

Near Flat Space Limit of Strings on $\text{AdS}_4 \times \text{CP}^3$

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Vienna, Preprint ESI 2057 (2008)

October 14, 2008

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via <http://www.esi.ac.at>

Near Flat Space limit of strings on $AdS_4 \times \mathbb{CP}^3$

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Abstract

The non-linear nature of string theory on non-trivial backgrounds related to the AdS/CFT correspondence suggests to look for simplifications. Two such simplifications proved to be useful in studying string theory. These are the pp-wave limit which describes point-like strings and the so called “near flat space” limit which connects two different sectors of string theory – pp-waves and “giant magnons”. Recently another example of AdS/CFT duality emerged - AdS_4/CFT_3 , which suggests duality between $\mathcal{N} = 6$ CS theory and superstring theory on $AdS_4 \times \mathbb{CP}^3$. In this paper we study the “near flat space” limit of strings on the $AdS_4 \times \mathbb{CP}^3$ background and discuss possible applications of the reduced theory.

PACS number: 11.25.Tq

1 Introduction

The idea of a correspondence between the large N limit of gauge theories and strings emerged over thirty years ago [1] and since then many attempts for the explicit realization have been made. One of the most promising so far was provided by Maldacena who conjectured the AdS/CFT correspondence [2]. Since then this became a topical area and many fascinating discoveries were made over the last decade. One of the predictions of the correspondence is a one-to-one map between the spectrum of string theory on $AdS_5 \times S^5$ and the spectrum of anomalous dimensions of gauge invariant operators in $\mathcal{N} = 4$ Supersymmetric Yang-Mills (SYM) theory.

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The vast majority of papers were on qualitative and quantitative description of $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group by making use of the string sigma model on $AdS_5 \times S^5$. This special case is remarkable in several ways. First of all the background preserves the maximal number of supersymmetries. Secondly, as indicated first in [42], the huge amount of symmetries of the relevant classical coset sigma model leads to its integrability and opens the possibility to study its integrability at the quantum level as well. Thus, one can put forward the important issue of derivation of the S-matrix and study the main features of the string model and its dual. This issue is important because one of the predictions of the correspondence is the equivalence between the spectrum of string theory on $AdS_5 \times S^5$ and the spectrum of anomalous dimensions of gauge invariant operators. There has been a good deal of success recently in comparing the energies of semiclassical strings and the anomalous dimensions of the gauge theory operators. However, on one side string theory on $AdS_5 \times S^5$ is highly non-linear and on another the tools at hand to study gauge theory are limited mainly to perturbation theory. Having in mind the duality, i.e. complementarity of the validity regions of the perturbation theories, the proof of the conjecture even in this quite well understood case still remains a real challenge. Fortunately, there exists a wide region where one can trust quasi-classical considerations and compare the corresponding spectra on both sides. Integrability provides powerful tools for calculating the key features on each side of the correspondence.

There are several approaches using simplifications of the non-linear system we are dealing with. An important proposal by Berenstein, Maldacena and Nastase [40] made the first step beyond the supergravity approximation in the AdS/CFT correspondence. They showed how certain operators in SYM theory can be related to string theory in the pp-wave limit in a certain way. Namely, it was suggested that string theory in pp-wave background gives a good approximation of a certain class of "nearly" chiral operators and vice versa. Since string theory in the pp-wave background is exactly solvable, a lot of information for the AdS/CFT correspondence was extracted¹.

Superstring theory on $AdS_5 \times S^5$ is integrable at the classical level and there exist strong indications that it happens also at the quantum level. Nevertheless, it is a highly non-trivial task to find the exact S-matrix of the theory and to study the duality in its full extent. The difficulties are not only of technical nature but also conceptual. Moreover, one must keep considerations such that one can check the results on both sides of the correspondence. The latter assumes certain limits [41], allowing reliability of the semiclassical results on both sides. In this case the techniques of integrable systems have become useful to study the AdS/CFT correspondence in detail. For instance, the remarkable observation by Minahan and Zarembo [43] made it possible to relate the gauge theory side of the correspondence to some integrable spin chain (at least to leading orders). Thus, the question of relating the string theory side to some spin chain and then comparing to the gauge theory side becomes very important. If so, the integrability of spin chains would provide a powerful tool for investigation of the AdS/CFT correspondence.

¹For extensive review see for instance [53] [54].

Discussing the integrability properties of the superstring on $AdS_5 \times S^5$, one should point out several key developments leading to the current understanding. One of them is the gauge theory S-matrix introduced in [44] and derived in its full form in [45]. In [46] the authors consider the Bethe ansatz for quantum strings and introduce the dressing phase as well as the "symplectic" form of the charges appearing in the exponential². The power of this approach was further utilized in [47, 48, 62, 49, 63] leading to important results.

An important step was done by Maldacena and Hofman [50] who were able to map spin chain "magnon" states to a specific class of rotating semiclassical strings on $R \times S^2$ [51]. This result was soon generalized to magnon bound states [55, 52, 56, 57], dual to strings on $R \times S^3$ with two and three non-vanishing angular momenta. Magnon solutions and the dispersion relations for strings in beta-deformed backgrounds were obtained in [65, 66]. The applicability of the considerations is restricted to the case of large quantum numbers, i.e. quasiclassical approximations. Although we assume infinitely large energy E and momentum J , the difference $E - J$ is finite like in the case of the pp-wave limit. Remarkably enough, due to the integrability on the gauge theory side one is able to obtain the corresponding S-matrix explicitly [62, 63].

