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Exact Categories

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EXACT CATEGORIES

THEO BÜHLER

ABSTRACT. We survey the basics of homological algebra in exact categories in the sense of Quillen. All diagram lemmas are proved directly from the axioms, notably the five lemma, the 3×3-lemma and the snake lemma. We briefly discuss exact functors, idempotent completion and weak idempotent completeness. We then show that it is possible to construct the derived category of an exact category without any embedding into abelian categories. The construction of classical derived functors with values in an abelian category painlessly translates to exact categories, i.e., we give proofs of the comparison theorem for projective resolutions and the horseshoe lemma. After discussing some examples we elaborate on Thomason's proof of the Gabriel-Quillen embedding theorem in an appendix.

CONTENTS

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1. Introduction

There are several notions of exact categories. On the one hand, there is the notion in the context of additive categories commonly attributed to Quillen [32] with which the present article is concerned; on the other hand, there is the non-additive notion due to Barr [2] to mention but the two most prominent ones. While Barr's definition is intrinsic and an additive category is exact in his sense if and only if it is abelian, Quillen's definition is extrinsic in that one has to specify a distinguished class of short exact sequences (an exact structure) in order to obtain an exact category.

From now on we shall only deal with additive categories, so functors are tacitly assumed to be additive. On every additive category $\mathscr{A}$ the class of all split exact sequences provides the smallest exact structure, i.e., every other exact structure must contain it. In general, an exact structure consists of kernel-cokernel pairs subject to some closure requirements, so the class of all kernel-cokernel pairs is a candidate for the largest exact structure. It is quite often the case that the class of all kernel-cokernel pairs is an exact structure, but this fails in general: Rump [34] constructs an example of an additive category with kernels and cokernels whose kernel-cokernel pairs fail to be an exact structure.

It is commonplace that basic homological algebra in categories of modules over a (sheaf of) rings extends to abelian categories. By using the Freyd-Mitchell full embedding theorem ([13] and [28]), diagram lemmas can be transferred from module categories to general abelian categories, i.e., one may argue by chasing elements around in diagrams. There is a point in proving the fundamental diagram lemmas directly, and be it only to familiarize oneself with the axioms. A careful study of what is actually needed in order to prove the fundamentals reveals that in most situations the axioms of exact categories are sufficient. An *a posteriori* reason is provided by the Gabriel-Quillen embedding theorem which reduces homological algebra in exact categories to the case of abelian categories, the slogan is "relative homological algebra made absolute", (Freyd [12]). In the appendix we present Thomason's proof of the Gabriel-Quillen embedding theorem for the sake of completeness, but we will not apply it in these notes. The author is convinced that the embedding theorem should be used to transfer the intuition from abelian categories to exact categories rather than to prove (simple) theorems with it. A direct proof from the axioms provides much more insight than a reduction to abelian categories.

That being said, we turn to a short description of the contents of this paper.

In section 2 we state and discuss the axioms and draw the basic consequences, in particular we give the characterization of pull-back squares and Keller's proof of the obscure axiom.

In section 3 we prove the (short) five lemma, the Noether isomorphism theorem and the 3×3-lemma.

Section 4 briefly discusses quasi-abelian categories, a source of many examples of exact categories. Contrary to the notion of an exact category, the property of being quasi-abelian is intrinsic.

Exact functors are briefly touched upon in section 5 and after that we treat the idempotent completion and the property of weak idempotent completeness in sections 6 and 7.

We come closer to the heart of homological algebra when discussing admissible morphisms, long exact sequences, the five lemma and the snake lemma in section 8. In order for the snake lemma to hold, it seems that the assumption of weak idempotent completeness is necessary.

After that we briefly remind the reader of the notions of chain complexes and chain homotopy in section 9, before we turn to acyclic complexes and quasi-isomorphisms in section 10. Notably, we give an elementary proof of Neeman's crucial result that the category of acyclic complexes is triangulated. We do not indulge in the details of the construction of the derived category of an exact category because this is well treated in the literature.

On a more leisurely level, projective and injective objects are treated in section 11 preparing the grounds for a treatment of classical derived functors (with values in an abelian category) in section 12, where we state and prove the resolution lemma, the comparison theorem and the horseshoe lemma, i.e., the three basic ingredients for the classical construction.

We end with a short list of examples in section 13.

In Appendix A we give Thomason's proof of the Gabriel-Quillen embedding theorem of an exact category into an abelian one. In a second appendix we give a proof of the folklore fact that under the assumption of weak idempotent completeness Heller's axioms for an "abelian" category are equivalent to Quillen's axioms for an exact category.

Historical Note. Quillen's notion of an exact category has its predecessors e.g. in Heller [19], Buchsbaum [7], Yoneda [40], Butler-Horrocks [9] and Mac Lane [26, XII.4]. It should be noted that Buchsbaum, Butler-Horrocks and Mac Lane assume the existence of an ambient abelian category and miss the crucial push-out and pull-back axioms, while Heller and Yoneda anticipate Quillen's definition. According to Quillen [32, p. "92/16/100"], assuming idempotent completeness, Heller's notion of an "abelian category" [19, § 3], i.e., an additive category equipped with an "abelian class of short exact sequences"[1] coincides with the present definition of an exact category. We give a proof of this assertion in appendix B. Yoneda's quasi-abelian $\mathcal{S}$-categories are nothing but Quillen's exact categories and it is a remarkable fact that Yoneda proves that Quillen's "obscure axiom" follows from his definition, see [40, p. 525, Corollary], a fact rediscovered thirty years later by Keller in [23, A.1].

Prerequisites. The prerequisites are kept at a minimum. The reader should know what an additive category is and be familiar with fundamental categorical concepts such as kernels, pull-backs, products and duality. Acquaintance with basic category theory as presented in Hilton-Stammbach [20, Chapter II] or Weibel [39, Appendix A] should amply suffice for a complete understanding of the text.

Disclaimer. This article is written for the reader who *wants* to learn about exact categories and knows *why*. Very few motivating examples are given in this text.

The author makes no claim to originality. All the results are well-known in some form and they are scattered around in the literature. The *raison d'être* of this article is the lack of a systematic *elementary* exposition of the theory. The works of Heller [19], Keller [23, 24] and Thomason [37] heavily influenced the present paper and many proofs given here can be found in their papers.

2. Definition and Basic Properties

In this section we introduce the notion of an exact category and draw the basic consequences of the axioms. We do not use the minimal axiomatics as provided by Keller [23, Appendix A] but prefer to use a convenient self-dual presentation of the axioms due to Yoneda [40, § 2] (modulo some of Yoneda's numerous 3×2-lemmas and our Proposition 2.12). The author hopes that the Bourbakists among

[1] It appears that Heller's article [19] was written independently of Grothendieck's influential Tôhoku paper [18] where today's notion of an abelian category was introduced.

the readers will pardon this *faux pas*. We will discuss that the present axioms are equivalent to Quillen's [32, § 2] in the course of events. The main points of this section are a characterization of push-out squares (Proposition 2.12) and the obscure axiom (Proposition 2.15).

2.1. Definition. Let $\mathscr{A}$ be an additive category. A *kernel-cokernel pair* (i,p) in $\mathscr{A}$ is a pair of composable morphisms

$$A' \xrightarrow{i} A \xrightarrow{p} A''$$

such that i is a kernel of p and p is a cokernel of i. If a class $\mathscr{E}$ of kernel-cokernel pairs on $\mathscr{A}$ is fixed, an *admissible monic* is a morphism i for which there exists a morphism p such that $(i,p) \in \mathscr{E}$. *Admissible epics* are defined dually. We depict admissible monics by $\rightarrowtail$ and admissible epics by $\twoheadrightarrow$ in diagrams.

An *exact structure* on $\mathscr{A}$ is a class $\mathscr{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms:

- [E0] For all objects $A \in \mathscr{A}$, the identity morphism 1_A is an admissible monic.
- [E0$^{\mathrm{op}}$] For all objects $A \in \mathscr{A}$, the identity morphism 1_A is an admissible epic.
- [E1] The class of admissible monics is closed under composition.
- [E1$^{\mathrm{op}}$] The class of admissible epics is closed under composition.
- [E2] The push-out of an admissible monic along an arbitrary morphism exists and yields an admissible monic.
- [E2$^{\mathrm{op}}$] The pull-back of an admissible epic along an arbitrary morphism exists and yields an admissible epic.

Axioms [E2] and [E2$^{\mathrm{op}}$] are subsumed in the diagrams

$$\begin{array}{ccc} A & \rightarrowtail & B \\ \downarrow & \text{PO} & \vdots \\ A' & \rightarrowtail & B' \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A' & \twoheadrightarrow & B' \\ \vdots & \text{PB} & \downarrow \\ A & \twoheadrightarrow & B \end{array}$$

respectively.

An *exact category* is a pair $(\mathscr{A},\mathscr{E})$ consisting of an additive category $\mathscr{A}$ and an exact structure $\mathscr{E}$ on $\mathscr{A}$. Elements of $\mathscr{E}$ are called *short exact sequences*.

2.2. Remark. Note that $\mathscr{E}$ is an exact structure on $\mathscr{A}$ if and only if $\mathscr{E}^{\mathrm{op}}$ is an exact structure on $\mathscr{A}^{\mathrm{op}}$. This allows for reasoning by dualization.

2.3. Remark. Isomorphisms are admissible monics and admissible epics. Indeed, this follows from the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow[\cong]{f} & B & \longrightarrow & 0 \\ {\scriptstyle 1_A}\downarrow\cong & & f^{-1}\downarrow\cong & & \cong\downarrow \\ A & \overset{1_A}{\rightarrowtail} & A & \twoheadrightarrow & 0, \end{array}$$

the fact that exact structures are assumed to be closed under isomorphisms and that the axioms are self-dual.

2.4. Remark (Keller [23, App. A]). The axioms are somewhat redundant and can be weakened. For instance, let us assume instead of [E0] and [E0$^{\mathrm{op}}$] that 1_0, the identity of the zero object, is an admissible epic. For any object A there is the pull-back diagram

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow & \text{PB} & \downarrow \\ 0 & \xrightarrow{1_0} & 0 \end{array}$$

so [E2$^{\mathrm{op}}$] together with our assumption on 1_0 shows that [E0$^{\mathrm{op}}$] holds. Since 1_0 is a kernel of itself, it is also an admissible monic, so we conclude by [E2] that [E0] holds as well. More importantly, Keller proves in *loc. cit.* (A.1, proof of the proposition, step 3), that one can also dispose of one of [E1] or [E1$^{\mathrm{op}}$]. Moreover, he mentions (A.2, Remark), that one may also weaken one of [E2] or [E2$^{\mathrm{op}}$]—this is a straightforward consequence of (the proof of) Proposition 3.1.

2.5. Remark. Keller [23, 24] uses *conflation*, *inflation* and *deflation* for what we call short exact sequence, admissible monic and admissible epic. This terminology stems from Gabriel-Roĭter [15, Ch. 9] who give a list of axioms for exact categories whose underlying additive category has weakly split idempotents in the sense of section 7, see Keller's appendix to [11] for a thorough comparison of the axioms.

2.6. Exercise. An admissible epic which is additionally monic is an isomorphism.

2.7. Lemma. *The sequence*

$$A \overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightarrowtail} A \oplus B \overset{[\,0\ 1\,]}{\twoheadrightarrow} B$$

is short exact.

Proof. The following diagram is a push-out square

$$\begin{array}{ccc} 0 & \rightarrowtail & B \\ \downarrow & \text{PO} & \downarrow {\scriptstyle \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \\ A & \overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightarrowtail} & A \oplus B. \end{array}$$

The top arrow and the left hand arrow are admissible monics by [E0$^{\mathrm{op}}$] while the bottom arrow and the right hand arrow are admissible monics by [E2]. The lemma now follows from the facts that the sequence in question is a kernel-cokernel pair and that $\mathscr{E}$ is closed under isomorphisms. ☐

2.8. Remark. Lemma 2.7 shows that Quillen's axiom a) [32, § 2] stating that split exact sequences belong to $\mathscr{E}$ follows from our axioms. Conversely, Quillen's axiom a) obviously implies [E0] and [E0$^{\mathrm{op}}$]. Quillen's axiom b) coincides with our axioms [E1], [E1$^{\mathrm{op}}$], [E2] and [E2$^{\mathrm{op}}$]. We will prove that Quillen's axiom c) follows from our axioms in Proposition 2.15.

