

The Constant Mean Curvature Slices of Asymptotically Flat Spherical Spacetimes

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The constant mean curvature slices of asymptotically flat spherical spacetimes

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We investigate the formation of trapped surfaces in asymptotically flat spherical spacetimes, using constant mean curvature slicing.

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I. INTRODUCTION

In the analysis of General Relativity as a Hamiltonian system [1] one chooses a time function and considers the foliation of the spacetime by the slices of constant time. Two natural geometrical quantities arise on such three slices. One is the intrinsic three metric, usually g_{ab} , and the other is the extrinsic curvature K^{ab} , the time derivative of g_{ab} . These are not independent: they are related by the constraints

$$\begin{aligned}\mathcal{R}^{(3)} - K^{ab}K_{ab} + (\text{tr}K)^2 &= 16\pi\rho \\ \nabla_a K^{ab} - g^{ab}\nabla_a \text{tr}K &= -8\pi j^a\end{aligned}$$

where $\mathcal{R}^{(3)}$ is the three scalar curvature, ρ is the energy density and j^a is the current density of the sources.

It is often useful to specify the foliation, and thus the time, by placing a condition on the extrinsic curvature. The most common choice in asymptotically flat spacetimes is the maximal slicing condition, $\text{tr}K = 0$. In cosmologies, the favoured slicing is the constant mean curvature (CMC) foliation with $\text{tr}K = \text{constant}$.

Such CMC slices have also been used in an asymptotically flat context [3]. They are everywhere spacelike, but at infinity they approach null infinity. Thus they are very useful in investigating the relationship between spatial and null infinity. A standard model of CMC hypersurfaces are the mass hyperboloids in Minkowski space [4].

In this paper we investigate a very special class of CMC, those which are spherically symmetric. Because of the absence of gravitational radiation, spherical spacetimes are particularly simple, yet realistic, models of general solutions to the Einstein equations.

If we have a spherically symmetric three surface, the intrinsic metric can be written as

$$ds^2 = dl^2 + R^2 d\Omega^2$$

where l is the proper distance in the radial direction and R is the Schwarzschild or areal radius. The geometry is encoded into the relationship between R and l and a useful object to use is the mean curvature of the spherical two surfaces, given by

$$p = \frac{2}{R} \frac{dR}{dl}$$

The constraints now can be written as

$$\begin{aligned}\partial_l p &= -8\pi\rho - \frac{3}{4}(K_r^r)^2 - \frac{3}{4}p^2 + \frac{1}{R^2} + \frac{1}{2}\text{tr}K K_r^r + \frac{1}{4}(\text{tr}K)^2 \\ \partial_l(K_r^r - \text{tr}K) &= -\frac{3}{2}p K_r^r + \frac{1}{2}p \text{tr}K - 8\pi j_l\end{aligned}\tag{1}$$

It has been recently shown [2] that the constraints of General Relativity in the spherically symmetric case can be expressed very simply by using the null expansions as subsidiary variables and the constraints can be expressed as a system of quasilinear first order O.D.E.'s. We apply this new formulation in the CMC case to investigate a number of interesting problems.

Much work has been carried out in recent years on how concentrations of matter may gravitationally collapse [2], [5], [6]. One of the motivations for repeating the calculation in various slicings of asymptotically flat spacetimes is

due to the fact that no covariant formulation of the question has been found. This article, in which we derive both necessary and sufficient conditions for the formation of trapped surfaces, can be regarded as an attempt to see how the criteria we obtain are more or less independent of the details of the slicing used. Let us emphasise that the appearance of trapped surfaces indicates that irreversible gravitational collapse has commenced.

We derive, for the sake of completeness, the general line element for the Reissner-Nordström spacetime in the slicing by constant mean curvature hypersurfaces. That is a generalization of the corresponding solution in the maximal slicing [7].