While similar in dealing with infinitely large quantum numbers, the two pictures, namely the pp-wave limit and the giant magnon sector, are a bit different. To see this one can use simple scaling arguments. While in the first case the angular momentum J goes to infinity as $\sqrt{\lambda} \rightarrow \infty$ and the product $p \cdot \lambda$ is kept fixed (for which p has to be vanishing), in the magnon case the momentum p is kept fixed. In the first case we have point-like strings and we are at the bottom of the $E - J$ scale, while in the second case the string spike is essentially not a point-like object with higher $E - J$. Following this logic it is natural to look for string solutions interpolating between these two sectors. In a recent paper Maldacena and Swanson [68] suggested another limit, called near flat space (NFS) limit, which is interpolating between the two regions above³. In doing a Penrose limit of the geometry we perform a certain boost in the vicinity of a given null geodesics. The limit is taken in such a way that the product $p \cdot \sqrt{\lambda}$ is fixed but the space becomes of pp-wave type. The authors of [68] suggest a weaker boosting and the limit is taken with $p^2 \sqrt{\lambda}$ fixed. The resulting sigma model is much simpler than that of the $AdS_5 \times S^5$ superstring and more complicated than in the pp-wave limit. A remarkable feature of the reduced model is that, as compared to the pp-wave case, it keeps much more information of the original theory. For instance, it is integrable with the Lax connection obtained from the original one by taking the limit, and it possesses the same supersymmetry algebra with a phase originating from the Hopf algebra structure of the problem. The corresponding S matrix was also conjectured. The worldsheet scattering of the reduced model and some other properties were subsequently studied in [69, 70, 71], see also [72].

The discussion above was focused on superstrings on $AdS_5 \times S^5$ background and

²See [46] for further details.

³One should note that the limit $p\lambda^{1/4} = \text{const.}$ was considered before in [46] in the context of [41], while the general discussion of this limit on which our study is based appeared in [68].

its reductions via certain limits and identifying the corresponding sectors of its dual $\mathcal{N} = 4$ SYM theory. It is of importance, however, to extend these techniques to other less symmetric backgrounds that have gauge theory duals. One step in this direction was given in [73] where the authors considered the NFS limit of spaces where the S^5 part of the geometry is replaced by certain Sasaki-Einstein spaces. For instance, it is known that spaces like Y^{pq} and $L^{p,q,r}$ have $\mathcal{N} = 1$ CFT duals. Remarkably, the near flat space limit of all these spaces is of the same type as in [68]. Since the latter is integrable, one can hope that the corresponding theory may contain, at least, an integrable subsector. Another step was done in [74] where the near flat space limit was studied in the Maldacena-Nunez background [75]⁴.

Recently, motivated by the possible description of the worldvolume dynamics of coincident membranes in M-theory in the context of the AdS/CFT correspondence, a new class of conformally invariant, maximally supersymmetric field theories in 2+1 dimensions has been found [4, 5]. The main feature of these theories is that they contain gauge fields with Chern-Simons-like kinetic terms. This development motivated Aharony, Bergman, Jafferis and Maldacena to propose a new gauge/string duality between an $\mathcal{N} = 6$ super-conformal Chern-Simons theory (ABJM theory) coupled with bi-fundamental matter (describing N membranes on S^7/\mathbb{Z}_k) and string theory on \mathbb{CP}^3 . The ABJM model [5] is believed to be holographically dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$.

The ABJM theory actually consists of two Chern-Simons theories of level k and $-k$, respectively, and each with gauge group $SU(N)$. The two pairs of chiral superfields transform in the bi-fundamental representation of $SU(N) \times SU(N)$ and the R-symmetry is $SU(4)$ as it should be for $\mathcal{N} = 6$ supersymmetry. It was observed in [5] that there exists a natural definition of a 't Hooft coupling, namely, $\lambda = N/k$. It was observed that in the 't Hooft limit $N \rightarrow \infty$ with λ held fixed, one has a continuous coupling λ and the ABJM theory is weakly coupled for $\lambda \ll 1$. The ABJM theory is conjectured to be dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$ with N units of four-form flux. In the scaling limit $N, k \rightarrow \infty$ with $k \ll N \ll k^5$ the theory can be compactified to type IIA string theory on $AdS_4 \times \mathbb{P}^3$. Thus, the AdS/CFT correspondence, which has led to many exciting developments in the duality between type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang-Mills theory, is now being extended to AdS_4/CFT_3 and is expected to constitute a new example of exact gauge/string theory duality.

The semi-classical string has played an important role in studying various aspects of the AdS_5/SYM_4 correspondence [39]-[67]. The developments and successes in this particular case suggest the methods and tools that should be used to investigate the new emergent duality. An important role in these studies plays the integrability. Superstrings on $AdS_4 \times \mathbb{CP}^3$ as a coset were first studied in [6]⁵ which opens the door for investigation of integrable structures in the theory. Various properties on the gauge theory side and

⁴This geometry was originally obtained in the context of non-abelian BPS monopoles in gauged supergravity and uplifted to ten dimensions in [76].

⁵See also [7]

tests on the string theory side, like rigid rotating strings, pp-wave limit, relations to spin chains, as well as a pure spinor formulation, have been considered [6]-[37]. In these intensive studies many properties were uncovered and impressive results obtained, but still the understanding of this duality is far from complete.

Inspired by the success of the study of the near flat space limit in the known cases mentioned above, in this paper we focus on the ABJM theory. The paper is organized as follows. First we give a brief review of the near-flat space limit of $AdS_5 \times S^5$. In the next section we give a concise definition of the Aharony-Bergman-Jafferis-Maldacena (ABJM) theory and show how one can make the reduction from membranes to strings on $AdS_4 \times \mathbb{CP}^3$. Next we derive the near-flat space limit of the string sigma model on $AdS_4 \times \mathbb{CP}^3$. We conclude with some comments on the results and future directions.

2 The Near flat space limit of $AdS_5 \times S^5$

In this short section we will review the procedure of taking the near flat space limit of type IIB string theory on $AdS_5 \times S^5$ as suggested in [68]. As we already discussed in the introduction, the idea is to weaken the Penrose limit so that more structures of the original theory get preserved. The first step is to find the null geodesics with respect to which the pp-wave limit can be taken. We start with the standard metric of $AdS_5 \times S^5$ in global coordinates

$$ds^2 = R^2 \left[-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\tilde{\Omega}_3^2 + d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\Omega_3^2 \right] \quad (2.1)$$

where each of the unit 3-spheres, we take below S^3 from the spherical part, can be parameterized by

$$d\Omega_3^2 = d\phi_1^2 + \cos^2 \phi_1 d\phi_2^2 + \sin^2 \phi_1 d\phi_3^2. \quad (2.2)$$

The light-like directions are characterized by covariantly constant null vectors

$$\nabla_\mu v_\nu = 0, \quad v_\mu v^\mu = 0.$$

The S^5 part of the null geodesics is parameterized by $\theta = 0$ and $\rho = 0$, i.e. in our case this is the equator of the five-sphere. Now we boost the worldsheet coordinates and expand about the geodesics ($\dot{\psi} = 1$). This amounts to a redefinition of the fields as follows,

$$\begin{aligned} t &= \sqrt{g}\sigma^+ + \frac{\tau}{\sqrt{g}}, & \psi &= \sqrt{g}\sigma^+ + \frac{\chi}{\sqrt{g}}, \\ \rho &= \frac{z}{\sqrt{g}}, & \theta &= \frac{y}{\sqrt{g}}, \end{aligned} \quad (2.3)$$

where we switched to light-cone worldsheet coordinates σ^+, σ^- . The parameter g is related to the radius of S^5 (and AdS_5 as well) via $g = R^2/4\pi$.