2.9. Proposition. *The direct sum of two short exact sequences is short exact.*

Proof. Let $A' \rightarrowtail A \twoheadrightarrow A''$ and $B' \rightarrowtail B \twoheadrightarrow B''$ be two short exact sequences. First observe that for every object C the sequence

$$A' \oplus C \rightarrowtail A \oplus C \twoheadrightarrow A''$$

is exact—the second morphism is an admissible epic because it is the composition of the admissible epics $[\,1\ 0\,] : A \oplus C \twoheadrightarrow A$ and $A \twoheadrightarrow A''$; the first morphism in the sequence is a kernel of the second one, hence an admissible monic. Now it follows from [E1] that

$$A' \oplus B' \rightarrowtail A \oplus B$$

is an admissible monic because it is the composition of the two admissible monics $A' \oplus B' \rightarrowtail A \oplus B'$ and $A \oplus B' \rightarrowtail A \oplus B$. It is obvious that

$$A' \oplus B' \rightarrowtail A \oplus B \twoheadrightarrow A'' \oplus B''$$

is a kernel-cokernel pair, hence the proposition is proved. □

2.10. Corollary. *The exact structure $\mathscr{E}$ is an additive subcategory of the additive category $\mathscr{A}^{\to\to}$ of composable morphisms of $\mathscr{A}$.* □

2.11. Remark. In Exercise 3.9 the reader is asked to show that $\mathscr{E}$ is exact with respect to a natural exact structure.

2.12. Proposition. *Consider a commutative square*

$$\begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ A' & \underset{i'}{\rightarrowtail} & B' \end{array}$$

in which the horizontal arrows are admissible monics. The following assertions are equivalent:

(i) The square is a push-out.

(ii) The sequence $A \overset{\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]}{\rightarrowtail} B \oplus A' \overset{[\,f'\ i'\,]}{\twoheadrightarrow} B'$ *is short exact.*

(iii) The square is bicartesian, i.e., both a push-out and a pull-back.

(iv) The square is part of a commutative diagram

$$\begin{array}{ccccc} A & \overset{i}{\rightarrowtail} & B & \overset{p}{\twoheadrightarrow} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & & \| \\ A' & \underset{i'}{\rightarrowtail} & B' & \underset{p'}{\twoheadrightarrow} & C \end{array}$$

with exact rows.

Proof. (i) ⇒ (ii): The push-out property is equivalent to the assertion that $[\,f'\ i'\,]$ is a cokernel of $\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]$, so it suffices to prove that the latter is an admissible monic. But this follows from [E1] since $\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]$ is equal to the composition of the morphisms

$$A \overset{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}{\rightarrowtail} A \oplus A' \underset{\cong}{\overset{\left[\begin{smallmatrix} 1 & 0 \\ -f & 1 \end{smallmatrix}\right]}{\longrightarrow}} A \oplus A' \overset{\left[\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right]}{\rightarrowtail} B \oplus A'$$

which are all admissible monics, see Remark 2.3.

(ii) ⇒ (iii) and (iii) ⇒ (i): obvious.

(i) ⇒ (iv): Let $p : B \twoheadrightarrow C$ be a cokernel of i. The push-out property of the square yields that there is a unique morphism $p' : B' \to C$ such that $p'f = p$ and $p'i' = 0$. Observe that $p'f = p$ implies that p' is epic. In order to see that p' is a cokernel of i', let $g : B' \to X$ be such that $gi' = 0$. Then $gf'i = gi'f = 0$, so gf' factors uniquely over a morphism $h : C \to X$ such that $gf' = hp$. We claim that $hp' = g$: this follows from the push-out property of the square because $hp'f' = hp = gf'$ and $hp'i' = 0 = gi'$. Since p' is epic, the factorization h of g is unique, so p' is a cokernel of i'.

(iv) $\Rightarrow$ (ii): Form the pull-back over p and p' in order to obtain the commutative diagram

$$\begin{array}{ccccc} & & A & = & A \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ A' & \overset{j'}{\rightarrowtail} & D & \overset{q'}{\twoheadrightarrow} & B \\ \| & & \downarrow{\scriptstyle q} & \text{PB} & \downarrow{\scriptstyle p} \\ A' & \overset{i'}{\rightarrowtail} & B' & \overset{p'}{\twoheadrightarrow} & C \end{array}$$

with exact rows and columns using the dual of the implication (i) $\Rightarrow$ (iv). Since the square

$$\begin{array}{ccc} B & = & B \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle p} \\ B' & \overset{p'}{\twoheadrightarrow} & C \end{array}$$

is commutative, there is a unique morphism $k : B \to D$ such that $q'k = 1_B$ and $qk = f'$. Since $q'(1_D - kq') = 0$, there is a unique morphism $l : D \to A'$ such that $j'l = 1_D - kq'$. Note that $lk = 0$ because $j'lk = (1_D - kq')k = 0$ and j' is monic. Furthermore

$$i'lj = (qj')lj = q(1_D - kq')j = -(qk)(q'j) = -f'i = -i'f$$

implies $lj = -f$ since i' is monic.

The morphisms

$$\left[\begin{smallmatrix} k & j' \end{smallmatrix}\right] : B \oplus A' \to D \qquad \text{and} \qquad \left[\begin{smallmatrix} q' \\ l \end{smallmatrix}\right] : D \to B \oplus A'$$

are mutually inverse since

$$\left[\begin{smallmatrix} k & j' \end{smallmatrix}\right] \left[\begin{smallmatrix} q' \\ l \end{smallmatrix}\right] = kq' + j'l = 1_D \qquad \text{and} \qquad \left[\begin{smallmatrix} q' \\ l \end{smallmatrix}\right] \left[\begin{smallmatrix} k & j' \end{smallmatrix}\right] = \left[\begin{smallmatrix} q'k & q'j' \\ lk & lj' \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_B & 0 \\ 0 & 1_{A'} \end{smallmatrix}\right].$$

Now

$$\left[\begin{smallmatrix} f' & i' \end{smallmatrix}\right] = q \left[\begin{smallmatrix} k & j' \end{smallmatrix}\right] \qquad \text{and} \qquad \left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right] = \left[\begin{smallmatrix} q' \\ l \end{smallmatrix}\right] j$$

show that $A \xrightarrow{\left[\begin{smallmatrix} i \\ -f \end{smallmatrix}\right]} B \oplus A' \xrightarrow{\left[\begin{smallmatrix} f' & i' \end{smallmatrix}\right]} B'$ is isomorphic to $A \xrightarrow{j} D \xrightarrow{q} B'$. $\square$

The following simple observation will only be used in the proof of Lemma 10.3. We state it here for ease of reference.

2.13. Corollary. *The surrounding rectangle in a diagram of the form*

$$\begin{array}{ccccc} A & \overset{f}{\twoheadrightarrow} & B & \overset{g}{\rightarrowtail} & C \\ \downarrow{\scriptstyle a} & \text{PB} & \downarrow{\scriptstyle b} & \text{PO} & \downarrow{\scriptstyle c} \\ A' & \overset{f'}{\twoheadrightarrow} & B' & \overset{g'}{\rightarrowtail} & C' \end{array}$$

is bicartesian and $A \overset{\left[\begin{smallmatrix} -a \\ gf \end{smallmatrix}\right]}{\rightarrowtail} A' \oplus C \overset{\left[\begin{smallmatrix} g'f' & c \end{smallmatrix}\right]}{\twoheadrightarrow} C'$ *is short exact.*

Proof. It follows from Proposition 2.12 and its dual that both squares are bicartesian. Gluing two bicartesian squares along a common arrow yields another bicartesian square, which entails the first part and the fact that the sequence of the second part is a kernel-cokernel pair. The equation $\left[\begin{smallmatrix} g'f' & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} g' & c \end{smallmatrix}\right] \left[\begin{smallmatrix} f' & 0 \\ 0 & 1_C \end{smallmatrix}\right]$ exhibits $\left[\begin{smallmatrix} g'f' & c \end{smallmatrix}\right]$ as a composition of admissible epics by Proposition 2.9 and Proposition 2.12. $\square$

2.14. Proposition. *The pull-back of an admissible monic along an admissible epic yields an admissible monic.*

Proof. Consider the diagram

$$\begin{array}{ccccc} A' & \xrightarrow{i'} & B' & \xrightarrow{pe} \!\!\!\!\rightarrow & C \\ {\scriptstyle e'}\downarrow\!\!\downarrow & \text{PB} & \downarrow\!\!\downarrow{\scriptstyle e} & & \| \\ A & \overset{i}{\rightarrowtail} & B & \xrightarrow{p}\!\!\!\!\rightarrow & C. \end{array}$$

The pull-back square exists by axiom [E2$^{\mathrm{op}}$] which also implies that e' is an admissible epic. Let p be a cokernel of i, so it is an admissible epic and pe is an admissible epic by axiom [E1$^{\mathrm{op}}$]. In any category, the pull-back of a monic is a monic (if it exists). In order to see that i' is an admissible monic, it suffices to prove that i' is a kernel of pe. Suppose that $g' : X \to B'$ is such that $peg' = 0$. Since i is a kernel of p, there exists a unique $f : X \to A$ such that $eg' = if$. Applying the universal property of the pull-back square, we find a unique $f' : X \to A'$ such that $e'f' = f$ and $i'f' = g'$. Since i' is monic, f' is the unique morphism such that $i'f' = g'$ and we are done. ☐

2.15. Proposition (Obscure Axiom). *Suppose that $i : A \to B$ is a morphism in $\mathscr{A}$ admitting a cokernel. If there exists a morphism $j : B \to C$ in $\mathscr{A}$ such that the composite $ji : A \rightarrowtail C$ is an admissible monic then i is an admissible monic.*

2.16. Remark. The statement of the previous proposition is given as axiom c) in Quillen's definition of an exact category [32, § 2]. At that time, it was already proved to be a consequence of the other axioms by Yoneda [40, Corollary, p. 525]. The redundancy of the obscure axiom was rediscovered by Keller [23, A.1]. Thomason baptized axiom c) the "obscure axiom" in [37, A.1.1].

A convenient and quite powerful strengthening of the obscure axiom holds under the rather mild additional hypothesis that $\mathscr{A}$ have weakly split idempotents, see Proposition 7.5.

Proof of Proposition 2.15 (Keller). Let $k : B \to D$ be a cokernel of i. From the push-out diagram

$$\begin{array}{ccc} A & \overset{ji}{\rightarrowtail} & C \\ {\scriptstyle i}\downarrow & \text{PO} & \downarrow \\ B & \rightarrowtail & E \end{array}$$

and Proposition 2.12 we conclude that

$$A \xrightarrow{\left[\begin{smallmatrix} i \\ ji \end{smallmatrix}\right]} B \oplus C$$

is an admissible monic. Because

$$B \oplus C \xrightarrow{\left[\begin{smallmatrix} 1_B & 0 \\ -j & 1_C \end{smallmatrix}\right]} B \oplus C$$

is an isomorphism it is in particular an admissible monic, hence

$$\left[\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1_B & 0 \\ -j & 1_C \end{smallmatrix}\right] \left[\begin{smallmatrix} i \\ ji \end{smallmatrix}\right] : A \longrightarrow B \oplus C$$

is an admissible monic as well. Because $\left[\begin{smallmatrix}k & 0\\ 0 & 1_C\end{smallmatrix}\right]$ is a cokernel of $\left[\begin{smallmatrix}i\\ 0\end{smallmatrix}\right]$, it is an admissible epic. Consider the following diagram

$$\begin{array}{ccccc} A & \xrightarrow{\;i\;} & B & \xrightarrow{\;k\;} & D \\ \| & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & \text{PB} & \downarrow{\scriptstyle \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \\ A & \underset{\left[\begin{smallmatrix}i\\0\end{smallmatrix}\right]}{\rightarrowtail} & B\oplus C & \underset{\left[\begin{smallmatrix}k&0\\0&1_C\end{smallmatrix}\right]}{\twoheadrightarrow} & D\oplus C. \end{array}$$

Since the right hand square is a pull-back, it follows that k is an admissible epic and that i is a kernel of k, so i is an admissible monic. □

2.17. Corollary. *Let (i,p) and (i',p') be two pairs of composable morphisms. If the direct sum $(i \oplus i', p \oplus p')$ is exact then (i,p) and (i',p') are both exact.*

Proof. It is clear that (i,p) and (i',p') are kernel-cokernel pairs. Since i has p as a cokernel and since

$$\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right] i = \left[\begin{smallmatrix}i & 0\\ 0 & i'\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$$

is an admissible monic, the obscure axiom implies that i is an admissible monic. □

2.18. Exercise. Suppose that the commutative square

$$\begin{array}{ccc} A' & \overset{f'}{\rightarrowtail} & B' \\ \downarrow{\scriptstyle a} & \text{PO} & \downarrow{\scriptstyle b} \\ A & \underset{f}{\rightarrowtail} & B \end{array}$$

is a push-out. Prove that a is an admissible monic.

Hint: Let $b' : B \twoheadrightarrow B''$ be a cokernel of $b : B' \rightarrowtail B$. Prove that $a' = b'f : A \to B''$ is a cokernel of a, then apply the obscure axiom.

