

$$0 \to D_{\sigma(n)} \xrightarrow{I_{\sigma(n)}} D^{\bullet}_{\sigma(n-1)} \circ Diff^{+}_{\sigma_n} \xrightarrow{\pi_{\sigma(n)}} D_{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)} \qquad (2)$$

where $I_{\sigma(n)}$ is the natural inclusion and $\pi_{\sigma(n)}$ arises from the "glueing" functorial morphism

$$g_{\sigma_{n-1},\sigma_n} : Diff^{+}_{\sigma_{n-1}} \circ Diff^{+}_{\sigma_n} \to Diff^{+}_{\sigma_{n-1}+\sigma_n}$$
$$\left[g_{\sigma_{n-1},\sigma_n}(P) \right] (\Delta)(a) \doteq [\Delta(a)](1)$$
$$\Delta \in Diff^{+}_{\sigma_{n-1}} \left(Diff^{+}_{\sigma_n}(P) \right), \ a \in A.$$

Let $\Re$ be a *differentially closed subcategory* of $A - \mathbf{Mod}$[3] ([2]) and $P \in Ob(\Re)$. Then, by definition, there exist $\mathbf{J}^k_{\Re}(P) \in Ob(\Re)$ and $j^{\Re}_k(P) \in Diff_k\left(P, \mathbf{J}^k_{\Re}(P)\right)$ such that the map

$$h \longmapsto h \circ j^{\Re}_k(P)$$

establishes an A-module isomorphism between $Hom_A\left(j^{\Re}_k(P), Q\right)$ and $Diff_k(P, Q)$, for any $Q \in Ob(\Re)$, which is natural in Q. $\mathbf{J}^k_{\Re}(P)$ is called the *k-jet module of* P in $\Re$. The (strictly) representative objects $\mathbf{\Lambda}^{\sigma(n)}_{\Re}$ of the functors $D_{\sigma(n)}$ in $\Re$ are also defined. They are higher order analogs of the standard modules of differential forms. We also put, for the sake of uniformity, $\mathbf{\Lambda}^{\emptyset}_{\Re} \doteq A$.

Examples 1.1 *(***i***) If* $\Re = A - \mathbf{Mod}$ *and* $\sigma(n) = (1, \ldots, 1)$ *(n times) then* $\mathbf{\Lambda}^{\sigma(n)}_{\Re}$ *coincides with* $\Omega^n_{A/K}$ *([4] and [5]).*

*(***ii***) If* $K = \mathbf{R}$, $A = C^{\infty}(M; \mathbf{R})$, *M being a smooth real manifold*[4], $\Re$ *is the category of geometric A-modules*[5] *and* $\sigma(n) = (1, \ldots, 1)$, *then* $\mathbf{\Lambda}^{\sigma(n)}_{\Re}$ *is the* $C^{\infty}(M; \mathbf{R})$*-module of n-th order differential forms on M.*

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[3]This means that $\Re$ is full and all the differential functors $A - \mathbf{Mod} \to A - \mathbf{Mod}$, when restricted to $\Re$, are strictly representable in $\Re$. $A - \mathbf{Mod}$ itself is a differentially closed subcategory.

[4]Our smooth manifolds are Haussdorff and with a countable basis.

[5]A $C^{\infty}(M; \mathbf{R})$-module P is called geometric if "each of its elements is univoquely defined by its values on the manifold M" i.e. if $\bigcap_{x \in M} I_x \cdot P = (0)$, where I_x is the maximal ideal of smooth functions vanishing at x.

Now we can associate to any $\sigma \in \mathbf{N}_+^\infty$ a de Rham-like complex of differential operators $\mathbf{dR}_\sigma(\Re)$ in $\Re$ as follows:

$$\mathbf{dR}_\sigma(\Re): \quad 0 \to A \xrightarrow{d_{\sigma(1)}^{\Re}} \mathbf{\Lambda}_{\Re}^{\sigma(1)} \to \cdots \to \mathbf{\Lambda}_{\Re}^{\sigma(n)} \xrightarrow{d_{\sigma(n+1)}^{\Re}} \mathbf{\Lambda}_{\Re}^{\sigma(n+1)} \to \cdots \tag{3}$$

with $d_{\sigma(n+1)}^{\Re} \doteq I_{\sigma(n+1)}^{\vee} \circ j_{\sigma(n+1)}^{\Re}\left(\mathbf{\Lambda}_{\Re}^{\sigma(n)}\right)$, $I_{\sigma(n+1)}^{\vee}: \mathbf{J}_{\Re}^{\sigma_{n+1}}\left(\mathbf{\Lambda}_{\Re}^{\sigma(n)}\right) \longrightarrow \mathbf{\Lambda}_{\Re}^{\sigma(n+1)}$ being the dual-representative (in the sense of duality between representable functors and representative objects) of $I_{\sigma(n+1)}$ in (2). The "higher" differential $d_{\sigma(n+1)}^{\Re}$ is a differential operator of order $\leq \sigma_{n+1}$ and $\mathbf{dR}_\sigma(\Re)$ is called the *higher de Rham complex of type* σ *in* $\Re$.

In the situations of the above examples, (3) coincides with the canonical "algebraic" and "differential-geometrical" de Rham complex, respectively. We emphasize also that the complexes $\mathbf{dR}_\sigma(\Re)$, $\Re$ being the category of geometric $C^\infty(M;\mathbf{R})$-modules, are natural in the category of smooth manifolds.