The near flat space limit is, as in the Penrose limit, taking $g \rightarrow \infty$. In the Lagrangian the leading terms are divergent, $g \cdot \partial_-(\tau - \chi)$, but since they are total derivatives they can be dropped. The relevant (bosonic) part of the Lagrangian thus becomes

$$S = 4 \left[-\partial_+ \tau \partial_- \tau + \partial_+ \chi \partial_- \chi + \partial_+ \vec{z} \partial_- \vec{z} + \partial_+ \vec{y} \partial_- \vec{y} - \vec{y}^2 \partial_- \chi - \vec{z}^2 \partial_- \tau \right] \quad (2.4)$$

where we skipped the fermionic part and used the notation \vec{z} for coordinates in AdS_5 and \vec{y} for those in the S^5 part. The reduced model has two conserved chiral currents

$$\begin{aligned} j_+^\chi &= \partial_+ \chi - \frac{\vec{y}^2}{2} + \text{fermions}, & \partial_- j_+^\chi &= 0, \\ j_+^\tau &= \partial_+ \tau + \frac{\vec{z}^2}{2} - \text{fermions}, & \partial_- j_+^\tau &= 0. \end{aligned} \quad (2.5)$$

The right-moving conformal invariance is preserved since its generator

$$T_{--} = -(\partial_- \tau)^2 + (\partial_- \chi)^2 + (\partial_- \vec{z})^2 + (\partial_- \vec{y})^2 + \text{fermions} \quad (2.6)$$

is conserved, i.e. $\partial_+ T_{--} = 0$. The left-moving conformal invariance however is broken, since $T_{++} \propto (j_+^\chi - j_+^\tau)$. One can still impose the Virasoro constraints requiring

$$\begin{aligned} j_+^\chi + j_+^\tau &= \partial_+ (\tau + \chi) + \frac{z^2 - y^2}{2} = 0, \\ T_{--} &= 0. \end{aligned} \quad (2.7)$$

This can also be considered as a gauge fixing condition.

It is useful now to make a change of variables

$$x^+ = \sigma^+, \quad x^- = 2(\tau + \chi). \quad (2.8)$$

For completeness we write down the complete gauge fixed Lagrangian

$$\begin{aligned} \mathcal{L} = & 4 \left\{ \partial_+ \vec{z} \partial_- \vec{z} + \partial_+ \vec{y} \partial_- \vec{y} - \frac{1}{4}(\vec{z}^2 + \vec{y}^2) + (\vec{y}^2 - \vec{x}^2)[(\partial_- \vec{z})^2 + (\partial_- \vec{y})^2] \right. \\ & + i\psi_+ \partial_- \psi_+ + i\psi_- \partial_+ \psi_- + i\psi_- \Pi \psi_+ + i(\vec{y}^2 - \vec{x}^2)\psi_- \partial_- \psi_- \\ & - \psi_- (\partial_- z^j \Gamma^j + \partial_- y^{j'} \Gamma^{j'}) (z^i \Gamma^i - y^{i'} \Gamma^{i'}) \psi_- \\ & \left. + \frac{1}{24} [\psi_- \Gamma^{ij} \psi_- \psi_- \Gamma^{ij} \psi_- \psi_- \Gamma^{i'j'} \psi_- \psi_- \Gamma^{i'j'} \psi_-] \right\}. \end{aligned} \quad (2.9)$$

In the last expression a simple rescaling of ψ_\pm was used. The indices i and i' correspond to the transverse directions in the anti-de Sitter and the spherical parts, respectively.

3 Near-flat space limit of string sigma model on $AdS_4 \times \mathbb{CP}^3$

ABJM and strings on $AdS_4 \times \mathbb{CP}^3$: To find the ABJM theory one starts with analysing the M2-brane dynamics governed by the 11-dimensional supergravity action [5]

$$S = \frac{1}{2\kappa_{11}^2} \int dx^{11} \sqrt{-g} \left(R - \frac{1}{2 \cdot 4!} F_{\mu\nu\rho\sigma} F^{\mu\nu\rho\sigma} \right) - \frac{1}{12\kappa_{11}^2} \int C^{(3)} \wedge F^{(4)} \wedge F^{(4)}, \quad (3.1)$$

where $\kappa_{11}^2 = 2^7 \pi^8 l_p^9$. Solving for the equations of motion

$$R_\nu^\mu = \frac{1}{2} \left(\frac{1}{3!} F^{\mu\alpha\beta\gamma} F_{\nu\alpha\beta\gamma} - \frac{1}{3 \cdot 4!} \delta_\nu^\mu F_{\alpha\beta\rho\sigma} F^{\alpha\beta\rho\sigma} \right), \quad (3.2)$$

and

$$\partial_\sigma (\sqrt{-g} F^{\sigma\mu\nu\xi}) = \frac{1}{2 \cdot (4!)^2} \epsilon^{\mu\nu\xi\alpha_1 \dots \alpha_8} F_{\alpha_1 \dots \alpha_4} F_{\alpha_5 \dots \alpha_8}, \quad (3.3)$$

one can find the M2-brane solutions whose near horizon limit becomes $AdS_4 \times S^7$

$$ds^2 = \frac{R^2}{4} ds_{AdS_4}^2 + R^2 ds_{S^7}^2. \quad (3.4)$$

In addition we have N' units of four-form flux

$$F^{(4)} = \frac{3R^3}{8} \epsilon_{AdS_4}, \quad R = l_p (2^5 N' \pi^2)^{\frac{1}{6}}. \quad (3.5)$$

