

Conformally Invariant Operators via Curved Casimirs: Examples

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CONFORMALLY INVARIANT OPERATORS VIA CURVED CASIMIRS: EXAMPLES

ANDREAS ČAP, A. ROD GOVER, AND V. SOUČEK

Dedicated to Professor J.J. Kohn on the occasion of his 75th birthday

ABSTRACT. We discuss a scheme for a construction of linear conformally invariant differential operators from curved Casimir operators; we then explicitly carry this out for several examples. Apart from demonstrating the efficacy of the approach via curved Casimirs, this shows that this method is general in that it applies both in regular and in singular infinitesimal character, and also that it can be used to construct standard as well as non-standard operators. (Nevertheless the scheme discussed here does not recover all operators.) The examples treated include conformally invariant operators with leading term, in one case, a square of the Laplacian, and in another case, a cube of the Laplacian.

1. INTRODUCTION

Curved Casimir operators were originally introduced in [7] in the setting of general parabolic geometries. For any natural vector bundle associated to such a geometry, there is a curved Casimir operator which acts on the space of smooth sections of the bundle. The name of the operator is due to the fact that on the homogeneous model of the geometry, it reduces to the canonical action of the quadratic Casimir element. The curved Casimir operators may be expressed by a simple (Laplacian like) formula in terms of the fundamental derivative from [3] and hence share the very strong naturality properties of the fundamental derivative. While on a general natural vector bundle the curved Casimir operator is of order at most one, it always acts by a scalar on a bundle associated to an irreducible representation. This scalar can be easily computed from representation theory data. It was already shown in [7] that using this and the naturality properties, one can use the curved Casimir operators systematically to construct higher order invariant differential operators. Namely, [7] contains a general construction of splitting operators, which are basic ingredients in all versions of the curved translation principle.

Essentially the same construction can be also used to directly obtain invariant differential operators acting between sections of bundles associated to irreducible representations. One considers the tensor product of a tractor bundle and an irreducible bundle. Such a bundle has an invariant filtration such that the quotients of subsequent filtrations components are completely reducible. Adapting the action of the centre of the structure group (which amounts to tensoring with a density bundle), one may force a coincidence of

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curved Casimir eigenvalues for irreducible components in different subquotients. As we shall see this leads to an invariant linear differential operator acting between the sections of these components. A more difficult issue is to prove, in some general context, that the resulting operator is nontrivial. General tools for doing this systematically are developed in [5].

The purpose of this article is to carry out the construction of invariant operators explicitly for a few examples in the realm of conformal structures. First, this shows that the general ideas can be made explicit rather easily. Secondly, it shows that the curved Casimir operators can be used to produce both standard and non-standard operators, and they work both in regular and in singular infinitesimal character; this is in contrast to the usual constructions of BGG sequences as developed in [6, 1].

Finally, we want to indicate how some of the well known and intriguing phenomena concerning conformally invariant powers of the Laplacian show up in the approach via curved Casimirs. In particular, this concerns the fact that the critical powers of the Laplacian are not strongly invariant and the non-existence of supercritical powers of the Laplacian.

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2. EXAMPLES OF CONFORMALLY INVARIANT OPERATORS CONSTRUCTED FROM CURVED CASIMIRS

2.1. Conformal structures, tractor bundles, and tractor connections. We shall use the conventions on conformal structures from [4]. We consider a smooth manifold M of dimension $n \geq 3$ endowed with a conformal equivalence class $[g]$ of pseudo-Riemannian metrics of some fixed signature (p, q) . We use Penrose abstract index notation, so \mathcal{E}^a will denote the tangent bundle TM and \mathcal{E}_a the cotangent bundle T^*M . Several upper or lower indices will indicate tensor products of these basic bundles, round brackets will denote symmetrisation, square brackets alternation, and the subscript 0 indicates a tracefree part.

For $w \in \mathbb{R}$ we denote by $\mathcal{E}[w]$ the bundle of $(-\frac{w}{n})$ -densities on M . For any choice of metric g in the conformal class, sections of $\mathcal{E}[w]$ can be identified with smooth functions but changing from g to $\hat{g} = f^2g$ (where f is a positive smooth function on M), this function changes by multiplication by f^w . Adding $[w]$ to the notation for a bundle indicates a tensor product by

$\mathcal{E}[w]$. Using these conventions, the conformal structure can be considered as a smooth section \mathbf{g}_{ab} of the bundle $\mathcal{E}_{(ab)}[2]$, called the *conformal metric*. Contraction with \mathbf{g}_{ab} defines an isomorphism $\mathcal{E}^a \cong \mathcal{E}_a[2]$, whose inverse can be viewed as a smooth section \mathbf{g}^{ab} of $\mathcal{E}^{(ab)}[-2]$. We shall use \mathbf{g}_{ab} and \mathbf{g}^{ab} to raise and lower tensor indices.

The *standard tractor bundle* of $(M, [g])$ will be denoted by \mathcal{E}^A . This is a vector bundle of rank $n+2$ canonically associated to the conformal structure. It is endowed with a canonical bundle metric h_{AB} of signature $(p+1, q+1)$ which will be used to raise and lower tractor indices. Further, there is a canonical linear connection ∇^T on \mathcal{E}^A which is equivalent to the conformal Cartan connection. Finally, there is a canonical inclusion $\mathcal{E}[-1] \hookrightarrow \mathcal{E}^A$ whose image is an isotropic line subbundle of \mathcal{E}^A . This can be viewed as a canonical section X^A of $\mathcal{E}^A[1]$ which satisfies $h_{AB}X^AX^B = 0$. Next, $X_A := h_{AB}X^B$ can be interpreted as a projection $\mathcal{E}^A \rightarrow \mathcal{E}[1]$. These data fit together to define a composition series for \mathcal{E}^A that we shall denote $\mathcal{E}[1] \leftarrow \mathcal{E}_a[1] \leftarrow \mathcal{E}[-1]$; the second \leftarrow indicates that $\mathcal{E}[-1]$ is a subbundle of \mathcal{E}^A while the first \leftarrow means $\mathcal{E}_a[1]$ is (isomorphic to) a subbundle of the quotient bundle $\mathcal{E}^A/\mathcal{E}[-1]$ and that $(\mathcal{E}^A/\mathcal{E}[-1])/\mathcal{E}_a[1] \cong \mathcal{E}[1]$. (The motivation for the notation is that summands include, while there is a projection onto direct summands). General tractor bundles then correspond to $SO(p+1, q+1)$ -invariant subspaces in tensor powers of $\mathbb{R}^{(p+1, q+1)}$, and we will also use abstract index notation for tractor indices.

Any choice of a metric g in the conformal class gives rise to a splitting $\mathcal{E}^A \cong \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ of the composition series. The change of this splitting caused by a conformal rescaling of the metric can be easily described explicitly, see [2], but we will not need these formulae here. What we will need is the expression of the tractor connection in the splitting associated to g in terms of the Levi-Civita connection ∇ of g . To formulate this efficiently, we need the *adjoint tractor bundle* of $(M, [g])$. By definition, this is the bundle $\mathfrak{so}(\mathcal{E}^A) \cong \mathcal{E}_{[AB]}$ of endomorphisms of \mathcal{E}^A which are skew symmetric with respect to the tractor metric. By definition, this bundle naturally acts on \mathcal{E}^A and hence (tensorially) on any tractor bundle.