II. CMC HYPERSURFACES IN MINKOWSKI SPACE.

Let us consider spherically symmetric CMC hypersurfaces in Minkowski space. We write the four metric as

$$ds^2 = -d\tau^2 + \tau^2[dr^2 + \sinh^2 r d\Omega^2], \quad (2)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the standard round two metric. The scalar curvature $\mathcal{R}^{(3)}$ of the three space defined by $\tau = \text{constant}$ is $\mathcal{R}^{(3)} = -\frac{6}{\tau^2}$.

The extrinsic curvature of this slice is pure trace, $K_{ab} \equiv \frac{1}{2}\partial_\tau g_{ab} = \frac{1}{\tau}g_{ab}$ which implies $\text{tr}K = K = \frac{3}{\tau}$. The proper radial distance l along the slice is related to the coordinate radius r by $\tau dr = dl$ which yields

$$r = \frac{l}{\tau} = \frac{Kl}{3}. \quad (3)$$

The Schwarzschild radius R is given by

$$\begin{aligned} R &= \tau \sinh r \\ &= \frac{3}{K} \sinh \frac{Kl}{3} \end{aligned} \quad (4)$$

and its derivative reads

$$R' = \frac{dR}{dl} = \cosh \frac{Kl}{3}. \quad (5)$$

The primary objects we deal with are the optical scalars, the expansion θ of the outgoing null rays and the convergence θ' of the ingoing light rays. These are given by

$$\begin{aligned} R\theta &= 2R' + \frac{2}{3}KR \\ &= 2 \cosh \frac{Kl}{3} + 2 \sinh \frac{Kl}{3} \\ &= 2e^{\frac{Kl}{3}} \end{aligned} \quad (6)$$

and similarly

$$R\theta' = 2e^{-\frac{Kl}{3}}, \quad (7)$$

therefore the product of $R\theta$ and $R\theta'$ remains constant, $R\theta R\theta' = 4$. We also have

$$R\theta = \frac{4RK}{3} + 2e^{-\frac{Kl}{3}}. \quad (8)$$

Thus at the origin we have

$$R\theta' = R\theta = 2 \quad (9)$$

and at infinity one of the scalars is divergent while the other vanishes

$$\begin{aligned} R\theta &\rightarrow \frac{4}{3}KR, \\ R\theta' &\rightarrow 0. \end{aligned} \quad (10)$$

An alternative form of the metric (2) is that in terms of the Schwarzschild radius

$$ds^2 = \frac{-\tau^2}{\tau^2 + R^2} d\tau^2 - \frac{2R\tau}{\tau^2 + R^2} dR d\tau + \frac{\tau^2}{\tau^2 + R^2} dR^2 + R^2 d\Omega^2. \quad (11)$$

The two divergences of null rays are given by

$$\omega_+ = R\theta = Rp - RK_r^r + RK, \quad (12)$$

$$\omega_- = R\theta' = Rp + RK_r^r - RK, \quad (13)$$

where

$$p = \frac{2}{R} \frac{dR}{dl}, \quad (14)$$

is the mean curvature of a surface of constant R in the slice where R is the Schwarzschild radius and l is the proper distance. The constraints now can be written as

$$\partial_l(\omega_+) = -8\pi R(\rho - j) - \frac{1}{4R}(2\omega_+^2 - \omega_+\omega_- - 4 - 4\omega_+RK), \quad (15)$$

$$\partial_l(\omega_-) = -8\pi R(\rho + j) - \frac{1}{4R}(2\omega_-^2 - \omega_+\omega_- - 4 + 4\omega_-RK), \quad (16)$$

$$\partial_l R = R' = \frac{1}{4}(\omega_+ + \omega_-). \quad (17)$$

We assume we are given ρ (the energy density), $j = \vec{j} \cdot \hat{n}$ (the current density), where \hat{n} is the outgoing radial normal and RK as functions of l and then solve the triplet of O.D.E's (15), (16) and (17) for (R, ω_+, ω_-) . The only conditions we assume are regularity at the origin ($R = 0, \omega_+ = \omega_- = 2$), asymptotic flatness and that the sources satisfy the dominant energy condition, $\rho \geq |j|$.

