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ON AUTOMORPHISM GROUPS OF SOME TYPES OF GENERIC DISTRIBUTIONS

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Abstract. To certain types of generic distributions (subbundles in a tangent bundle) one can associate canonical Cartan connections. Many of these constructions fall into the class of parabolic geometries. The aim of this article is to show how strong restrictions on the possibles sizes of automorphism groups of such distributions can be deduced from the existence of canonical Cartan connections. This needs no information on how the Cartan connections are actually constructed and only very basic information on their properties. In particular, we discuss the examples of generic distributions of rank two in dimension five, rank three in dimension six, and rank four in dimension seven.

1. Introduction

This article deals with geometric questions on subbundles in the tangent bundles of smooth manifolds. While such structures always have been of interest in control theory, their importance in various parts of differential geometry and geometric analysis has increased a lot during the last years. This refers to, for example, sub–Riemannian structures, questions related to Carnot groups and Carnot–Caratheodory manifolds, as well as analytical properties of differential operators obtained as sums of squares of sections of such subbundles.

Since integrable subbundles can be removed by passing to leaves of the corresponding foliation, one usually restricts the attention to *bracket generating* distributions. This condition means that sections of the distribution together with their iterated Lie brackets span the full tangent bundle.

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Even under this assumption, different types of distributions can have entirely different behavior. For example, consider automorphisms of a distribution, i.e. diffeomorphisms of the manifold whose derivatives in all points respect the distribution. On the one hand, there are examples like contact distributions, which admit a local normal form and always have infinite dimensional families of automorphisms. The first example of the other possible behavior was found independently by F. Engel and E. Cartan in their work on exceptional Lie algebras of type G_2. It was studied in detail in Cartan's famous "five variables paper" [6]. In this article, he studied distributions of rank two and three on five dimensional manifolds, which, in addition to being bracket generating, satisfy a genericity condition. He associated to such distributions a canonical *Cartan connection* on a certain principal bundle. This immediately implies that such distributions have local invariants (similar to the curvature of a Riemannian metric) and hence cannot admit simple local normal forms. Further, it implies that the automorphisms of such a distribution form a finite dimensional Lie group and each automorphism is determined by some finite jet (actually the two–jet) in one point.

Cartan's result has been (much later) extended to various other generic types of distributions. Many of these examples fall into the class of so called *parabolic geometries*, since the homogeneous model of the geometry is the quotient of a semisimple Lie group by a parabolic subgroup. Motivated by the examples of conformal structures and CR structures, these geometries have been intensively studied during the last years, and many striking results have been achieved.

The results on existence of canonical Cartan connections are usually difficult and Cartan connections themselves are often considered as being hard to use. In this article we want to show that Cartan connections lead to interesting results on distributions in a rather simple way. For these applications, no knowledge about the actual construction of the Cartan connections but only some basic information about their properties is needed. We show that simple algebraic computations (mostly linear algebra) can be used to obtain surprising restrictions on the possible dimensions of the automorphism groups of certain types of distributions.

The basic ideas we use certainly go back to Cartan, the more specific version for parabolic subalgebras has been implicitly used in [19] in the study of automorphism groups of CR structures. They have been explicitly formulated in [4] in the context of general parabolic geometries. The algebraic computations needed to apply these ideas in the cases of generic distributions discussed here as well as some of the realizations of automorphism groups are part of the second author's diploma thesis, see [13].

2. Distributions which are equivalent to parabolic geometries

In this section, we give a brief description of the relation between certain types of distributions and parabolic geometries, i.e. Cartan connections with homogeneous model the quotient of a semisimple Lie group by a parabolic subgroup.

2.1. **Parabolic subalgebras.** These are a special type of subalgebras in semisimple Lie algebras, which can be defined in several equivalent ways. In terms of structure theory, one best defines a parabolic subalgebra in a complex semisimple Lie algebra as one which contains a maximal solvable subalgebra (which usually is called a *Borel subalgebra*). Then one defines parabolic subalgebras in real semisimple Lie algebras via complexification. Equivalently, one may define a subalgebra in an arbitrary semisimple Lie algebra to be parabolic if its nilradical coincides with its annihilator under the Killing form, see [3].

For our purposes, the most useful definition is the one in terms of $|k|$–gradings, which also handles the real and complex cases simultaneously.

Definition. Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra, and let k be a positive integer.
(1) A $|k|$–grading on $\mathfrak{g}$ is a vector space decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for all i and j, with the convention that $\mathfrak{g}_\ell = \{0\}$ for $|\ell| > k$, and such that the subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$.
(2) Given a $|k|$–grading as in (1), we put $\mathfrak{g}^i := \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$ for all i as well as $\mathfrak{p} := \mathfrak{g}^0$ and $\mathfrak{p}_+ := \mathfrak{g}^1$.
(3) A Lie subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *parabolic* if it can be realized as $\mathfrak{g}^0$ for some $|k|$–grading of $\mathfrak{g}$.

The subspaces $\mathfrak{g}^i$ from (2) define a decreasing filtration of $\mathfrak{g}$, which makes $\mathfrak{g}$ into a *filtered Lie algebra*, i.e. $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ for all i, j. In particular, this implies that $\mathfrak{p} = \mathfrak{g}^0$ is a Lie subalgebra of $\mathfrak{g}$, and that $\mathfrak{p}_+ = \mathfrak{g}^1$ is an ideal in $\mathfrak{p}$, which is nilpotent since the grading has finite length. Likewise, $\mathfrak{g}_- \subset \mathfrak{g}$ is a nilpotent Lie subalgebra. It turns out, see [20], that the Lie algebras $\mathfrak{g}_-$ and $\mathfrak{p}_+$ are always isomorphic. From the filtration property it also follows that each of the filtration components $\mathfrak{g}^i$ is $\mathfrak{p}$–invariant.

By the grading property, $\mathfrak{g}_0 \subset \mathfrak{p} \subset \mathfrak{g}$ is a Lie subalgebra, and each of the grading components $\mathfrak{g}_i$ is $\mathfrak{g}_0$–invariant. It turns out that the Lie algebra $\mathfrak{g}_0$ is always *reductive*, so it is the direct sum of a semisimple Lie algebra and a center. In particular, the representation theory of $\mathfrak{g}_0$ is significantly easier than the one of the parabolic $\mathfrak{p}$, which makes $\mathfrak{g}_0$ a valuable technical tool in the theory. Moreover, it turns out that completely reducible representations of $\mathfrak{p}$ always come from representation of $\mathfrak{g}_0$ via the quotient map $\mathfrak{p} \to \mathfrak{p}/\mathfrak{p}_+ \cong \mathfrak{g}_0$.

A crucial feature of the theory is the computability of the Lie algebra cohomology groups $H^*(\mathfrak{g}_-, \mathfrak{g})$ via Kostant's version of the Bott–Borel–Weil theorem, see [8]. The standard complex for computing these cohomologies consists of spaces of multilinear alternating maps from $\mathfrak{g}_-$ to $\mathfrak{g}$. In particular, $\mathfrak{g}_0$ acts naturally on these spaces and it is easy to see that the differentials in the standard complex are $\mathfrak{g}_0$–equivariant. Hence the cohomologies naturally are representations of $\mathfrak{g}_0$, and Kostant's results describes them as such representations. The computation of the cohomologies is completely algorithmic and it is also implemented by J. Šilhan as an extension to the Lie software system, see [17], so the computation of the cohomologies can be left to a computer.

2.2. **The symbol algebra of a distribution.** Let M be a smooth manifold and let $H \subset TM$ be a distribution, i.e. a smooth subbundle. We assume that H is bracket generating, i.e. that sections of H together with their iterated Lie brackets span the whole tangent bundle. We will also write $T^{-1}M$ for H. Next we require that sections of H together with Lie brackets of two such sections span a smooth subbundle $T^{-2}M \subset TM$, which by construction contains $T^{-1}M$. Inductively, we require that we get a filtration $TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$ of the tangent bundle by smooth subbundles, such that for each $i < 0$ sections of T^iM together with Lie brackets of one section of T^iM and one section of $T^{-1}M$ span $T^{i-1}M$. The sequence of the ranks of the subbundles T^iM is usually called the *small growth vector* of the distribution H.

Having extended the distribution H to the filtration $\{T^iM\}$, one can next encode the non–integrability properties of H. Namely, for each $i = -k, \dots, -1$, one defines $\mathrm{gr}_i(TM) := T^iM/T^{i+1}M$, and then $\mathrm{gr}(TM) = \oplus\, \mathrm{gr}_i(TM)$ is the associated graded to the filtered vector bundle TM. One immediately verifies, that for sections $\xi \in \Gamma(T^iM)$ and $\eta \in \Gamma(T^jM)$, the Lie bracket $[\xi, \eta]$ is a section of $T^{i+j}M$, where $T^\ell M = TM$ for $\ell \leq -k$. Hence the Lie bracket of vector fields induces a bilinear bundle map $\mathrm{gr}_i(TM) \times \mathrm{gr}_j(TM) \to \mathrm{gr}_{i+j}(TM)$, which for each $x \in M$ makes $\mathrm{gr}(T_xM)$ into a nilpotent graded Lie algebra. This is called the *symbol algebra* of the distribution H at x.