Now the composition series of \mathcal{E}^A gives rise to a composition series $\mathcal{E}_{[AB]} = \mathcal{E}^a \leftarrow (\mathcal{E}_{[ab]}[2] \oplus \mathcal{E}[0]) \leftarrow \mathcal{E}_a$, so the adjoint tractor bundle contains T^*M as a natural subbundle and has TM as a natural quotient. A choice of metric in the conformal class also splits this composition series, so we obtain an isomorphism $\mathcal{E}_{[AB]} \cong \mathcal{E}^a \oplus (\mathcal{E}_{[ab]}[2] \oplus \mathcal{E}[0]) \oplus \mathcal{E}_a$ depending on the choice of metric. In particular, we can view elements of T^*M naturally as elements of the adjoint tractor bundle and, choosing a metric in the conformal class, we can also view elements of TM as elements in the adjoint tractor bundle.

There are explicit formulae how the identifications of tractor bundles behave under a conformal change of metric, see e.g. Theorem 1.3 of [2]. However, we will not need this formulae here, since we will always deal with operations which are known to be invariant in advance and use the splittings only to compute explicit formulae for these operations. We shall only need the formula for the canonical tractor connection in a splitting, which also can be found in Theorem 1.3 of [2]. This formula is given in the proposition below. Note that, comparing with [2], the difference in the sign of the

term involving the Rho tensor (also sometimes called the Schouten tensor) is due to the fact that [2] uses a different sign convention for the Rho-tensor than [4].

Proposition. *Consider a tractor bundle $\mathcal{T} \rightarrow M$ for a conformal structure $[g]$ on M , and let $\nabla^{\mathcal{T}}$ be the canonical tractor connection on \mathcal{T} . Choose a metric g in the conformal class with Rho tensor \mathbf{P} and let ∇ be its Levi Civita connection, acting on \mathcal{T} via the isomorphism with a direct sum of weighted tensor bundles induced by the choice of metric. Further let us denote by \bullet both the actions of T^*M and of TM (the latter depending on the choice of metric) coming from the inclusion of the bundles into the adjoint tractor bundle. Then for any vector field $\xi \in \mathfrak{X}(M)$ and any section $s \in \Gamma(\mathcal{T})$ we have*

$$\nabla_{\xi}^{\mathcal{T}} s = \nabla_{\xi} s + \xi \bullet s - \mathbf{P}(\xi) \bullet s.$$

2.2. A formula for the curved Casimir operator. The main tool used to efficiently treat examples is a new formula for the curved Casimir operator acting on the tensor product of a tractor bundle and an irreducible bundle. Consider the group $G := SO(p+1, q+1)$ and let $P \subset G$ be the stabiliser of an oriented isotropic line in the standard representation $\mathbb{R}^{(p+1, q+1)}$ of G . Then it is well known that P is the semidirect product of the (orientation preserving) conformal group $CSO(p, q)$ and a normal vector subgroup $P_+ \cong \mathbb{R}^{n^*}$. It is also well known that a conformal structure of signature (p, q) on a smooth manifold M determines a canonical Cartan geometry of type (G, P) , so in particular there is a canonical principal bundle on M with structure group P . Forming associated bundles, any representation of the group P gives rise to a natural vector bundle on conformal manifolds.

The conformal group $CSO(p, q)$ is naturally a quotient of P , so any representation of $CSO(p, q)$ gives rise to a representation of P . The resulting representations turn out to be exactly those representations of P which are completely reducible, so they split into direct sums of irreducibles. The corresponding bundles are called *completely reducible bundles* and they split into direct sums of *irreducible bundles*. The completely reducible bundles are exactly the usual tensor and density bundles. On the other hand, one can look at restrictions to P of representations of G , and these give rise to tractor bundles. The standard tractor bundle \mathcal{E}^A and the adjoint tractor bundle $\mathcal{E}_{[AB]}$ from 2.1 above correspond to the standard representation $\mathbb{R}^{(p+1, q+1)}$ respectively the adjoint representation $\mathfrak{so}(p+1, q+1)$ of G in this way.

Now recall first from Theorem 3.4 of [7] that the curved Casimir operator on an irreducible bundle $W \rightarrow M$ acts by a real multiple of the identity, and we denote the corresponding scalar by β_W . This scalar can be computed in terms of weights of the representation which induces W . If the lowest weight of this representation is $-\nu$, then $\beta_W = \langle \nu, \nu + 2\rho \rangle$, where ρ is half the sum of all positive roots. On a completely reducible bundle, the action of the curved Casimir is tensorial and can be obtained by decomposing the bundle into irreducible pieces, multiplying each piece by the corresponding factor and then adding back up.

Proposition. *Let $(M, [g])$ be a conformal manifold of signature (p, q) and let $\mathcal{T} \rightarrow M$ be a bundle which can be written as the tensor product of a tractor bundle and an irreducible bundle. Choose a metric g in the conformal class and let ∇ be its Levi-Civita connection, acting on \mathcal{T} via the identification with a completely reducible bundle induced by the choice of g . Further, let $\beta : \mathcal{T} \rightarrow \mathcal{T}$ be the bundle map which, in this identification, acts on each irreducible component $W \subset \mathcal{T}$ by multiplication by β_W . Let \bullet denote the action of T^*M on \mathcal{T} coming from the natural action on the tractor bundle. Then for a local orthonormal frame ξ_ℓ for TM with dual frame φ^ℓ for T^*M , the curved Casimir operator \mathcal{C} acts on $s \in \Gamma(\mathcal{T})$ by*

$$\mathcal{C}(s) = \beta(s) - 2 \sum_\ell \varphi^\ell \bullet (\nabla_{\xi_\ell} s - P(\xi_\ell) \bullet s)$$

Proof. We use the formula for \mathcal{C} in terms of an adapted local frame for the adjoint tractor bundle from Proposition 3.3 of [7]. Having chosen the metric g , the adjoint tractor bundle splits as $TM \oplus \mathfrak{so}(TM) \oplus T^*M$, and for any local frame $\{A_r\}$ for $\mathfrak{so}(TM)$, the local frame $\{\xi_\ell, A_r, \varphi^\ell\}$ for the adjoint tractor bundle is evidently adapted. According to Proposition 3.3 of [7], one may write $\mathcal{C}(s)$ as the sum of $-2 \sum_\ell \varphi^\ell \bullet D_{\xi_\ell} s$ (with D denoting the fundamental derivative) and a tensorial term, in which only actions of elements of $\mathfrak{so}(TM)$ show up. Hence the latter term preserves any irreducible summand of \mathcal{T} , and the proof of Theorem 3.4 of [7] shows that, on such a summand W , $\mathcal{C}(s)$ acts by multiplication by β_W . To complete the proof, it thus suffices to show that

$$D_{\xi_\ell} s = \nabla_{\xi_\ell} s - P(\xi_\ell) \bullet s.$$

If \mathcal{T} is a tractor bundle, then this follows immediately from the formula for the fundamental derivative in section 1.7 of [2]. The formula there (applied to standard tractors) shows that D_{ξ_ℓ} equals ∇_{ξ_ℓ} on the tangent bundle and on a non-trivial density bundle. By naturality, this is true for arbitrary irreducible bundles, and the result follows. \square

This formula shows that to compute explicitly the curved Casimir on the tensor product of a tractor bundle with an irreducible bundle, only two ingredients are needed: first we need to systematically compute the numbers β_W , and second we need an explicit formula for the action of T^*M on the tractor bundle, since this can be first used to compute $P(\xi) \bullet s$ and then the action of φ^ℓ .