Combining Eq. (15) and (16) we can write

$$\partial_l(\omega_-\omega_+) = -8\pi R(\rho(\omega_+ + \omega_-) + j(\omega_+ - \omega_-)) - \frac{1}{4R}(\omega_-\omega_+ - 4)(\omega_+ + \omega_-) \quad (18)$$

and by regularity and asymptotic flatness we have that $\lim_{R \rightarrow 0} \omega_-\omega_+ = 4$ also $\lim_{R \rightarrow \infty} \omega_-\omega_+ = 4$.

Suppose that $\omega_-\omega_+ > 4$, if both are positive we have that the right hand side of Eq. (18) is strictly negative and if both are negative the right hand side is positive. Thus we have

$$\omega_-\omega_+ \leq 4. \quad (19)$$

IV. CONSTRAINTS ON ASYMPTOTICALLY FLAT CMC HYPERSURFACES.

When we consider asymptotically flat CMC hypersurfaces, it is useful to use variables that are finite at the origin and infinity. From the Minkowski analysis, it is clear that we need as boundary conditions that $\omega_+ \rightarrow 2e^{Kl/3}$ and $\omega_- \rightarrow 2e^{-Kl/3}$. Thus the natural variables to use are $A = \omega_+e^{-Kl/3}$ and $B = \omega_-e^{Kl/3}$. Using these the equations (15) and (16) become

$$\partial_l A = -8\pi R e^{-Kl/3}(\rho - j) - \frac{e^{Kl/3}}{4R}[2A^2 - \frac{8}{3}KR e^{-Kl/3}A - AB e^{-2Kl/3} - 4e^{-2Kl/3}], \quad (20)$$

$$\partial_l B = -8\pi R e^{Kl/3}(\rho + j) - \frac{e^{-Kl/3}}{4R}[2B^2 + \frac{8}{3}KR e^{Kl/3}B - AB e^{2Kl/3} - 4e^{2Kl/3}]. \quad (21)$$

We know, from the previous Section (inequality (19)) that $AB = \omega_+\omega_-$ is bounded above by 4. Let us write the expression which does not depend on the sources in Eq. (20) as

$$-\frac{e^{Kl/3}}{2R} \left[A^2 - \frac{4}{3} KR e^{-Kl/3} A - 4e^{-2Kl/3} \right] - \frac{e^{-Kl/3}}{4R} [4 - AB]. \quad (22)$$

Consider

$$\begin{aligned} \alpha &= 2\sqrt{\frac{K^2 R^2}{9} + 1} + \frac{2}{3} KR, \\ \beta &= 2\sqrt{\frac{K^2 R^2}{9} + 1} - \frac{2}{3} KR, \end{aligned} \quad (23)$$

these are essentially the roots of the quadratic equation in A in (22). If A lies outside the range

$$[-\beta e^{-Kl/3}, \alpha e^{-Kl/3}], \quad (24)$$

then every term on the right hand side of (20) is negative and therefore $\partial_l A < 0$.

It is clear that $\alpha > \frac{4}{3}KR$ and also $\alpha \geq 2$, and these are the limiting values of A at infinity and at the origin respectively. At any maximum of A we have that $\partial_l A = 0$ which implies that $A \leq \alpha e^{-Kl/3}$ at that point. We can show that this is a global upper bound, or equivalently

$$\omega_+ \leq 2\sqrt{\frac{K^2 R^2}{9} + 1} + \frac{2}{3} KR. \quad (25)$$

Consider the function

$$f(l) = 2\sqrt{\frac{K^2 R^2}{9} + 1} + \frac{2}{3} KR - \omega_+, \quad (26)$$

$f(l)$ is zero at the origin and by asymptotic flatness it is positive at infinity. We can show that it is always positive. Let us assume, to the contrary, that $f(l)$ is negative somewhere. This means that there must exist a negative minimum, i. e., a point where

(i) $\omega_+ > 2\sqrt{\frac{K^2 R^2}{9} + 1} + \frac{2}{3} KR$ and

(ii) $f'(l) = 0$.