Suppose that $f : M \to M$ is a diffeomorphism such that $Tf(H) \subset H$. Then by construction Tf preserves each of the subbundles T^iM and hence is compatible with the filtration of TM. Hence for each $x \in M$, the tangent map T_xf induces a linear isomorphism $\mathrm{gr}(T_xM) \to \mathrm{gr}(T_{f(x)}M)$. Compatibility of Tf with the Lie bracket of vector fields immediately implies that this map actually is an isomorphism of the symbol algebras at x and $f(x)$. Hence the symbol algebra is a fundamental invariant of the distribution.

In general, the isomorphism type of the symbol algebra may change from point to point, but we will be only interested in distributions for which $\mathrm{gr}(TM)$ is locally trivial as a bundle of nilpotent graded Lie algebras. If $\mathfrak{n} = \mathfrak{n}_{-k} \oplus \cdots \oplus \mathfrak{n}_{-1}$ is the modelling nilpotent graded Lie algebra, then $\mathrm{gr}(TM)$ has an obvious natural frame bundle with structure group $\mathrm{Aut}_{gr}(\mathfrak{n})$, the group of automorphisms of the graded Lie algebra $\mathfrak{n}$, compare with [11].

2.3. **From parabolics to distributions.** Consider a $|k|$–graded Lie algebra $\mathfrak{g} = \oplus_{i=-k}^{k}\mathfrak{g}_i$ as in 2.1 and the corresponding filtration $\{\mathfrak{g}^i\}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then one shows, see [20] and [5], that

$$P := \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}^i) \subset \mathfrak{g}^i \quad \forall i\}$$
$$G_0 := \{g \in G : \mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i \quad \forall i\}$$

are closed subgroups of G with Lie algebras $\mathfrak{p} = \mathfrak{g}^0$ and $\mathfrak{g}_0$, respectively. In particular, restricting the action of G_0 to the nilpotent subalgebra $\mathfrak{g}_-$ we obtain an action by Lie algebra automorphisms, i.e. a homomorphism $G_0 \to \mathrm{Aut}_{gr}(\mathfrak{g}_-)$.

The homogeneous spaces of the form G/P are the so–called *generalized flag manifolds*, which are of central interest in representation theory. It turns out that they are always compact.

Proposition 2.3. *The generalized flag manifold G/P carries a natural distribution $H \subset T(G/P)$ of rank $\dim(\mathfrak{g}_{-1})$, whose bundle of symbol algebras is locally trivial with modelling algebra $\mathfrak{g}_-$. The natural left action of G on G/P preserves this distribution.*

Proof. The tangent bundle of G/P can be identified with the associated bundle $G \times_P (\mathfrak{g}/\mathfrak{p})$. Now the P–invariant filtration $\{\mathfrak{g}^i\}$ of $\mathfrak{g}$ induces a P–invariant filtration $\mathfrak{g}/\mathfrak{p} = \mathfrak{g}^{-k}/\mathfrak{p} \supset \cdots \supset \mathfrak{g}^{-1}/\mathfrak{p}$. For each $i = -k, \dots, -1$, we get a smooth subbundle $G \times_P (\mathfrak{g}^i/\mathfrak{p}) =: T^i(G/P) \subset T(G/P)$, i.e. a filtration of the tangent bundle of G/P. Explicitly, denoting by $p : G \to G/P$ the natural projection, the subbundle $T^i(G/P)$

is given by

$$T^i_{gP}(G/P) = T_g p(\{L_X(g) : X \in \mathfrak{g}^i\}),$$

where L_X denotes the left invariant vector field generated by X. This immediately shows that the left actions of elements of G preserve the filtration $\{T^i(G/P)\}$ of the tangent bundle.

By construction, the component $\mathrm{gr}_i(T(G/P))$ of the associated graded is the bundle induced by the representation $(\mathfrak{g}^i/\mathfrak{p})/(\mathfrak{g}^{i+1}/\mathfrak{p}) \cong \mathfrak{g}^i/\mathfrak{g}^{i+1}$. Consider sections $\xi \in T^i(G/P)$ and $\eta \in T^j(G/P)$ with $i < j$. By construction, there are local lifts $\tilde{\xi}, \tilde{\eta} \in \mathfrak{X}(G)$ which can be written in the form $\tilde{\xi} = \sum_a \varphi_a L_{X_a}$ and $\tilde{\eta} = \sum_b \psi_b L_{Y_b}$ for smooth functions φ_a and ψ_b and elements $X_a \in \mathfrak{g}^i$ and $Y_b \in \mathfrak{g}^j$. This shows that $[\tilde{\xi}, \tilde{\eta}]$ can be written as the sum of

$$\textstyle\sum_{a,b} \varphi_a \psi_b [L_{X_a}, L_{Y_b}] = \sum_{a,b} \varphi_a \psi_b L_{[X_a, Y_b]}$$

and a linear combination of left invariant vector fields with generators in $\mathfrak{g}^i$. Hence we conclude that $[\xi, \eta] \in \Gamma(T^{i+j}(G/P))$. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$ we conclude that the distribution $H := T^{-1}(G/P)$ is bracket generating and that $\{T^i(G/P)\}$ is the associated filtration as described in 2.2. Finally, it also shows that under the natural identification $\mathfrak{g}^i/\mathfrak{g}^{i+1} \cong \mathfrak{g}_i$, the symbol algebra of H in each point is isomorphic to $\mathfrak{g}_-$. $\square$

2.4. **Canonical Cartan connections.** Since the algebras $\mathfrak{g}$ are always semisimple, there is a natural choice of a Lie group G with Lie algebra $\mathfrak{g}$, namely the automorphism group $\mathrm{Aut}(\mathfrak{g})$. Then the subgroups $G_0 \subset P \subset G$ are the groups $\mathrm{Aut}_{gr}(\mathfrak{g}) \subset \mathrm{Aut}_f(\mathfrak{g})$ of automorphisms preserving the grading respectively the filtration of $\mathfrak{g}$. Now we have to assume an additional (cohomological) condition on the grading of $\mathfrak{g}$. The first Lie algebra cohomology group $H^1(\mathfrak{g}_-, \mathfrak{g})$ consists of equivalence classes of linear maps $\mathfrak{g}_- \to \mathfrak{g}$, and there is an obvious notion of homogeneity for such maps. The Lie algebra differentials are compatible with homogeneity, so the cohomology group naturally splits according to homogeneity.

Assuming that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in negative homogeneities, it turns out (see [15]) that the homomorphism $G_0 \to \mathrm{Aut}_{gr}(\mathfrak{g}_-)$ is actually an isomorphism, and then the general prolongation procedures of [18, 10, 5] imply

Theorem 2.4. *Let M be a smooth manifold such that $\dim(M) = \dim(G/P)$ endowed with a bracket generating distribution $H \subset TM$ of rank $\dim(\mathfrak{g}_{-1})$, whose bundle of symbol algebras is locally trivial with modelling Lie algebra $\mathfrak{g}_-$. Then the natural frame bundle for M with* *structure group* $\mathrm{Aut}_{gr}(\mathfrak{g}_-) \cong G_0$ *can be canonically extended to a principal* P*–bundle* $\mathcal{G} \to M$*, which can be endowed with a regular normal Cartan connection* $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$*.*

The pair $(\mathcal{G}, \omega)$ *is uniquely determined up to isomorphism, and the construction actually establishes an equivalence of categories between manifolds endowed with appropriate distributions and regular normal Cartan geometries.*

Let us explain this a bit. First of all, it is easy to see (compare with [20, 5]) that G_0 can be naturally viewed as a quotient of P. Indeed, the exponential map defines a diffeomorphism from $\mathfrak{g}^1$ onto a closed subgroup $P_+ \subset P$ such that $P/P_+ \cong G_0$. The statement that $\mathcal{G}$ extends the natural frame bundle then simply means that the quotient $\mathcal{G}/P_+$ is (via forms induced by the Cartan connection) isomorphic to this frame bundle.

A Cartan connection on $\mathcal{G}$ by definition is a one–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ such that $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for each $u \in \mathcal{G}$. Further, ω has to be equivariant for the principal right action of P, i.e. $(r^g)^*\omega = \mathrm{Ad}(g^{-1}) \circ \omega$ for all $g \in P$, and it has to reproduce the generators of fundamental vector fields. The pair $(\mathcal{G}, \omega)$ is then referred to as a *Cartan geometry* of type (G, P). Cartan geometries can be viewed as "curved analogs" of the homogeneous space G/P, which determines a Cartan geometry via the natural principal bundle $G \to G/P$ and the left Maurer–Cartan form on G. In this context, G/P is referred to as the *homogeneous model* of geometries of type (G, P).

The amount to which a general Cartan geometry differs from the homogeneous model is measured by its *curvature*. This is the two–form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ defined by

$$K(\xi, \eta) = d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

From the defining properties of a Cartan connection it immediately follows that K is horizontal and P–equivariant. In particular, its value on ξ and η depends only on the projections of the tangent vectors to M. The uniqueness of the Cartan connection in the theorem is ensured by the conditions of regularity and normality on the curvature. Regularity means that if the image of ξ and η in TM lie in the subbundles T^iM and T^jM, respectively, then $K(\xi, \eta) \in \mathfrak{g}^{i+j+1}$.