2.3. The construction principle. The construction principle we use is actually very close to the construction of splitting operators in section 3.5 of [7]. Let \mathcal{T} be the tensor product of a tractor bundle and a tensor bundle. The natural filtration of the tractor bundle (inherited from the filtration of the standard tractor bundle from 2.1) induces a natural filtration of \mathcal{T} , which we write as $\mathcal{T} = \mathcal{T}^0 \supset \mathcal{T}^1 \supset \dots \supset \mathcal{T}^N$. Each of the subquotients $\mathcal{T}^i / \mathcal{T}^{i+1}$ splits into a direct sum of irreducible tensor bundles. On sections of each of these bundles, the curved Casimir operator acts by a scalar by Theorem 3.4 of [7], and this scalar is computable from the highest (or lowest) weight of the inducing representation. We denote by $\beta_i^1, \dots, \beta_i^{n_i}$ the different scalars that occur in this way.

Now define $L_i := \prod_{\ell=1}^{n_i} (\mathcal{C} - \beta_i^\ell)$. This can be viewed as a differential operator of order $\leq n_i$ acting on sections of \mathcal{T} . Moreover, naturality of the curved Casimir operator implies that L_i preserves each of the subspaces formed by sections of one filtration component. Moreover, for each j , the operator induced on sections of $\mathcal{T}^j/\mathcal{T}^{j+1}$ is given by the same formula, but with \mathcal{C} being the curved Casimir operator for that quotient bundle. In particular, this implies that L_i induces the zero operator on $\Gamma(\mathcal{T}^i/\mathcal{T}^{i+1})$ and hence $L_i(\Gamma(\mathcal{T}^i)) \subset \Gamma(\mathcal{T}^{i+1})$.

Now fix indices $i < j$ and an irreducible component $W \subset \mathcal{T}^i/\mathcal{T}^{i+1}$. Consider the composition $\pi_j \circ L_j \circ \dots \circ L_{i+1}$, where π_j is the tensorial operator induced by the projection $\mathcal{T}^i \rightarrow \mathcal{T}^i/\mathcal{T}^{j+1}$. Evidently, this composition defines a differential operator mapping sections of \mathcal{T}^i to sections of $\mathcal{T}^i/\mathcal{T}^{j+1}$. However, by construction, sections of \mathcal{T}^{i+1} are mapped to sections of \mathcal{T}^{i+2} by L_{i+1} , which are mapped to sections of \mathcal{T}^{i+3} by L_{i+2} , and so on. Hence our operator factors to sections of $\mathcal{T}^i/\mathcal{T}^{i+1}$ and restricting to sections of W , we obtain an operator $L : \Gamma(W) \rightarrow \Gamma(\mathcal{T}^i/\mathcal{T}^{j+1})$.

In section 3.5 of [7], it is then assumed that the Casimir eigenvalue β corresponding to the irreducible bundle W is different from all the β_ℓ^k for $i < k \leq j$ and all ℓ . In that case, composing the projection $\mathcal{T}^i/\mathcal{T}^j \rightarrow \mathcal{T}^i/\mathcal{T}^{i+1}$ with L , one obtains a non-zero multiple of the identity, and hence L is a splitting operator.

But now let us assume that (with appropriate numeration) $\beta = \beta_j^1$, and let $\tilde{W} \subset \mathcal{T}^j/\mathcal{T}^{j+1}$ be the sum of the irreducible components corresponding to this eigenvalue. Then we can write L_j as $(\mathcal{C} - \beta) \circ \tilde{L}_j$ where operator \tilde{L}_j is a polynomial in \mathcal{C} . Next, since all polynomials in \mathcal{C} commute, we can also write the composition $\pi_j \circ L_j \circ \dots \circ L_{i+1}$ as $\pi_j \circ \tilde{L}_j \circ \dots \circ L_{i+1} \circ (\mathcal{C} - \beta)$. But the latter composition evidently maps a section of \mathcal{T}^i , whose image in $\mathcal{T}^i/\mathcal{T}^{i+1}$ has values in W to a section of \mathcal{T}^j . Hence in this case, L has values in sections of $\mathcal{T}^j/\mathcal{T}^{j+1}$. Moreover, since

$$(\mathcal{C} - \beta) \circ \pi_j \circ L_j \circ \dots \circ L_{i+1} = \pi_j \circ L_j \circ \dots \circ L_{i+1} \circ (\mathcal{C} - \beta)$$

evidently induces the zero operator on $\Gamma(W)$, we conclude that L actually has values in $\Gamma(\tilde{W})$, so we have obtained an operator $L : \Gamma(W) \rightarrow \Gamma(\tilde{W})$.

2.4. Computing the Casimir eigenvalues. We need a systematic notation for weights and their relation to irreducible bundles. Since these issues are slightly different in even and odd dimensions, we will restrict our attention to the case of even dimension $n = 2m$ from now on; in many senses conformally invariant powers of the Laplacian are more interesting in even dimensions. Note that the weights involved are actually defined on the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2m+2, \mathbb{C})$ of $\mathfrak{g} = \mathfrak{so}(p+1, q+1)$. The process of assigning weights to real representations of \mathfrak{g} and $\mathfrak{g}_0 = \mathfrak{co}(p, q)$ is discussed in section 3.4 of [7].

We use the notation from chapter 19 of [9] for weights for $\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(2m+2, \mathbb{C})$. Hence weights will be denoted by tuples $(a_1, a_2, \dots, a_{m+1})$, and the (highest weights of) irreducible tensor representations (we will not require any spin representations) correspond to tuples in which all the a_i are integers and $a_1 \geq a_2 \geq \dots \geq a_{m-1} \geq \pm a_m$. For example, for $i < m$, the i th exterior power $\Lambda^i \mathbb{C}^{2m+2}$ is irreducible and corresponds to the tuple $a_1 = \dots = a_i = 1$

and $a_{i+1} = \dots = a_{m+1} = 0$. In this notation, the half sum of all positive roots is given by $\rho = (m, m-1, \dots, 1, 0)$.

Weights for the complexification of \mathfrak{g}_0 can be viewed as functionals on the same space, the conditions on dominance and integrality are different, however. Since this difference concerns the first entry only, we use the notation $(a_1|a_2, \dots, a_{m+1})$ for these weights.

The formula for the Casimir eigenvalues is in terms of lowest weights. For weights of tensor representations of $\mathfrak{g}_{\mathbb{C}}$ this coincides with the highest weight since any such representation is isomorphic to its dual. It will be helpful to keep in mind that the lowest weight of a representation of $\mathfrak{g}_{\mathbb{C}}$ coincides with the lowest weight of the irreducible quotient representation of $(\mathfrak{g}_0)_{\mathbb{C}}$. This is sufficient to understand the correspondence between weights and irreducible bundles. For example, the standard representation of $\mathfrak{g}_{\mathbb{C}}$ corresponds to the weight $(1, 0, \dots, 0)$ and the standard tractor bundle \mathcal{E}^A , whose irreducible quotient is $\mathcal{E}[1]$. Hence $\mathcal{E}[1]$ corresponds to the weight $(1|0, \dots, 0)$ and therefore $\mathcal{E}[w]$ corresponds to $(w|0, \dots, 0)$ for $w \in \mathbb{R}$.