However, we can show that if (i) holds then $f'(l) > 0$. Using Eq. (15) and (17), we can calculate

$$\begin{aligned} f'(l) &= \frac{1}{4R} \left[\frac{KR \alpha(\omega_+ + \omega_-)}{3 \sqrt{\frac{K^2 R^2}{9} + 1}} \right. \\ &\quad \left. + 2\omega_+^2 - \omega_+ \omega_- - 4\omega_+ KR - 4 \right] + 8\pi R(\rho - j). \end{aligned} \quad (27)$$

The coefficient of ω_- is

$$\frac{1}{4R} \left[-\omega_+ + \frac{KR\alpha}{3\sqrt{\frac{K^2 R^2}{9} + 1}} \right]. \quad (28)$$

This is obviously negative since $\omega_+ \geq \alpha$. Therefore we minimize $f'(l)$ by choosing the maximum value of ω_- . There is a condition, Eq. (19), that constrains the product of both optical scalars, $\omega_+ \omega_- \leq 4$. Therefore the maximum of ω_- is $\frac{4}{\omega_+}$.

Now consider the function

$$\begin{aligned} \tilde{f}(l) &= \frac{1}{4R} \left[\frac{KR \alpha(\omega_+ + \frac{4}{\omega_+})}{3 \sqrt{\frac{K^2 R^2}{9} + 1}} \right. \\ &\quad \left. + 2\omega_+^2 - 8 - 4\omega_+ KR \right]. \end{aligned} \quad (29)$$

The derivative of this function with respect to ω_+ is positive (assuming $\omega_+ \geq \alpha$) and therefore its minimum is achieved at the minimum of ω_+ , i. e., at $\omega_+ = \alpha$.

Hence

$$f'(l) \geq \frac{1}{4R} \left[\frac{KR}{3} \frac{\alpha^2 + 4}{\sqrt{\frac{K^2 R^2}{9} + 1}} + 2\alpha^2 - 8 - 4\alpha KR \right] + 8\pi R(\rho - j). \quad (30)$$

It is easy to see that this expression is positive. Hence we get a contradiction. We could show, in a similar vein, the existence of the upper bound on B as well of the lower bounds. Therefore we have the following global bounds on the optical scalars

$$-\beta \leq \omega_+ \leq \alpha, \quad (31)$$

$$-\alpha \leq \omega_- \leq \beta. \quad (32)$$

Those bounds are valid for both signs, positive and negative, of the trace of the extrinsic curvature K .

V. SUFFICIENT CONDITION FOR TRAPPED SURFACES IN CMC HYPERSURFACES

We can use the formulae (15) and (17) in Section II to derive

$$\partial_l(\omega_+ R) = -8\pi R^2(\rho - j) + 1 + \frac{1}{4}(2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK) \quad (33)$$

and we have the bounds on ω_+ and ω_- from the previous section. It is easy to show that the maximum value of $(2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK)$ occurs when i) $\omega_+ = \alpha$ and $\omega_- = \beta$ if $K > 0$, and ii) $\omega_+ = -\beta$ and $\omega_- = -\alpha$. Hence we get in both cases

$$2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK \leq 4 + \frac{16}{9}K^2 R^2 + \frac{16}{3}|K|R\sqrt{\frac{K^2 R^2}{9} + 1}. \quad (34)$$

We can complete the square in the square root to finally get

$$\frac{1}{4}(2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK) \leq 1 + \frac{8}{9}K^2 R^2 + \frac{4}{3}|K|R, \quad (35)$$

thus we get

$$\partial_l(\omega_+ R) \leq -8\pi R^2(\rho - j) + 2 + \frac{8}{9}K^2 R^2 + \frac{4}{3}|K|R. \quad (36)$$