The condition of normality is crucial for the uniqueness question, and finding appropriate normalization conditions often is a very difficult step in the construction of canonical Cartan connections. For the purposes of this article, we do not need any details on the form of this condition. The only information we need (and also this is only needed to deal with the non–flat case) is that K determines a natural quantity called the *harmonic curvature*, which is a section κ_h of the associated bundle $\mathcal{G}\times_P(H^2(\mathfrak{g}_-,\mathfrak{g}))$. Vanishing of the harmonic curvature is equivalent to vanishing of K and to local isomorphism of the geometry with G/P, see [5].

The theorem on existence and uniqueness of Cartan connections has important consequences for the homogeneous model.

Corollary 2.4. *Suppose that $H^1(\mathfrak{g}_-,\mathfrak{g})$ is concentrated in negative homogeneities. Then for any connected open subset $U\subset G/P$, the automorphism group of the distribution $T^{-1}U := T^{-1}(G/P)|_U \subset TU$ is the subgroup of G consisting of all elements whose left action on G/P maps the subset U to itself. In particular, the automorphism group of $T^{-1}(G/P)$ itself is the group $G = \mathrm{Aut}(\mathfrak{g})$.*

Proof. Of course, any element of G whose left action preserves U gives rise to an automorphism. Conversely, let $p : G \to G/P$ be the natural projection and let ω be the left Maurer Cartan form on G. Then ω restricts to a Cartan connection on the principal P–bundle $p^{-1}(U)\to U$, which is flat by the Maurer–Cartan equation. Hence it must be the normal Cartan connection determined by $T^{-1}U$. The equivalence of categories stated in the theorem implies that any automorphism of U lifts to an automorphism of the Cartan geometry, i.e. to a P–equivariant diffeomorphism on $p^{-1}(U)$ which is compatible with the Cartan connection. Then [16, Theorem 5.2] applied to the given automorphism and the identity shows that on each connected component of $p^{-1}(U)$ the automorphism is given by the left action of some element of G. Since U is connected, P acts transitively on the set of connected components of $p^{-1}(U)$, so P–equivariancy implies that we always get the same element $g\in G$. □

2.5. **Automorphism groups.** For distributions which are equivalent to Cartan geometries, any automorphism of the distributions canonically lifts to the Cartan geometry, so the automorphism groups of the distribution can be identified with the one of the Cartan geometry. Of course, an automorphism of a Cartan geometry $(\mathcal{G},\omega)$ is a P–equivariant diffeomorphism $\Phi:\mathcal{G}\to\mathcal{G}$ such that $\Phi^*\omega=\omega$. There is an obvious infinitesimal analog of this concept. Namely, the flow of a complete vector field on $\mathcal{G}$ consists of automorphisms if and only if the field lies in

$$\mathfrak{inf}(\mathcal{G},\omega) := \{\xi\in\mathfrak{X}(\mathcal{G}) : (r^g)^*\xi=\xi \text{ for all } g\in P \text{ and } \mathcal{L}_\xi\omega=0\},$$

where (r^g) is the principal right action of g and $\mathcal{L}$ denotes the Lie derivative. This is called the set of *infinitesimal automorphisms* of the geometry. Notice that by definition $\mathfrak{inf}(\mathcal{G}, \omega)$ is closed under Lie brackets.

Theorem 2.5. *(1) Let $(p : \mathcal{G} \to M, \omega)$ be a Cartan geometry of type (G, P) with connected base M. Then the automorphism group $\mathrm{Aut}(\mathcal{G}, \omega)$ can be made into a Lie group of dimension $\leq \dim(G)$, whose Lie algebra $\mathfrak{aut}(\mathcal{G}, \omega)$ consists of all complete vector fields contained in $\mathfrak{inf}(\mathcal{G}, \omega)$.*
(2) For any point $u \in \mathcal{G}$, the map $\xi \mapsto \omega(\xi(u))$ induces an injection $\mathfrak{aut}(\mathcal{G}, \omega) \hookrightarrow \mathfrak{g}$. Denoting by $\mathfrak{a} \subset \mathfrak{g}$ the image, the Lie bracket on $\mathfrak{aut}(\mathcal{G}, \omega)$ is mapped to the operation

$$(X, Y) \mapsto [X, Y] - K(\omega^{-1}(X), \omega^{-1}(Y))(u)$$

on $\mathfrak{a}$.
(3) If the Cartan geometry is regular, then restricting the filtration $\{\mathfrak{g}^i\}$ of $\mathfrak{g}$ to the subspace $\mathfrak{a}$ makes $\mathfrak{aut}(\mathcal{G}, \omega)$ into a filtered Lie algebra. The associated graded of this Lie algebra is isomorphic to a graded Lie subalgebra of $\mathfrak{g}$.

Proof. We first claim that $\xi \mapsto \omega(\xi(u))$ defines an injection $\mathfrak{inf}(\mathcal{G}, \omega) \to \mathfrak{g}$, i.e. that any infinitesimal automorphism is determined by its value in one point. Using the standard formula for the Lie derivative and inserting the definition of the exterior derivative, we get

$$0 = (\mathcal{L}_\xi \omega)(\eta) = d\omega(\xi, \eta) + \eta \cdot \omega(\xi) = \xi \cdot \omega(\eta) - \omega([\xi, \eta]).$$

for $\xi \in \mathfrak{inf}(\mathcal{G}, \omega)$ and $\eta \in \mathfrak{X}(\mathcal{G})$. In particular, if $\omega(\eta)$ is constant, then injectivity of ω implies $[\xi, \eta] = 0$. But this implies that ξ is invariant under the flows of such fields. Since such fields span each tangent space, we see that $\xi(u)$ determines ξ locally around u. Since ξ is P–invariant, this determines the restriction of ξ to $p^{-1}(U)$ for some open neighborhood U of $p(u)$ in M. Now connectedness of M implies that ξ is globally determined by $\xi(u)$. In particular, $\mathfrak{inf}(\mathcal{G}, \omega)$ is a finite dimensional Lie subalgebra of $\mathfrak{X}(\mathcal{G})$, and (1) follows from R. Palais' characterization of Lie transformation groups, see [14].
(2) We already know that $\mathfrak{aut}(\mathcal{G}, \omega)$ injects into $\mathfrak{g}$, so it remains to verify the formula for the Lie bracket. It is well known that the Lie bracket on $\mathfrak{aut}(\mathcal{G}, \omega)$ is induced by the negative of the Lie bracket of vector fields. Now for $\xi, \eta \in \mathfrak{inf}(\mathcal{G}, \omega)$, we get

$$0 = d\omega(\xi, \eta) + \eta \cdot \omega(\xi) = K(\xi, \eta) - [\omega(\xi), \omega(\eta)] + \omega([\eta, \xi]),$$

where we have used the definition of K to rewrite the first term and the infinitesimal automorphism equation for η to rewrite the second one. This evidently implies that claimed formula.
(3) The regularity of the geometry implies that for $X \in \mathfrak{g}^i$ and $Y \in \mathfrak{g}^j$ we get $K(\omega^{-1}(X), \omega^{-1}(Y))(u) \in \mathfrak{g}^{i+j+1}$. This implies that the modified

bracket on $\mathfrak{a}$ still respects the filtration and that the K–term does not contribute to the bracket on the associated graded. Since the filtration of $\mathfrak{g}$ comes from a grading, the associated graded of $\mathfrak{g}$ is isomorphic to $\mathfrak{g}$ itself, and the last statement follows. □

Part (3) of this theorem is the main input for getting restrictions on possible automorphism groups. To get additional information for geometries which are non–flat (i.e. not locally isomorphic to G/P) we need an additional bit of information. We have briefly discussed in 2.4 the harmonic curvature κ_h of ω which is a section of $\mathcal{G} \times_P H^2(\mathfrak{g}_-, \mathfrak{g})$. Hence it corresponds to a P–equivariant function $\mathcal{G} \to H^2(\mathfrak{g}_-, \mathfrak{g})$. Now it turns out that the latter representation is completely reducible, so the P–action factorizes through G_0. Naturality of the construction of κ_h implies that any infinitesimal automorphism of $(\mathcal{G}, \omega)$ has to preserve κ_h, i.e. annihilate the corresponding function. Using this, we formulate

Corollary 2.5. *Let $\mathfrak{g} = \oplus_{i=-k}^{k} \mathfrak{g}_i$ be a $|k|$–graded Lie algebra such that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in negative homogeneous degrees, and let $G_0 \subset P \subset G$ be the corresponding groups. Let M be a smooth manifold of dimension $\dim(G/P)$ endowed with a distribution $H \subset TM$ of rank $\dim(\mathfrak{g}_{-1})$ such that the bundle of symbol algebras is locally trivial and modelled on $\mathfrak{g}_-$. Let $\mathrm{Aut}(M, H)$ be the group of all diffeomorphisms of M which preserve the distribution H.*
(1) $\mathrm{Aut}(M, H)$ is a Lie group of dimension $\leq \dim(G)$.
(2) If $\ell := \dim(\mathrm{Aut}(M, H)) < \dim(G)$, then ℓ equals the dimension of a proper graded subalgebra $\mathfrak{b} = \oplus_{i=-k}^{k} \mathfrak{b}_i$ of $\mathfrak{g}$.
(3) If (M, H) is not locally isomorphic to the canonical distribution on G/P induced by $\mathfrak{g}^{-1}/\mathfrak{p} \subset \mathfrak{g}/\mathfrak{p}$, then the graded Lie subalgebra in (2) has the additional property that there is a nonzero element in $H^2(\mathfrak{g}_-, \mathfrak{g})$ which is annihilated by all elements of $\mathfrak{b}_0$.