More generally, for $i < m$, the i th exterior power of the standard representation corresponds to $(1, \dots, 1, 0, \dots, 0)$ (with i entries equal to 1) and is also a notation for $\Lambda^i \mathcal{E}^A$, which clearly has $\Lambda^{i-1} \mathcal{E}_a \otimes \mathcal{E}[i]$ as an irreducible quotient. Hence \mathcal{E}_a and \mathcal{E}^a correspond to $(-1|1, 0, \dots, 0)$ and $(1|1, 0, \dots, 0)$, respectively, and $\mathcal{E}_{[ab]}[w]$ corresponds to $(w-2|1, 1, 0, \dots, 0)$. The highest weight of $S_0^k \mathcal{E}_a$ is just k times the highest weight of \mathcal{E}_a , so $S_0^k \mathcal{E}_a[w]$ corresponds to $(w-k|k, 0, \dots, 0)$, and so on.

The final ingredient needed to apply the formula for Casimir eigenvalues is the inner product on weights. Taking as our invariant bilinear form half the trace form on the Lie algebra (which leads to the nicest conventions), one simply obtains the standard inner product. For example, for $W = S_0^k \mathcal{E}_a[w]$ the corresponding weight $\lambda = (w-k|k, 0, \dots, 0)$ and

$$\beta_W = \langle \lambda, \lambda + 2\rho \rangle = (w-k)(w+2m-k) + k(2m+k-2).$$

2.5. Standard tractors twisted by one-forms. We now have all the technical input at hand, so we look at the first example. Consider the tensor product $\mathcal{E}_a[w] \otimes \mathcal{E}^A$ of the standard tractor bundle with the bundle of weighted one-forms. We will describe the curved Casimir operator on this bundle and find basic splitting operators and all the invariant differential operators between irreducible bundles that can be constructed from this curved Casimir. From the composition series for \mathcal{E}^A from 2.1 we get a composition series $\mathcal{E}_a[w+1] \oplus \mathcal{E}_{ab}[w+1] \oplus \mathcal{E}_a[w-1]$ for our bundle. We use the convention that in the middle slot the first indices come from $\mathcal{E}_a[w]$ and the second ones from the tractor bundle. The middle term decomposes as $\mathcal{E}_{(ab)0}[w+1] \oplus \mathcal{E}[w-1] \oplus \mathcal{E}_{[ab]}[w+1]$, and if $n \geq 6$ then each of the summands is irreducible. For $n = 4$, the bundle $\mathcal{E}_{[ab]}[w+1]$ splits into the sum of self-dual and anti-self-dual two forms, which then are irreducible. As we shall see below, however, this does not cause any change, so we can treat all even dimensions ≥ 4 uniformly. According to these decompositions,

sections $\mathcal{E}_a[w] \otimes \mathcal{E}^A$ will be written as vectors of the form

$$\begin{pmatrix} \sigma_a \\ A_{ab} \mid \alpha \mid B_{ab} \\ \rho_a \end{pmatrix}$$

with $A_{ab} = A_{(ab)_0}$ and $B_{ab} = B_{[ab]}$. Following the usual conventions the top slot is the projecting slot, so σ_a has weight $w + 1$ while ρ_a has weight $w - 1$. The action of $\varphi^i \in \Omega^1(M)$ on the standard tractor bundle can be immediately computed from the matrix representation of \mathfrak{g} , and using this, we obtain

$$\varphi_i \cdot \begin{pmatrix} \sigma_a \\ A_{ab} \mid \alpha \mid B_{ab} \\ \rho_a \end{pmatrix} = \begin{pmatrix} 0 \\ -\sigma_{(a}\varphi_{b)_0} \mid -\sigma^i\varphi_i \mid -\sigma_{[a}\varphi_{b]} \\ A_{ab}\varphi^b + \frac{1}{n}\alpha\varphi_a + B_{ab}\varphi^b \end{pmatrix}.$$

The Casimir eigenvalues β_W for the irreducible components in our bundle can be computed using the formulae from 2.4. In dimension four, the self-dual and anti-self-dual parts in $\mathcal{E}_{[ab]}[w + 1]$ correspond to the weights $(w - 1|1, 1)$ and $(w - 1|1, -1)$, respectively. This shows that, for any choice of the weight w , the curved Casimir operator acts by the same scalar on sections of the two bundles. Hence in our constructions schemes for operators we may always treat the sum of these two bundles as if it were a single irreducible component, which shows that the general discussion applies to dimension four as well. The numbers β_W are given by

$$(1) \quad \begin{pmatrix} a_0 + n - 1 \\ a_0 - 2w + n + 1 \mid a_0 - 2w - n + 1 \mid a_0 - 2w + n - 3 \\ a_0 - 4w - n + 3 \end{pmatrix},$$

where $a_0 = w(w + n)$. We will denote the eigenvalue in the top slot by β_0 , the one in the bottom slot by β_2 , and the three middle ones by β_1^1, β_1^2 and β_1^3 . Using this, we can now write out the curved Casimir operator explicitly. Acting by $\nabla - \mathbf{P}\bullet$ on a typical element, we get

$$\begin{pmatrix} \nabla_a\sigma_b \\ \nabla_a A_{bc} + \mathbf{P}_{a(b}\sigma_{c)_0} \mid \nabla_a\alpha + \mathbf{P}_a{}^d\sigma_d \mid \nabla_a B_{bc} - \mathbf{P}_{a[b}\sigma_{c]} \\ \nabla_a\rho_b - \mathbf{P}_a{}^d A_{db} - \frac{1}{n}\alpha\mathbf{P}_{ab} + \mathbf{P}_a{}^d B_{db} \end{pmatrix}.$$

Via Proposition 2.2 we can compute \mathcal{C} by applying to this the action of the index a , multiplying the result by -2 , and adding the components of the original element multiplied by the appropriate scalar. This gives

$$\begin{pmatrix} \beta_0\sigma_a \\ \beta_1^1 A_{ab} + 2\nabla_{(a}\sigma_{b)_0} \mid \beta_1^2\alpha + 2\nabla^c\sigma_c \mid \beta_1^3 B_{ab} + 2\nabla_{[a}\sigma_{b]} \\ \beta_2\rho_a - 2\nabla^c A_{ca} - 2\mathbf{P}^c{}_{(c}\sigma_{a)_0} - \frac{2}{n}\nabla_a\alpha - \frac{2}{n}\mathbf{P}_a{}^c\sigma_c - 2\nabla^c B_{ca} - 2\mathbf{P}^c{}_{[c}\sigma_{a]} \end{pmatrix}.$$

From this formula, we can immediately read off a number of invariant first order splitting operators as well as invariant first order operators between irreducible bundles. For example, elements with $\sigma_a = \alpha = B_{ab} = 0$ form a natural subbundle of $\mathcal{E}_A \otimes \mathcal{E}_a[w]$ for each w . On sections of this natural subbundle, $\mathcal{C} - \beta_2 \text{id}$ defines a natural operator given by

$$\begin{pmatrix} 0 \\ A_{ab} \mid 0 \mid 0 \\ \rho_a \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ (\beta_1^1 - \beta_2)A_{ab} \mid 0 \mid 0 \\ -2\nabla^c A_{ca} \end{pmatrix}$$

Since the value is independent of ρ_a , it descends to a natural operator defined on $\mathcal{E}_{(ab)_0}[w+1]$. If $\beta_1^1 - \beta_2 \neq 0$ or equivalently $w \neq 1 - n$, this is the splitting operator $\Gamma(\mathcal{E}_{(ab)_0}[w+1]) \rightarrow \Gamma(\mathcal{E}_a^A[w])$ as constructed in [7]. However, for $w = 1 - n$, the operator has values in the natural subbundle $\mathcal{E}_a[-n] \subset \mathcal{E}_a^A[1-n]$, so we obtain a natural differential operator $\Gamma(\mathcal{E}_{(ab)_0}[2-n]) \rightarrow \Gamma(\mathcal{E}_a[-n])$ given by $A_{ab} \mapsto -2\nabla^b A_{ba}$. This is the adjoint of the conformal Killing operator.