If we integrate this equation out to some surface S we get

$$\omega_+ R|_S \leq -2(M - P) + 2L + \frac{2K^2}{9\pi}V + \frac{4}{3}|K| \int_0^l R dl, \quad (37)$$

where $M = \int 4\pi R^2 \rho dl$ is the total amount of matter inside S , $P = \int 4\pi R^2 j dl$ is the total outward radial momentum of the matter, L is the proper radius and V is the volume inside S (notice that $4\pi R^2 \rho dl = dV$ is the proper volume element). Therefore we have that if

$$(M - P)(S) \geq L + \frac{K^2}{9\pi}V + \frac{2}{3}|K| \int R dl,$$

we must have that $\omega_+ R|_S$ is negative and so the surface at S is a future trapped surface. We can estimate $\int R dl$ as follows:

$$\int R dl \leq \left[\int R^2 dl \right]^{\frac{1}{2}} \left[\int dl \right]^{\frac{1}{2}} = \left(\frac{VL}{4\pi} \right)^{\frac{1}{2}}. \quad (38)$$

Therefore a sufficient condition for the appearance of a future trapped surface on a slice with constant trace of the extrinsic curvature is that

$$(M - P)(S) \geq L + \frac{K^2}{9\pi}V + \frac{|K|}{3} \left(\frac{VL}{\pi} \right)^{\frac{1}{2}}. \quad (39)$$

Let us return to the equality (33) we derived in Section V and again integrate it out to some surface S

$$\omega_+ R|_S = -2(M - P) + L(S) + \frac{1}{4} \int_0^L (2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK) dl, \quad (40)$$

but now we wish to minimize the integral rather than maximize it. We assume that no future trapped surface exists within S , i.e. $\omega_+ \geq 0$. We also assume that no past trapped surface exists in S . Not only that, but that ω_- is strongly bounded away from zero, i.e. $\omega_- \geq C > 0$; that means that all radially ingoing null rays are converging.

In other words we want to minimize the function $f(\omega_+, \omega_-) = 2\omega_+\omega_- - \omega_+^2 + 4\omega_+ RK$ in the region given by $0 \leq \omega_+ \leq \alpha$ and $C \leq \omega_- \leq \beta$. A simple calculation gives $f_{\min} = \min(f(\alpha, C), 0)$.

Because α is a function of R , we need to study the function

$$\begin{aligned} \tilde{f}(R) &= f(\alpha(R), C) \\ &= 2\alpha C - \alpha^2 + 4\alpha RK \end{aligned}$$

in order to find f_{\min} .

We will consider separately two cases, with the positive and negative trace of the extrinsic curvature.

i) Let $K > 0$. By inspection we obtain that this function is an increasing function in the variable R , therefore

$$\min(\tilde{f}) = f(\alpha(0), C) = 4C - 4.$$

Clearly when $C \geq 1$ $f_{\min} = 0$ otherwise $f_{\min} \geq 4C - 4$. Inserting this into Eq. 40 we obtain two estimates

$$\omega_+ R|_S \geq \begin{cases} -2(M - P)(S) + L(S) & \text{for } C \geq 1 \\ -2(M - P)(S) + CL(S) & \text{for } C \leq 1, \end{cases} \quad (41)$$

that is, since $\theta(S) = 0$,

$$M(S) - P(S) \geq \begin{cases} \frac{L}{2} & \text{for } C \geq 1 \\ \frac{CL}{2} & \text{for } C \leq 1 \end{cases}$$

is the necessary condition for the existence of a trapped surface.

The above result obviously applies to maximal slices. In connection with that, two of us have to admit that Theorem 2 in [2] should be stated as above; the actual statement of [2] that the necessary condition for future trapped surfaces is $M(S) - P(S) \geq \frac{CL}{2}$ can be wrong.

ii) Let $K < 0$. In this case one easily estimates $\tilde{f}(R)$ from below by

$$\begin{cases} 4(C - 1) + \frac{4KR(4+C)}{3} & \text{for } C \leq 1 \\ \frac{4KR(4+C)}{3} & \text{for } C \geq 1 \end{cases}$$

That leads to a necessary condition for S to be trapped

$$M(S) - P(S) + \frac{(4+C)|K|}{6} \int_0^{L(S)} dl R(l) \geq \begin{cases} \frac{L}{2} & \text{for } C \geq 1 \\ \frac{CL}{2} & \text{for } C \leq 1 \end{cases}$$