Proof. In view of the equivalence of categories proved in theorem 2.4 parts (1) and (2) follow directly from parts (1) and (3) of theorem 2.5.

For part (3), we can choose our point $u \in \mathcal{G}$ in such a way that $\kappa_h(u) \neq 0$. Then the value in u of the corresponding equivariant function determines an element of $H^2(\mathfrak{g}_-, \mathfrak{g})$. Elements of $\mathfrak{a}^0$ correspond to infinitesimal automorphisms ξ such that $\omega(\xi)(u) \in \mathfrak{g}^0 = \mathfrak{p}$, i.e. such that $\xi(u)$ is vertical. Differentiating an equivariant function along ξ, the result in u therefore coincides with the algebraic action of $\omega(\xi)(u)$ on the value of the function. By complete reducibility of $H^2(\mathfrak{g}_-, \mathfrak{g})$, this action depends only on the component in $\mathfrak{g}_0 = \mathfrak{g}^0/\mathfrak{g}^1$. □

Remark 2.5. (1) By definition any infinitesimal automorphism ξ of a Cartan geometry of type (G, P) is a right invariant vector field and

hence projects to an vector field $\underline{\xi}$ on M. For geometries equivalent to distributions, this is the "actual" infinitesimal automorphism, i.e. its local flows preserves the distribution. The filtration from part (3) of the theorem has a nice interpretation in terms of $\underline{\xi}$. Namely, for $u \in \mathcal{G}$ and $x = p(u) \in M$, we have $\omega(\xi)(u) \in \mathfrak{g}^i$ if and only if $\underline{\xi}(u) \in T^i_x M$ for all $i < 0$. Likewise $\omega(\xi)(u) \in \mathfrak{g}^0$ if and only if $\underline{\xi}(u) = 0$, so we have a fix point for the one–parameter group of automorphisms generated by ξ. The fact that $\omega(\xi)(u) \in \mathfrak{g}^i$ for some $i > 0$ can be interpreted (in a certain sense) as higher order vanishing of $\underline{\xi}$ in x.
(2) Part (3) of Corollary 2.5 shows that the dimension of $\mathfrak{b}_0$ is bounded by the largest possible dimension of an annihilator of a nonzero element in $H^2(\mathfrak{g}_-, \mathfrak{g})$, which can be easily determined using representation theory as follows. The Lie algebra $\mathfrak{g}_0$ is always reductive, and in the cases studied in this paper its center is one–dimensional and acts non–trivially on $H^2(\mathfrak{g}_-, \mathfrak{g})$.

For a complex semisimple Lie group H and a complex irreducible representation V, one can consider the action of H on the projectivization $\mathbb{P}(V)$. It is well known (see [7, chapter 23]) that the orbit of the line through a highest weight vector is the unique H–orbit of smallest dimension, and the stabilizer of this line is a parabolic subgroup of H, whose type can be read off the highest weight of V. Consequently, the stabilizer in H of a highest weight vector in V has the largest possible dimension among all stabilizers of non–zero vectors. Since the parabolic subgroup acts non–trivially on the highest weight line, this stabilizer has codimension one in the parabolic subgroup.

If H is reductive with one–dimensional center which acts non–trivially (and by Schur's lemma by a scalar) on V, then one can look at the corresponding parabolic subgroup in the semisimple part of H. Since both this parabolic subgroup and the center act non–trivially on the highest weight line, it follows that the stabilizer of a highest weight vector in H has the same dimension as the parabolic subgroup in the semisimple part.

To apply this to our situation, we only have to notice that via complexification, the complex dimension of the stabilizer of a non–zero element gives an upper bound for the real dimension of the stabilizer of a non–zero element. Hence to obtain the upper bounds for $\dim(\mathfrak{b}_0)$ one only has to compute the dimensions of parabolic subalgebras in the complexification of the semisimple part of $\mathfrak{g}_0$, whose type can be read off the highest weight of $H^2(\mathfrak{g}_-, \mathfrak{g})$, which is the output of Kostant's theorem.

3. Examples

3.1. **Generic rank n distributions in dimension $\frac{n(n+1)}{2}$.** For $n \geq 3$ consider $\mathbb{R}^{2n+1}$ with the (indefinite) inner product

$$\langle v, w\rangle = v_{n+1}w_{n+1} + \sum_{i=1}^{n} v_i w_{n+1+i} + \sum_{i=1}^{n} v_{n+1+i} w_i.$$

By definition, the subspaces generated by the first n respectively by the last n vectors in the standard basis are isotropic, which shows that the inner product has split signature $(n+1, n)$. The orthogonal Lie algebra $\mathfrak{g} \cong \mathfrak{so}(n+1, n)$ for this inner product has the form

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & v & B \\ w & 0 & -v^t \\ C & -w^t & -A^t \end{pmatrix} : \begin{array}{l} A \in \mathfrak{gl}(n, \mathbb{R}), C, B \in \mathfrak{o}(n) \\ v \in \mathbb{R}^n, w \in \mathbb{R}^{n*} \end{array} \right\}.$$

This admits an obvious grading of the form

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix},$$

with blocks of sizes n, 1, and n. The associated parabolic subalgebra $\mathfrak{p} = \mathfrak{g}^0$ is the stabilizer of the n–dimensional isotropic subspace generated by the first n basis vectors. From this representation it is evident that $\mathfrak{g}_0 \cong \mathfrak{gl}(n, \mathbb{R})$. The adjoint action makes each $\mathfrak{g}_i$ into a $\mathfrak{g}_0$–module and the bracket on $\mathfrak{g}$ induces homomorphisms of $\mathfrak{g}_0$–modules. From the matrix representation it is evident, that $\mathfrak{g}_1 \cong \mathbb{R}^n$, $\mathfrak{g}_{-1} \cong \mathbb{R}^{n*}$ as $\mathfrak{g}_0$–modules. Further, the bracket induces isomorphisms $\Lambda^2\mathfrak{g}_{\pm1} \to \mathfrak{g}_{\pm2}$ and $\mathfrak{g}_{-1} \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$. Finally, the restrictions $\mathfrak{g}_{\pm1} \times \mathfrak{g}_{\mp2} \to \mathfrak{g}_{\mp1}$ of the bracket are induced by $(w, B) \mapsto wB$ and $(v, C) \mapsto Cv$, respectively.

The general tools mentioned in 2.1 show that $H^1(\mathfrak{g}_-, \mathfrak{g})$ is concentrated in negative homogeneities in this case, see also [15]. Putting $G = \mathrm{Aut}(\mathfrak{g})$ and $P = \mathrm{Aut}_f(\mathfrak{g})$ we can thus apply the results from 2.4. The corresponding bracket generating distributions are rank n distributions on manifolds of dimension $n(n+1)/2$. If $H \subset TM$ is such a distribution, then the condition that the bundle of symbol algebras is locally trivial and modelled on $\mathfrak{g}_-$ simply means that for each $x \in M$, the Levi bracket induces an isomorphism $\Lambda^2 H_x \to T_xM/H_x$. Since we know that one such distribution exists (on G/P, which can be viewed as the Grassmannian of all isotropic n–dimensional subspaces in $\mathbb{R}^{2n+1}$), this is evidently a generic condition. This clearly is the only generic type of rank n distributions in dimension $\frac{n(n+1)}{2}$. For the minimal value

$n = 3$, we obtain generic rank 3 distributions in dimension 6. These have been studied by R. Bryant in his thesis, see [1, 2].

Theorem 3.1. *Let M be a smooth manifold of dimension $n(n+1)/2$, $H \subset TM$ a generic distribution of rank n, and let $\mathrm{Aut}(M,H)$ be the group of all diffeomorphisms of M which are compatible with H.*
(1) $\mathrm{Aut}(M,H)$ *is a Lie group of dimension at most $2n^2+n$.*
(2) If $\dim(\mathrm{Aut}(M,H)) < 2n^2+n$, *then* $\dim(\mathrm{Aut}(M,H)) \le 2n^2-n+1$, *i.e. the dimension has to drop by at least $2n-1$.*
(3) If (M,H) is not locally isomorphic to the canonical distribution on G/P, then

$$\dim(\mathrm{Aut}(M,H)) \le \begin{cases} 13 & n=3 \\ 27 & n=4 \\ 2n^2-3n+6 & 5 \le n \le 8 \\ 2n^2-3n+5 & n \ge 9 \end{cases}$$

Proof. (1) follows immediately from part (1) of Corollary 2.5. In view of part (2) of that Corollary, we can prove (2) by showing that any proper graded subalgebra $\mathfrak{b} = \oplus_{i=-2}^{2}\mathfrak{b}_i$ of $\mathfrak{g}$ has dimension at most $2n^2-n+1$. Suppose that $\mathfrak{b}$ is such a subalgebra and put $d_j = \dim(\mathfrak{b}_j)$ for all $j = -2,\dots,2$.