3. Some Diagram Lemmas

In this section we will prove variants of diagram lemmas which are well-known in the context of abelian categories, in particular we will prove the five lemma and the 3×3-lemma. Further familiar diagram lemmas will be proved in section 8. The proofs will be based on the following simple observation:

3.1. Proposition. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. A morphism from a short exact sequence $A' \rightarrowtail B' \twoheadrightarrow C'$ to another short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ factors over a short exact sequence $A \rightarrowtail D \twoheadrightarrow C'$*

$$\begin{array}{ccccc} A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ \downarrow{\scriptstyle a} & \text{BC} & \downarrow{\scriptstyle b'} & & \| \\ A & \overset{m}{\rightarrowtail} & D & \overset{e}{\twoheadrightarrow} & C' \\ \| & & {\scriptstyle b''}\downarrow & \text{BC} & \downarrow{\scriptstyle c} \\ A & \underset{f}{\rightarrowtail} & B & \underset{g}{\twoheadrightarrow} & C \end{array}$$

in such a way that the two squares marked BC *are bicartesian. In particular there is a canonical isomorphism of the push-out $A \cup_{A'} B'$ and the pull-back $B \times_C C'$.*

Proof. Form the push-out under f' and a in order to obtain the object D and the morphisms m and b'. Let $e : D \to C'$ be the unique morphism such that $eb' = g'$ and $em = 0$ and let $b'' : D \to B$ be the unique morphism $D \to B$ such that $b''b' = b : B' \to B$ and $b''m = f$. It is easy to see that e is a cokernel of

m (see the proof of Proposition 2.12 (i) $\Rightarrow$ (iv)) and hence the result follows from Proposition 2.12 since the square $DC'BC$ is commutative [this is because a and $b''b'$ determine c uniquely]. ☐

3.2. Corollary (Five Lemma, I). *Consider a morphism of short exact sequences*

$$\begin{array}{ccccc} A' & \rightarrowtail & B' & \twoheadrightarrow & C' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A & \rightarrowtail & B & \twoheadrightarrow & C. \end{array}$$

If a and c are isomorphisms (or admissible monics, or admissible epics) then so is b.

Proof. Assume first that a and c are isomorphisms. Because isomorphisms are preserved by push-outs and pull-backs, it follows from the diagram of Proposition 3.1 that b is the composition of two isomorphisms $B' \to D \to B$. If a and c are both admissible monics, it follows from the diagram of Proposition 3.1 together with [E2] and Proposition 2.14 that b is the composition of two admissible monics. The case of admissible epics is dual. ☐

3.3. Exercise. If in a morphism

$$\begin{array}{ccccc} A' & \rightarrowtail & B' & \twoheadrightarrow & C' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A & \rightarrowtail & B & \twoheadrightarrow & C. \end{array}$$

of short exact sequences two out of a, b, c are isomorphisms then so is the third.

Hint: Use e.g. that c is uniquely determined by a and b.

3.4. Remark. The reader insisting that Corollary 3.2 should be called "three lemma" rather than "five lemma" is cordially invited to give the details of the proof of Lemma 8.9 and to solve Exercise 8.10. We will however use the more customary name five lemma.

3.5. Lemma ("Noether Isomorphism $C/B \cong (C/A)/(B/A)$"). *Consider the commutative diagram*

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & X \\ \| & & \downarrow & & \vdots \\ A & \rightarrowtail & C & \twoheadrightarrow & Y \\ & & \downarrow & & \vdots \\ & & Z & = & Z \end{array}$$

in which the first two horizontal rows and the middle column are short exact. Then the third column exists, is short exact, and is uniquely determined by the requirement that it makes the diagram commutative. Moreover, the upper right hand square is bicartesian.

Proof. The morphism $X \to Y$ exists since the first row is exact and the composition $A \to C \to Y$ is zero while the morphism $Y \to Z$ exists since the second row is exact and the composition $B \to C \to Z$ vanishes. By Proposition 2.12 the square containing $X \to Y$ is bicartesian. It follows that $X \to Y$ is an admissible monic and that $Y \to Z$ is its cokernel. The uniqueness assertion is obvious. ☐

3.6. Corollary (3×3-Lemma). *Consider a commutative diagram*

$$\begin{array}{ccccc}
A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\
A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\
\downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle c'} \\
A'' & \overset{f''}{\rightarrowtail} & B'' & \overset{g''}{\twoheadrightarrow} & C''
\end{array}$$

in which the rows are exact and assume in addition that one of the following conditions holds:

(i) the two outer columns are short exact and $b'b = 0$;
(ii) the middle column and either one of the outer columns is short exact.

Then the remaining column is short exact as well.

Proof. Assume that condition (i) holds. We apply Proposition 3.1 to the morphism between the first two rows in order to obtain a commutative diagram

$$\begin{array}{ccccc}
A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\
\downarrow{\scriptstyle a} & \text{BC} & \downarrow{\scriptstyle \bar{b}} & & \| \\
A & \overset{\bar{f}}{\rightarrowtail} & D & \overset{\bar{g}}{\twoheadrightarrow} & C' \\
\| & & \downarrow{\scriptstyle \hat{b}} & \text{BC} & \downarrow{\scriptstyle c} \\
A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C
\end{array}$$

with $b = \hat{b}\bar{b}$—it follows from the five lemma that $\bar{b}$ and $\hat{b}$ are admissible monics, hence so is b. If we can prove that b' is a cokernel of b we are done.

Note that the morphism $\bar{a} : D \to A''$ satisfying $\bar{a}\bar{f} = a'$ and $\bar{a}\bar{b} = 0$ is a cokernel of $\bar{b}$ while the morphism $c'g : B \to C''$ is a cokernel of $\hat{b}$.

We will need to know in a moment that the square

$$(*) \qquad \begin{array}{ccc}
D & \overset{\bar{a}}{\longrightarrow} & A'' \\
\downarrow{\scriptstyle \hat{b}} & & \downarrow{\scriptstyle f''} \\
B & \overset{b'}{\longrightarrow} & B''
\end{array}$$

is commutative. Indeed, we have on the one hand $(f''\bar{a})\bar{f} = f''a' = b'f = (b'\hat{b})\bar{f}$ and on the other $(f''\bar{a})\bar{b} = 0 = b'b = (b'\hat{b})\bar{b}$ so that this follows from the push-out property of the square $A'B'AD$—note that the hypothesis $b'b = 0$ enters at this point of the argument.

Let $e : B \twoheadrightarrow E$ be a cokernel of b. Noether's isomorphism 3.5 yields the commutative diagram

$$\begin{array}{ccccc}
B' & \overset{\bar{b}}{\rightarrowtail} & D & \overset{\bar{a}}{\twoheadrightarrow} & A'' \\
\| & & \downarrow{\scriptstyle \hat{b}} & \text{BC} & \vdots{\scriptstyle d} \\
B' & \overset{b}{\rightarrowtail} & B & \overset{e}{\twoheadrightarrow} & E \\
 & & \downarrow{\scriptstyle c'g} & & \vdots{\scriptstyle d'} \\
 & & C'' & = & C''.
\end{array}$$

with exact rows and columns. Applying the push-out property of upper right hand square to the commutative square $(*)$ yields a unique morphism $\hat{e} : E \to B''$ such that $f'' = \hat{e}d$ and $b' = \hat{e}e$. Moreover, the push-out property of the square $DA''BE$ together with $g''\hat{e}e = g''b' = c'g = d'e$ and $g''\hat{e}d = g''f'' = 0 = d'd$ implies that the diagram

$$\begin{array}{ccccc} A'' & \overset{d}{\rightarrowtail} & E & \overset{d'}{\twoheadrightarrow} & C'' \\ \| & & \downarrow{\scriptstyle \hat{e}} & & \| \\ A'' & \overset{f''}{\rightarrowtail} & B'' & \overset{g''}{\twoheadrightarrow} & C'' \end{array}$$

is commutative and hence the two sequences are isomorphic by the five lemma. This finally establishes that $b' = \hat{e}e$ is a cokernel of b and settles case (i).

The two possibilities in case (ii) are dual to each other, so we need only consider the case that the middle and the right hand column are exact. By the obscure axiom 2.15 it suffices to prove that a has a' as a cokernel because $fa = bf'$ is an admissible monic. Observe right away that $a'a = 0$ because $f''a'a = b'bf' = 0$ and f'' is monic.

Again, Proposition 3.1 yields a commutative diagram

$$\begin{array}{ccccc} A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ \downarrow{\scriptstyle a} & \text{BC} & \downarrow{\scriptstyle \bar{b}} & & \| \\ A & \overset{\bar{f}}{\rightarrowtail} & D & \overset{\bar{g}}{\twoheadrightarrow} & C' \\ \| & & \downarrow{\scriptstyle \hat{b}} & \text{BC} & \downarrow{\scriptstyle c} \\ A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

such that $b = \hat{b}\bar{b}$. Note that $\hat{b}$ is an admissible monic by the five lemma and that it has $c'g : B \to C''$ as a cokernel. By the dual of the Noether isomorphism 3.5 we obtain the commutative diagram

$$\begin{array}{ccccc} B' & \overset{\bar{b}}{\rightarrowtail} & D & \overset{\bar{a}}{\twoheadrightarrow} & A'' \\ \| & & \downarrow{\scriptstyle \hat{b}} & & \downarrow{\scriptstyle f''} \\ B' & \overset{b}{\rightarrowtail} & B & \overset{b'}{\twoheadrightarrow} & B'' \\ & & \downarrow{\scriptstyle c'g} & & \downarrow{\scriptstyle g''} \\ & & C'' & = & C''. \end{array}$$

with exact rows and columns. Observe that $f''(\bar{a}\bar{f}) = b'\hat{b}\bar{f} = b'f = f''a'$ implies that $\bar{a}\bar{f} = a'$ since f'' is monic.

We now prove that a' is a cokernel of a, so let a morphism $x : A \to X$ with $xa = 0$ be given. The push-out property of the square $A'B'AD$ yields a unique morphism $\bar{x} : D \to X$ such that $\bar{x}\bar{f} = x$ and $\bar{x}\bar{b} = 0$. But then the exactness of the dotted row in the last diagram shows that $\bar{x} = y\bar{a}$ for a unique morphism $y : A'' \to X$ and this morphism satisfies $ya' = y\bar{a}\bar{f} = \bar{x}\bar{f} = x$. In order to see that the factorization $x = ya'$ is unique, we prove that a' is epic. To this end, consider the diagram

$$\begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle a'} & \text{BC} & \downarrow{\scriptstyle \beta} & & \| \\ A'' & \rightarrowtail & D' & \twoheadrightarrow & C \\ \| & & \downarrow{\scriptstyle \beta'} & \text{BC} & \downarrow{\scriptstyle c'} \\ A'' & \overset{f''}{\rightarrowtail} & B'' & \overset{g''}{\twoheadrightarrow} & C'' \end{array}$$

obtained from Proposition 3.1. By the five lemma β' is an admissible epic and its kernel is isomorphic to C' by Proposition 2.12. It is easy to see that there is a commutative diagram

$$\begin{array}{ccccc} B' & \overset{b}{\rightarrowtail} & B & \overset{b'}{\twoheadrightarrow} & B'' \\ {\scriptstyle g'}\big\downarrow\!\!\!\!\downarrow & & \big\downarrow{\scriptstyle \beta} & & \big\| \\ C' & \overset{\ker \beta'}{\rightarrowtail} & D' & \overset{\beta'}{\twoheadrightarrow} & B'' \end{array}$$

which implies that β is an admissible epic by the five lemma, and hence a' is an admissible epic by the pull-back property of the square $ABA''D'$. $\square$

3.7. Exercise. Consider the solid arrow diagram

$$\begin{array}{ccccc} A' & \rightarrowtail & B' & \twoheadrightarrow & C' \\ \big\downarrow & & \big\downarrow & & \vdots \\ A & \rightarrowtail & B & \twoheadrightarrow & C \\ \big\downarrow & & \big\downarrow & & \vdots \\ A'' & \rightarrowtail & B'' & \twoheadrightarrow & C'' \end{array}$$

with exact rows and columns. Strengthen the Noether isomorphism 3.5 to the statement that there exist unique maps $C' \to C$ and $C \to C''$ making the diagram commutative and the sequence $C' \rightarrowtail C \twoheadrightarrow C''$ is short exact.

3.8. Exercise. In the situation of the 3×3-lemma prove that there are two exact sequences $A' \rightarrowtail A \oplus B' \to B \twoheadrightarrow C''$ and $A' \rightarrowtail B \to B'' \oplus C \twoheadrightarrow C''$ in the sense that the morphism $\to$ factors as $\twoheadrightarrow\rightarrowtail$ in such a way that consecutive $\rightarrowtail\twoheadrightarrow$ are short exact [compare also with Definition 8.8].

3.9. Exercise. Let $(\mathscr{A}, \mathscr{E})$ be an exact category and consider $\mathscr{E}$ as a full subcategory of $\mathscr{A}^{\to\to}$. We have shown that $\mathscr{E}$ is additive in Corollary 2.10. Let $\mathscr{F}$ be the class of short sequences of type

$$\begin{array}{ccccc} (A' & \rightarrowtail & B' & \twoheadrightarrow & C') \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (A & \rightarrowtail & B & \twoheadrightarrow & C) \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ (A'' & \rightarrowtail & B'' & \twoheadrightarrow & C'') \end{array}$$

with short exact columns [we write $(A \rightarrowtail B \twoheadrightarrow C)$ to indicate that we think of the sequence as an object of $\mathscr{E}$]. Prove that $(\mathscr{E}, \mathscr{F})$ is an exact category.