The A-modules $\mathbf{\Lambda}_{\Re}^{\sigma(n)}$ are generated by elements $d_{\sigma(n)}\left(a_1 d_{\sigma(n-1)}\left(a_2 \ldots d_{\sigma(1)}(a_n)\ldots\right)\right)$, $a_1, a_2, \ldots, a_n \in A$ (reference to $\Re$ will be omitted, unless it will be necessary). If $X \in D_{(1)}(A)$, then the *Lie derivative* operators $L_X : \mathbf{\Lambda}^{\sigma(n)} \to \mathbf{\Lambda}^{\sigma(n)}$, $n = 1, 2, \ldots$, can be defined as those operators commuting with higher differentials $d_{\sigma(n)}$ and satisfying the Leibniz rule with respect to the product by elements of A. In particular, the action of L_X on generators is ($a_0 \in A$):

$$\begin{aligned} & L_X\left(a_0 d_{\sigma(n)}\left(a_1 d_{\sigma(n-1)}\left(a_2 ... d_{\sigma_1}(a_n)...\right)\right)\right) = \\ & = X(a_0)\, d_{\sigma(n)}\left(a_1 d_{\sigma(n-1)}\left(a_2 ... d_{\sigma_1}(a_n)...\right)\right) + \\ & + \textstyle\sum_{i=1}^n a_0 d_{\sigma(n)}\left(a_1 ... d_{\sigma(n-i+1)}\left(X(a_i)\, d_{\sigma(n-i)}\left(a_{i+1} ... d_{\sigma_1}(a_n)...\right)\right)\right). \end{aligned} \tag{4}$$

If $\sigma, \tau \in \mathbf{N}_+^\infty$ and $\tau \geq \sigma$, i.e. $\tau_k \geq \sigma_k\ \forall k \in \mathbf{N}_+$, there is a canonical monomorphism in $[\Re, A-\mathbf{Mod}]$, $D_{\sigma(n)} \hookrightarrow D_{\tau(n)}$, $\forall n \in \mathbf{N}_+$, whose dual-representative is an $A-\mathbf{Mod}$-epimorphism $\mathbf{\Lambda}^{\tau(n)} \to \mathbf{\Lambda}^{\sigma(n)}$, commuting with differentials of $\mathbf{dR}$-complexes. So we have an inverse system (over $\mathbf{N}_+^\infty$) of complexes of K-modules:

$$\mathbf{dR}_\tau \longrightarrow \mathbf{dR}_\sigma \qquad \text{if } \tau \geq \sigma. \tag{5}$$

This allows us to define the ∞*-order de Rham complex in* $\Re$ as:

$$\mathbf{dR}_\infty^{\Re} \doteq inv \lim_{\sigma \in \mathbf{N}_+^\infty} \mathbf{dR}_\sigma^{\Re}. \tag{6}$$

2 The generalized Infinitesimal Stokes' Formula

We call $\sigma \in \mathbf{N}_+^\infty$ *regular* if $\sigma_n \leq \sigma_{n+1}$, $\forall n > 0$; a complex $\mathbf{dR}_\sigma$ is likewise called *regular* if so is σ. In this Section we first define, for regular $\mathbf{dR}_\sigma$'s, the insertion generalize the Infinitesimal Stokes' Formula to regular **dR**-complexes. In what follows we adopt the convention of writing $\nabla(a_n)(a_{n-1})\cdots(a_1)$ in place of the correct but cumbersome $(\cdots((\nabla(a_n))(a_{n-1}))\cdots)(a_1)$ for $\nabla \in Diff^+_{\sigma_1,\ldots,\sigma_n}(P)$ and $a_1, a_2, \ldots, a_n \in A$, $P \in Ob(\Re)$. Let's associate to such a ∇, the operators ∇_r, $0 \le r \le n$, defined by ($a_0 \in A$):

$$\begin{gathered}\nabla_r(a_n)\ldots(a_1)(a_0) \doteq \nabla(a_n)\ldots(a_r a_{r-1})\ldots(a_0), \quad 0 < r \le n \\ \nabla_0(a_n)\ldots(a_1)(a_0) \doteq a_0 \cdot \nabla(a_n)\ldots(a_1), \quad a_0,\ldots,a_n \in A.\end{gathered}$$

It is a long but straightforward verification, to prove that $\nabla_r \in Diff^+_{\mu_1^{(r)},\mu_2^{(r)},\ldots,\mu_{n+1}^{(r)}}(P)$, with

$$\mu^{(r)}_{n-r+1} = 0 \text{ and } \left(\mu^{(r)}_1,\ldots,\mu^{(r)}_{n-r},\mu^{(r)}_{n-r+2},\ldots,\mu^{(r)}_{n+1}\right) = \sigma(n), \ \forall r.$$

Definition 2.1 *If $X \in D_{(1)}(A)$, $l = 0,1,\ldots,n$; $r = 0,1,\ldots,l$ and $\nabla \in Diff^+_{\sigma_1,\sigma_2,\ldots,\sigma_n}(P)$, we define $(\nabla, X)_{r,l} \in Diff^+_{\nu_1^{(r,l)},\ldots,\nu_{n+1}^{(r,l)}}(P)$, as:*

$$(\nabla, X)_{r,l}(a_n)\ldots(a_1)(a_0) \doteq ([\nabla(a_n)\ldots]_r \circ X)(a_l)\ldots(a_0),$$

$a_0,\ldots,a_n \in A$.