Let us first assume that $d_{-1} = n$, i.e. $\mathfrak{b}_{-1} = \mathfrak{g}_{-1}$. Then $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1},\mathfrak{g}_{-1}] \subset \mathfrak{b}$, so all of $\mathfrak{g}_-$ is contained in $\mathfrak{b}$. Then we must have $\mathfrak{b}_1 \neq \mathfrak{g}_1$, since otherwise also $\mathfrak{g}_2 = [\mathfrak{g}_1,\mathfrak{g}_1]$ and $\mathfrak{g}_0 = [\mathfrak{g}_{-1},\mathfrak{g}_1]$ would be contained in $\mathfrak{g}$.

Hence $\ell := d_1 < n$. The group $G_0 = \mathrm{Aut}_{gr}(\mathfrak{g}_-)$ is isomorphic to $GL(\mathfrak{g}_{-1})$ and conjugating $\mathfrak{b}$ by an appropriate element of this group, we may assume that $\mathfrak{b}_1 = \mathbb{R}^\ell \subset \mathbb{R}^n = \mathfrak{g}_1$. Now $[\mathfrak{b}_{-1},\mathfrak{b}_2] \subset \mathfrak{b}_1$ and since $\mathfrak{b}_{-1} = \mathfrak{g}_{-1}$ the description of the bracket shows that the matrices in $\mathfrak{b}_2$ may have nonzero entries only in the first ℓ rows, so $d_2 \le \ell(\ell-1)/2$ by skew symmetry. In particular, if $d_1 = 0$ then $d_2 = 0$ and we have already lost $n(n+1)/2 \ge 2n$ dimensions. So we may assume that $1 \le \ell \le n-1$. Then $\mathfrak{b}_0 \subset \mathfrak{g}_0 = \mathfrak{gl}(n,\mathbb{R})$ must stabilize $\mathfrak{b}_1$, so $d_0 \le n^2-(n-\ell)\ell$, and this becomes maximal for $\ell = 1$ and $\ell = n-1$, whence $d_0 \le n^2-n+1$. From above, we know that $d_2 \le \frac{(n-1)(n-2)}{2} = \frac{n^2-3n+2}{2}$, so we conclude that

$$\dim(\mathfrak{b}) \le \frac{n(n-1)}{2} + n + (n^2-(n-\ell)\ell) + i + \frac{\ell(\ell-1)}{2} \le 2n^2-n+1,$$

with equality attained for $\ell = n-1$.

If $d_1 = n$, the situation is completely symmetric and we get the same bound on $\dim(\mathfrak{b})$, so it remains to consider the case that both d_1 and d_{-1} are smaller than n. We claim, that in these cases already

$d_{-1}+d_0+d_1 \leq \dim(\mathfrak{g}_{-1})+\dim(\mathfrak{g}_0)+\dim(\mathfrak{g}_1)-2n+1$. This is evidently true if $d_{-1}=d_1=0$, so by symmetry we may assume that for $d_1=\ell$ we have $0<\ell<n$. Then we know that $d_0 \leq n^2-(n-\ell)\ell$, so we have lost at least $n-1$ dimensions already. If $d_{-1} \leq n-\ell$ then we have also lost at least n dimensions from $d_{\pm 1}$ and we are done again.

Hence we are left with the case that $\ell' := d_{-1} > n-\ell$. Since $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are dual $\mathfrak{g}_0$–modules, the annihilator of $\mathfrak{b}_{-1}$ in $\mathfrak{g}_1$ is a subspace of dimension $n-\ell'<\ell$ which by construction must be invariant under $\mathfrak{b}_0$. Hence $\mathfrak{b}_0$ has to preserve two subspaces of different dimensions. Preserving the ℓ–dimensional subspace forces $d_0 \leq \dim(\mathfrak{g}_0)-(n-1)$. If the second subspace is contained in the first one, we loose $\ell-1$ more dimensions, so the total loss adds up to $(n-1)+\ell-1+n-\ell+n-\ell' = 3n-2-\ell'$, and since $\ell'<n$, this is at least $2n-1$. If the two subspaces are not nested but have non–trivial intersection, then the same argument applies to one of them and the intersection. Finally, if the two subspaces are transverse, then each of them causes a reduction of $(n-1)$ dimensions for $\mathfrak{b}_0$.

(3) Denote by $\mathfrak{b}=\oplus_{i=-2}^{2}\mathfrak{b}_i$ the graded subalgebra of $\mathfrak{g}$ associated to the Lie algebra of the automorphism group, cf. Corollary 2.5. From part (3) of this Corollary we know that $\mathfrak{b}_0$ annihilates a nonzero element in $H^2(\mathfrak{g}_-,\mathfrak{g})$.

If $n=3$ the $\mathfrak{g}_0$-module $H^2(\mathfrak{g}_-,\mathfrak{g})$ is the irreducible component of highest weight in $\mathbb{R}^3\otimes\mathbb{R}^3\otimes\Lambda^2\mathbb{R}^3\otimes(\mathbb{R}^3)^*$. Following Remark 2.5 (2), $\dim(\mathfrak{b}_0)$ is bounded by the dimension of the Borel subalgebra of the complexification of $\mathfrak{g}_0$, so $\dim(\mathfrak{b}_0)\leq 5$.

Suppose first that $d_{-1}=3$. Since $\mathfrak{g}_0\cong\mathfrak{g}_{-1}\otimes\mathfrak{g}_1$ and $d_0\leq 5$, we obtain $d_1\leq 1$. Hence $d_2=0$ and $\dim(\mathfrak{b})\leq 12$. The same argument applies to $d_1=3$. If $d_{-1}\leq 2$, then $\mathfrak{g}_0\cong\mathfrak{g}_{-1}\otimes\mathfrak{g}_1$ and $d_0\leq 5$ imply $d_1\leq 2$. If $d_{-1}=2$, we may assume that $\mathfrak{b}_{-1}=\mathbb{R}^2\subset\mathbb{R}^3$. Since $d_1\leq 2$ and $\mathfrak{g}_2\times\mathfrak{b}_{-1}\to\mathfrak{g}_1$ is surjective, we must have $d_2\leq 2$. For $d_1\leq 1$ we are done, since we already lost 8 dimensions. If $d_1=2$ we conclude from $[\mathfrak{b}_{-2},\mathfrak{b}_1]\subset\mathfrak{b}_{-1}$ that $d_{-2}\leq 2$ and therefore $\dim(\mathfrak{b})\leq 13$ also in this case. For $d_{-1}\leq 1$ and $d_1\leq 1$ one already has $\dim(\mathfrak{b})\leq 13$. If $d_{-1}\leq 1$ and $d_1>1$ we can use the arguments above to see $\dim(\mathfrak{b})\leq 13$, thus completing the proof for $n=3$.

For $n>3$ the $\mathfrak{g}_0$-module $H^2(\mathfrak{g}_-,\mathfrak{g})$ is the irreducible component of highest weight in $\mathbb{R}^n\otimes\Lambda^2\mathbb{R}^n\otimes\Lambda^2(\mathbb{R}^n)^*$. Following Remark 2.5 (2), we can compute the dimension of parabolic subalgebra in the complexification of $\mathfrak{g}_0$ to conclude that $\dim(\mathfrak{b}_0)\leq n^2-4n+10$. If $d_{-1}=n$, then we have $d_1\leq\frac{n^2-4n+10}{n}$, since $\mathfrak{g}_0\cong\mathfrak{g}_{-1}\otimes\mathfrak{g}_1$. Further we know that $d_2\leq\frac{(d_1-1)d_1}{2}$. Putting this together we conclude

$\dim(\mathfrak{b}) \leq 2n^2 - 7n + 26 - \frac{35}{n} + \frac{100}{n^2}$. The same argument applies if $d_1 = n$, so it remains to consider the case $d_{-1}, d_1 \leq n-1$.

If $0 \leq d_{-1} \leq n-1$ and $2 \leq d_1 \leq n-1$, then the surjectivity of $\mathfrak{g}_{-2} \times \mathfrak{b}_1 \to \mathfrak{g}_{-1}$ forces $d_{-2} < \dim(\mathfrak{g}_{-2})$. A straightforward analysis of the possible cases using $d_1 \cdot d_{-1} \leq n^2 - 4n + 10$ shows that we get $\dim(\mathfrak{b}) \leq 2n^2 - 3n + 5 + \frac{7}{n-1}$. By the symmetry of the grading one obtains the same bound for $d_{-1} \geq 2$. Finally, if $d_{-1}, d_1 \leq 1$, we already have $\dim(\mathfrak{b}) \leq 2n^2 - 5n + 12$.