4. Quasi-Abelian Categories

4.1. Definition. An additive category $\mathscr{A}$ is called *quasi-abelian* if

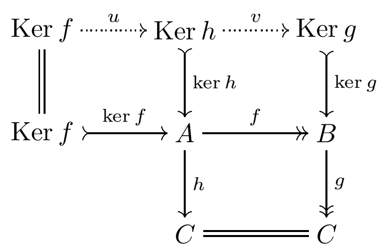
(i) Every morphism has a kernel and a cokernel.
(ii) The class of kernels is stable under push-out along arbitrary morphisms and the class of cokernels is closed under pull-back along arbitrary morphisms.

4.2. Remark. The concept of a quasi-abelian category is self-dual, that is to say $\mathscr{A}$ is quasi-abelian if and only if $\mathscr{A}^{\mathrm{op}}$ is quasi-abelian.

4.3. Exercise. Let $\mathscr{A}$ be an additive category with kernels. Prove that every pullback of a kernel is a kernel.

4.4. Proposition (Schneiders [35, 1.1.7]). *The class $\mathscr{E}_{\max}$ of all kernel-cokernel pairs on a quasi-abelian category is an exact structure.*

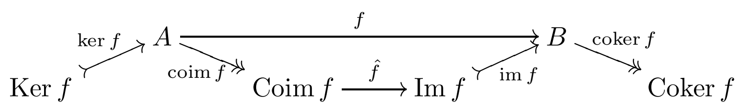
Proof. It is clear that $\mathscr{E}_{\max}$ is closed under isomorphisms and that the classes of kernels and cokernels contain the identity morphisms. The pull-back and push-out axioms are part of the definition of quasi-abelian categories. By duality it only remains to show that the class of cokernels is closed under composition. So let $f : A \twoheadrightarrow B$ and $g : B \twoheadrightarrow C$ be cokernels and put $h = gf$. In the diagram



there exist unique morphisms u and v making it commutative. The upper right hand square is a pull-back, so v is a cokernel and u is its kernel. But then it follows by duality that the upper right hand square is also a push-out and this together with the fact that h is epic implies that h is a cokernel of $\ker h$. $\square$

4.5. Remark. Note that we have just re-proved the Noether isomorphism 3.5 in the special case of quasi-abelian categories.

4.6. Definition. The *coimage* of a morphism f in a category with kernels and cokernels is $\operatorname{Coker}(\ker f)$, while the *image* is defined to be $\operatorname{Ker}(\operatorname{coker} f)$. The *analysis* (cf. [26, IX.2]) of f is the commutative diagram

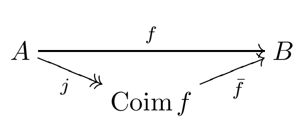


in which $\hat{f}$ is uniquely determined by requiring that the diagram is commutative.

4.7. Remark. The difference between quasi-abelian categories and abelian categories is that in the quasi-abelian case the canonical morphism $\hat{f}$ in the analysis f is not in general an isomorphism. Indeed, it is easy to see that a quasi-abelian is abelian *provided* that $\hat{f}$ is always an isomorphism. Equivalently, not every monic is a kernel and not every epic is a cokernel.

4.8. Proposition ([35, 1.1.5]). *Let f be a morphism in the quasi-abelian category $\mathscr{A}$. The canonical morphism $\hat{f} : \operatorname{Coim} f \to \operatorname{Im} f$ is monic and epic.*

Proof. By duality it suffices to check that the morphism $\bar{f}$ in the diagram



is monic. Let $x : X \to \operatorname{Coim} f$ be a morphism such that $\bar{f}x = 0$. The pull-back $y : Y \to A$ of x along j satisfies $fy = 0$, so y factors over $\operatorname{Ker} f$ and hence $jy = 0$. But then the map $Y \twoheadrightarrow X \to \operatorname{Coim} f$ is zero as well, so $x = 0$. $\square$

4.9. Remark. Every morphism f in a quasi-abelian category $\mathscr{A}$ has two epic-monic factorizations, one over $\operatorname{Coim} f$ and one over $\operatorname{Im} f$. The quasi-abelian category $\mathscr{A}$ is abelian if and only if the two factorizations coincide for all morphisms f.

4.10. Remark. An additive category with kernels and cokernels is called *semi-abelian* if the canonical morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is always monic and epic. We have just proved that quasi-abelian categories are semi-abelian. It may seem obvious that the concept of semi-abelian categories is strictly weaker than the concept of a quasi-abelian category. However, it is surprisingly delicate to come up with an explicit example. This led Raĭkov to conjecture that every semi-abelian category is quasi-abelian. A counterexample to this conjecture was recently found by Rump [34].

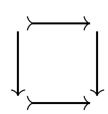
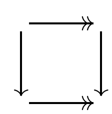
4.11. Remark. We do not develop the theory of quasi-abelian categories any further. The interested reader may consult Schneiders [35], Rump [33] and the references therein.

5. Exact Functors

5.1. Definition. Let $(\mathscr{A}, \mathscr{E})$ and $(\mathscr{A}', \mathscr{E}')$ be exact categories. An (additive) functor $F : \mathscr{A} \to \mathscr{A}'$ is called *exact* if $F(\mathscr{E}) \subset \mathscr{E}'$. The functor F *reflects exactness* if $F(\sigma) \in \mathscr{E}'$ implies $\sigma \in \mathscr{E}$ for all $\sigma \in \mathscr{A}^{\to\to}$.

5.2. Proposition. *An exact functor preserves push-outs along admissible monics and pull-backs along admissible epics.*

Proof. An exact functor preserves admissible monics and admissible epics, in particular it preserves diagrams of type

 and 

so the result follows immediately from Proposition 2.12 and its dual. □

6. Idempotent Completion

An additive category $\mathscr{A}$ is *idempotent complete* [21, 1.2.1, 1.2.2] if for every idempotent $p : A \to A$, *i.e.* $p^2 = p$, there is a decomposition $A \cong K \oplus I$ of A such that $p \cong \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Notice that $\mathscr{A}$ is idempotent complete if and only if every idempotent has a kernel.

For every additive category $\mathscr{A}$ there is a fully faithful embedding $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^{\wedge}$ into an idempotent complete additive category. Let $\mathscr{A}^{\wedge}$ be the following category: objects are pairs (A, p) consisting of an object in $\mathscr{A}$ and an idempotent $p : A \to A$; the morphisms are given by

$$\operatorname{Hom}_{\mathscr{A}^{\wedge}}((A,p),(B,q)) = q \circ \operatorname{Hom}_{\mathscr{A}}(A,B) \circ p$$

with the obvious composition. It is easy to see that $\mathscr{A}^{\wedge}$ is additive with biproduct $(A,p) \oplus (A',p') = (A \oplus A', p \oplus p')$ and that the functor $\mathscr{A} \to \mathscr{A}^{\wedge}$ given on objects by $A \mapsto (A, 1_A)$ is fully faithful. If $p : A \to A$ is an idempotent then $(A, 1_A)$ is isomorphic to $(A, 1-p) \oplus (A, p)$ in $\mathscr{A}^{\wedge}$ and, more generally, $\mathscr{A}^{\wedge}$ is idempotent complete [21, 1.2.2]. If $\mathscr{A}$ is already idempotent complete then $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^{\wedge}$ is easily seen to be an equivalence of categories.

6.1. Proposition. *The functor $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^{\wedge}$ is 2-universal among functors from $\mathscr{A}$ to idempotent complete categories:*

(i) *Let $F : \mathscr{A} \to \mathscr{I}$ be a functor to an idempotent complete category $\mathscr{I}$. There exists a functor $\widetilde{F} : \mathscr{A}^{\wedge} \to \mathscr{I}$ and a natural isomorphism $\widetilde{\alpha} : F \Rightarrow \widetilde{F} i$.*

(ii) *For every pair $(\bar{F}, \bar{\alpha})$ having the property of point (i) there is a unique natural isomorphism $\beta : \widetilde{F} \Rightarrow \bar{F}$ such that $\bar{\alpha} = (\beta i_{\mathscr{A}}) \circ \widetilde{\alpha}$.*

Proof. There is only one way to extend F to a functor $F^{\wedge} : \mathscr{A}^{\wedge} \to \mathscr{I}^{\wedge}$, it is given on objects by $F^{\wedge}(A,p) = (F(A), F(p))$. Since $\mathscr{I}$ is idempotent complete, the functor $i_{\mathscr{I}} : \mathscr{I} \to \mathscr{I}^{\wedge}$ is an equivalence, so we can choose a quasi-inverse $j_{\mathscr{I}} : \mathscr{I}^{\wedge} \to \mathscr{I}$ of $i_{\mathscr{I}}$ and we obtain the desired functor by setting $\widetilde{F} = j_{\mathscr{I}} F^{\wedge}$. The natural isomorphism $j_{\mathscr{I}} i_{\mathscr{I}} \Rightarrow \mathrm{id}_{\mathscr{I}}$ yields the natural isomorphism $\widetilde{\alpha} : F \Rightarrow \widetilde{F} i_{\mathscr{A}}$. This settles point (i), we leave point (ii) as an exercise for the reader. □

6.2. Remark. Another way of phrasing the proposition is: Let $\mathscr{A}$ be a small additive category and let $\mathscr{I}$ be an idempotent complete category. The inclusion functor $i_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}^{\wedge}$ induces an equivalence of functor categories

$$(i_{\mathscr{A}})^* : \mathrm{Hom}\,(\mathscr{A}^{\wedge}, \mathscr{I}) \xrightarrow{\simeq} \mathrm{Hom}\,(\mathscr{A}, \mathscr{I}).$$

6.3. Example. Let $\mathscr{F}$ be the category of free modules over a ring R. Its idempotent completion $\mathscr{F}^{\wedge}$ is equivalent to the category of projective modules over R.

Let now $(\mathscr{A}, \mathscr{E})$ be an exact category. Call a sequence in $\mathscr{A}^{\wedge}$ short exact if it is a direct summand in $\mathscr{A}^{\wedge}$ of a sequence in $\mathscr{E}$ and denote the class of short exact sequences in $\mathscr{A}^{\wedge}$ by $\mathscr{E}^{\wedge}$.

6.4. Proposition. *The class $\mathscr{E}^{\wedge}$ is an exact structure on $\mathscr{A}^{\wedge}$. The inclusion functor $i_{\mathscr{A}} : (\mathscr{A}, \mathscr{E}) \to (\mathscr{A}^{\wedge}, \mathscr{E}^{\wedge})$ preserves and reflects exactness and is 2-universal among exact functors to idempotent complete exact categories:*

(i) *Let $F : \mathscr{A} \to \mathscr{I}$ be an exact functor to an idempotent complete exact category $\mathscr{I}$. There exists an exact functor $\widetilde{F} : \mathscr{A}^{\wedge} \to \mathscr{I}$ and a natural isomorphism $\widetilde{\alpha} : F \Rightarrow \widetilde{F} i_{\mathscr{A}}$.*

(ii) *For every pair $(\bar{F}, \bar{\alpha})$ having the property of point (i) there is a unique natural isomorphism $\beta : \widetilde{F} \Rightarrow \bar{F}$ such that $\bar{\alpha} = (\beta i_{\mathscr{A}}) \circ \widetilde{\alpha}$.*

Proof. To prove that $\mathscr{E}^{\wedge}$ is an exact structure is straightforward but rather tedious, so we skip it.[2] Given this, it is clear that the functor $\mathscr{A} \to \mathscr{A}^{\wedge}$ is exact and reflects exactness. If $F : \mathscr{A} \to \mathscr{I}$ is an exact functor to an idempotent complete exact category then $F^{\wedge} : \mathscr{A}^{\wedge} \to \mathscr{I}^{\wedge}$ is exact. Finally, $\mathscr{I}$ and $\mathscr{I}^{\wedge}$ are equivalent as exact categories, so we are done by appealing to the proof of Proposition 6.1. □

6.5. Remark. One can interpret Proposition 6.4 by saying that the equivalence of categories of Remark 6.2 restricts to an equivalence of the full subcategories of exact functors $(i_{\mathscr{A}})^* : \mathrm{Hom}_{\mathrm{Ex}}\,((\mathscr{A}^{\wedge}, \mathscr{E}^{\wedge}), \mathscr{I}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Ex}}\,((\mathscr{A}, \mathscr{E}), \mathscr{I})$.

7. Weak Idempotent Completeness

Thomason introduced in [37, A.5.1] the notion of an exact category with "weakly split idempotents". It turns out that this is a property of the underlying additive category rather than the exact structure.

Recall that in an arbitrary category a morphism $r : B \to C$ is called a *retraction* if there exists a *section* $s : C \to B$ of r in the sense that $rs = 1_C$. Dually, a morphism $c : A \to B$ is a *coretraction* if it admits a section $s : B \to A$, i.e., $sc = 1_A$. Observe that retractions are epics and coretractions are monics. Moreover, a section of a retraction is a coretraction and a section of a coretraction is a retraction.

7.1. Lemma. *In an additive category $\mathscr{A}$ the following are equivalent:*

(i) *Every coretraction has a cokernel.*

(ii) *Every retraction has a kernel.*

[2] Thomason [37, A.9.1 (b)] gives a short argument relying on the embedding into an abelian category, but it can be done by completely elementary means as well.