In the same way, one obtains splitting operators for the other middle slots, and first order operators $\mathcal{E}[0] \rightarrow \mathcal{E}_a[0]$ (the exterior derivative from functions to one-forms) and $\mathcal{E}_{[ab]}[4-n] \rightarrow \mathcal{E}_a[2-n]$ (the divergence or equivalently the exterior derivative from $(n-2)$ -forms to $(n-1)$ -forms).

To construct invariant operators defined on the quotient bundle $\mathcal{E}_a[w+1]$, consider the differences of the β 's from β_0 , which are given by

$$\begin{pmatrix} & 0 & & \\ c_1^1 & | & c_1^2 & | & c_1^3 \\ & c_2 & & & \end{pmatrix} := \begin{pmatrix} & 0 & & \\ 2w-2 & | & 2w+2n-2 & | & 2w+2 \\ & & 4w+2n-4 & & \end{pmatrix}$$

From the formula for \mathcal{C} from above, we can read off the three first order invariant operators obtained in the case that $c_1^i = 0$. For $c_1^1 = 0$, i.e. $w = 1$ we get the conformal Killing operator $\mathcal{E}_a[2] = \mathcal{E}^a \rightarrow \mathcal{E}_{(ab)_0}[2]$. For $c_1^2 = 0$ we get $w = 1 - n$ and we obtain the divergence $\mathcal{E}_a[2-n] \rightarrow \mathcal{E}[-n]$ (or equivalently the exterior derivative from $(n-1)$ -forms to n -forms. Finally, $c_1^3 = 0$ corresponds to $w = -1$ as this gives the exterior derivative from one-forms to two forms.

To construct the full splitting operator defined on $\mathcal{E}_a[w+1]$ respectively an operator from this bundle to $\mathcal{E}_a[w-1]$ (for a special value of w), we have to form $(\mathcal{C} - \beta_2) \circ (\mathcal{C} - \beta_1^1) \circ (\mathcal{C} - \beta_1^2) \circ (\mathcal{C} - \beta_1^3)$. This gives a splitting operator provided that all c_1^i and c_2 are nonzero by Theorem 2 of [7]. For $c_2 = 0$, i.e. $w = 1 - \frac{n}{2}$, we see from 2.3 that we obtain an invariant differential operator $\Gamma(\mathcal{E}_a[2 - \frac{n}{2}]) \rightarrow \Gamma(\mathcal{E}_a[-\frac{n}{2}])$ of order at most two. We can immediately calculate this operator using the above formula for \mathcal{C} . Its value on σ_a reads as

$$\begin{pmatrix} & c_2 c_1^1 c_1^2 c_1^3 \sigma_a & \\ 2c_2 c_1^2 c_1^3 \nabla_{(a} \sigma_{b)} & | & 2c_2 c_1^1 c_1^3 \nabla^i \sigma_i \\ & A_a(\sigma) & -2c_2 c_1^1 c_1^2 \nabla_{[a} \sigma_{b]} \end{pmatrix},$$

where

$$\begin{aligned} A_a(\sigma) = & -2c_1^2 c_1^3 (2\nabla^i \nabla_{(i} \sigma_{a)_0} + c_1^1 P^i_{(i} \sigma_{a)_0}) - \frac{2}{n} c_1^1 c_1^3 (2\nabla_a \nabla^i \sigma_i + c_1^2 P_a^i \sigma_i) \\ & + 2c_1^1 c_1^2 (2\nabla^i \nabla_{[i} \sigma_{a]} - c_1^3 P^i_{[a} \sigma_{i]}) \end{aligned}$$

In particular, we see that for $c_2 = 0$, only the bottom slot is non-zero, and, as expected, we obtain an invariant operator $\sigma \mapsto A_a(\sigma)$. We can easily compute the principal part of this operator by looking only at the second order terms and commuting derivatives. This shows that, up to a non-zero factor, the principal part is given by

$$\sigma_a \mapsto (n-2)(n\Delta\sigma_a - 4\nabla_a \nabla^i \sigma_i).$$

In particular, except for the case $n = 2$, which is geometrically irrelevant, we obtain a true second order operator.

Collecting our results, we see that from curved Casimirs on the bundle $\mathcal{E}_a[w] \otimes \mathcal{E}^A$ we obtain seven invariant operators between irreducible bundles. Six of these are first order, while one is of order two. The first order operators belong to two different BGG sequences. The two exterior derivatives and the two divergences are part of the de–Rham sequence, i.e. the BGG sequence of the trivial representation. The conformal Killing operator and its adjoint are well known to be part of the BGG sequence corresponding to the adjoint representation. Finally, for $n \geq 6$ the second order operator $\Gamma(\mathcal{E}_a[2 - \frac{n}{2}]) \rightarrow \Gamma(\mathcal{E}_a[-\frac{n}{2}])$ is not part of any BGG sequence, since the corresponding representations (or rather the Verma modules associated to their duals) have singular infinitesimal character. Moreover, the resulting operator is a non–standard operator. Hence we see that even for this simple example, we obtain both standard and non–standard operators both in regular and singular infinitesimal character. In dimension four, the situation is slightly different, since the two critical weights $w = -1$ and $w = 1 - \frac{n}{2}$ coincide. This means that the second order operator is obtained as the composition of the divergence and the exterior derivative. Hence for $n = 4$, we obtain the Maxwell operator, which is a standard operator in the BGG–sequence of the trivial representation.

3. CONFORMALLY INVARIANT POWERS OF THE LAPLACIAN

In this section, we show how to construct the conformally invariant square and cube of the Laplacian from curved Casimir operators. There are some well known subtle phenomena concerning these operators. As shown in [11] in dimension four and in [10] in general, there are no conformally invariant powers of the Laplacian in even dimensions $n = 2m$ whose order exceeds n . Moreover, the m th power (called the critical power) is of much more subtle nature than the lower powers. As shown in [8], for all lower powers of the Laplacian (as well as all operators occurring in BGG–sequences) there are formulae which are strongly invariant (induced from homomorphisms on semi–holonomic jet modules), while the critical powers do not have this property. As we shall see, these phenomena are reflected very nicely in the constructions via curved Casimir operators. For the square of the Laplacian, a different construction has to be used in the critical dimension four. On the other hand, the construction for the cube of the Laplacian completely breaks down in dimension four.