Remark 2.1 *The positive integers $\nu_i^{(r,l)}$, $i = 1,\ldots,n+1$, can easily be determined in terms of $\sigma(n)$ but we will not need their expressions in the sequel.*

In the following key proposition, the preceding definitions are applied to elements of $D_{\sigma(n)}(P)$, thanks to the natural inclusion (in $[\Re, K - \mathbf{Mod}]$) $D_{\sigma(n)} \subset Diff^+_{\sigma_1,\sigma_2,\ldots,\sigma_n}$.

Proposition 2.1 *Let $P \in Ob(\Re)$, $\sigma \in \mathbf{N}^\infty_+$, $n > 0$, $\Delta \in D_{\sigma(n)}(P)$, $p \in P$ and $a_0,\ldots,a_n \in A$. We define, by induction on n,:*

$$\begin{gathered}i^X(P) : Hom_A(A,P) \simeq P \to D(P) \hookrightarrow D_{\sigma(1)}(P) \\ \left[i^X(P)(p)\right](a_0) \doteq X(a_0)\cdot p \text{ , for } n = 0 \text{ ;} \\ \left[i^X(P)(\Delta)\right](a_n)\ldots(a_1)(a_0) \doteq \left[i^X(P)(\Delta(a_n))\right](a_{n-1})\ldots(a_1)(a_0) + \\ + \left[\sum_{r=0}^n (-1)^r(\Delta_r \circ X)\right](a_n)\ldots(a_1)(a_0) \text{ , for } n \ge 1.\end{gathered} \tag{7}$$

Then:

(i) $P \longmapsto i^X(P)$ is a well defined functorial morphism in $[\Re, A - \mathbf{Mod}]$:

$$i^X : D_{\sigma(n)} \to D_{(\mu_1,\ldots,\mu_n,\mu_{n+1})}$$

where $\mu_{n+1} \doteq \sigma_n$, $\mu_n \doteq \max\{\sigma_n, \sigma_{n-1}\}$, ... , $\mu_2 \doteq \max\{\sigma_2, \sigma_1\}$, $\mu_1 \doteq \sigma_1$ (if $n = 1$, to clear misunderstandings, we declare explicitly that $i^X : D_{\sigma(1)} \to D_{(\sigma_1,\sigma_1)}$);

(ii) the inductive definition (7) can be resolved in the following:

$$i^X(P)(\Delta) \doteq \sum_{l=0}^{n}\sum_{r=0}^{l}(-1)^r(\Delta, X)_{r,l}\,. \tag{8}$$

<u>Dim</u>. (ii) is a straightforward calculation.

(i) is proved by induction on n. For $n=0$ the proof is trivial; let us suppose (i) true for $0 \le k < n$ and let's prove it for n.

We begin by showing that $\forall a_n \in A$, $\left[i^X(\Delta)\right](a_n) \in D_{(\mu_2,\ldots,\mu_{n+1})}(P)$ (where we have written $i^X(P) \equiv i^X$). By the inductive definition we have:

$$\left[i^X(\Delta)\right](a_n) = i^X(\Delta(a_n)) + \sum_{r=0}^{n}(-1)^r\Delta_r(X(a_n))\,.$$

But by the inductive hypothesis (i), $i^X(\Delta(a_n)) \in D_{(\sigma_2,\mu_3,\ldots,\mu_{n+1})}(P)$, since $\Delta(a_n) \in D_{(\sigma_2,\ldots,\sigma_n)}(P)$. After some combinatorics we also get:

$$\left[\sum_{r=0}^{n}(-1)^r(\Delta_r \circ X)\right](a_n) \in D_{(\mu_2,\ldots,\mu_{n+1})}(P).$$

Remembering that $\mu_2 \doteq \max\{\sigma_1, \sigma_2\}$ and putting the two previous results together we obtain

$$\left[i^X(\Delta)\right](a_n) \in D_{(\mu_2,\ldots,\mu_{n+1})}(P)$$

which is our first step.

To fill the proof of (i), for n, we are left to prove:

(A) $i^X(\Delta) \in Diff_{\sigma_1}\left(Diff^+_{\mu_2,\ldots,\mu_{n+1}}(P)\right)$;

(B) $\left[i^X(\Delta)\right](1) = 0$.

(B) is obvious from the definition and the fact that $X(1) = \Delta(1) = 0$. To prove (A) we must show that $\delta_{b_0,\ldots,b_{\sigma_1}}\left[i^X(\Delta)\right] = 0$, $\forall b_0,\ldots,b_{\sigma_1} \in A$. By direct calculations, one proves that, $\forall b \in A$:

$$\delta_b\left(i^X(\Delta)\right) = i^X(\delta_b\Delta) + \sum_{r=0}^{n-1}(-1)^r\left[\delta_{X(b)}\Delta\right]_r + (-1)^n\left[\Delta_n \circ (\delta_b X) - [\delta_b\Delta]_n \circ X\right].$$

This can be iterated to give:

$$\begin{aligned}\delta_{b_0,\ldots,b_{\sigma_1}}\left(i^X(\Delta)\right) &= i^X\left(\delta_{b_0,\ldots,b_{\sigma_1}}\Delta\right) + \textstyle\sum_{r=0}^{n-1}(-1)^r\sum_{i=0}^{\sigma_1}\left[\delta_{b_0,\ldots,X(b_i),\ldots b_{\sigma_1}}\Delta\right]_r + \\ &+ (-1)^n\left[\Delta_n \circ \left(\delta_{b_0,\ldots,b_{\sigma_1}}X\right) - \left[\delta_{b_0,\ldots,b_{\sigma_1}}\Delta\right]_n \circ X\right]\end{aligned}$$

which, finally, proves (A), because $\sigma_1 \ge 1$.