7.2. Definition. If the conditions of the previous lemma hold then $\mathscr{A}$ is said to be *weakly idempotent complete.*

7.3. Remark. Assume that $r : B \to C$ is a retraction with section $s : C \to B$. Then $sr : B \to B$ is an idempotent. Let us prove that this idempotent gives rise to a splitting of B if r admits a kernel $k : A \to B$.

Indeed, since $r(1_B - sr) = 0$, there is a unique morphism $t : B \to A$ such that $kt = 1_B - sr$. It follows that k is a coretraction because $ktk = (1_B - sr)k = k$ implies that $tk = 1_A$. Moreover $kts = 0$, so $ts = 0$, hence $[\,k\ s\,] : A \oplus C \to B$ is an isomorphism with inverse $\left[\begin{smallmatrix} t \\ r \end{smallmatrix}\right]$. In particular, the sequences $A \to B \to C$ and $A \to A \oplus C \to C$ are isomorphic.

Proof of Lemma 7.1. By duality it suffices to prove that (ii) implies (i).

Let $c : C \to B$ be a coretraction with section s. Then s is a retraction and, assuming (ii), it admits a kernel $k : A \to B$. By the discussion in Remark 7.3, k is a coretraction with section $t : B \to A$ and it is obvious that t is a cokernel of c. □

7.4. Corollary. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. The following are equivalent:*

(i) The additive category $\mathscr{A}$ is weakly idempotent complete.
(ii) Every coretraction is an admissible monic.
(iii) Every retraction is an admissible epic.

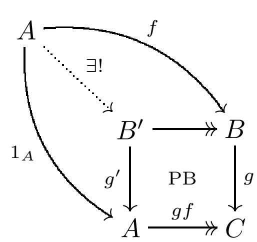
Proof. It follows from Remark 7.3 that every retraction $r : B \to C$ admitting a kernel gives rise to a sequence $A \to B \to C$ which is isomorphic to the split exact sequence $A \rightarrowtail A \oplus C \twoheadrightarrow C$, hence r is an admissible epic by Lemma 2.7, whence (i) implies (iii). By duality (i) implies (ii) as well. Conversely, every admissible monic has a cokernel and every admissible epic has a kernel, hence (ii) and (iii) both imply (i). □

In a weakly idempotent complete exact category the obscure axiom (Proposition 2.15) has an easier statement—this is Heller's cancellation axiom [19, (P2), p. 492]:

7.5. Proposition. *Let $(\mathscr{A}, \mathscr{E})$ be an exact category. The following are equivalent:*

(i) The additive category $\mathscr{A}$ is weakly idempotent complete.
(ii) Consider two morphisms $g : B \to C$ and $f : A \to B$. If $gf : A \twoheadrightarrow C$ is an admissible epic then g is an admissible epic.

Proof. (i) ⇒ (ii): Form the pull-back over g and gf and consider the diagram



which proves g' to be a retraction, so g' has a kernel $K' \to B'$. Because the diagram is a pull-back, the composite $K' \to B' \to B$ is a kernel of g and now the dual of Proposition 2.15 applies to yield that g is an admissible epic.

For the implication (ii) ⇒ (i) simply observe that (ii) implies that retractions are admissible epics. □

7.6. Remark ([29, 1.12]). Every small additive category $\mathscr{A}$ has a *weak idempotent completion* $\mathscr{A}'$. Objects of $\mathscr{A}'$ are the pairs (A, p), where $p : A \to A$ is an idempotent factoring as $p = cr$ for some retraction $r : A \to X$ and coretraction $c : X \to A$ with $rc = 1_B$, while the morphisms are given by

$$\mathrm{Hom}_{\mathscr{A}'}((A,p),(B,q)) = q \circ \mathrm{Hom}_{\mathscr{A}}(A,B) \circ p.$$

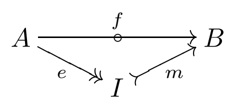
It is easy to see that the functor $\mathscr{A} \to \mathscr{A}'$ given on objects by $A \mapsto (A, 1_A)$ is 2-universal among functors from $\mathscr{A}$ to a weakly idempotent complete category. Moreover, if $(\mathscr{A}, \mathscr{E})$ is exact then so is $(\mathscr{A}', \mathscr{E}')$, where the sequences in $\mathscr{E}'$ are the direct summands in $\mathscr{A}'$ of sequences in $\mathscr{E}$, and the functor $\mathscr{A} \to \mathscr{A}'$ preserves and reflects exactness and is 2-universal among exact functors to weakly idempotent complete categories.

7.7. Remark. Contrary to the construction of the idempotent completion, there is the set-theoretic subtlety that the weak idempotent completion might not be well-defined if $\mathscr{A}$ is not small: it is *not* clear a priori that the objects (A, p) form a class—essentially for the same reason that the monics in a category need not form a class, see e.g. the discussion in Borceux [4, p. 373f].

8. Admissible Morphisms and the Snake Lemma

Throughout this section $(\mathscr{A}, \mathscr{E})$ denotes an exact category.

8.1. Definition. A morphism $f : A \to B$ is called *admissible* if it factors

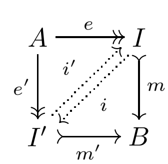


as a composition of an admissible monic with an admissible epic. Admissible morphisms will sometimes be displayed as $\longrightarrow\!\!\!\!\!\!\circ\;\;$ in diagrams.

8.2. Remark. Let f be an admissible morphism. If e' is an admissible epic and m' is an admissible monic then $m'fe'$ is admissible if the composition is defined. However, admissible morphisms are *not* closed under composition in general. Notice also that every zero morphism is admissible.

8.3. Remark. We choose the terminology *admissible morphism* even though *strict morphism* seems to be more standard (see e.g. [33, 35]). By Exercise 2.6 an admissible monic is the same thing as an admissible morphism which happens to be monic.

8.4. Lemma ([19, 3.4]). *The factorization of an admissible morphism is unique up to unique isomorphism. More precisely: In a commutative diagram of the form*



there exist unique morphisms i, i' making the diagram commutative. In particular, i and i' are mutually inverse isomorphisms.

Proof. Let k be a kernel of e. Since $m'e'k = mek = 0$ and m' is monic we have $e'k = 0$, hence there exists a unique morphism $i : I \to I'$ such that $e' = ie$. Moreover, $m'ie = m'e' = me$ and e epic imply $m'i = m$. Dually for i'. $\square$

8.5. Remark. An admissible morphism has an *analysis* (cf. [26, IX.2])

$$K \overset{k}{\rightarrowtail} A \xrightarrow{\;f\;} B \overset{c}{\twoheadrightarrow} C, \qquad A \overset{e}{\twoheadrightarrow} I \overset{m}{\rightarrowtail} B$$

where k is a kernel, c is a cokernel, e is a coimage and m is an image of f and the isomorphism classes of K, I and C are well-defined by Lemma 8.4.

8.6. Exercise. If $\mathscr{A}$ is an exact category in which every morphism is admissible then $\mathscr{A}$ is abelian. [A solution is given by Freyd in [14, Proposition 3.1]].

8.7. Lemma. *Admissible morphisms are stable under push-out along admissible monics and pull-back along admissible epics.*

Proof. Let $A \twoheadrightarrow I \rightarrowtail B$ be an admissible epic-admissible monic factorization of an admissible morphism. To prove the claim about push-outs construct the diagram

$$\begin{array}{ccccc} A & \twoheadrightarrow & I & \rightarrowtail & B \\ \downarrow & \text{PO} & \downarrow & \text{PO} & \downarrow \\ A' & \twoheadrightarrow & I' & \rightarrowtail & B'. \end{array}$$

Proposition 2.14 yields that $A' \to I'$ is an admissible epic and the rest is clear. □

8.8. Definition. A sequence of admissible morphisms

$$A' \xrightarrow{\;f\;} A \xrightarrow{\;f'\;} A'', \qquad A' \overset{e}{\twoheadrightarrow} I \overset{m}{\rightarrowtail} A, \quad A \overset{e'}{\twoheadrightarrow} I' \overset{m'}{\rightarrowtail} A''$$

is *exact* if $I \rightarrowtail A \twoheadrightarrow I'$ is short exact. Longer sequences of admissible morphisms are exact if the sequence given by any two consecutive morphisms is exact. Since the term "exact" is heavily overloaded, we also use the synonym *"acyclic"*, in particular in connection with chain complexes.

8.9. Lemma (Five Lemma, II). *If the commutative diagram*

$$\begin{array}{ccccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\ \downarrow{\cong} & & \downarrow{\cong} & & \downarrow{f} & & \downarrow{\cong} & & \downarrow{\cong} \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5 \end{array}$$

has exact rows then f is an isomorphism.

Sketch of the Proof. Choose factorizations $A_i \twoheadrightarrow I_i \rightarrowtail A_{i+1}$ of $A_i \to A_{i+1}$ and $B_i \twoheadrightarrow J_i \rightarrowtail B_{i+1}$ of $B_i \to B_{i+1}$ for $i = 1, \dots, 4$. Using Lemma 8.4 and Exercise 3.3 there are isomorphisms $I_1 \cong J_1$ and $I_2 \cong J_2$ which one may insert into the diagram without destroying its commutativity. Dually for $I_4 \cong J_4$ and $I_3 \cong J_3$. The five lemma 3.2 then implies that f is an isomorphism. □

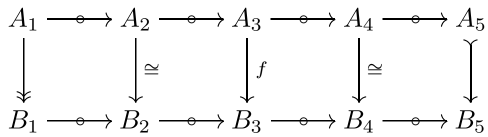
8.10. Exercise. Assume that $\mathscr{A}$ is weakly idempotent complete (Definition 7.2).

(i) (Sharp Four Lemma) Consider a commutative diagram

$$\begin{array}{ccccccc} A_1 & \to & A_2 & \to & A_3 & \to & A_4 \\ \downarrow\!\!\!\downarrow & & \downarrow{\cong} & & \downarrow{f} & & \downarrow \\ B_1 & \to & B_2 & \to & B_3 & \to & B_4 \end{array}$$

with exact rows. Prove that f is an admissible monic. Dualize.

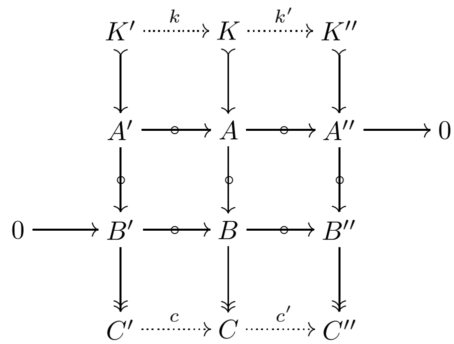
(ii) (Sharp Five Lemma) If the commutative diagram



has exact rows then f is an isomorphism.

Hint: Use Proposition 7.5, Exercise 2.6, Exercise 3.3 as well as Corollary 3.2.

8.11. Proposition (Snake Lemma [19, 4.3]). *Assume that $\mathscr{A}$ is weakly idempotent complete (Definition 7.2). For every commutative diagram*



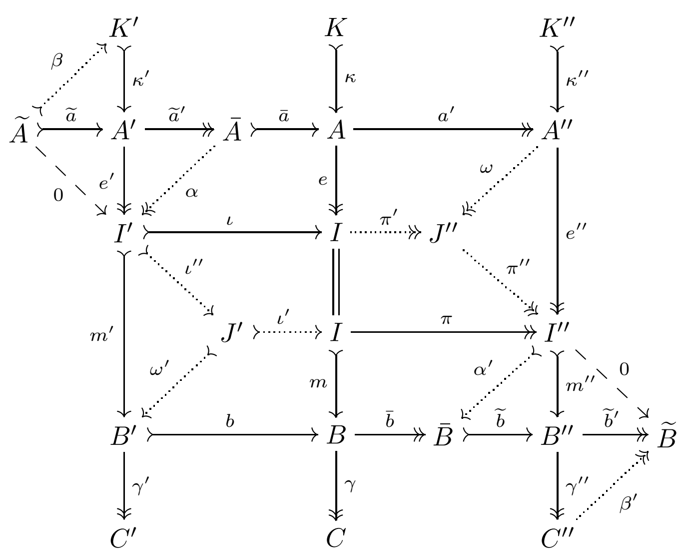
with exact rows and columns there are morphisms k, k', c, c' and $\delta : K'' \to C'$ fitting into an exact sequence

$$K' \xrightarrow{k} K \xrightarrow{k'} K'' \xrightarrow{\delta} C' \xrightarrow{c} C \xrightarrow{c'} C''$$

depending naturally on the diagram.

Proof (Heller). First observe that the morphisms k, k', c, c' in the statement of the proposition are uniquely determined by the requirement that the resulting diagram must commute.

Unfolding the definition of admissible morphisms and introducing names for the morphisms we obtain the following commutative diagram



with exact rows and columns—ignore the dotted arrows for the moment.