Now one proceeds with considering the quotient S^7/\mathbb{Z}_k acting as $z_i \rightarrow e^{i\frac{2\pi}{k}} z_i$. It is convenient first to write the metric on S^7 as

$$ds_{S^7}^2 = (d\varphi' + \omega)^2 + ds_{CP^3}^2, \quad (3.6)$$

where

$$ds_{CP^3}^2 = \frac{\sum_i dz_i d\bar{z}_i}{r^2} - \frac{|\sum_i z_i d\bar{z}_i|^2}{r^4}, \quad r^2 \equiv \sum_{i=1}^4 |z_i|^2, \\ d\varphi' + \omega \equiv \frac{i}{2r^2} (z_i d\bar{z}_i - \bar{z}_i dz_i), \quad d\omega = J = id \left(\frac{z_i}{r} \right) d \left(\frac{\bar{z}_i}{r} \right). \quad (3.7)$$

and then to perform the \mathbb{Z}_k quotient identifying $\varphi' = \varphi/k$ with $\varphi \sim \varphi + 2\pi$ (J is proportional to the Kähler form on \mathbb{CP}^3). The resulting metric becomes

$$ds_{S^7/\mathbb{Z}_k}^2 = \frac{1}{k^2} (d\varphi + k\omega)^2 + ds_{CP^3}^2. \quad (3.8)$$

One observes that the first volume factor on the right hand side is divided by a factor of k^2 as compared to the initial one. In order to have consistent quantized flux one must

impose $N' = kN$ where N is the number of quanta of the flux on the quotient. One should note that the spectrum of the supergravity fields of the final theory is just the projection of the initial $AdS_4 \times S^7$ onto the \mathbb{Z}_k invariant sector. In this setup there is a natural definition of the 't Hooft coupling $\lambda \equiv N/k$. The decoupling limit should be taken as $N, k \rightarrow \infty$ while N/k is kept fixed.

One can follow now [5] to make a reduction to type IIA with the following final result

$$ds_{string}^2 = \frac{R^3}{k} \left(\frac{1}{4} ds_{AdS_4}^2 + ds_{\mathbb{CP}^3}^2 \right), \quad (3.9)$$

$$e^{2\phi} = \frac{R^3}{k^3} \sim \frac{N^{1/2}}{k^{5/2}} = \frac{1}{N^2} \left(\frac{N}{k} \right)^{5/2}, \quad (3.10)$$

$$F_4 = \frac{3}{8} R^3 \epsilon_4, \quad F_2 = k d\omega = kJ, \quad (3.11)$$

We end up then with the $AdS_4 \times \mathbb{CP}^3$ compactification of type IIA string theory with N units of F_4 flux on AdS_4 and k units of F_2 flux on the $\mathbb{CP}^1 \subset \mathbb{CP}^3$ 2-cycle.

The radius of curvature in string units is $R_{str}^2 = \frac{R^3}{k} = 2^{5/2} \pi \sqrt{\lambda}$. It is important to note that the type IIA approximation is valid in the regime where $k \ll N \ll k^5$.

To fix the notations, we write down the explicit form of the metric on $AdS_4 \times \mathbb{CP}^3$ in spherical coordinates. The metric on $AdS_4 \times \mathbb{CP}^3$ can be written as [38]

$$ds^2 = R^2 \left\{ \frac{1}{4} [-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2] + d\mu^2 + \sin^2 \mu \left[d\alpha^2 + \frac{1}{4} \sin^2 \alpha (\sigma_1^2 + \sigma_2^2 + \cos^2 \alpha \sigma_3^2) + \frac{1}{4} \cos^2 \mu (d\chi + \sin^2 \mu \sigma_3)^2 \right] \right\}. \quad (3.12)$$

Here R is the radius of the AdS_4 , and $\sigma_{1,2,3}$ are left-invariant 1-forms on an S^3 , parameterized by (θ, ϕ, ψ) ,

$$\begin{aligned} \sigma_1 &= \cos \psi d\theta + \sin \psi \sin \theta d\phi, \\ \sigma_2 &= \sin \psi d\theta - \cos \psi \sin \theta d\phi, \\ \sigma_3 &= d\psi + \cos \theta d\phi. \end{aligned} \quad (3.13)$$

The range of the coordinates is

$$0 \leq \mu, \alpha \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \chi, \psi \leq 4\pi.$$

Taking near-flat space limit of $AdS_4 \times \mathbb{CP}^3$: There are several ways to parametrize the \mathbb{CP}^3 part of the spacetime. One of them is presented in (3.12) and makes use of

the embedding $SU(2) \subset SU(3) \subset SU(4)$. Another way, which puts an emphasis on an eventual limit to the $S^2 \times S^2$ sigma model, is [12]

$$ds_{CP^3}^2 = d\xi^2 + \cos^2 \xi \sin^2 \xi (d\psi + \frac{\cos \theta_1}{2} d\varphi_1 - \frac{\cos \theta_2}{2} d\varphi_2)^2 \quad (3.14)$$

$$+ \frac{1}{4} \cos^2 \xi (d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2) + \frac{1}{4} \sin^2 \xi (d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2). \quad (3.15)$$

The anti-de-Sitter part is described in a standard way

$$ds_{AdS_4}^2 = -\cosh^2 \rho dt + d\rho^2 + \sinh^2 \rho d\Omega_2^2. \quad (3.16)$$

The aim of this Section is to study the near-flat space limit of the spacetime. While for the AdS_4 par the procedure is well known from the $AdS_5 \times S^5$ case, we will focus mainly on the details for the \mathbb{CP}^3 part. To this end we introduce the coordinate $\tilde{\psi}$ by

$$\tilde{\psi} = \psi + \frac{\varphi_1 - \varphi_2}{2}, \quad (3.17)$$

and make a rescaling of the other coordinates as follows

$$\rho = \frac{r}{R}, \quad \theta_i = \frac{\sqrt{2}y_i}{R}, \quad \xi = \frac{\pi}{4} + \frac{y_3}{2R}, \quad i = 1, 2. \quad (3.18)$$

Here $\tilde{\psi}$ is the direction of the null geodesic obtained by taking $\rho = \theta_1 = \theta_2 = 0$.