Using relations $dl = \frac{2}{Rp} dR$ and $pR = \frac{1}{2}(\omega_+ + \omega_-) \geq \frac{C}{2}$ one obtains

$$\int_0^{L(S)} dl R(l) \leq \frac{S}{2\pi C}$$

and the necessary condition

$$M(S) - P(S) + \frac{(4+C)|K|S}{12\pi C} \geq \begin{cases} \frac{L}{2} & \text{for } C \geq 1 \\ \frac{CL}{2} & \text{for } C \leq 1 \end{cases}$$

A similar necessary condition, under a somewhat stronger condition, has been obtained by Zannias ([6]). Thus negative values of the trace of the extrinsic curvature can help to form trapped surfaces.

Let us recall that yet another necessary result has been derived in [8], where the following equation has been proven

$$\begin{aligned} \frac{R^3}{8}\theta(S)\theta'(S) + m - \frac{S}{16\pi} &= \\ \pi \int_r^\infty \sqrt{a}R^3 [\rho_0(\theta + \theta') + j(\theta - \theta')] &. \end{aligned} \tag{42}$$

(42) has been derived on maximal slices, but it holds true on any slicing, assuming a quick enough falloff of matter fields. Under the dominant energy condition one concludes that an outermost trapped surface S (future or past) must have a radial radius R not greater than $2m$. This conclusion is slicing-independent.

VII. REISSNER - NORDSTRÖM GEOMETRY IN CMC FOLIATIONS

In this Section we will present an explicit line element for electrovacuum in constant mean curvature foliations. The most general spherically symmetric line element can be put

$$ds^2 = -N^2 dt^2 + a dr^2 + R^2 d\Omega^2. \tag{43}$$

We assume that the trace of the extrinsic curvature

$$K = \frac{\partial_t(\sqrt{a}R^2)}{2N\sqrt{a}R^2} \tag{44}$$

is constant on a particular slice and, moreover, is time independent. The three nonzero components of the extrinsic curvature are

$$K_r^r = \frac{\partial_t(\sqrt{a})}{2N\sqrt{a}}, \quad K_\phi^\phi = K_\theta^\theta = \frac{\partial_t R}{NR} = \frac{1}{2}(K - K_r^r). \tag{45}$$

The spherically symmetric Einstein equations consist of constraint equations (1), the evolution equation

$$\partial_t(K_r^r - trK) = -\frac{p^3 R^2}{2N} \frac{\partial_r}{\sqrt{a}} \left(\frac{N}{pR}\right)^2 + \frac{3N}{2}(K_r^r)^2 + 8\pi(T_r^r + \rho) + \frac{N}{2}K^2 - 2NK_r^r \tag{46}$$

and the lapse equation

$$\Delta^{(3)}N = N\left[\frac{3}{2}(K_r^r)^2 + \frac{1}{2}K^2 - KK_r^r + 4\pi(T_i^i + \rho)\right]. \tag{47}$$

In electrovacuum we have $\frac{q^2}{8\pi R^4} = \rho = T_i^i = -T_r^r$, where q is the electric charge. The mean curvature p of nested two spheres and the extrinsic curvatures are easily found from the constraints (1) and they read

$$\begin{aligned} pR &= 2\sqrt{1 + \frac{C}{R} + \frac{q^2}{R^2} + \left(\frac{KR}{3} + \frac{C_1}{2R^2}\right)^2}, \\ K_r^r &= \frac{K}{3} + \frac{C_1}{R^3}. \end{aligned} \tag{48}$$

The lapse equation becomes now

$$\Delta^{(3)}N = N\left[\frac{3C_1^2}{2R^6} + \frac{q^2}{R^4} + \frac{K^2}{3}\right] \quad (49)$$

and one easily finds out that it is solved by

$$N = \gamma \frac{pR}{2}, \quad (50)$$

where γ is given by

$$\gamma(r, t) = 1 + C_2 \int_{R(r)}^{\infty} d\tilde{R} \frac{1}{(\tilde{R})^2 (p\tilde{R})^3}. \quad (51)$$