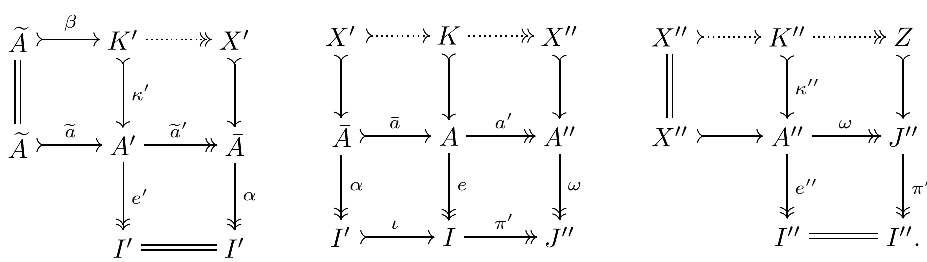
The assumption that $\mathscr{A}$ be weakly idempotent complete allows us to use Proposition 7.5 and its dual in order to recognize admissible monics and epics. Thus it follows from $m\iota = bm'$ that ι is an admissible monic and, dually, $\pi e = e''a'$ implies that π is an admissible epic. Let $\pi' : I \twoheadrightarrow J''$ be a cokernel of ι and let $\iota' : J' \rightarrowtail I$ be a kernel of π. Because e' is epic (or because m'' is monic) we have $\pi\iota = 0$, hence there are factorizations $\iota = \iota'\iota''$ and $\pi = \pi''\pi'$. Proposition 7.5 yields that ι'' is an admissible monic and that π'' is an admissible epic.

Next, $\pi' e \bar{a} = 0$ because $\widetilde{a}'$ is epic, so there exist $\alpha : \bar{A} \to I'$ such that $\iota\alpha = e\bar{a}$ and $\omega : A'' \to J''$ such that $\omega a' = \pi' e$. Since ι is monic we have $e' = \alpha\widetilde{a}'$, hence α is an admissible epic. Since $\omega a' = \pi' e$ it follows that ω is an admissible epic and we have $e'' = \pi''\omega$ since a' is epic. Finally note that $e' = \alpha\widetilde{a}'$ implies that $e'\widetilde{a} = 0$, hence there exists $\beta : \widetilde{A} \to K'$ such that $\widetilde{a}' = \kappa'\beta$, in particular β is an admissible monic.

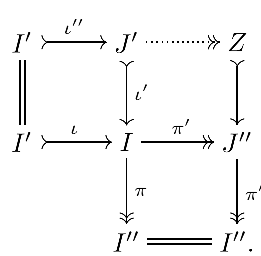
Dually, we have $\bar{b}m\iota' = 0$ and this implies the existence of morphisms α', ω' and finally β' making the diagram commutative and which are admissible monics and epics as indicated in the diagram.

We have thus constructed all the dotted arrows and argued why the resulting diagram is commutative.

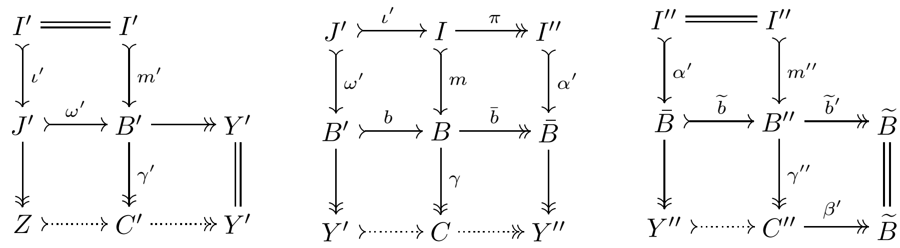
Let $X' \rightarrowtail \bar{A}$ be a kernel of $\alpha : \bar{A} \twoheadrightarrow I'$, let $X'' \rightarrowtail A''$ be a kernel of $\omega : A'' \twoheadrightarrow J''$ and let $Z \rightarrowtail J''$ be a kernel of $\pi'' : J'' \twoheadrightarrow I''$. Now use the Noether isomorphism 3.5 and the 3×3-lemma 3.6 in order to construct the following commutative diagrams with exact rows and columns:



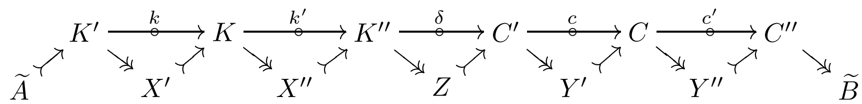
Notice that the dotted arrows in the diagrams above already yield half of the desired exact sequence. We now need to construct an admissible monomorphism $Z \rightarrowtail C'$. Before doing this, we apply the Noether isomorphism 3.5 to the diagram



Now let $B' \twoheadrightarrow Y'$ be a cokernel of $\omega' : J' \rightarrowtail B'$ and let $\bar{B} \twoheadrightarrow Y''$ be a cokernel of $\alpha' : I'' \rightarrowtail \bar{B}$. Again, the Noether isomorphism 3.5 and the 3×3-lemma 3.6 give the commutative diagrams



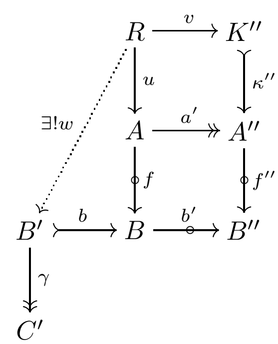
with exact rows and columns. Finally, the diagram



exhibits the desired exact sequence and the naturality assertion follows easily from the construction. ☐

8.12. Remark. The author does not know how to avoid the assumption of weak idempotent completeness in the proof of the snake lemma. It entered crucially in the guise of Heller's cancellation axiom (Proposition 7.5 (ii)).

8.13. Exercise ([19, 4.4]). Retain the assumptions of the snake lemma 8.11. The connecting morphism $\delta : K'' \to C'$ has the following property: Given a commutative square $RK''AA''$



there exists a unique morphism $w : R \to B'$ making the diagram above commutative, and, moreover, $\delta v = \gamma w$.

Hint: Consider the map $eu : R \to I$, the short exact sequence $J' \rightarrowtail I \twoheadrightarrow I''$ and the three small commutative squares involving Z in the proof of the snake lemma.

8.14. Remark. In Exercise 8.13 consider the special case that $\mathscr{A}$ is the category of modules over a ring R. The morphism v corresponds to the element $v(1) \in K''$ and $u(1) \in A''$ is *some* lift of $\kappa''(v(1))$ over a'. Moreover, the usual diagram chase in the proof of the snake lemma shows that there is an element $w(1) \in B'$ such that $\gamma(w(1))$ is independent of the choice of $u(1)$, hence it makes sense to put $\delta(v(1)) = \gamma(w(1))$. Thus, Exercise 8.13 provides the link to the classical proof of the snake lemma.

9. Chain Complexes and Chain Homotopy

The notion of chain complexes makes sense in every additive category $\mathscr{A}$. A *(chain) complex* is a diagram $(A^{\bullet}, d_A^{\bullet})$

$$\cdots \to A^{n-1} \xrightarrow{d_A^{n-1}} A^n \xrightarrow{d_A^n} A^{n+1} \to \cdots$$

subject to the condition that $d^n d^{n-1} = 0$ for all n and a *chain map* is a morphism of such diagrams. The category of complexes and chain maps is denoted by $\mathbf{Ch}(\mathscr{A})$. Obviously, the category $\mathbf{Ch}(\mathscr{A})$ is additive.

9.1. Lemma. *If $(\mathscr{A}, \mathscr{E})$ is an exact category then $\mathbf{Ch}(\mathscr{A})$ is an exact category with respect to the class $\mathbf{Ch}(\mathscr{E})$ of short sequences of chain maps which are exact in each degree. If $\mathscr{A}$ is abelian then so is $\mathbf{Ch}(\mathscr{A})$.*

Proof. The point is that (as in every functor category) limits and colimits of diagrams in $\mathbf{Ch}(\mathscr{A})$ are obtained by taking the limits and colimits pointwise (in each degree), in particular push-outs under admissible monics and pull-backs over admissible epics exist and yield admissible monics and epics. The rest is obvious. □

9.2. Definition. The *mapping cone* of a chain map $f : A \to B$ is the complex

$$\operatorname{cone}(f)^n = A^{n+1} \oplus B^n \qquad \text{with differential} \qquad d_f^n = \begin{bmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{bmatrix}.$$

Notice that $d_f^{n+1} d_f^n = 0$ precisely because f is a chain map. It is plain that the mapping cone defines a functor from the category of morphisms in $\mathbf{Ch}(\mathscr{A})$ to $\mathbf{Ch}(\mathscr{A})$.

The *translation functor* on $\mathbf{Ch}(\mathscr{A})$ is defined to be $\Sigma A = \operatorname{cone}(A \to 0)$. More explicitly, ΣA is the complex with components $(\Sigma A)^n = A^{n+1}$ and differentials $d_{\Sigma A}^n = -d_A^{n+1}$. If f is a chain map, its translate is given by $(\Sigma f)^n = f^{n+1}$. Clearly, Σ is an additive automorphism of $\mathbf{Ch}(\mathscr{A})$.

The *strict triangle* over the chain map $f : A \to B$ is the 3-periodic (or rather 3-helicoidal, if you insist) sequence

$$A \xrightarrow{f} B \xrightarrow{i_f} \operatorname{cone}(f) \xrightarrow{j_f} \Sigma A \xrightarrow{\Sigma f} \Sigma B \xrightarrow{\Sigma i_f} \Sigma \operatorname{cone}(f) \xrightarrow{\Sigma j_f} \cdots,$$

where the chain map i_f has components $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and j_f has components $\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]$.

9.3. Remark. Let $f : A \to B$ be a chain map. Observe that the sequence of chain maps

$$B \xrightarrow{i_f} \operatorname{cone}(f) \xrightarrow{j_f} \Sigma A$$

splits in each degree, however it need not be a split exact sequence in $\mathbf{Ch}(\mathscr{A})$, because the degreewise splitting maps need not assemble to chain maps. In fact, it is straightforward to verify that the above sequence is split exact in $\mathbf{Ch}(\mathscr{A})$ if and only if f is chain homotopic to zero in the sense of Definition 9.5.

9.4. Exercise. Assume that $\mathscr{A}$ is an abelian category. Prove that the strict triangle over the chain map $f : A \to B$ gives rise to a long exact homology sequence

$$\cdots \to H^n(A) \xrightarrow{H^n(f)} H^n(B) \xrightarrow{H^n(i_f)} H^n(\operatorname{cone}(f)) \xrightarrow{H^n(j_f)} H^{n+1}(A) \to \cdots.$$

Deduce that f induces an isomorphism of $H^*(A)$ with $H^*(B)$ if and only if $\operatorname{cone}(f)$ is acyclic.

9.5. Definition. A chain map $f : A \to B$ is *chain homotopic to zero* if there exist morphisms $h^n : A^n \to B^{n-1}$ such that $f^n = d_B^{n-1} h^n + h^{n+1} d_A^n$. A chain complex A is called *null-homotopic* if 1_A is chain homotopic to zero.

9.6. Remark. The maps which are chain homotopic to zero form an ideal in $\mathbf{Ch}(\mathscr{A})$, that is to say if $h : B \to C$ is chain homotopic to zero then so are hf and gh for all morphisms $f : A \to B$ and $g : C \to D$, if h_1 and h_2 are chain homotopic to zero then so is $h_1 \oplus h_2$. The set $N(A, B)$ of chain maps $A \to B$ which are chain homotopic to zero is a subgroup of the abelian group $\operatorname{Hom}_{\mathbf{Ch}(\mathscr{A})}(A, B)$.

9.7. Definition. The *homotopy category* $\mathbf{K}(\mathscr{A})$ is the category with the chain complexes over $\mathscr{A}$ as objects and $\operatorname{Hom}_{\mathbf{K}(\mathscr{A})}(A,B) := \operatorname{Hom}_{\mathbf{Ch}(\mathscr{A})}(A,B)/N(A,B)$ as morphisms.

9.8. Remark. Notice that the null-homotopic complexes are isomorphic to the zero object in $\mathbf{K}(\mathscr{A})$ (the converse is *not* true if $\mathscr{A}$ fails to be idempotent complete, see Proposition 10.9). It turns out that $\mathbf{K}(\mathscr{A})$ is additive, but it is very rarely abelian or exact with respect to a non-trivial exact structure (see Verdier [38, Ch.II, 1.3.6]). However, $\mathbf{K}(\mathscr{A})$ has the structure of a *triangulated category* induced by the *strict triangles* in $\mathbf{Ch}(\mathscr{A})$, see e.g. Verdier [38], Beĭlinson-Bernstein-Deligne [3], Gelfand-Manin [17], Grivel [6, Chapter I], Kashiwara-Schapira [22], Keller [24], Neeman [30] or Weibel [39].

9.9. Remark. For each object $A \in \mathscr{A}$, define $\operatorname{cone}(A) = \operatorname{cone}(1_A)$. Notice that $\operatorname{cone}(A)$ is null-homotopic with $\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ as contracting homotopy.