3.1. The square of the Laplacian in dimensions $\neq 4$. We consider the tracefree part in the symmetric square of the standard tractor bundle twisted by a weight, i.e. the bundle $\mathcal{E}^{(AB)_0}[w]$. From the composition series of the standard tractor bundle in 2.1 we see that

$$\mathcal{E}^{(AB)_0}[w] = \mathcal{E}[w + 2] \oplus \mathcal{E}_a[w + 2] \oplus (\mathcal{E}_{(ab)_0}[w + 2] \oplus \mathcal{E}[w]) \oplus \mathcal{E}_a[w] \oplus \mathcal{E}[w - 2].$$

We will again use a vector notation with the projecting slot on top. To compute the action of \mathfrak{p}_+ , one has to represent typical elements in each slot by tensor products of standard tractors, and then compute the tensorial action. It is obvious how to get such representatives, except for the two components in the middle. Using \vee to denote the symmetric tensor product,

the representatives for $\mathcal{E}[w]$ are the multiples of the element

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{n} \sum_j \begin{pmatrix} 0 \\ e_j \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ e^j \\ 0 \end{pmatrix}$$

for dual bases $\{e_j\}$ and $\{e^j\}$. On the other hand, typical representatives for the elements in $\mathcal{E}_{(ab)_0}[w+2]$ are given by the sum of $\begin{pmatrix} 0 \\ \mu_a \\ 0 \end{pmatrix} \vee \begin{pmatrix} 0 \\ \nu_b \\ 0 \end{pmatrix}$ and an appropriate multiple of the \mathfrak{g} -invariant expression representing the tractor metric. Using these facts, one easily computes that the \mathfrak{p}_+ -action as a map $\mathcal{E}_a \otimes \mathcal{E}^{(AB)_0} \rightarrow \mathcal{E}^{(AB)_0}$ is in vector notation given by

$$\varphi_i \cdot \begin{pmatrix} \sigma \\ \mu_a \\ A_{ab} \mid \alpha \\ \nu_a \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ -2\sigma\varphi_a \\ -\varphi_{(a}\mu_{b)_0} \mid \varphi^i\mu_i \\ 2\varphi^i A_{ia} - \frac{n+2}{n}\alpha\varphi_a \\ \varphi^i\nu_i \end{pmatrix}.$$

From this, we can determine the formula for the curved Casimir operator as in 2.5 to obtain

$$\mathcal{C} \begin{pmatrix} \sigma \\ \mu_a \\ A_{ab} \mid \alpha \\ \nu_a \\ \rho \end{pmatrix} = \begin{pmatrix} \beta_0\sigma \\ \beta_1\mu_a + 4\nabla_a\sigma \\ \beta_2^1 A_{ab} + 2\nabla_{(a}\mu_{b)_0} + 4\mathbf{P}_{(ab)_0}\sigma \mid \beta_2^2\alpha - 2\nabla^c\mu_c - 4\mathbf{P}\sigma \\ \beta_3\nu_a - 4\nabla^c A_{ca} - 4\mathbf{P}^c_{(c}\mu_{a)_0} + 2\frac{n+2}{n}\nabla_a\alpha - 2\frac{n+2}{n}\mathbf{P}_a{}^c\mu_c \\ \beta_4\rho - 2\nabla_c\nu^c + 4\mathbf{P}^{cd}A_{cd} - 2\frac{n+2}{n}\mathbf{P}\alpha \end{pmatrix}$$

Computing the Casimir eigenvalues corresponding to the irreducible components which occur in that formula is straightforward and gives

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2^1 \mid \beta_2^2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{pmatrix} w(w+n) + 4w + 2n + 4 \\ w(w+n) + 2w + 2n \\ w(w+n) + 2n \mid w(w+n) \\ w(w+n) - 2w \\ w(w+n) - 4w - 2n + 4 \end{pmatrix}.$$

The differences of β_0 from these numbers are given by

$$(2) \quad \begin{pmatrix} 0 \\ 2w + 4 \\ 4w + 4 \mid 4w + 2n + 4 \\ 6w + 2n + 4 \\ 8w + 4n \end{pmatrix}$$

The critical weight for which we can expect an operator from the top slot to the bottom slot is therefore given by $w = -m$ in dimension $n = 2m$. Inserting this into (2), we obtain

$$(3) \quad \begin{pmatrix} 0 \\ 4 - n \\ 4 - 2n \mid 4 \\ 4 - n \\ 0 \end{pmatrix}.$$

This already shows that something special will happen in dimension four, since there we obtain a coincidence of four (rather than two) of the Casimir eigenvalues. There would be another potential speciality (a coincidence of

three of the eigenvalues) in dimension $n = 2$, but this is not geometrically relevant.

According to 2.3, an operator from the top slot to the bottom slot is induced by $(\mathcal{C} - \beta_4) \circ (\mathcal{C} - \beta_3) \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2) \circ (\mathcal{C} - \beta_1)$. To compute the principal part of this induced operator, one can apply this composition to an element for which only the top component is nonzero. Moreover, observe that any derivative moves down one level, so terms in lower levels which contain only few derivatives can be ignored. Finally, one can freely commute derivatives when determining the principal part. Using this simplifications and computing the composition in the opposite order as written above, it is easy to verify directly that up to a nonzero factor, the principal part equals $(n - 4)\Delta^2\sigma$. In particular, for $n \neq 4$ the principal part is nonzero and we have constructed a conformally invariant square of the Laplacian.

3.2. The square of the Laplacian in dimension 4. In dimension four, the operator considered in 3.1 reads as $(\mathcal{C} - \beta_4)^3 \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2)$ because of the additional coincidences of eigenvalues. From 3.1 we see that the (fourth order) principal part of the induced operator $\mathcal{E} \rightarrow \mathcal{E}[-4]$ vanishes, and indeed we shall see from the further discussion, that this operator is identically zero. Still we can obtain a conformally invariant square of the Laplacian in dimension four from curved Casimirs. Namely, we will show that actually the operator $(\mathcal{C} - \beta_4)^2 \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2)$ induces such a square, but this needs some verifications.

Indeed, let us write the natural filtration of the bundle $\mathcal{T} = \mathcal{E}^{(AB)0}[w]$ as $\mathcal{T} = \mathcal{T}^0 \supset \mathcal{T}^1 \supset \dots \supset \mathcal{T}^4 \supset \{0\}$. Now by construction, $(\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2)$ maps sections of \mathcal{T}^2 to sections of \mathcal{T}^3 , and each occurrence of $\mathcal{C} - \beta_4$ maps sections of \mathcal{T} to sections of \mathcal{T}^1 , sections of \mathcal{T}^1 to sections of \mathcal{T}^2 , sections of \mathcal{T}^3 to sections of \mathcal{T}^4 , and sections of \mathcal{T}^4 to zero. Thus the composition $(\mathcal{C} - \beta_4)^2 \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2)$ vanishes on $\Gamma(\mathcal{T}^2)$, maps $\Gamma(\mathcal{T}^1)$ to $\Gamma(\mathcal{T}^4)$ and all of $\Gamma(\mathcal{T})$ to $\Gamma(\mathcal{T}^3)$. In particular, it induces operators

$$\begin{aligned}\Gamma(\mathcal{E}) &= \Gamma(\mathcal{T}/\mathcal{T}^1) \rightarrow \Gamma(\mathcal{T}^3/\mathcal{T}^4) = \Gamma(\mathcal{E}_a[-2]) \\ \Gamma(\mathcal{E}_a) &= \Gamma(\mathcal{T}^1/\mathcal{T}^2) \rightarrow \Gamma(\mathcal{T}^4) = \Gamma(\mathcal{E}[-4]).\end{aligned}$$

If we can prove that both these operators vanish, then we get an induced operator $\Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E}[-4])$ as required. Since this is induced by a composition of four curved Casimirs, it follows immediately that the symbol is induced by the four-fold action of \mathfrak{p}_+ and hence we have found an invariant square of the Laplacian.

It turns out that we can write the two operators whose vanishing we want to prove as compositions. Since $\beta_0 = \beta_1 = \beta_3 = \beta_4$, the operator $\mathcal{C} - \beta_4$ induces invariant operators $\Gamma(\mathcal{T}/\mathcal{T}^1) \rightarrow \Gamma(\mathcal{T}^1/\mathcal{T}^2)$ as well as $\Gamma(\mathcal{T}^3/\mathcal{T}^4) \rightarrow \Gamma(\mathcal{T}^4)$, and these are just the exterior derivative d mapping functions to 1-forms, respectively the divergence δ , which is a formal adjoint to this. On the other hand, the composition $(\mathcal{C} - \beta_1) \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2)$ induces an invariant operator $T : \Gamma(\mathcal{T}^1/\mathcal{T}^2) \rightarrow \Gamma(\mathcal{T}^3/\mathcal{T}^4)$, so this maps 1-forms to 3-forms. The two operators we have to study are the compositions $T \circ d$ and $\delta \circ T$, so we have to prove that these vanish. We do this by showing that T is the Maxwell operator (as expected).