From (8) it follows that $i^X(P)$ is a homomorphism. Functoriality in P is evident. □

morphism:

$$i^X : D_{\sigma(n)} \longrightarrow D_{\sigma(n+1)} \tag{9}$$

understood as the composition:

$$D_{\sigma(n)} \longrightarrow D_{(\sigma_1,\sigma_2,\ldots,\sigma_n,\sigma_n)} \hookrightarrow D_{\sigma(n+1)}.$$

(the last inclusion is allowed by regularity of σ: in fact $\sigma_n \leq \sigma_{n+1}$).

In explicit terms, i^X is given by:

$$\begin{gathered}\left[i^X(\Delta)\right](a_n)\ldots(a_1)(a_0) = X(a_0)\cdot\Delta(a_n)\ldots(a_1) + \sum_{l=1}^n a_0\Delta(a_n)\ldots(X(a_l))\ldots(a_1) + \\ + \sum_{l=1}^n (-1)^l \Delta(a_n)\ldots(a_{l+1})(X(a_l)a_{l-1})(a_{l-2})\ldots(a_0) + \\ + \sum_{l=2}^n \sum_{i=1}^{l-1} (-1)^i \Delta(a_n)\ldots(a_{l+1})(X(a_l))(a_{l-1})\ldots(a_i a_{i-1})(a_{i-2})\ldots(a_0).\end{gathered} \tag{10}$$

We are now able to give the basic definition of this Section:

Definition 2.2 *If $\sigma \in \mathbf{N}_+^\infty$ is regular and $X \in D_{(1)}(A)$, we define the X-*insertion $A-\mathbf{Mod}$-morphism *(or interior product by X):*

$$i_X : \mathbf{\Lambda}^{\sigma(n+1)} \rightarrow \mathbf{\Lambda}^{\sigma(n)}, \qquad n \geq 0, \tag{11}$$

to be the dual-representative of i^X: $i_X \doteq \left(i^X\right)^\vee$.

Since the functorial isomorphism $D_{\sigma(n)} \simeq Hom_A\left(\mathbf{\Lambda}^{\sigma(n)}, \cdot\right)$ is realized by $\Delta \longmapsto f^\Delta$, with:

$$f^\Delta\left(a_0 d_{\sigma(n)}\left(a_1 d_{\sigma(n-1)}\left(a_2\ldots d_{\sigma(1)}(a_n)\ldots\right)\right)\right) = a_0\Delta(a_n)\ldots(a_1),$$

(11) implies the following expression of $i_X : \mathbf{\Lambda}^{\sigma(n+1)} \rightarrow \mathbf{\Lambda}^{\sigma(n)}$ on generators:

$$\begin{gathered} i_X\left(d_{\sigma(n+1)}\left(a_0 d_{\sigma(n)}\left(a_1 d_{\sigma(n-1)}\left(a_2\ldots d_{\sigma(1)}(a_n)\ldots\right)\right)\right)\right) = \\ = X(a_0)\cdot d_{\sigma(n)}\left(a_1 d_{\sigma(n-1)}\left(a_2\ldots d_{\sigma(1)}(a_n)\ldots\right)\right) + \\ + \sum_{l=1}^n a_0 d_{\sigma(n)}\left(a_1\ldots d_{\sigma(n-l+1)}\left(X(a_l)\ldots d_{\sigma(1)}(a_n)\ldots\right)\right) + \\ + \sum_{l=1}^n (-1)^l d_{\sigma(n)}\left(a_0 d_{\sigma(n-1)}\left(a_1\ldots d_{\sigma(n-l+2)}(a_{l-2} d_{\sigma(n-l+1)}(a_{l-1}X(a_l)\ldots d_{\sigma(1)}(a_n)\ldots)\ldots\right)\right) + \\ + \sum_{l=2}^n \sum_{i=1}^{l-1} (-1)^i d_{\sigma(n)}\left(a_0\left(\ldots d_{\sigma(n-i+2)}(a_{i-2} d_{\sigma(n-i+1)}(a_i a_{i-1}\ldots d_{\sigma(n-l+1)}(X(a_l)\ldots d_{\sigma(1)}(a_n)\ldots\right)\right) \end{gathered} \tag{12}$$

the following diagram in $[\Re, A - \mathbf{Mod}]$ *is commutative:*

$$\begin{array}{ccc} D_{\sigma(n)} & \xrightarrow{i^X} & D_{\sigma(n+1)} \\ \uparrow & & \uparrow \\ D_{\tau(n)} & \xrightarrow[i^X]{} & D_{\tau(n+1)} \end{array}$$

(where the vertical arrows are natural inclusions). Its dual-representative is therefore commutative:

$$\begin{array}{ccc} \mathbf{\Lambda}^{\sigma(n+1)} & \xrightarrow{i_X} & \mathbf{\Lambda}^{\sigma(n)} \\ \downarrow & & \downarrow \\ \mathbf{\Lambda}^{\tau(n+1)} & \xrightarrow[i_X]{} & \mathbf{\Lambda}^{\tau(n)} \end{array}$$

and the vertical (reversed) arrows are now $A - \mathbf{Mod}$ *epimorphisms.*

Now we can extend the Infinitesimal Stokes' Formula to higher de Rham complexes by a straightforward verification:

Proposition 2.3 *("Generalized Infinitesimal Stokes' Formula")*
If $\mathbf{dR}_\sigma$ *is regular then,* $\forall n \geq 1$, *in:*

$$\mathbf{\Lambda}^{\sigma(n-1)} \underset{i_X}{\overset{d_{\sigma(n)}}{\rightleftarrows}} \mathbf{\Lambda}^{\sigma(n)} \underset{i_X}{\overset{d_{\sigma(n+1)}}{\rightleftarrows}} \mathbf{\Lambda}^{\sigma(n+1)}$$

we have:

$$L_X = i_X \circ d_{\sigma(n+1)} + d_{\sigma(n)} \circ i_X . \tag{13}$$

From formula (13) and commutativity of higher order de Rham's differentials with pullbacks, as it is well known (see for example [7]), it descends the homotopy-invariance property of the (standard and) higher de Rham cohomologies when $A = C^\infty(M; \mathbf{R})$, M being a smooth real manifold, and $\Re$ is the category of geometric A-modules. The other Eilenberg-Steenrod's axioms being easily verified for higher de Rham cohomologies, we get then that standard de Rham's and higher de Rham's cohomologies coincide for smooth real manifolds. In the next Section we outline the key points of a purely algebraic proof of a slightly more general result.

It is also worth mentioning that (13) is of great interest in trying to build a C-spectral sequence ([8]) by working from the very beginning with higher or infinite de Rham complexes (see [8], 12.2 p.126).

Definition 3.1 *A differentially closed subcategory* $\Re$ *of* $A-\mathbf{Mod}$ *is called* smooth *if* $\mathbf{\Lambda}_{\Re}^{(1)}$ *is a projective* A*-module of finite type.*

Examples 3.1 (**i**) *If* M *is a smooth real manifold,* $A=C^{\infty}(M;\mathbf{R})$*, by the Serre-Swan theorem, the subcategory* $\Re=A-\mathbf{Mod}_{geom}$ *of geometric* A*-modules, is smooth.*

(**ii**) *If* K *is an algebraically closed field and* A *is the coordinate ring of a regular affine variety over* K*, then the whole* $A-\mathbf{Mod}$ *is smooth.*

In the rest of this Section A is a K-algebra of zero characteristic, containing K as a subring and $\Re$ will be a differentially closed smooth subcategory of $A-\mathbf{Mod}$. All representative objects (unless otherwise stated) will be considered in $\Re$, so we will omit any reference to $\Re$ in them.

Smoothness of $\Re$ implies that, in the exact sequence (2), $\pi_{\sigma(n)}$ is epic, so that we get, $\forall\sigma\in\mathbf{N}_{+}^{\infty}$ and $\forall n\geq 1$, the following short exact sequence in $[\Re, A-\mathbf{Mod}]$:

$$0\rightarrow D_{\sigma(n)}\hookrightarrow D^{\bullet}_{\sigma(n-1)}\circ Diff^{+}_{\sigma_n}\rightarrow D_{(\sigma_1,\ldots,\sigma_{n-2},\sigma_{n-1}+\sigma_n)}\rightarrow 0. \tag{14}$$

The n-th cohomology K-module of the complex

$$\mathbf{dR}_{\sigma}:\quad 0\rightarrow A\overset{d_{\sigma(1)}}{\longrightarrow}\mathbf{\Lambda}^{\sigma(1)}\rightarrow\ldots\rightarrow\mathbf{\Lambda}^{\sigma(n)}\overset{d_{\sigma(n+1)}}{\longrightarrow}\mathbf{\Lambda}^{\sigma(n+1)}\rightarrow\ldots$$

is denoted by:

$$H^n_{\sigma}\doteq\frac{\ker\left(d_{\sigma(n+1)}\right)}{im\left(d_{\sigma(n)}\right)}\equiv H^n\left(\mathbf{dR}_{\sigma}\right).$$

Since H^n_{σ} only depends on $\sigma(n+1)$, we will write also $H^n_{\sigma(n+1)}$ in place of H^n_{σ}. We recall to reader's attention that in the situation of Examples 1.1 (**ii**), $H^n_{\sigma(n+1)}$ is nothing but the n-th de Rham cohomology $\mathbf{R}$-vector space of the smooth manifold M.

In the rest of this Section we will sketch the proof of the following:

Theorem 3.1 *("Smooth" rigidity of higher de Rham cohomologies)*

If $\Re$ *is a smooth subcategory of* $A-\mathbf{Mod}$*, then,* $\forall\ \tau,\sigma\in\mathbf{N}_{+}^{\infty}$ *with* $\tau\geq\sigma$*, the canonical epimorphism (5):*

$$\mathbf{dR}_{\tau}\rightarrow\mathbf{dR}_{\sigma}$$

is a quasi-isomorphism. Then:

$$H^n_{\sigma}\simeq H^n_{\tau},\ \forall n\geq 0. \tag{15}$$

Corollary 3.2 (**i**) *If M is a* <u>*smooth*</u> *manifold, $A = C^{\infty}(M;\mathbf{R})$ and $\Re = C^{\infty}(M;\mathbf{R}) - \mathbf{Mod}_{geom}$, then the higher de Rham cohomologies coincide with the standard one.*

(**ii**) *If K is an algebraically closed field of zero characteristic and A is the coordinate ring of a* <u>*regular*</u> *affine variety over K, then the higher de Rham cohomologies coincide with the standard algebraic one.*

We note that the last Corollary is false, in general, for a non-smooth manifold or a non-regular affine variety.