Comparing the three bounds obtained so far, we see that $\dim(\mathfrak{b}) \leq 2n^2 - 3n + 5 + \frac{7}{n-1}$ is always valid for $n \geq 5$ while $\dim(\mathfrak{b}) \leq 27$ for $n = 4$. $\square$

Remark 3.1. (1) From the proof we see that for each $k = 1, \dots, n-1$

$$\mathfrak{b}^k := \Lambda^2(\mathbb{R}^n)^* \oplus (\mathbb{R}^n)^* \oplus \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \oplus \mathbb{R}^k \oplus \Lambda^2\mathbb{R}^k$$

is a graded subalgebra of $\mathfrak{g}$. The algebra $\mathfrak{b}^{n-1}$ is a graded subalgebra of the maximal possible dimension $2n^2 - n + 1$. It turns out that, up to conjugation, it is the unique graded subalgebra of this dimension. For each k, one can actually realize $\mathfrak{b}^k$ as the Lie algebra of the automorphism group of a generic distribution. Namely, consider the canonical distribution on the homogeneous model G/P. Then G/P is the space of all maximal isotropic subspaces of $\mathbb{R}^{n+1,n}$. For each $k = 1, \dots, n$ let W_k be the isotropic subspace spanned by the last $(n-k)$ vectors in the standard basis of $\mathbb{R}^{n+1,n}$. Then the maximal isotropic subspaces which intersect W_k only in 0 form an open subset U_k of G/P. Restricting the canonical distribution to this subspace, Corollary 2.4 shows that the automorphism group of this restriction is the subgroup of G consisting of all elements whose action on G/P preserves U_k. It is elementary to show that this coincides with the stabilizer of W_k in G, and hence has Lie algebra isomorphic to $\mathfrak{b}^k$.
(2) The fact that $G_0 = GL(\mathfrak{g}_{-1})$ implies that sub–Riemannian metrics on generic distributions of the type we consider have no pointwise invariants. Hence for any sub–Riemannian structure on such a distribution [12] constructs a canonical Cartan connection.

3.2. **Generic rank two distributions in dimension five.** This is the classical example studied in Cartan's article [6] from 1910. The simple Lie algebra in question is the split real form of the exceptional Lie algebra of type G_2. Although it is not difficult to describe an explicit matrix representation for this Lie algebra, all the information we need can be directly obtained from the root system of type G_2. There are two simple roots, α_1 and α_2, and the other positive roots are

$\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$, $3\alpha_1+\alpha_2$, and $3\alpha_1+2\alpha_2$. The grading we are interested in comes from the coefficient of the short simple root α_1. Hence this is a $|3|$–grading, and the dimensions of the grading components are $\dim(\mathfrak{g}_{\pm 3}) = \dim(\mathfrak{g}_{\pm 1}) = 2$, $\dim(\mathfrak{g}_{\pm 2}) = 1$, and $\dim(\mathfrak{g}_0) = 4$. This decomposition works for the complex simple Lie algebra of type G_2 as well as for its split real form. The root decomposition also implies immediately that the Lie bracket induces isomorphisms $\mathfrak{g}_0 \cong \mathfrak{gl}(\mathfrak{g}_{-1})$, $\Lambda^2\mathfrak{g}_{\pm 1} \cong \mathfrak{g}_{\pm 2}$, and $\mathfrak{g}_{\pm 1}\otimes\mathfrak{g}_{\pm 2} \to \mathfrak{g}_{\pm 3}$. Note that together with the dimensions of the components, the last two statements completely determine the structure of the subalgebra $\mathfrak{g}_-$. Finally, for $i = 1,2,3$ the components $\mathfrak{g}_i$ and $\mathfrak{g}_{-i}$ are dual $\mathfrak{g}_0$–modules. Using this, we can identify $\mathfrak{g}_{\pm 3}$ with $\mathfrak{g}_{\pm 2}\otimes\mathfrak{g}_{\pm 1}$ and $\mathfrak{g}_{\mp 1}$ with $\mathfrak{g}^*_{\pm 1}$, and under these identifications, the bracket $\mathfrak{g}_{\pm 3}\otimes\mathfrak{g}_{\mp 1} \to \mathfrak{g}_{\pm 2}$ is induced by the dual pairing $\mathfrak{g}_{\pm 1}\otimes\mathfrak{g}^*_{\pm 1} \to \mathbb{R}$.

Now suppose that M is a smooth manifold of dimension five and $H \subset TM$ is a distribution of rank 2. Since $\Lambda^2 H$ then has rank one, we see that for $x \in M$ the subspace spanned by sections of H and brackets of two such sections can have dimension at most three, and forming the bracket with another section of H, one can get at most two additional dimensions. The distribution H is called generic if and only if brackets of sections of H of length at most three span all of the tangent space, i.e. if and only if H has small growth vector $(2,3,5)$. This is evidently equivalent to the fact that the Levi bracket induces isomorphisms $\Lambda^2 H_x \to T^{-2}_x M/H_x$ and $(T^{-2}_x M/H_x)\otimes H_x \to T_x M/T^{-2}_x M$, i.e. that each symbol algebra is isomorphic to $\mathfrak{g}_-$. Putting $G := \mathrm{Aut}(\mathfrak{g})$ and $P := \mathrm{Aut}_f(\mathfrak{g})$, Theorem 2.4 implies that regular normal parabolic geometries of type (G,P) are equivalent to generic rank two distributions on five–manifolds.

Theorem 3.2. *Let M be a smooth manifold of dimension five, $H \subset TM$ a generic distribution of rank two, and $\mathrm{Aut}(M,H)$ the automorphism group of this distribution.*
(1) $\mathrm{Aut}(M,H)$ is a Lie group of dimension at most 14.
(2) If $\dim(\mathrm{Aut}(M,H)) < 14$, then $\dim(\mathrm{Aut}(M,H)) \leq 9$.
(3) If (M,H) is not locally isomorphic to the canonical distribution on G/P, then $\dim(\mathrm{Aut}(M,H)) \leq 8$.

Proof. For (2) it remains to show that any proper graded subalgebra of $\mathfrak{g}$ has dimension at most nine. Suppose that $\mathfrak{b} = \oplus_{i=-3}^{3}\mathfrak{b}_i$ is such a subalgebra and put $d_j := \dim(\mathfrak{b}_j)$ for $j = -3,...,3$. Let us first assume that $\mathfrak{g}_{-1} \subset \mathfrak{b}$, i.e. that $d_{-1} = 2$. Since $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$, this implies that $\mathfrak{g}_- \subset \mathfrak{b}$. Hence $\mathfrak{g}_1$ cannot be contained in $\mathfrak{b}$, so $d_1 < 2$.

Now the bracket induces an isomorphism $\mathfrak{g}_2\otimes\mathfrak{g}_{-1} \to \mathfrak{g}_1$, so we must have $d_2 = 0$. But then also $d_3 = 0$, since bracketing with a nonzero element of $\mathfrak{g}_3$ is a surjection $\mathfrak{g}_{-1} \to \mathfrak{g}_2$. If $d_1 = 0$, then we conclude $\dim(\mathfrak{b}) \leq 9$. If $d_1 = 1$, then the fact the the adjoint action of $\mathfrak{b}_0$ on $\mathfrak{g}_1$ must preserve the one dimensional subspace $\mathfrak{b}_1 \subset \mathfrak{g}_1$ implies that $d_0 \leq 3$, and hence we again get $\dim(\mathfrak{b}) \leq 9$, so the case $d_{-1} = 2$ is complete. By symmetry this also applies if $d_1 = 2$, so we are left with the case that $d_{-1}, d_1 \leq 1$.

If $d_{-1} = 0$, then either $d_2 = 0$ or $d_{-3} = 0$, and likewise $d_1 = 0$ implies that $d_{-2} = 0$ or $d_3 = 0$. In particular, $d_{-1} = d_1 = 0$ implies $\dim(\mathfrak{b}) \leq 8$. If $d_{-1} = 0$ and $d_1 = 1$, then $d_0 \leq 3$ implies that $\dim(\mathfrak{b}) \leq 9$. By symmetry, this also holds if $d_1 = 0$ and $d_{-1} = 1$. Finally, if $d_{-1} = d_1 = 1$, then $d_0 \leq 3$, either $d_{-3} = 1$ or $d_2 = 0$, and either $d_3 = 1$ or $d_{-2} = 0$, so again $\dim(\mathfrak{b}) \leq 9$.

(3) Let $\mathfrak{b} = \oplus_{i=-3}^{3} \mathfrak{b}_i$ be the graded subalgebra of $\mathfrak{g}$ associated to the Lie algebra of the automorphism group. By part (3) of Corollary 2.5, $\mathfrak{b}_0$ stabilizes a nonzero element in the irreducible $\mathfrak{g}_0$- module $H^2(\mathfrak{g}_-, \mathfrak{g}) \cong S^4(\mathfrak{g}_1)$. Following Remark 2.5 (2), we get $\dim(\mathfrak{b}_0) \leq 2$ and the arguments from the proof of (2) show that $\dim(\mathfrak{b}) \leq 8$. $\square$

The simplest example of a proper graded subalgebra of the maximal possible dimension 9 is given by $\mathfrak{g}_- \oplus \mathfrak{g}_0$. This can be realized as the Lie algebra of the automorphism group of the complement of a point in the homogeneous model G/P. Similarly, $\mathfrak{g}_- \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ for a line $\mathfrak{b}_1 \subset \mathfrak{g}_1$ and its stabilizer $\mathfrak{b}_0$ in $\mathfrak{g}_0$ can be easily realized. The homogeneous model G/P can be viewed as the space of null lines in the seven dimensions space of purely imaginary split octonions and $\mathfrak{b}$ can be realized as the Lie algebra of the automorphism group of the complement of a fixed null line. It is easy to give a complete description of all the graded subalgebras of $\mathfrak{g}$ of dimension 9, and all of them can be realized, see [13].