Using the formula for \mathcal{C} from 3.1, a simple direct computation shows that the operator T maps μ_a to

$$-4\nabla^c\nabla_{(c\mu_a)_0} + 3\nabla_a\nabla^c\mu_c + 8\mathbb{P}^c_{(c\mu_a)_0} + 6\mathbb{P}_a{}^c\mu_c.$$

Now expanding the definition of the tracefree symmetric part respectively of the Rho-tensor immediately leads to the identities

$$\begin{aligned} -4\nabla^c\nabla_{(c\mu_a)_0} &= -2\nabla^c\nabla_c\mu_a - 2\nabla^c\nabla_a\mu_c + \nabla_a\nabla^c\mu_c \\ 8\mathbb{P}^c_{(c\mu_a)_0} &= 4\mathbb{P}\mu_a + 2\mathbb{P}_a{}^c\mu_c \\ \nabla_a\nabla^c\mu_c &= \nabla^c\nabla_a\mu_c - 2\mathbb{P}_a{}^c\mu_c - \mathbb{P}\mu_a. \end{aligned}$$

Putting this together, we immediately get $T(\mu_a) = 2\nabla^c\nabla_{[a\mu_c]}$ and this completes the argument.

While we do not intend to discuss the concept of strong invariance in detail in this paper, we want to make a brief comment on these issues. The curved Casimir operators themselves are of course strongly invariant in every sense, since they are of first order. Consequently, any operator directly induced by a polynomial in curved Casimirs is strongly invariant, too. In particular, the construction of 3.1 provides strongly invariant squares of the Laplacian in dimensions different from 4. The construction in dimension four however depends on vanishing of the compositions $T \circ d$ and $\delta \circ T$, which (like the equation $d \circ d = 0$) are not valid in a strong sense. Hence in dimension 4 we cannot conclude that we get a strongly invariant operator.

3.3. The cube of the Laplacian. To conclude this article, we briefly outline what happens for the cube of the Laplacian. The relevant bundle to obtain a cube of the Laplacian is of course $S_0^3\mathcal{E}^A$, which has composition series

$$\begin{aligned} \mathcal{E}[w+3] \oplus \mathcal{E}_a[w+3] \oplus \left(\begin{array}{c} \mathcal{E}_{(ab)_0}^{[w+3]} \\ \mathcal{E}_{[w+1]} \end{array} \right) \oplus \left(\begin{array}{c} \mathcal{E}_{(abc)_0}^{[w+3]} \\ \mathcal{E}_a^{[w+1]} \end{array} \right) \oplus \\ \left(\begin{array}{c} \mathcal{E}_{(ab)_0}^{[w+1]} \\ \mathcal{E}_{[w-1]} \end{array} \right) \oplus \mathcal{E}_a[w-1] \oplus \mathcal{E}[w-3] \end{aligned}$$

We use a vector notation similar as before. Computing the Casimir eigenvalues is straightforward, and shows that the weight for which one may expect an operator from the top slot to the bottom slot is again $w = \frac{-n}{2}$. For this the differences of the Casimir eigenvalue for the top slot from the other Casimir eigenvalues form the pattern

$$\left(\begin{array}{c} 0 \\ 6-n \\ 2(4-n) \quad | \quad 8 \\ 6-3n \quad | \quad 10-n \\ 2(4-n) \quad | \quad 8 \\ 6-n \\ 0 \end{array} \right),$$

which shows that additional coincidences of Casimir eigenvalues occur in dimensions 4, 6, and 10. While the special role of dimensions 4 (for which non-existence of a conformally invariant power of the Laplacian is proved in [11]) and 6 (for which the cube is the critical power of the Laplacian) has to be expected, the special role of dimension 10 comes as a surprise.

To compute the curved Casimir, the main input is again the action of \mathfrak{g}_1 which, viewed as a map $\mathcal{E}_a \otimes S_0^3 \mathcal{E}^A \rightarrow S_0^3 \mathcal{E}^A$, is given by

$$\varphi_i \cdot \begin{pmatrix} \sigma \\ \mu^a \\ A_{ab} \mid \alpha \\ \Phi_{abc} \mid \nu^a \\ B_{ab} \mid \beta \\ \tau^a \\ \rho \end{pmatrix} = \begin{pmatrix} 0 \\ -3\varphi_a \\ -2\varphi_{(a}\mu_{b)_0} \mid \varphi^i \mu_i \\ -\varphi_{(a}A_{bc)_0} \mid -2\frac{n+2}{n}\alpha\varphi_a + 2\varphi^i A_{ia} \\ -\frac{n+4}{n+2}\varphi_{(a}\nu_{b)_0} + 3\varphi^i \Phi_{iab} \mid \varphi^i \nu_i \\ -\frac{n+4}{n}\beta\varphi_a + 2\varphi^i B_{ia} \\ \varphi^i \tau_i \end{pmatrix}.$$

From this, one easily derives the full formula for the curved Casimir operator on the bundle $S_0^3 \mathcal{E}^A[w]$. According to 2.3, the operator to consider is

$$(4) \quad (\mathcal{C} - \beta_0) \circ (\mathcal{C} - \beta_1)^2 \circ (\mathcal{C} - \beta_2^1)^2 \circ (\mathcal{C} - \beta_2^2)^2 \circ (\mathcal{C} - \beta_3^1) \circ (\mathcal{C} - \beta_3^2),$$

where the squares are due to the fact that $\beta_5 = \beta_1$ and $\beta_4^i = \beta_2^i$ for $i = 1, 2$. To compute the principal part of the induced operator, one proceeds in a manner similar to 3.1 above. That is by working through the composition starting with the factor $\mathcal{C} - \beta_0$ and then working down level by level. One takes only terms of high enough order in each level, and freely commutes derivatives. This shows that, up to a nonzero factor, the principal part is given by

$$\sigma \mapsto (n-4)(n-6)(n-10)\Delta^3 \sigma.$$

We want to point out however, that while the factors $(n-4)$, $(n-6)$, and $(n-10)$ occur as differences of Casimir eigenvalues, the fact that they arise in the principal part is not at all straightforward, but has to be verified by rather nasty computations. In all dimensions except for these three critical ones, our operator directly defines a conformally invariant cube of the Laplacian.