The strategy of the proof of Theorem 3.1 is the following:

keeping $n \geq 0$ fixed, we will prove the thesis by reducing, step by step, each entry of $\sigma(n+1)$ to 1, starting from σ_{n+1} :

$$H^n_{\sigma(n+1)} \simeq H^n_{(\sigma(n),1)} \simeq H^n_{(\sigma(n-1),1,1)} \simeq \cdots \simeq H^n_{(\sigma_1,1,...,1)} \simeq H^n_{dR} \tag{16}$$

where H^n_{dR} stands for the n-th standard de Rham cohomology $H^n_{(1,...,1)}$, $(1,...,1) \in \mathbf{N}^{n+1}_+$.

The first step in the chain (16) is accomplished by the following[6]:

Lemma 3.3 *Let $n \in \mathbf{N}_+$. If $\sigma, \tau \in \mathbf{N}^{\infty}_+$ are such that $\sigma(n) = \tau(n)$, then:*

(i) $\ker d_{\sigma(n+1)} = \ker d_{\tau(n+1)}$ *;*

(ii) $im\left(d_{\sigma(n+1)}\right) \simeq im\left(d_{\tau(n+1)}\right)$

(where $\simeq$ means $K - \mathbf{Mod}$-isomorphism).

<u>**Dim**</u>. Let $\sigma \in \mathbf{N}^n_+$, $k > 1$. Let us consider the functorial short exact sequence:

$$0 \to D_{(\sigma,k-1,1)} \hookrightarrow D_{(\sigma,k-1)} \circ Diff^+_1 \xrightarrow{i} D_{(\sigma,k)} \to 0$$

whose dual-representative:

$$0 \to \mathbf{\Lambda}^{(\sigma,k)} \xrightarrow{i^{\vee}} \mathbf{J}^1\left(\mathbf{\Lambda}^{(\sigma,k-1)}\right) \longrightarrow \mathbf{\Lambda}^{(\sigma,k-1,1)} \to 0$$

is likewise exact (in $\Re$). We embed this last one in the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathbf{\Lambda}^{(\sigma,k)} & \xrightarrow{i^{\vee}} & \mathbf{J}^1\left(\mathbf{\Lambda}^{(\sigma,k-1)}\right) & \longrightarrow & \mathbf{\Lambda}^{(\sigma,k-1,1)} & \to & 0 \\
 & & d_{(\sigma,k)} \uparrow & & \uparrow j_1 & & & & \\
 & & \mathbf{\Lambda}^{\sigma} & \underset{d_{(\sigma,k-1)}}{\longrightarrow} & \mathbf{\Lambda}^{(\sigma,k-1)} & & & &
\end{array}$$

(on generators, in fact, we have:

$$i^{\vee}\left(d_{(\sigma,k)}(a \cdot \omega)\right) = j_1\left(d_{(\sigma,k-1)}(a \cdot \omega)\right),\ a \in A,\ \omega \in \mathbf{\Lambda}^{\sigma},$$

which proves commutativity).

[6]This Lemma has been proved, independently, also by Yu. Torkhov.

and, by commutativity, $\left(i^{\vee} \circ d_{(\sigma,k)}\right)(\omega) = 0$. But $i^{\vee}$ is a monomorphism, so:

$$\ker d_{(\sigma,k-1)} \subseteq \ker d_{(\sigma,k)},\ \forall k > 1.$$

The reversed inclusion being obvious, we have proven (i).

(ii) follows from (i) and the canonical exact sequences:

$$0 \to \ker d_{(\sigma,k)} \to \mathbf{\Lambda}^{\sigma} \overset{d_{(\sigma,k)}}{\longrightarrow} im\left(d_{(\sigma,k)}\right) \to 0$$
$$0 \to \ker d_{(\sigma,k-1)} \to \mathbf{\Lambda}^{\sigma} \overset{d_{(\sigma,k-1)}}{\longrightarrow} im\left(d_{(\sigma,k-1)}\right) \to 0.$$

□

To prove the k-th step of chain (16), it is enough to prove that:

$$H^{n}_{(\sigma(k-1),\sigma_k+1,1,\dots,1)_{n+1}} \simeq H^{n}_{(\sigma(k-1),\sigma_k,1,\dots,1)_{n+1}} \tag{17}$$

where we write $(\rho)_r$ to stress the fact that $\rho \in \mathbf{N}^{r}_{+}$. To prove (17) we use an auxiliary complex.

Definition 3.2 *Let $P \in Ob(\Re)$ and $\tau \in \mathbf{N}^{\infty}_{+}$. We define the following functors $\Re \to A - \mathbf{Mod}$:*

$$Hol^{\tau(1)}_{P} \doteq Diff_{\tau_1}(P,\cdot)$$

$$Hol^{\tau(n)}_{P} \doteq Diff^{\bullet}_{\tau_1}\left(P, D_{(\tau_2,\dots,\tau_n)} \subset Diff^{+}_{\tau_2,\dots,\tau_n}\right)$$

where, if (Q,Q^{+}) is an (A,A)-bimodule, Q, $Q^{+} \in Ob(\Re)$ and $S \underset{\Re}{\subset} Q$, we have put:

$$Diff^{\bullet}_{k}\left(P, S \subset Q^{+}\right) \doteq \left\{\Delta \in Diff_k\left(P,Q^{+}\right) \mid \Delta(P) \subset S\right\}$$

the A-module structure being that inherited by $Diff_k(P,Q)$, $\forall k \geq 0$.