3.3. **Generic rank four distributions in dimension seven.** The last example we consider is rank four distributions with small growth vector $(4, 7)$ on manifolds of dimension seven. For such a manifold, the Levi bracket in a point x is a map $\Lambda^2 H_x \to T_x M/H_x$. Choosing an isomorphism $H_x \to \mathbb{R}^4$ and $T_x M/H_x \to \mathbb{R}^3$, we have to deal with the space $L(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$ of linear maps, which has dimension 18. Changing the identifications of H_x with $\mathbb{R}^4$ and of $T_x M/H_x$ with $\mathbb{R}^3$ is expressed by the natural action of the group $GL(4, \mathbb{R}) \times GL(3, \mathbb{R})$, which has dimension 25. Thus the Levi bracket in x corresponds to a well defined orbit of the natural action of $GL(4, \mathbb{R}) \times GL(3, \mathbb{R})$ on $L(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$.

In this picture, a distribution is stable (up to isomorphism) under small perturbations if and only if the corresponding orbit is open in $L(\Lambda^2\mathbb{R}^4, \mathbb{R}^3)$. It turns out (see [9, 7.12]) that there are exactly two open orbits, and hence two types of generic distributions. From the dimension count, we see that this is equivalent to the fact that the stabilizer in $GL(4,\mathbb{R})\times GL(3,\mathbb{R})$ of the corresponding map has dimension seven. We can also view an element of $L(\Lambda^2\mathbb{R}^4,\mathbb{R}^3)$ as defining the structure of a graded Lie algebra of $\mathbb{R}^4\oplus\mathbb{R}^3$, and then the stabilizer in $GL(4,\mathbb{R})\times GL(3,\mathbb{R})$ is exactly the automorphism group of this graded Lie algebra. Hence generic distributions are in bijective correspondence with graded Lie algebras structures on $\mathbb{R}^4\oplus\mathbb{R}^3$ whose automorphism group has dimension seven (which is the minimal possible dimension).

To relate this to parabolic geometries, consider the complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}} := \mathfrak{sp}(6,\mathbb{C})$ of endomorphisms of $\mathbb{C}^6$ which preserve a non–degenerate skew symmetric bilinear form. For this form, we use $(z,w)\mapsto\sum_{j=1}^6(-1)^j z_j w_{6-j}$. For a matrix $A\in\mathfrak{gl}(2,\mathbb{C})$ we denote by $\overline{A}\in\mathfrak{gl}(2,\mathbb{C})$ the classical adjoint, so $\overline{A}A=A\overline{A}=\det(A)\mathbb{I}$. Using this, we can realize $\mathfrak{g}^{\mathbb{C}}$ as

$$\left\{\begin{pmatrix} A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & -\overline{A}_{12}\\ A_{31} & -\overline{A}_{21} & -\overline{A}_{11}\end{pmatrix} : \begin{array}{l} A_{13},A_{22},A_{31}\in\mathfrak{sl}(2,\mathbb{C})\\ A_{11},A_{12},A_{21}\in\mathfrak{gl}(2,\mathbb{C})\end{array}\right\}$$

In this realization, we have an evident $|2|$–grading, with $\mathfrak{g}_0^{\mathbb{C}}$ corresponding to the entries A_{11} and A_{22}, $\mathfrak{g}_{-2}^{\mathbb{C}}$ to A_{31}, $\mathfrak{g}_{-1}^{\mathbb{C}}$ to A_{21}, $\mathfrak{g}_1^{\mathbb{C}}$ to A_{12}, and $\mathfrak{g}_2^{\mathbb{C}}$ to A_{13}. Hence the complex dimensions of the ±1–components are four, the ±2–components have complex dimension three, while the 0–component has complex dimension seven. The Lie bracket of $\mathfrak{g}_-^{\mathbb{C}}$ thus defines a map $\Lambda^2\mathbb{C}^4\to\mathbb{C}^3$. Now the cohomological condition from 2.4 is satisfied for any real form of the given grading on $\mathfrak{g}^{\mathbb{C}}$. From 2.4 we know that if $\mathfrak{g}=\oplus_{i=-2}^2\mathfrak{g}_i$ is such a real form and $G=\mathrm{Aut}(\mathfrak{g})$, then the subgroup $G_0\subset G$ with Lie algebra $\mathfrak{g}_0$ is isomorphic to the automorphism group of the graded Lie algebra $\mathfrak{g}_-$. Since $\dim(\mathfrak{g}_0)=7$, any such real form corresponds to a generic rank four distribution in dimension seven, and by Theorem 2.4 such distributions then are equivalent to regular normal parabolic geometries of type (G,P).

It turns out that there are two real forms of this grading. One is the split real form $\mathfrak{sp}(6,\mathbb{R})$. With respect to the obvious real analog of the bilinear form used above, we get the same matrix representation as for $\mathfrak{g}^{\mathbb{C}}$ above but with the blocks lying in $\mathfrak{gl}(2,\mathbb{R})$ respectively $\mathfrak{sl}(2,\mathbb{R})$. The other real form is obtained by passing to quaternionic matrices. If we identity $\mathbb{C}^6$ with $\mathbb{H}^3$, then our skew symmetric form comes from a quaternionic Hermitian form of signature $(2,1)$. Maps preserving both the quaternionic structure and the skew form also preserve the quaternionic Hermitian form, and hence form a Lie algebra isomorphism to

$\mathfrak{sp}(2,1)$. The matrix representation in this case is exactly as above but with the quaternions $\mathbb{H}$ replacing $\mathfrak{gl}(2,\mathbb{R})$, the purely imaginary quaternions $\mathrm{im}(\mathbb{H})$ replacing $\mathfrak{sl}(2,\mathbb{R})$, and the quaternionic conjugation instead of the classical adjoint.

The deeper reason for this dichotomy is that up to isomorphism there are exactly two real composition algebras of dimension four, namely the quaternions with their standard positive definite quadratic form, and the split–quaternions, which are isomorphic to the algebra of 2×2–matrices, with the quadratic form given by the determinant. This quadratic form is preserved by the action of G_0 up to scale. Passing to distributions this means that any generic rank four distribution in dimension seven admits a canonical conformal class of inner products (i.e. a canonical conformal sub–Riemannian structure) which is either definite or of split signature. We refer to these cases as elliptic type respectively hyperbolic type.

3.4. **Results for elliptic type.** Here we have $\mathfrak{g}=\mathfrak{sp}(2,1)$, $\mathfrak{g}_{\pm1}\cong\mathbb{H}$, and $\mathfrak{g}_{\pm2}\cong\mathrm{im}(\mathbb{H})$. The bracket $\Lambda^2\mathfrak{g}_{\pm1}\to\mathfrak{g}_{\pm2}$ is given by $(p,q)\mapsto p\bar{q}-q\bar{p}$ for $p,q\in\mathbb{H}$. The brackets $\mathfrak{g}_{\pm2}\times\mathfrak{g}_{\mp1}\to\mathfrak{g}_{\pm_1}$ is given by $(p,q)\mapsto p\bar{q}$. Finally, the adjoint action on $\mathfrak{g}_{-1}\cong\mathbb{H}$ identifies G_0 with the conformal group $CSO(4)$.

Theorem 3.4. *Let M be a smooth manifold of dimension seven, let $H\subset TM$ be a generic rank four distribution of elliptic type, and let $\mathrm{Aut}(M,H)$ be the automorphism group of H.*
(1) $\mathrm{Aut}(M,H)$ is a Lie group of dimension at most 21.
(2) If $\dim(\mathrm{Aut}(M,H))<21$, then $\dim(\mathrm{Aut}(M,H))\leq 14$.
(3) If (M,H) is not locally isomorphic to the canonical distribution on G/P, then $\dim(\mathrm{Aut}(M,H))\leq 12$.

Proof. As before, we only have to prove (2) and (3), and for (2) we have to show that any proper graded subalgebra $\mathfrak{b}\subset\mathfrak{g}$ has dimension at most 14. We put $\mathfrak{b}=\oplus_{j=-2}^{2}\mathfrak{b}_j$ and $d_j:=\dim(\mathfrak{b}_j)$.

Let us first assume that $\mathfrak{b}_{-1}=\mathfrak{g}_{-1}$, i.e. that $d_{-1}=4$. Then $\mathfrak{b}_{-2}=\mathfrak{g}_{-2}$, and to obtain a proper subalgebra, we have to have $d_1<4$. Since the bracket with a non–zero element of $\mathfrak{g}_2$ defines a surjection $\mathfrak{g}_{-1}\to\mathfrak{g}_1$, we see that we must have $d_2=0$, which in turn implies that $d_1\leq 1$. If $d_1=0$, then we have $\dim(\mathfrak{b})\leq 14$. If $d_1=1$, then $\mathfrak{b}_0$ has to stabilize a line in $\mathfrak{g}_1$, which forces $d_0\leq 4$, so we get $\dim(\mathfrak{b})\leq 12$. If $d_1=4$, then the result follows in the same way by symmetry.

Let us next assume that $d_{-1}=3$. Conjugating by an element of G_0, we may assume that $\mathfrak{b}_{-1}$ is spanned by $1,i,j\in\mathbb{H}$. This shows that $\mathfrak{b}_{-2}=\mathfrak{g}_{-2}$, and then the fact that $[\mathfrak{b}_1,\mathfrak{g}_{-2}]\subset\mathfrak{b}_{-1}$ forces $d_1\leq 1$. This is only possible if $d_2 = 0$, and $\dim(\mathfrak{b}) \leq 14$ readily follows. The same argument applies if $d_1 = 3$.