Concerning the critical dimensions, the situation is the following. The easiest of these cases is dimension 10. Here there is an additional coincidence of Casimir eigenvalues, since $\beta_3^2 = \beta_0$. Let us write $\mathcal{T} = S_0^3 \mathcal{E}^A$ and us denote the canonical filtration of \mathcal{T} by $\mathcal{T} = \mathcal{T}^0 \supset \dots \supset \mathcal{T}^6 \supset \{0\}$. Now consider the composition

$$(\mathcal{C} - \beta_3^2) \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2) \circ (\mathcal{C} - \beta_1).$$

This maps $\Gamma(\mathcal{T})$ to $\Gamma(\mathcal{T}^3)$, and if we project to $\mathcal{T}^3/\mathcal{T}^4$ and then further to the component $\mathcal{E}_a[-4]$ (which corresponds to the eigenvalue β_3^2), then the composition vanishes on $\Gamma(\mathcal{T}^1)$. Hence it induces an operator from sections of $\mathcal{T}/\mathcal{T}^1 \cong \mathcal{E}[-2]$ to sections of $\mathcal{E}_a[-4]$. (It is known from the classification of conformally invariant operators, that this has to vanish in the conformally flat case.) Now a direct computation shows that this operator actually is always identically zero. This shows that

$$(\mathcal{C} - \beta_3^1) \circ (\mathcal{C} - \beta_3^2) \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2) \circ (\mathcal{C} - \beta_1)$$

maps all of $\Gamma(\mathcal{T})$ to $\Gamma(\mathcal{T}^4)$. Hence if we further apply $(\mathcal{C} - \beta_5) \circ (\mathcal{C} - \beta_4^1) \circ (\mathcal{C} - \beta_4^2)$, the result maps all of $\Gamma(\mathcal{T})$ to $\Gamma(\mathcal{T}^6)$.

Similarly, we can consider the composition

$$(\mathcal{C} - \beta_5) \circ (\mathcal{C} - \beta_4^1) \circ (\mathcal{C} - \beta_4^2) \circ (\mathcal{C} - \beta_3^2)$$

on the space of those sections of \mathcal{T}^3 whose image in $\mathcal{T}^3/\mathcal{T}^4$ is a section of the component $\mathcal{E}_a[-4]$ only. As before, this clearly maps all such sections to sections of \mathcal{T}^6 , and since $\beta_3^2 = \beta_6$ it vanishes on sections of the subbundle \mathcal{T}^4 . Hence we get an induced operator from sections of $\mathcal{E}_a[-4]$ to sections of $\mathcal{T}^6 = \mathcal{E}[-8]$. Once again, a direct computation shows that this operator vanishes identically (which in the conformally flat case follows from the known classification results). Now on the other hand, the composition

$$(\mathcal{C} - \beta_3^1) \circ (\mathcal{C} - \beta_2^1) \circ (\mathcal{C} - \beta_2^2) \circ (\mathcal{C} - \beta_1)$$

maps $\Gamma(\mathcal{T}^1)$ to $\Gamma(\mathcal{T}^3)$ and projecting to $\mathcal{T}^3/\mathcal{T}^4$ the result lies in $\Gamma(\mathcal{E}_a[-4])$ only. Together with the above observation we conclude that if in the composition (4) we leave out one of the two factors $(\mathcal{C} - \beta_0)$, then the result still maps sections of \mathcal{T} to sections of \mathcal{T}^6 and vanishes on sections of \mathcal{T}^1 . Hence we again get an induced operator mapping sections of $\mathcal{T}/\mathcal{T}^1 \cong \mathcal{E}[-2]$ to sections of $\mathcal{E}[-8] \cong \mathcal{T}^6$. Of course, this also implies that the original composition (4) induces the zero operator in dimension 10.

A similar computation as for general dimensions now shows that the principal part of this operator is a nonzero multiple of $\sigma \mapsto \Delta^3 \sigma$. Hence we have obtained a cube of the Laplacian in dimension 10, although we cannot conclude that this is strongly invariant.

Next, let us discuss dimension $n = 4$, for which there is no conformally invariant cube of the Laplacian by [11]. Due to the coincidences of Casimir eigenvalues, the composition (4) here specialises to

$$(5) \quad (\mathcal{C} - \beta_0)^3 \circ (\mathcal{C} - \beta_1)^2 \circ (\mathcal{C} - \beta_2^2)^2 \circ (\mathcal{C} - \beta_3^1) \circ (\mathcal{C} - \beta_3^2).$$

One might hope that one can define a cube of the Laplacian in dimension four, at last for a certain class of conformal manifolds by leaving out one of the three factors $(\mathcal{C} - \beta_0)$. This turns out to work however, only on the subcategory of locally conformally flat structures.

The pattern is similar to that arising for the square of the Laplacian in dimension four. The composition $(\mathcal{C} - \beta_0) \circ (\mathcal{C} - \beta_3^1) \circ (\mathcal{C} - \beta_3^2)$ is easily seen to induce a second order operator Φ mapping sections of $\mathcal{E}_{(ab)_0}[1] \subset \mathcal{T}^2/\mathcal{T}^3$ to sections of $\mathcal{E}_{(ab)_0}[-1] \subset \mathcal{T}^4/\mathcal{T}^5$. Likewise, the composition $(\mathcal{C} - \beta_0) \circ (\mathcal{C} - \beta_1)$ induces an operator Ψ_1 mapping sections of $\mathcal{E}[1] \cong \mathcal{T}/\mathcal{T}^1$ to sections of $\mathcal{E}_{(ab)_0}[1] \subset \mathcal{T}^2/\mathcal{T}^3$ as well as an operator Ψ_2 , which maps sections of $\mathcal{E}_{(ab)_0}[-1] \subset \mathcal{T}^4/\mathcal{T}^5$ to sections of $\mathcal{E}[-5] \cong \mathcal{T}^6$. To get an induced operator $\Gamma(\mathcal{E}[1]) \rightarrow \Gamma(\mathcal{E}[-5])$ after leaving out one of the three factors $(\mathcal{C} - \beta_0)$ in (5), one needs the compositions $\Phi \circ \Psi_1$ and $\Psi_2 \circ \Phi$ to vanish identically. However, it turns out that both these compositions actually are second order operators with Weyl curvature in the principal symbol and a tensorial part involving the Bach tensor. Further, from the explicit form for the principal symbol one may see that it vanishes only in the locally flat case (where this also follows from the classification results). In the latter case, one can then compute the principal part similarly as before to see that one indeed does obtain a conformally invariant cube of the Laplacian on locally conformally flat 4-manifolds, but not for a larger class.

Finally, in the critical dimension $n = 6$ some details remain unresolved. Due to the coincidences of Casimir eigenvalues, the composition (4) specialises to

$$(6) \quad (\mathcal{C} - \beta_0)^3 \circ (\mathcal{C} - \beta_2^1)^2 \circ (\mathcal{C} - \beta_2^2)^2 \circ (\mathcal{C} - \beta_3^1) \circ (\mathcal{C} - \beta_3^2).$$

As for the square of the Laplacian in dimension four, the hope would be to leave out one of the three factors $(\mathcal{C} - \beta_0)$ and still get an induced operator. Also, the verifications to be made are analogous to ones from 3.2. The composition

$$(\mathcal{C} - \beta_0) \circ (\mathcal{C} - \beta_2^1)^2 \circ (\mathcal{C} - \beta_2^2)^2 \circ (\mathcal{C} - \beta_3^1) \circ (\mathcal{C} - \beta_3^2)$$

induces a fourth order operator $T : \Gamma(\mathcal{E}_a) \rightarrow \Gamma(\mathcal{E}_a[-4])$. On the other hand, $(\mathcal{C} - \beta_0)$ induces the exterior derivative $d : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E}_a)$ as well as the divergence $\delta : \Gamma(\mathcal{E}_a[-4]) \rightarrow \Gamma(\mathcal{E}[-6])$. Leaving out one of the three factors $(\mathcal{C} - \beta_0)$ in (6), the result induces an operator $\Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{E}[-6])$ if and only if the compositions $T \circ d$ and $\delta \circ T$ vanish identically. Of course, this is true in the flat case, so there the construction again works. While we have been able to compute a complete formula for T in the curved case, computing the two compositions explicitly seems to be a serious task. To sort out this problem new ideas would be helpful.

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