We call these functors *Hol-functors*[7] of the A-module P. By an easy diagram chasing we get:

Proposition 3.4 *The functorial sequence:*

$$0 \to Hol^{\tau(n)}_{P} \longrightarrow \left[Hol^{\tau(n-1)}_{P}\right]^{\bullet} \circ Diff^{+}_{\tau_n} \longrightarrow Hol^{(\tau(n-2),\tau_{n-1}+\tau_n)}_{P} \to 0 \tag{18}$$

(in which the morphisms are natural extensions of that of (2)) is exact in $[\Re, A - \mathbf{Mod}]$.

[7] Hol stands for "Holonomy".

$\Re$. We shall indicate its representative objects by:

$$Hol_{\tau(n)}(P)$$

*and call it Hol-*object *of type $\tau(n)$ of the A-module P.*

Dim. $Hol_P^{\tau(1)}$ is strictly representable since $\Re$ is differentially closed. From Proposition 3.4 we get the thesis by induction on n. $\square$

In fact, taking the dual-representative of (17) we get:

$$Hol_{\tau(n)}(P) \simeq \frac{\mathbf{J}^{\tau_n}\left(Hol_{\tau(n-1)}(P)\right)}{Hol_{(\tau(n-2),\tau_{n-1}+\tau_n)}(P)} \tag{19}$$

in $\Re$.

Let us now define the differential operator (natural in $P \in Ob(\Re)$):

$$\delta_{\tau(n+1)}(P) : Hol_{\tau(n)}(P) \longrightarrow Hol_{\tau(n+1)}(P) \tag{20}$$

as the composition:

$$Hol_{\tau(n)}(P) \overset{j_{\tau_{n+1}}\left(Hol_{\tau(n)}(P)\right)}{\longrightarrow} \mathbf{J}^{\tau_{n+1}}\left(Hol_{\tau(n)}(P)\right) \overset{p_{\tau(n+1)}(P)}{\longrightarrow} \frac{\mathbf{J}^{\tau_{n+1}}\left(Hol_{\tau(n)}(P)\right)}{Hol_{(\tau(n-1),\tau_n+\tau_{n+1})}(P)} \simeq Hol_{\tau(n+1)}(P)$$

$p_{\tau(n+1)}(P)$ being the canonical quotient projection. $\delta_{\tau(n)}$, $n > 0$, will be the "differential" of our auxiliary complex; in fact one can prove the following:

Proposition 3.6 *$\forall P \in Ob(\Re)$, $\forall \tau \in \mathbf{N}_+^\infty$, the sequence:*

$$\begin{aligned}\mathbf{Hol}_\tau(P):\ 0 \to Hol_\emptyset(P) \doteq P \overset{\delta_{\tau(1)}(P)}{\longrightarrow} Hol_{\tau(1)}(P) \equiv J^{\tau_1}(P) \to \dots \\ \dots \to Hol_{\tau(n)}(P) \overset{\delta_{\tau(n+1)}(P)}{\longrightarrow} Hol_{\tau(n+1)}(P) \to \dots\end{aligned}$$

*is a complex in $\Re$, called Hol_τ-*complex of *P. Furthermore $\mathbf{Hol}_\tau$ is a functor $\Re \to \mathbf{K}_{diff}(\Re)$*[8].

We now want to show that if $\tau \in \mathbf{N}_+^\infty$ is regular, then, $\forall P \in Ob(\Re)$, $\mathbf{Hol}_\tau(P)$ is acyclic; in fact we are going to exhibit (functorially in P) a trivializing homotopy. With the notations of Proposition 3.6, we define the following $\Re$-morphisms:

$$\begin{aligned}\varphi_\emptyset(P) \doteq 0 : Hol_\emptyset(P) \doteq P \to 0 \\ \varphi_{\tau(1)}(P) \doteq \pi_{\tau_1}(P) : Hol_{\tau(1)}(P) \equiv J^{\tau_1}(P) \to P \equiv Hol_\emptyset(P).\end{aligned}$$

[8]$\mathbf{K}_{diff}(\Re)$ denotes the category of complexes of differential operators formed by objects of $\Re$.

$$\varphi_{\tau(n)}(P) : Hol_{\tau(n)}(P) \to Hol_{\tau(n-1)}(P)$$

let's consider the differential operator of order $\leq \tau_n$:

$$\Delta_{\tau(n+1)}(P) \doteq Id\left(Hol_{\tau(n)}(P)\right) - \delta_{\tau(n)}(P) \circ \varphi_{\tau(n)}(P) : Hol_{\tau(n)}(P) \to Hol_{\tau(n)}(P).$$

By duality, it defines a $\Re$-morphism

$$\hat{\varphi}_{\tau(n+1)}(P) : \mathbf{J}^{\tau_{n+1}}\left(Hol_{\tau(n)}(P)\right) \to Hol_{\tau(n)}(P)$$

(τ being regular).

Since

$$\ker\left[\hat{\varphi}_{\tau(n+1)}(P)\right] \supseteq Hol_{(\tau(n-1),\tau_n+\tau_{n+1})}(P),$$

$\hat{\varphi}_{\tau(n+1)}(P)$ induces a quotient morphism:

$$\varphi_{\tau(n+1)}(P) : Hol_{\tau(n+1)}(P) \to Hol_{\tau(n)}(P).$$

This accomplishes the inductive definition of the family $\left\{\varphi_{\tau(n)}(P)\right\}_{n\geq 1}$.