Taking the near-flat space limit means taking $R \rightarrow \infty$, hence we obtain for the metric the following expansion,

$$\begin{aligned} 4ds_{CP^3}^2 &= \frac{dy_3^2}{R^2} + (1 - \frac{y_3^2}{R^2})(d\tilde{\psi}^2 - \frac{y_1^2}{R^2} d\tilde{\psi} d\varphi_1 + \frac{y_2^2}{R^2} d\tilde{\psi} d\varphi_2) \\ &\quad + (\frac{dy_1^2}{R^2} + \frac{y_1^2}{R^2} d\varphi_1^2) + (\frac{dy_2^2}{R^2} + \frac{y_2^2}{R^2} d\varphi_2^2) + \mathcal{O}(R^{-4}) \\ ds_{AdS_4}^2 &= -(1 - \frac{r^2}{R^2})dt^2 + \frac{dr^2}{R^2} + \frac{r^2}{R^2} d\Omega_2^2 + \mathcal{O}(R^{-4}). \end{aligned} \quad (3.19)$$

To take the limit we have to restore the dependence of the metric on the radius R , which in the string frame metric is

$$ds_{IIA}^2 = R^2(ds_{AdS_4}^2 + 4ds_{CP^3}^2). \quad (3.20)$$

In addition it is useful to diagonalize the metric and to make a redefinition of the angles as follows

$$\varphi_1 = \phi_1 + \frac{\tilde{\psi}}{2}, \quad \varphi_2 = \phi_2 - \frac{\tilde{\psi}}{2}. \quad (3.21)$$

After all these we end up with

$$\begin{aligned}
ds_{IIA}^2 = & R^2 \left\{ -\left(1 - \frac{r^2}{R^2}\right) dt^2 + \frac{dr^2}{R^2} + \frac{r^2}{R^2} d\Omega_2^2 + \frac{dy_3^2}{R^2} + \left(1 - \frac{y_3^2}{R^2}\right) d\tilde{\psi}^2 - \frac{y_1^2}{4R^2} d\tilde{\psi}^2 - \frac{y_2^2}{4R^2} d\tilde{\psi}^2 \right. \\
& \left. + \left(\frac{dy_1^2}{R^2} + \frac{y_1^2}{R^2} d\phi_1^2\right) + \left(\frac{dy_2^2}{R^2} + \frac{y_2^2}{R^2} d\phi_2^2\right) + \mathcal{O}(R^{-3}) \right\}.
\end{aligned} \tag{3.22}$$

As it was argued in [68, 73, 74], in order to describe the region connecting the giant magnon sector and the pp-wave limit of the string sigma model one must use a specific ansatz for the variables describing the null geodesic. We use the following ansatz for t and $\tilde{\psi}$

$$t = k_t R \sigma^+ + \frac{\tau}{R}, \quad \tilde{\psi} = k R \sigma^+ + \frac{\chi}{R}. \tag{3.23}$$

Substituting into the Lagrangian we find $\mathcal{L} = \partial_+ X^\mu \partial_- X^\nu G_{\mu\nu}$

$$\begin{aligned}
\mathcal{L} = & -r^2 k_t \partial_- \tau - \partial_+ \tau \partial_- \tau + \partial_+ r \partial_- r + r^2 (\partial_+ \theta_3 \partial_- \theta_3 + \sin^2 \theta_3 \partial_+ \varphi_3 \partial_- \varphi_3) \\
& + \sum_{i=1,3} \partial_+ y_i \partial_- y_i + y_1^2 \partial_+ \phi_1 \partial_- \phi_1 + \\
& + \partial_+ \chi \partial_- \chi - \left(\frac{y_2^2}{4} + \frac{y_1^2}{4} + y_3^2\right) k \partial_- \chi + y_2^2 \partial_+ \phi_2 \partial_- \phi_2.
\end{aligned}$$

The Virasoro constraints fix $k = k_t$, which, by making use of the residual conformal symmetry, is set to one. After obvious redefinitions of coordinates we obtain for the reduced Lagrangian

$$\begin{aligned}
\mathcal{L} = & -r^2 \partial_- \tau - \partial_+ \tau \partial_- \tau + \partial_+ \vec{r} \partial_- \vec{r} + \partial_+ y_3 \partial_- y_3 + \partial_+ \vec{y}_1 \partial_- \vec{y}_1 \\
& + \partial_+ \vec{y}_2 \partial_- \vec{y}_2 + \partial_+ \chi \partial_- \chi - \left(\frac{\vec{y}_2^2}{4} + \frac{\vec{y}_1^2}{4} + y_3^2\right) \partial_- \chi.
\end{aligned}$$

The equations of motion following from the reduced Lagrangian are

$$\begin{aligned}
0 &= \partial_+ \partial_- \tau + \frac{1}{2} \partial_- r^2, \\
0 &= \partial_+ \partial_- \chi - \left(\frac{\partial_- y_2^2}{8} + \frac{\partial_- y_1^2}{8} + \frac{\partial_- y_3^2}{2}\right), \\
0 &= \partial_+ \partial_- r_i + r_i \partial_- \tau, \quad i = 1, 2, 3 \\
0 &= \partial_+ \partial_- y_{a_j} + \frac{y_{a_j}}{4} \partial_- \chi, \quad a, j = 1, 2 \\
0 &= \partial_+ \partial_- y_3 + y_3 \partial_- \chi.
\end{aligned} \tag{3.24}$$

As in the case studied in [68, 73, 74], one obtains the following chiral conserved currents

$$j_+^\tau = \partial_+ \tau + \frac{1}{2} r^2, \tag{3.25}$$

$$j_+^\chi = \partial_+ \chi - \left(\frac{\vec{y}_2^2}{8} + \frac{\vec{y}_1^2}{8} + \frac{y_3^2}{2}\right). \tag{3.26}$$

Now let us examine the Virasoro constraints. We start with the energy momentum tensor of the original theory

$$T_{++} = G_{\mu\nu}\partial_+X^\mu\partial_+X^\nu, \quad T_{--} = G_{\mu\nu}\partial_-X^\mu\partial_-X^\nu. \quad (3.27)$$