Inserting the whole information into (46) and using the relation $\frac{\partial r}{\sqrt{a}} = \frac{Rp}{2} \partial_R$ one obtains that the constant C_2 depends on the rate of change of the radial - radial component of the extrinsic curvature,

$$C_2 = 4\partial_t C_1. \quad (52)$$

The change of the coordinate variable r into the areal radius R transforms the line element (43) into

$$\begin{aligned} ds^2 &= dt^2 \left[-N^2 + \frac{\gamma^2}{4} \left(\frac{C_1}{R^2} - \frac{2KR}{3} \right)^2 \right] + 2\gamma \frac{C_1 - \frac{2K}{3}}{p} dt dR + \frac{4}{(pR)^2} dR^2 + R^2 d\Omega^2, \\ &= -\frac{\gamma^2 \omega_+ \omega_-}{4} dt^2 + 2\gamma \frac{\omega_- - \omega_+}{\omega_- + \omega_+} dR dt + \frac{16}{(\omega_- + \omega_+)^2} dR^2 + R^2 d\Omega^2, \end{aligned} \quad (53)$$

with N and p defined above. Let us point out that the parameter C that appears in the expression for p may be identified with $-2m_B$, where m_B is the Bondi mass.

VIII. "CMC SURFACES AVOID SINGULARITIES"

In the article [2] an argument was advanced as to how foliations with bounded trace of the extrinsic curvature might avoid singularities. In this section we wish to produce a different (and sharper) argument to the same end. This argument works for essentially any slicing, but we present it here specifically for CMC slices. Let us consider a model of a collapsing system where the support of the matter becomes ever smaller as the collapse continues so that eventually the star is confined to a region much smaller than that enclosed by the apparent horizon. If the star were to be compressed inside a boundary which satisfies $R \ll m$, where m is the conserved ADM mass of the star, before any singularity appears then regular CMC foliations will be excluded from this part of the spacetime.

From the inequalities (31) and (32) we can show that on any regular CMC slice

$$\begin{aligned} \omega_+ \omega_- &\geq -4 \left(\frac{2K^2 R^2}{9} + 1 + \frac{2|K|R}{3} \sqrt{\frac{1 + K^2 R^2}{9}} \right) \\ &\geq -4 \left(\frac{2|K|R}{3} + 1 \right)^2. \end{aligned} \quad (54)$$

However, both the Schwarzschild radius R , and the product $\omega_+ \omega_-$ are four-scalars and we have

$$\omega_+ \omega_- = 4 \left(1 - \frac{2m_H}{R} \right). \quad (55)$$

where m_H is the so-called Hawking mass, which equals the constant ADM mass outside the support of the matter. Inequality (54) and equality (55) can be combined to give the following inequality

$$m_H \leq \frac{2K^2 R^3}{9} + \frac{2|K|R}{3} + R. \quad (56)$$

This means that for a fixed positive m_H we have a lower bound for R . Let us assume that we are considering a spherical collapse and viewing it using a CMC foliation. Let us also assume that during this collapse a two-surface appears which violates this inequality (56) before the CMC slices become singular. This means that the CMC foliation

cannot progress past this point, the lapse collapses. Since the lapse equation is elliptical, not only does the lapse go to zero at this point, it becomes small on the whole interior and the CMC slicing freezes.

This means that if we wish to find a solution where CMC slices run right up to the singularity we cannot allow a large accumulation of matter near the center before the singularity appears. A possible way for this to happen is that if the collapse were such that in addition to the infall of matter, one also had an explosion that pushed significant amounts of the star outwards, away from the horizon.

This bound is valid for solutions which have the spatial topology $R^1 \times S^2$ as in the extended Schwarzschild solution as well as topology R^3 . Maximal slicing can be viewed as a special case of CMC slicing and it was observed many years ago (see [7]) that the regular maximal slicing of the Schwarzschild solution saturates at $R = 3m/2$, in agreement with the bound stated above.

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