It is a direct calculation to prove:

Proposition 3.7 *$\forall P \in Ob(\Re)$, $\forall \tau \in \mathbf{N}_+^\infty$ regular, $\left\{\varphi_{\tau(n)}(P)\right\}_{n\geq 1}$ is a trivializing homotopy for $\mathbf{Hol}_\tau(P)$. Furthermore, $\left\{\varphi_{\tau(n)}(P)\right\}_{n\geq 1}$ is natural in P.*

We now use both the $\mathbf{Hol_1}$-complex and the functorial homotopy $\{\varphi_{\mathbf{1}_n}(P)\}_{n\geq 1}$, $\mathbf{1} = (1,...,1,...) \in \mathbf{N}_+^\infty$, $\mathbf{1}_n = (1,...,1) \in \mathbf{N}_+^n$, to prove formula (16). Let us consider the epimorphism of fragments of higher de Rham complexes:

$$\begin{array}{ccccccc}
\mathbf{\Lambda}^{(\sigma(k-1),\sigma_k+1,1,...,1)_{n-1}} & \to & \mathbf{\Lambda}^{(\sigma(k-1),\sigma_k+1,1,...,1)_{n}} & \to & \mathbf{\Lambda}^{(\sigma(k-1),\sigma_k+1,1,...,1)_{n+1}} & \to & \mathbf{\Lambda}^{(\sigma(k-1),\sigma_k+1,1,...,1)_{n+2}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{\Lambda}^{(\sigma(k-1),\sigma_k,1,...,1)_{n-1}} & \to & \mathbf{\Lambda}^{(\sigma(k-1),\sigma_k,1,...,1)_{n}} & \to & \mathbf{\Lambda}^{(\sigma(k-1),\sigma_k,1,...,1)_{n+1}} & \to & \mathbf{\Lambda}^{(\sigma(k-1),\sigma_k,1,...,1)_{n+2}}
\end{array} \tag{21}$$

(where, as before, we write $(\tau)_s$ if $\tau \in \mathbf{N}_+^s$). If we prove that the kernel of this epimorphism is acyclic, then (16) will follow from the cohomology long exact sequence. We need more sinthetic notations. For each $(\mu)_s = (\mu_1,...,\mu_s) \in \mathbf{N}_+^s$, $1 \leq r \leq s$, $r, s \in \mathbf{N}_+$:

$$K_{(\mu)_s}^{(r)} \doteq \ker\left(\mathbf{\Lambda}^{(\mu_1,...,\mu_r,...\mu_s)} \to \mathbf{\Lambda}^{(\mu_1,...,\mu_r-1,...\mu_s)}\right).$$

Then, the kernel of (21) may be rewritten as:

$$K^{(k)}_{[\sigma]_{n-1}} \to K^{(k)}_{[\sigma]_{n}} \to K^{(k)}_{[\sigma]_{n+1}} \to K^{(k)}_{[\sigma]_{n+2}}. \qquad (22)$$

where we have put:

$$K^{(k)}_{[\sigma]_r} \doteq K^{(k)}_{(\sigma_1,\dots,\sigma_{k-1},\sigma_k+1,1,\dots,1)_r} \quad \text{for } r = n-1, n, n+1, n+2.$$

Finally, we will write $K^{(k)}_{(\sigma,s)}$ in place of $K^{(k)}_{(\sigma_1,\dots,\sigma_{k-1},\sigma_k+s)}$, for $1 \leq s \leq n-k+3$.

Now, using:

(i) the dual-representatives of (18);
(ii) the exactness of functor $\mathbf{J}^k$, $k \geq 0$, in $\Re$;
(iii) the "3×3" Lemma;

we can prove the commutativity of the following diagram:

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & 0 & \to & K^{(k)}_{(\sigma,n-k+3)} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & K^{(k)}_{(\sigma,n-k+2)} & \to & Hol_{(1)}\left(K^{(k)}_{(\sigma,n-k+2)}\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & K^{(k)}_{(\sigma,n-k+1)} & \to & Hol_{(1)}\left(K^{(k)}_{(\sigma,n-k+1)}\right) & \to & Hol_{(1,1)}\left(K^{(k)}_{(\sigma,n-k+1)}\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^{(k)}_{(\sigma,n-k)} & \to & Hol_{(1)}\left(K^{(k)}_{(\sigma,n-k)}\right) & \to & Hol_{(1,1)}\left(K^{(k)}_{(\sigma,n-k)}\right) & \to & Hol_{(1,1,1)}\left(K^{(k)}_{(\sigma,n-k)}\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Hol_{\mathbf{1}_{n-k-1}}\left(K^{(k)}_{(\sigma,1)}\right) & \to & Hol_{\mathbf{1}_{n-k}}\left(K^{(k)}_{(\sigma,1)}\right) & \to & Hol_{\mathbf{1}_{n-k+1}}\left(K^{(k)}_{(\sigma,1)}\right) & \to & Hol_{\mathbf{1}_{n-k+2}}\left(K^{(k)}_{(\sigma,1)}\right) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^{(k)}_{[\sigma]_{n-1}} & \to & K^{(k)}_{[\sigma]_{n}} & \to & K^{(k)}_{[\sigma]_{n+1}} & \to & K^{(k)}_{[\sigma]_{n+2}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0
\end{array}$$

The bottom row is (22) while the other rows are fragments of different **Hol**-complexes. With a little diagram chasing, we can prove that if in this diagram we substitute the differentials of the **Hol**-complexes by their trvializing homotopies[9], we still end up with a commutative diagram. Therefore (22) is isomorphic to a quotient of a **Hol**-complex,

[9]Whose arrows are reversed, with respect to differentials.

motopies pass to this quotient, thus yielding a trivializing homotopy for this quotient, isomorphic to (22): this proves acyclicity of (22) and also Theorem 3.1.

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