9.10. Remark. If f and g are chain homotopy equivalent, i.e., $f - g$ is chain homotopic to zero, then $\operatorname{cone}(f)$ and $\operatorname{cone}(g)$ are isomorphic in $\mathbf{Ch}(\mathscr{A})$ but the isomorphism and its homotopy class will generally depend on the choice of a chain homotopy. In particular, the mapping cone construction does not yield a functor defined on morphisms of $\mathbf{K}(\mathscr{A})$.

9.11. Remark. A chain map $f : A \to B$ is chain homotopic to zero if and only if it factors as $hi_A = f$ over $h : \operatorname{cone}(A) \to B$, where $i_A = i_{1_A} : A \to \operatorname{cone}(A)$. Moreover, h has components $\left[\begin{smallmatrix} h^{n+1} & f^n \end{smallmatrix}\right]$, where the family of morphisms $\{h^n\}$ is a chain homotopy of f to zero. Similarly, f is chain homotopic to zero if and only if f factors through $j_{\Sigma^{-1}B} = j_{1_{\Sigma^{-1}B}} : \operatorname{cone}(\Sigma^{-1}B) \to B$.

9.12. Remark. The mapping cone construction yields the push-out diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle i_A}\downarrow & \text{PO} & \downarrow{\scriptstyle i_f} \\ \operatorname{cone}(A) & \xrightarrow[\left[\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right]]{} & \operatorname{cone}(f) \end{array}$$

in $\mathbf{Ch}(\mathscr{A})$. Now suppose that $g : B \to C$ is a chain map such that gf is chain homotopic to zero. By Remark 9.11, gf factors over i_A and using the push-out property of the above diagram it follows that g factors over i_f. This construction will depend on the choice of an explicit chain homotopy $gf \simeq 0$ in general. In particular, $\operatorname{cone}(f)$ is a *weak cokernel* in $\mathbf{K}(\mathscr{A})$ of the homotopy class of f in that it has the factorization property of a cokernel but without uniqueness. Similarly, $\Sigma^{-1}\operatorname{cone}(f)$ is a *weak kernel* of f in $\mathbf{K}(\mathscr{A})$.

10. Acyclic Complexes and Quasi-Isomorphisms

The present section is probably only of interest to readers acquainted with triangulated categories or at least with the construction of the derived category of an abelian category. After giving the fundamental definition of acyclicity of a complex over an exact category, we may formulate the intimately connected notion of quasi-isomorphisms.

We will give an elementary proof of the fact that the homotopy category $\mathbf{Ac}(\mathscr{A})$ of acyclic complexes over an exact category $\mathscr{A}$ is a triangulated category. It turns out that $\mathbf{Ac}(\mathscr{A})$ is a strictly full subcategory of the homotopy category of chain complexes $\mathbf{K}(\mathscr{A})$ if and only if $\mathscr{A}$ is idempotent complete, and in this case $\mathbf{Ac}(\mathscr{A})$

is even thick in $\mathbf{K}(\mathscr{A})$. Since thick subcategories are strictly full by definition, $\mathbf{Ac}(\mathscr{A})$ is thick if and only if $\mathscr{A}$ is idempotent complete.

By [30, Chapter 2], the Verdier quotient $\mathbf{K}/\mathscr{T}$ is defined for any strictly full triangulated subcategory $\mathscr{T}$ of a triangulated category $\mathbf{K}$ and it coincides with the Verdier quotient $\mathbf{K}/\bar{\mathscr{T}}$, where $\bar{\mathscr{T}}$ is the *thick closure* of $\mathscr{T}$. The case we are interested in is $\mathbf{K} = \mathbf{K}(\mathscr{A})$ and $\mathscr{T} = \mathbf{Ac}(\mathscr{A})$. The Verdier quotient $\mathbf{D}(\mathscr{A}) = \mathbf{K}(\mathscr{A})/\overline{\mathbf{Ac}(\mathscr{A})}$ is the *derived category* of $\mathscr{A}$. If $\mathscr{A}$ is idempotent complete then $\overline{\mathbf{Ac}(\mathscr{A})} = \mathbf{Ac}(\mathscr{A})$ and it is clear that quasi-isomorphisms are then precisely the chain maps with acyclic mapping cone. If $\mathscr{A}$ fails to be idempotent complete, it turns out that the thick closure $\overline{\mathbf{Ac}(\mathscr{A})}$ of $\mathbf{Ac}(\mathscr{A})$ is the same as the closure of $\mathbf{Ac}(\mathscr{A})$ under isomorphisms in $\mathbf{K}(\mathscr{A})$, so a chain map f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is homotopy equivalent to an acyclic complex.

Similarly, the derived categories of bounded, left bounded or right bounded complexes are constructed as in the abelian setting. It is useful to notice that for $* \in \{+, -, b\}$ the category $\mathbf{Ac}^*(\mathscr{A})$ is thick in $\mathbf{K}^*(\mathscr{A})$ if and only if $\mathscr{A}$ is weakly idempotent complete, which leads to an easier description of quasi-isomorphisms.

10.1. **The Homotopy Category of Acyclic Complexes.**

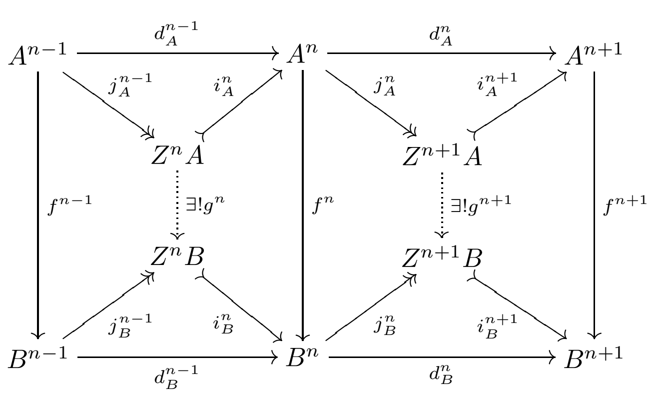
10.1. Definition. A chain complex A over an exact category is called *acyclic* if each differential factors as $A^n \twoheadrightarrow Z^{n+1}A \rightarrowtail A^{n+1}$ in such a way that each sequence $Z^nA \rightarrowtail A^n \twoheadrightarrow Z^{n+1}A$ is exact.

10.2. Remark. An acyclic complex is a complex with admissible differentials (Definition 8.1) which is exact in the sense of Definition 8.8. In particular, Z^nA is a kernel of $A^n \to A^{n+1}$, an image and coimage of $A^{n-1} \to A^n$ and a cokernel of $A^{n-2} \to A^{n-1}$.

The following lemma is due to Neeman. His proof relies on the embedding theorem for exact categories. We prefer to give an elementary proof, which should be compared to the proof of Theorem 12.8.

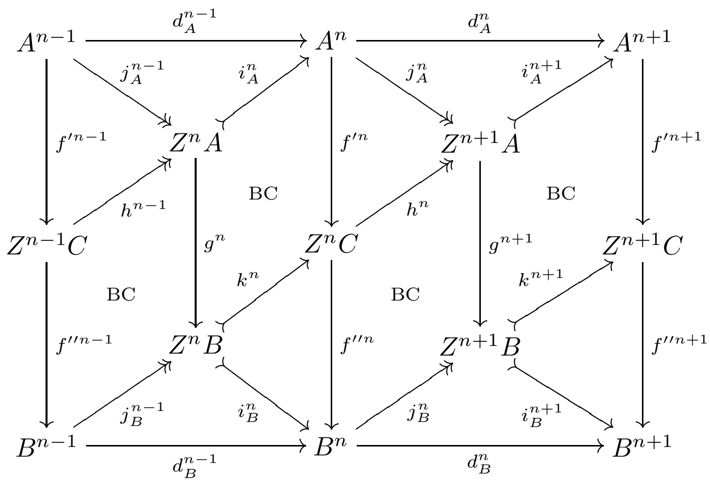
10.3. Lemma ([29, 1.1]). *The mapping cone of a chain map $f : A \to B$ between acyclic complexes is acyclic.*

Proof. An easy diagram chase shows that the dotted morphisms in the diagram



exist and are the unique morphisms g^n making the diagram commutative.

By Proposition 3.1 we find objects $Z^n C$ fitting into a commutative diagram

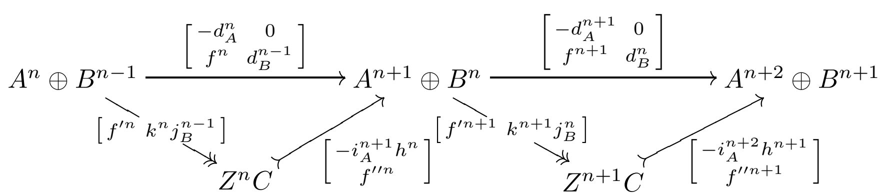


where $f^n = f''^n f'^n$ and the quadrilaterals marked BC are bicartesian. Recall that the objects $Z^n C$ are obtained by forming the push-outs under i_A^n and g^n (or the pull-backs over j_B^n and g^{n+1}) and that $Z^n B \rightarrowtail Z^n C \twoheadrightarrow Z^{n+1} A$ is short exact.

It follows from Corollary 2.13 that for each n the sequence

$$Z^{n-1}C \xrightarrow{\left[\begin{smallmatrix} -i_A^n h^{n-1} \\ f''^{n-1} \end{smallmatrix}\right]} A^n \oplus B^{n-1} \xrightarrow{\left[\begin{smallmatrix} f'^n & k^n j_B^{n-1} \end{smallmatrix}\right]} Z^n C$$

is short exact and the commutative diagram



proves that cone (f) is acyclic. □

10.4. Remark. Retaining the notations of the proof we have a short exact sequence

$$Z^n B \rightarrowtail Z^n C \twoheadrightarrow Z^{n+1} A.$$

This sequence exhibits $Z^n C = \operatorname{Ker} \left[\begin{smallmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{smallmatrix}\right]$ as an extension of $Z^{n+1} A = \operatorname{Ker} d_A^{n+1}$ by $Z^n B = \operatorname{Ker} d_B^n$.

Let $\mathbf{Ac}(\mathscr{A})$ be the full subcategory of the homotopy category $\mathbf{K}(\mathscr{A})$ consisting of acyclic complexes over the exact category $\mathscr{A}$. It follows from Proposition 2.9 that the direct sum of two acyclic complexes is acyclic. Thus $\mathbf{Ac}(\mathscr{A})$ is a full additive subcategory of $\mathbf{K}(\mathscr{A})$. The previous lemma implies that even more is true:

10.5. Corollary. *The homotopy category of acyclic complexes $\mathbf{Ac}(\mathscr{A})$ is a triangulated subcategory of $\mathbf{K}(\mathscr{A})$.* □

10.6. Remark. For reasons of convenience, many authors assume that triangulated subcategories are not only full but *strictly full*. We do not do so because $\mathbf{Ac}(\mathscr{A})$ is closed under isomorphisms in $\mathbf{K}(\mathscr{A})$ if and only if $\mathscr{A}$ is idempotent complete, see Proposition 10.9.

10.7. Lemma. *Assume that $(\mathscr{A}, \mathscr{E})$ is idempotent complete. Every retract in $\mathbf{K}(\mathscr{A})$ of an acyclic complex A is acyclic.*

PROOF (CF. [23, 2.3 a)]). Let the chain map $f : X \to A$ be a retraction, i.e., there is a chain map $s : A \to X$ such that $s^n f^n - 1_{X^n} = d_X^{n-1} h^n + h^{n+1} d_X^n$ for some morphisms $h^n : X^n \to X^{n-1}$. Obviously, the complex IX with components

$$(IX)^n = X^n \oplus X^{n+1} \qquad \text{and differential} \qquad \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$$

is acyclic. There is a chain map $i_X : X \to IX$ given by

$$i_X^n = \begin{bmatrix} 1_{X^n} \\ d_X^n \end{bmatrix} : X^n \to X^n \oplus X^{n+1}$$

and the chain map

$$\begin{bmatrix} f \\ i_X \end{bmatrix} : X \to A \oplus IX$$

has the chain map

$$\left[\begin{smallmatrix} s^n & -d_X^{n-1} h^n & -h^{n+1} \end{smallmatrix}\right] : A^n \oplus X^n \oplus X^{n+1} \to X^n$$

as a left inverse. Hence, on replacing the acyclic complex A by the acyclic complex $A \oplus IX$, we may assume that $f : X \to A$ has s as a right inverse in $\mathbf{Ch}(\mathscr{A})$. But then $e = fs : A \to A$ is an idempotent in $\mathbf{Ch}(\mathscr{A})$ and it induces an idempotent on the exact sequences $Z^n A \rightarrowtail A^n \twoheadrightarrow Z^{n+1} A$ witnessing that A is acyclic as in the first diagram of the proof of Lemma 10.3. This means that $Z^n A \rightarrowtail A^n \twoheadrightarrow Z^{n+1} A$ decomposes as a direct sum of two short exact sequences (Corollary 2.17) since $\mathscr{A}$ is idempotent complete. Therefore the acyclic complex $A = X' \oplus Y'$ is a direct sum of the acyclic complexes X' and Y', and $f :$ induces an isomorphism from X to X' in $\mathbf{Ch}(\mathscr{A})$. The details are left to the reader. ☐

10.8. EXERCISE. Prove that the sequence $X \to \operatorname{cone}(X) \to \Sigma X$ from Remark 9.3 is isomorphic to a sequence $X \to IX \to \Sigma X$ in $\mathbf{Ch}(\mathscr{A})$.