If $d_{-1} = 2$, then we may assume that $\mathfrak{b}_{-1}$ is spanned by 1 and i, and then conclude that $d_{-2} \geq 1$. If $d_{-2} = 1$, then we must have $d_1 \leq 2$, and this is only possible if $d_2 \leq 1$, so we have lost eight dimensions already. If $d_{-1} > 1$, then $d_1 \leq 1$ and $d_2 = 0$, and again we have lost eight dimensions. The case $d_1 = 2$ can be treated in the same way.

If $d_{-1} = d_1 = 1$, then both d_2 and d_{-2} must be ≤ 1, so we have lost 10 dimensions. If $d_{-1} = 1$ and $d_1 = 0$, then we also must have $d_2 = 0$, and again we are done. Of course, $d_{-1} = d_1 = 0$ already implies a loss of eight dimensions.

(3) In the proof of (2) we have used restrictions on d_0 only in one point, and there we obtained $\dim(\mathfrak{b}) \leq 12$. Hence it suffices to prove that in the setting of (3) we have to have $d_0 \leq 5$, since this causes a loss of two more dimensions compared to (2). But this follows immediately from Remark 2.5 (2) since the two irreducible components in $H^2(\mathfrak{g}_-, \mathfrak{g})$ are non–trivial, and the maximal parabolic subalgebras in the complexification of $\mathfrak{g}_0$ have dimension 5. $\square$

The simplest example of a proper graded subalgebra of $\mathfrak{g}$ of the maximal possible dimension 14 is of course the parabolic subalgebra $\mathfrak{p}$. Since the group P is the stabilizer of the base point o in the homogeneous model G/P, it is the subgroup of all elements that preserve the open subset $G/P \setminus \{o\}$. From Corollary 2.4 we conclude that P is the automorphism group of the canonical distribution on this open subset.

3.5. **Results for hyperbolic type.** The description of the brackets is as in the elliptic case but replacing the quaternions $\mathbb{H}$ by the algebra $M_2(\mathbb{R})$ of real 2×2–matrices, the imaginary part by the subspace $\mathfrak{sl}(2, \mathbb{R})$ of tracefree matrices, and the conjugation of quaternions by $\overline{A} = \mathcal{C}A$, the classical adjoint of A. The adjoint action of $\mathfrak{g}_{\pm 1} \cong \mathbb{R}^4$ identifies G_0 with the conformal group $CSO(2,2)$ of split signature $(2,2)$, and the adjoint action on $\mathfrak{g}_{\pm 2} \cong \mathbb{R}^3$ maps G_0 onto $CSO(1,2)$.

Theorem 3.5. *Let M be a smooth manifold of dimension seven, let $H \subset TM$ be a generic rank four distribution of hyperbolic type, and let* $\mathrm{Aut}(M,H)$ *be the automorphism group of H.*
(1) $\mathrm{Aut}(M,H)$ *is a Lie group of dimension at most* 21*.*
(2) If $\dim(\mathrm{Aut}(M,H)) < 21$*, then* $\dim(\mathrm{Aut}(M,H)) \leq 16$*.*
(3) If (M,H) is not locally isomorphic to the canonical distribution on G/P, then $\dim(\mathrm{Aut}(M,H)) \leq 15$*.*

Proof. To prove (2) we have to show that any proper graded subalgebra $\mathfrak{b} \subset \mathfrak{g}$ has dimension at most 15. We put $\mathfrak{b} = \oplus_{j=-2}^{2} \mathfrak{b}_j$ and $d_j := \dim(\mathfrak{b}_j)$.

If $d_{-1} = 4$, then $\mathfrak{g}_- \subset \mathfrak{b}$ and we must have $d_1 < 4$. We have seen that the bracket $\mathfrak{g}_2 \times \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is given by $(A, B) \mapsto A\mathcal{C}(B)$. Hence in contrast to the elliptic case, the map $\mathfrak{g}_{-1} \to \mathfrak{g}_1$ defined by the bracket with a fixed element of $\mathfrak{g}_2$ is only surjective for invertible elements. Since any subspace of $\mathfrak{sl}(2, \mathbb{R})$ of dimension at least two contains an invertible matrix, we obtain $d_2 \leq 1$. Moreover, we know that the bracket $\mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_2$ is given by $(A, B) \mapsto A\mathcal{C}(B) - B\mathcal{C}(A)$, the imaginary part of $A\mathcal{C}(B)$. Therefore we obtain $d_1 \leq 2$, since otherwise we would have $\dim([\mathfrak{b}_1, \mathfrak{b}_1])) > 1$. Now since $\mathfrak{b}_0$ has to stabilize the subspace $\mathfrak{b}_1 \subset \mathfrak{g}_1$, we conclude that $d_0 \leq 6$ and hence $\dim(\mathfrak{b}) \leq 16$. The case $d_1 = 4$ can be treated in the same way, so it remains to consider the case that $d_{\pm 1} \leq 3$.

If $d_{-1} = 3$, then $\mathfrak{b}_{-1} \subset \mathfrak{g}_{-1}$ is three dimensional, and we have to distinguish cases according to the signature (up to sign) of the restriction of the inner product to this subspace. The possible signatures are $(2, 1)$, $(2, 0)$, and $(1, 1)$. Since $G_0 \cong CSO(2, 2)$, any two subspaces with the same signature are conjugate. Using this, one verifies that in the first two cases, $[\mathfrak{b}_{-1}, \mathfrak{b}_{-1}] = \mathfrak{g}_{-2}$, so $d_{-2} = 3$. Then the fact that $[\mathfrak{g}_{-2}, \mathfrak{b}_1] \subset \mathfrak{b}_{-1}$ implies that $d_1 \leq 2$, and then $[\mathfrak{b}_2, \mathfrak{b}_{-1}] \subset \mathfrak{b}_1$ shows that $d_2 \leq 1$. This already shows that $\dim(\mathfrak{b}) \leq 16$.

For the remaining signature $(1, 1)$, we may assume that $\mathfrak{b}_{-1}$ consists of all matrices of the form $\{\left(\begin{smallmatrix} a & b \\ -a & c \end{smallmatrix}\right)\}$. Computing the bracket of two matrices of this form, we see that $\mathfrak{b}_{-2}$ has to contain all matrices of the form $\left\{\left(\begin{smallmatrix} \alpha & \beta \\ 0 & -\alpha \end{smallmatrix}\right)\right\}$ and $d_{-2} \geq 2$. If $d_{-2} = 3$, then the results follows as above. Otherwise, the arguments on brackets as above show that $d_1 \leq 3$ and $d_2 \leq 2$ and since $\mathfrak{b}_0$ has to preserve non–trivial subspaces, we get $d_0 \leq 6$ and hence $\dim(\mathfrak{b}) \leq 16$. This complete the case $d_{-1} = 3$ and hence also the case $d_1 = 3$, so we are left with the case $d_{-1}, d_1 \leq 2$. Suppose $0 < d_{-1} \leq 2$. Since $\mathfrak{b}_0$ has to stabilize $\mathfrak{b}_{-1}$, we obtain at least $d_0 \leq 6$. Hence in any case $\dim(\mathfrak{b}) \leq 16$. If $d_{-1} = 0$ we are already done, since $d_1 \leq 2$ by assumption.

(3) As in the proof of part (3) of Theorem 3.4, we obtain $d_0 \leq 5$, and going through the proof of part (2), we see that in each case we loose at least one more dimension. □

Remark 3.5. The algebras $\mathfrak{g}$ from 3.4 and here as well as their gradings have the same complexification, so we are just dealing with two different real forms of one complex object. Remarkably, we obtain different bounds on the sizes of possible automorphism groups in the elliptic and hyperbolic case. The maximal possible dimension 16 of an automorphism group in the hyperbolic can actually be realized, so the two case are truly different. A realization is obtained as follows. In the notation from 3.3, let $\mathfrak{b} \subset \mathfrak{sp}(6, \mathbb{R})$ be the subspace formed by all matrices for which $A_{11} = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$, $A_{12} = \{\left(\begin{smallmatrix} * & * \\ 0 & 0 \end{smallmatrix}\right)\}$, and $A_{13} = \{\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)\}$. One immediately verifies that this is a subalgebra of dimension 16. Let B be the corresponding subgroup of $Sp(6, \mathbb{R})$, and consider the B–orbit of $o = eP$ in $Sp(6, \mathbb{R})/P$. Since $\mathfrak{b}$ contains $\mathfrak{g}_-$, this orbit is open. The homogeneous space $Sp(6, \mathbb{R})/P$ admits a second interpretation, namely the space of all isotropic two planes in the symplectic vector space $\mathbb{R}^6$. In this picture, B is the stabilizer of a hyperplane in $\mathbb{R}^6$, which shows that B cannot act transitively on $Sp(6, \mathbb{R})/P$. Hence the B–orbit of eP in this space is a proper open subset, so its automorphism group must be of dimension strictly less then the dimension of $Sp(6, \mathbb{R})$. Since on the other hand, B is contained in this automorphism group, Theorem 3.5 implies that we must have an automorphism group of dimension 16.

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