We use the rescalings (3.18,3.23) and definitions (3.17) and (3.21) to obtain the corresponding expression for the reduced model

$$T_{++} \rightarrow -2\partial_+\tau - r^2 + 2\partial_+\chi - \left(\frac{y_2^2}{4} + \frac{y_1^2}{4} + y_3^2\right), \quad (3.28)$$

$$T_{--} \rightarrow \frac{1}{R^2}(-\partial_-\tau\partial_-\tau + \partial_-\chi\partial_-\chi + \partial_-\vec{r}\partial_-\vec{r} \quad (3.29)$$

$$+ \partial_-\vec{y}_3\partial_-\vec{y}_3 + \partial_-\vec{y}_1\partial_-\vec{y}_1 + \partial_-\vec{y}_2\partial_-\vec{y}_2). \quad (3.30)$$

We observe that $T_{++} \propto j_+^\chi - j_+^\tau$. The meaning of this observation is that the left-moving conformal symmetry is broken and is replaced by chiral symmetries which are generated by j_+^τ and j_+^χ . We note that at the same time the right-moving conformal symmetry remains unbroken. The right conformal symmetry is generated by the Virasoro generator having in our case the form

$$T_{--} := -\partial_-\tau\partial_-\tau + \partial_-\chi\partial_-\chi + \partial_-\vec{r}\partial_-\vec{r} + \partial_-\vec{y}_3\partial_-\vec{y}_3 + \partial_-\vec{y}_1\partial_-\vec{y}_1 + \partial_-\vec{y}_2\partial_-\vec{y}_2. \quad (3.31)$$

The next step is to obtain the gauge fixed action. This can be done in a standard way, namely, to fix the gauge, we impose the following conditions

$$T_{--} = 0, \quad (3.32)$$

$$j_+^\tau + j_+^\chi = \partial_+\tau + \partial_+\chi + \frac{1}{2}r^2 - \left(\frac{\vec{y}_2^2}{8} + \frac{\vec{y}_1^2}{8} + \frac{y_3^2}{2}\right) = 0. \quad (3.33)$$

To this end we choose coordinates

$$x^+ \equiv \sigma^+, \quad x^- \equiv 2(\tau + \chi), \quad (3.34)$$

so that the derivatives take the form

$$\partial_{\sigma^+} = \partial_{x^+} + \left[-r^2 + \left(\frac{\vec{y}_2^2}{4} + \frac{\vec{y}_1^2}{4} + y_3^2\right)\right]\partial_{x^-}, \quad \partial_{\sigma^-} = 2[\partial_{\sigma^-}(\tau + \chi)]\partial_{x^-}. \quad (3.35)$$

With this redefinitions we can rewrite the EOM for dynamical variables $\vec{r}, \vec{y}_1, \vec{y}_2$ and

y_3 . The latter take the form

$$0 = \partial_{x^-}(\partial_{x^+} + [-r^2 + (\frac{y_2^2}{4} + \frac{y_1^2}{4} + y_3^2)]\partial_{x^-})r_i + r_i(\frac{1}{4} + \partial_{x^-}\vec{r}\partial_{x^-}\vec{r} + \partial_{x^-}y_3\partial_{x^-}y_3 + \partial_{x^-}\vec{y}_1\partial_{x^-}\vec{y}_1 + \partial_{x^-}\vec{y}_2\partial_{x^-}\vec{y}_2), \quad i = 1, 2, 3 \quad (3.36)$$

$$0 = \partial_{x^-}(\partial_{x^+} + [-r^2 + (\frac{y_2^2}{4} + \frac{y_1^2}{4} + y_3^2)]\partial_{x^-})y_{a_j} + \frac{y_{a_j}}{4}(\frac{1}{4} - \partial_{x^-}\vec{r}\partial_{x^-}\vec{r} - \partial_{x^-}y_3\partial_{x^-}y_3 - \partial_{x^-}\vec{y}_1\partial_{x^-}\vec{y}_1 - \partial_{x^-}\vec{y}_2\partial_{x^-}\vec{y}_2), \quad a, j = 1, 2 \quad (3.37)$$

$$0 = \partial_{x^-}(\partial_{x^+} + [-r^2 + (\frac{y_2^2}{4} + \frac{y_1^2}{4} + y_3^2)]\partial_{x^-})y_3 + y_3(\frac{1}{4} - \partial_{x^-}\vec{r}\partial_{x^-}\vec{r} - \partial_{x^-}y_3\partial_{x^-}y_3 - \partial_{x^-}\vec{y}_1\partial_{x^-}\vec{y}_1 - \partial_{x^-}\vec{y}_2\partial_{x^-}\vec{y}_2), \quad (3.38)$$

where we used the gauge fixing condition

$$T_{--} = 0 \implies \frac{1}{2}\partial_{x^-}(\tau - \chi) = \partial_{x^-}\vec{r}\partial_{x^-}\vec{r} + \partial_{x^-}y_3\partial_{x^-}y_3 + \partial_{x^-}\vec{y}_1\partial_{x^-}\vec{y}_1 + \partial_{x^-}\vec{y}_2\partial_{x^-}\vec{y}_2. \quad (3.39)$$

These equations can be thought of as derived from the following effective Lagrangian

$$\begin{aligned} \mathcal{L}_{eff} = & \partial_+\vec{r}\partial_-\vec{r} + \partial_+\vec{y}_2\partial_-\vec{y}_2 + \partial_+\vec{y}_1\partial_-\vec{y}_1 + \partial_+y_3\partial_-\vec{y}_3 + \\ & + [-r^2 + (\frac{\vec{y}_2^2}{4} + \frac{\vec{y}_1^2}{4} + y_3^2)](\partial_-\vec{r}\partial_-\vec{r} + \partial_-\vec{y}_2\partial_-\vec{y}_2 + \partial_-\vec{y}_1\partial_-\vec{y}_1 + \partial_-\vec{y}_3\partial_-\vec{y}_3) + \\ & - \frac{r^2}{4} - \frac{\vec{y}_2^2}{16} - \frac{\vec{y}_1^2}{16} - \frac{y_3^2}{4}. \end{aligned} \quad (3.40)$$

For this effective Lagrangian we introduce $\sigma^\pm = \tau \pm \sigma$ and hence we obtain the following momenta (with $4\mathcal{L} = \mathcal{L}'$)

$$\begin{aligned} p_{r_i} &= \frac{\partial \mathcal{L}'}{\partial(\partial_\tau r_i)} = 2\partial_\tau r_i + [-r^2 + (\frac{y_2^2}{4} + \frac{y_1^2}{4} + y_3^2)](2\partial_\tau r_i - 2\partial_\sigma r_i), \\ p_{y_i} &= \frac{\partial \mathcal{L}'}{\partial(\partial_\tau y_i)} = 2\partial_\tau y_i + [-r^2 + (\frac{y_2^2}{4} + \frac{y_1^2}{4} + y_3^2)](2\partial_\tau y_i - 2\partial_\sigma y_i), \end{aligned}$$

with the canonical Poisson brackets

$$\{r^i(\sigma), p_{rj}(\sigma')\} = \delta_j^i \delta(\sigma - \sigma'), \quad \{y^i(\sigma), p_{yj}(\sigma')\} = \delta_j^i \delta(\sigma - \sigma'). \quad (3.41)$$

From here we find

$$\begin{aligned} \partial_\tau \vec{r} &= \frac{\frac{1}{2}\vec{p}_r + (y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4})\partial_\sigma \vec{r}}{1 + y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4}}, \\ \partial_\tau \vec{y} &= \frac{\frac{1}{2}\vec{p}_y + (y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4})\partial_\sigma \vec{y}}{1 + y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4}}. \end{aligned}$$

The Hamiltonian then reads

$$\mathcal{H} = \frac{1}{4} \frac{\vec{p}_r^2 + \vec{p}_y^2}{1 + y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4}} + \frac{(y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4})(\vec{p}_r \cdot \partial_\sigma \vec{r} + \vec{p}_y \cdot \partial_\sigma \vec{y})}{1 + y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4}} + \frac{(\partial_\sigma \vec{r})^2 + (\partial_\sigma \vec{y})^2}{1 + y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4}} + \frac{r^2}{4} + \frac{y_2^2}{16} + \frac{y_1^2}{16} + \frac{y_3^2}{4}. \quad (3.42)$$

or, equivalently

$$\mathcal{H} = \frac{\frac{\vec{p}_r^2 + \vec{p}_y^2}{4} + (y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4})(\vec{p}_r \cdot \partial_\sigma \vec{r} + \vec{p}_y \cdot \partial_\sigma \vec{y}) + (\partial_\sigma \vec{r})^2 + (\partial_\sigma \vec{y})^2}{1 + y_3^2 - r^2 + \frac{y_1^2 + y_2^2}{4}} + \frac{r^2}{4} + \frac{y_2^2}{16} + \frac{y_1^2}{16} + \frac{y_3^2}{4}. \quad (3.43)$$

One can use the Hamiltonian to further investigate the properties of the reduced model and this would be an interesting line to proceed.

4 Conclusions

As we discussed in the Introduction, string backgrounds other than $AdS_5 \times S^5$ are of great interest, especially because their gauge theory duals are closely related to QCD or are of particular importance for the prove of the holographic conjecture. Typically the string backgrounds define highly non-linear sigma models difficult to deal with. It is then natural to look for limiting procedures providing useful simplification but still allowing to extract non-trivial information for the models. One of the limits widely used in the last years is the so-called pp-wave limit. It is very attractive because the resulting theory is exactly solvable, describes non-trivial properties and one be quantized. The issues related to the validity of the correspondence are have been also addressed. Although this limit possesses very nice properties like integrability, some of the important properties and important structures of the original theory get lost. One possible way out is to weaken the pp-wave limit and to preserve many of the key features lost on the way. Such a procedure for the superstring theory on the $AdS_5 \times S^5$ background was proposed by Maldacena and Swanson. They called this procedure near-flat space limit.

In this paper we tested the near-flat space limit for the case of string theory on $AdS_4 \times \mathbb{CP}^3$. Being the string dual of a very non-trivial gauge theory, $\mathcal{N} = 6$ CS theory, superstring theory on this background is of particular importace. As in the $AdS_5 \times S^5$ case, it is conjectured that many of the key features of the original theory survive the near-flat space limit and can be studied in the reduced model. Our considerations show that the reduced model in this background is of the same type as the one obtained from the NFS limit of $AdS_5 \times S^5$ [68], Sasaki-Einstein [73] and Maldacena-Nunez [74]. The

bosonic part of the gauge fixed lagrangian of the reduced model contains again “mass terms”. The latter are composed differently from the case considered in [68, 73, 74]. In the Appendix we obtain the Lax pair for the reduced \mathbb{CP}^3 model which proves the integrability at the classical level. The similarity of the structure gives more arguments towards the conjectured universality of the near flat space limit of string theory on such backgrounds and their integrability. Therefore, the techniques developed in [69, 70, 71] will be also universal within the class of the NFS reduced models and one can apply them here.

There are several directions for further generalizations of this study. First of all, it would be interesting, following the close similarity of the structure of the Lagrangian obtained here and in [68], to find the explicit form of the Lax connection for the full string gauge fixed model and to perform an analysis along the lines of, say [72]. Another direction would be to apply these techniques to the study of string sigma models in so called beta-deformed backgrounds introduced first in [64], assuming that the NFS limit interpolates between pp-wave and magnon sectors of the theory [65, 66]. In this paper we considered only the bosonic part of the theory, which is analogous to the $AdS_5 \times S^5$ case in many ways. The near-flat space limit of the fermionic part is much more difficult due to the presence of non-trivial RR fields and it would be very important to extend the consideration to that case. Although there is some progress in our understanding of the role of the integrable structures in AdS/CFT, or more generally string/gauge theory duality, much has still to be done.

Acknowledgments

We thank C. Mayrhofer for useful discussion at an early stage of the project. R.R. acknowledges the warm hospitality and stimulating atmosphere at the ITP of TU Wien and the Erwin Schrödinger International Institute for Mathematical Physics (ESI). M.S. acknowledges also ESI for the Junior Research Fellowship during which this studies were initiated. This work was supported in part by the Austrian Research Fund FWF grant # P19051-N16. R.R. is partly supported by the Bulgarian Research Fund NSF grant VU-F-201/06.

Appendix: Remark on the Lax pair of the reduced \mathbb{CP}^n sigma model

The construction of the Lax pair for the reduced $O(n)$ model was dicussed in [68]. One way to do that in our case is to take the near-flat space limit of the connection in the \mathbb{CP}^n sigma model. We will pursue this issue elsewhere, but in this appendix we will take advantage of what has been done for the $O(n+2)$ case considering \mathbb{CP}^n as $\mathbb{CP}^n \simeq S^{n+1}/U(1)$. Here we will restrict the considerations to the case of cp sigma model as embedded into a seven sphere. We choose the generator J^{12} which mixes (1,2) plane with four of the directions,

say $J^{\pm i}$, $i = 1, \dots, 4$. In addiditon, one of the directions, say y_5^6 , is associated with a generator $J^{\pm 5}$ commuting with all $J^{\pm i}$ exept J^{12} . To construct the Lax connection we will use the following commutation relations:

$$\begin{aligned} [J^{12}, J^{\pm i}] &= \pm J^{\pm i}, & [J^{12}, J^{\pm 5}] &= \pm J^{\pm 5} \\ [J^{+i}, J^{-j}] &= -\delta_{ij} J^{12} - J^{ij}, & [J^{-i}, J^{+j}] &= \delta_{ij} J^{12} - J^{ij} \\ [J^{+5}, J^{-5}] &= -J^{12}, & [J^{\pm}, J^{\mp 5}] &= 0. \end{aligned} \quad (\text{A.1})$$

Now we can construct the flat connection with spectral parameter ω

$$d + \mathcal{A} = d + A_+ d\sigma^+ + A_- d\sigma^-, \quad d^2 + A \wedge A = 0.$$

$$A_+ = \frac{i}{\sqrt{2}} \left[-e^{-i\sigma^+\omega} \frac{y^i}{4} J^{+i} + e^{i\sigma^+\omega} \frac{y^i}{4} J^{-i} - e^{-i4\sigma^+\omega} y^5 J^{+5} + e^{i4\sigma^+\omega} y^5 J^{-5} \right] \quad (\text{A.2})$$

$$\begin{aligned} A_- = \frac{1}{4\omega} & \left[i\partial_- \chi J^{12} + \frac{1}{\sqrt{2}} e^{-i\sigma^+\omega} \partial_- y^i J^{+i} + \frac{1}{\sqrt{2}} e^{-i4\sigma^+\omega} \partial_- y^5 J^{+5} \right. \\ & \left. + \frac{1}{\sqrt{2}} e^{i\sigma^+\omega} \partial_- y^i J^{-i} + \frac{1}{\sqrt{2}} e^{i4\sigma^+\omega} \partial_- y^5 J^{-5} \right] \end{aligned} \quad (\text{A.3})$$

A simple check shows that the equations of motion are equivalent to the flatness of the connection. An interesting feature of the model is that one can use the new variables introduced for the gauge fixed effective Lagrangian to obtain its Lax connection. Indeed, having in mind that

$$d\sigma^+ = dx^+, \quad 2 d\sigma^-\left(\frac{1}{4} + \partial_{\sigma^-} \chi\right) = dx^- - \left(\frac{y_i^2}{4} + y_5^2\right) dx^+ \quad (\text{A.4})$$

and using the constraints we find the new connection in terms of the transverse degrees of freedom⁷

$$d + A_+ d\sigma^+ + A_- d\sigma^- \rightarrow d + A_+ dx^+ + \tilde{A} \left(dx^- - \left(\frac{y_i^2}{4} + y_5^2\right) dx^+ \right), \quad (\text{A.5})$$

where

$$\begin{aligned} \tilde{A} := \frac{1}{4\omega} & \left[i \left(\frac{1}{4} - (\partial_x y_5)^2 - (\partial_x y_i)^2 \right) J^{12} + \frac{1}{\sqrt{2}} e^{-i\sigma^+\omega} \partial_- y^i J^{+i} \right. \\ & \left. + \frac{1}{\sqrt{2}} e^{-i4\sigma^+\omega} \partial_- y^5 J^{+5} + \frac{1}{\sqrt{2}} e^{i\sigma^+\omega} \partial_- y^i J^{-i} + \frac{1}{\sqrt{2}} e^{i4\sigma^+\omega} \partial_- y^5 J^{-5} \right]. \end{aligned} \quad (\text{A.6})$$

A simple gauge transformation with $g = \exp(-i\frac{x^-}{4\omega} J^{12})$ removes the constant part and one ends up with

$$d + A'_+ dx^+ + \tilde{A}' \left(dx^- - \left(\frac{y_i^2}{4} + y_5^2\right) dx^+ \right), \quad (\text{A.7})$$

⁶Here we rename y_{a_i} as y_i and y_3 becomes y_5 .

⁷In the case under consideration ($j_\tau \equiv 0$), after adding a trivial time direction, the Virasoro constraints are $j_\chi = 0$ and $T_{--} = \frac{1}{16}$.

where

$$A'_+ = \frac{i}{\sqrt{2}} \left[-e^{-ix^+\omega+i\frac{x^-}{4\omega}} \frac{y^i}{4} J^{+i} - e^{-i4x^+\omega+i\frac{x^-}{4\omega}} y^5 J^{+5} + e^{ix^+\omega-i\frac{x^-}{4\omega}} \frac{y^i}{4} J^{-i} + e^{i4x^+\omega-i\frac{x^-}{4\omega}} y^5 J^{-5} \right], \quad (\text{A.8})$$

$$\begin{aligned} \tilde{A}' = \frac{1}{4\omega} \left[-i \left((\partial_{x^-} y_5)^2 + (\partial_{x^-} y_i)^2 \right) J^{12} + \frac{1}{\sqrt{2}} e^{-ix^+\omega+i\frac{x^-}{4\omega}} \partial_- y^i J^{+i} \right. \\ \left. + \frac{1}{\sqrt{2}} e^{-i4x^+\omega+i\frac{x^-}{4\omega}} \partial_- y^5 J^{+5} + \frac{1}{\sqrt{2}} e^{ix^+\omega-i\frac{x^-}{4\omega}} \partial_- y^i J^{-i} \right. \\ \left. + \frac{1}{\sqrt{2}} e^{i4x^+\omega-i\frac{x^-}{4\omega}} \partial_- y^5 J^{-5} \right]. \quad (\text{A.9}) \end{aligned}$$

This form of the Lax connection is not unique and maybe not be the most useful one, but we will further study this issue in another paper.

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