10.9. PROPOSITION ([24, 11.2]). *The following are equivalent:*

(i) Every null-homotopic complex in $\mathbf{Ch}(\mathscr{A})$ is acyclic.
(ii) The category $\mathscr{A}$ is idempotent complete.
(iii) The class of acyclic complexes is closed under isomorphisms in $\mathbf{K}(\mathscr{A})$.

PROOF (KELLER). Let us prove that (i) implies (ii). Let $e : A \to A$ be an idempotent of $\mathscr{A}$. Consider the complex

$$\cdots \xrightarrow{1-e} A \xrightarrow{e} A \xrightarrow{1-e} A \xrightarrow{e} \cdots$$

which is null-homotopic. By (i) this complex is acyclic. This means by definition that e has a kernel and hence $\mathscr{A}$ is idempotent complete.

Let us prove that (ii) implies (iii). Assume that X is isomorphic in $\mathbf{K}(\mathscr{A})$ to an acyclic complex A. Using the construction in the proof of Lemma 10.7 one shows that X is a direct summand in $\mathbf{Ch}(\mathscr{A})$ of the acyclic complex $A \oplus IX$ and we conclude by Lemma 10.7.

That (iii) implies (i) follows from the fact that a null-homotopic complex X is isomorphic in $\mathbf{K}(\mathscr{A})$ to the (acyclic) zero complex and hence X is acyclic. ☐

10.10. REMARK. Recall that a subcategory $\mathscr{T}$ of a triangulated category $\mathbf{K}$ is called *thick* if it is strictly full and $X \oplus Y \in \mathscr{T}$ implies $X, Y \in \mathscr{T}$.

10.11. COROLLARY. *The triangulated subcategory $\mathbf{Ac}(\mathscr{A})$ of $\mathbf{K}(\mathscr{A})$ is thick if and only if $\mathscr{A}$ is idempotent complete.* ☐

10.2. **Boundedness Conditions.** A complex A is called *left bounded* if $A^n = 0$ for $n \ll 0$, *right bounded* if $A^n = 0$ for $n \gg 0$ and *bounded* if $A^n = 0$ for $|n| \gg 0$.

10.12. Definition. Denote by $\mathbf{K}^+(\mathscr{A})$, $\mathbf{K}^-(\mathscr{A})$ and $\mathbf{K}^b(\mathscr{A})$ the full subcategories of $\mathbf{K}(\mathscr{A})$ generated by the left bounded complexes, right bounded complexes and bounded complexes over $\mathscr{A}$.

Observe that $\mathbf{K}^b(\mathscr{A}) = \mathbf{K}^+(\mathscr{A}) \cap \mathbf{K}^-(\mathscr{A})$. Note further that $\mathbf{K}^*(\mathscr{A})$ is *not* closed under isomorphisms in $\mathbf{K}(\mathscr{A})$ for $* \in \{+,-,b\}$ unless $\mathscr{A} = 0$.

10.13. Definition. For $* \in \{+,-,b\}$ we define $\mathbf{Ac}^*(\mathscr{A}) = \mathbf{K}^*(\mathscr{A}) \cap \mathbf{Ac}(\mathscr{A})$.

Plainly, $\mathbf{K}^*(\mathscr{A})$ is a full triangulated subcategory of $\mathbf{K}(\mathscr{A})$ and $\mathbf{Ac}^*(\mathscr{A})$ is a full triangulated subcategory of $\mathbf{K}^*(\mathscr{A})$ by Lemma 10.3.

10.14. Proposition. *The following assertions are equivalent:*

(i) *The subcategories $\mathbf{Ac}^+(\mathscr{A})$ and $\mathbf{Ac}^-(\mathscr{A})$ of $\mathbf{K}^+(\mathscr{A})$ and $\mathbf{K}^-(\mathscr{A})$ are thick.*
(ii) *The subcategory $\mathbf{Ac}^b(\mathscr{A})$ of $\mathbf{K}^b(\mathscr{A})$ is thick.*
(iii) *The category $\mathscr{A}$ is weakly idempotent complete.*

Proof. Since $\mathbf{Ac}^b(\mathscr{A}) = \mathbf{Ac}^+(\mathscr{A}) \cap \mathbf{Ac}^-(\mathscr{A})$, we see that (i) implies (ii). Let us prove that (ii) implies (iii). Assume that $\mathscr{A}$ is not weakly idempotent complete, so there is an idempotent $p : A \to A$ which factors as $p = st$ with $ts = 1_B$ in such a way that s has no cokernel and t has no kernel. The complex X given by

$$\cdots \to 0 \to B \xrightarrow{s} A \xrightarrow{1-st} A \xrightarrow{t} B \to 0 \to \cdots$$

is not acyclic because s has no cokernel and t has no kernel by hypothesis. However, X is a direct summand of $X \oplus \Sigma X$ and we claim that $X \oplus \Sigma X$ is acyclic, so if $\mathscr{A}$ is not weakly idempotent complete, (ii) does not hold. Indeed, there is an isomorphism in $\mathbf{Ch}(\mathscr{A})$

$$\begin{array}{ccccccccc}
B & \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & B \oplus A & \xrightarrow{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]} & A \oplus A & \xrightarrow{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]} & A \oplus B & \xrightarrow{[0\ 1]} & B \\
{\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix}0&-t\\s&1-st\end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}-1+st&st\\st&1-st\end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix}1-st&-s\\t&0\end{smallmatrix}\right]} & & \downarrow{\scriptstyle 1} \\
B & \xrightarrow[\left[\begin{smallmatrix}0\\s\end{smallmatrix}\right]]{} & B \oplus A & \xrightarrow[\left[\begin{smallmatrix}-s&0\\0&1-st\end{smallmatrix}\right]]{} & A \oplus A & \xrightarrow[\left[\begin{smallmatrix}-1+st&0\\0&t\end{smallmatrix}\right]]{} & A \oplus B & \xrightarrow[[-t\ 0]]{} & B
\end{array}$$

where the upper row is obviously acyclic and the lower row is isomorphic to $X \oplus \Sigma X$.

Let us prove that (iii) implies (i). Assume that X is a direct summand in $\mathbf{K}^+(\mathscr{A})$ of a complex $A \in \mathbf{Ac}^+(\mathscr{A})$. This means in particular that we are given a chain map $f : X \to A$ for which there exists a chain map $s : A \to X$ and morphisms $h^n : X^n \to X^{n-1}$ such that $s^n f^n - 1_{X^n} = d_X^{n-1} h^n + h^{n+1} d_X^n$. On replacing A by the acyclic complex $A \oplus IX$ as in the proof of Proposition 10.9, we may assume that s is a left inverse of f in $\mathbf{Ch}^+(\mathscr{A})$. In particular, since $\mathscr{A}$ is assumed to be weakly idempotent complete, Proposition 7.5 implies that each f^n is an admissible monic and that each s^n is an admissible epic. Moreover, as both complexes X and A are left bounded, we may assume that $A^n = 0 = X^n$ for $n < 0$. It follows that $d_A^0 : A^0 \rightarrowtail A^1$ is an admissible monic since A is acyclic. But then $d_A^0 f^0 = d_X^0 f^1$ is an admissible monic, hence Proposition 7.5 implies that d_X^0 is an admissible monic as well. Let $e_X^1 : X^1 \twoheadrightarrow Z^2 X$ be a cokernel of d_X^0 and let $e_A^1 : A^1 \twoheadrightarrow Z^2 A$ be a cokernel of d_A^0. The dotted morphisms in the diagram

$$\begin{array}{ccccc}
X^0 & \overset{d_X^0}{\rightarrowtail} & X^1 & \overset{e_X^1}{\twoheadrightarrow} & Z^2X \\
\downarrow f^0 & & \downarrow f^1 & & \vdots\, g^2 \\
A^0 & \overset{d_A^0}{\rightarrowtail} & A^1 & \overset{e_A^1}{\twoheadrightarrow} & Z^2A \\
\downarrow s^0 & & \downarrow s^1 & & \vdots\, t^2 \\
X^0 & \overset{d_X^0}{\rightarrowtail} & X^1 & \overset{e_X^1}{\twoheadrightarrow} & Z^2X
\end{array}$$

are uniquely determined by requiring that the resulting diagram be commutative. By the 3×3-lemma 3.6 the third column is short exact. Since $s^0 f^0 = 1_{X^0}$ and $s^1 f^1 = 1_{X^1}$ it follows that $t^2 g^2 = 1_{Z^2 X}$.

Now since A and X are complexes, there are unique maps $m_X^2 : Z^2X \to X^2$ and $m_A^2 : Z^2A \to A^2$ such that $d_X^1 = m_X^2 e_X^1$ and $d_A^1 = m_A^2 e_A^1$. Note that m_A^2 is an admissible monic since A is acyclic. The upper square in the diagram

$$\begin{array}{ccc}
Z^2X & \xrightarrow{m_X^2} & X^2 \\
\downarrow g^2 & & \downarrow f^2 \\
Z^2A & \overset{m_A^2}{\rightarrowtail} & A^2 \\
\downarrow t^2 & & \downarrow s^2 \\
Z^2X & \xrightarrow{m_X^2} & X^2
\end{array}$$

is commutative because e_X^1 is epic and the lower square is commutative because e_A^2 is epic. From the commutativity of the upper square it follows in particular that m_X^2 is an admissible monic by Proposition 7.5. An easy induction now shows that X is acyclic. The assertion about $\mathbf{Ac}^-(\mathscr{A})$ follows by duality. □

10.3. **Quasi-Isomorphisms.** In abelian categories, quasi-isomorphisms are defined to be chain maps inducing an isomorphism in homology. Taking the observation in Exercise 9.4 and Proposition 10.9 into account, one arrives at the following generalization for general exact categories:

10.15. Definition. A chain map $f : A \to B$ is called a *quasi-isomorphism* if its mapping cone is homotopy equivalent to an acyclic complex.

10.16. Remark. Assume that $\mathscr{A}$ is idempotent complete. By Proposition 10.9, a chain map f is a quasi-isomorphism if and only if cone(f) is acyclic. In particular, for abelian categories, the a quasi-isomorphism is the same thing as a chain map inducing an isomorphism on homology.

10.17. Remark. If $p : \mathscr{A} \to \mathscr{A}$ is an idempotent in $\mathscr{A}$ which does not split, then the complex C given by

$$\cdots \xrightarrow{1-p} A \xrightarrow{p} A \xrightarrow{1-p} A \xrightarrow{p} \cdots$$

is null-homotopic but *not* acyclic. However, $f : 0 \to C$ is a chain homotopy equivalence, hence it should be a quasi-isomorphism, but cone$(f) = C$ fails to be acyclic.

10.4. **The Definition of the Derived Category.** The *derived category* of the exact category of $\mathscr{A}$ is defined to be the *Verdier quotient*

$$\mathbf{D}(\mathscr{A}) = \mathbf{K}(\mathscr{A}) / \mathbf{Ac}(\mathscr{A})$$

as described e.g. in Neeman [30, Chapter 2] or Keller [24, §§ 10, 11].

When dealing with the boundedness conditions $* \in \{+, -, b\}$ we define

$$\mathbf{D}^* (\mathscr{A}) = \mathbf{K}^* (\mathscr{A}) / \mathbf{Ac}^* (\mathscr{A}).$$

It is not difficult to prove that the canonical functor $\mathbf{D}^* (\mathscr{A}) \to \mathbf{D} (\mathscr{A})$ is an equivalence between $\mathbf{D}^* (\mathscr{A})$ and the full subcategory of $\mathbf{D} (\mathscr{A})$ generated by the complexes satisfying the boundedness condition $*$, see Keller [24, 11.7].

10.18. Remark. If $\mathscr{A}$ is idempotent complete then a chain map becomes an isomorphism in $\mathbf{D} (\mathscr{A})$ if and only if its cone is acyclic by Corollary 10.11. If $\mathscr{A}$ is weakly idempotent complete then a chain map in $\mathbf{Ch}^* (\mathscr{A})$ becomes an isomorphism in $\mathbf{D}^* (\mathscr{A})$ if and only if its cone is acyclic by Proposition 10.14.

10.19. Remark. With these constructions at hand one can now introduce (total) derived functors in the sense of Grothendieck-Verdier, see e.g. Keller [24, §§13-15] or any one of the references given in Remark 9.8.

11. Projective and Injective Objects

11.1. Definition. An object P of an exact category $\mathscr{A}$ is called *projective* if the represented functor $\mathrm{Hom}_{\mathscr{A}} (P, -) : \mathscr{A} \to \mathbf{Ab}$ is exact. An object I of an exact category is called *injective* if the corepresented functor $\mathrm{Hom}_{\mathscr{A}} (-, I) : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ is exact.

11.2. Remark. The concepts of projectivity and injectivity are dual to each other in the sense that P is projective in $\mathscr{A}$ if and only if P is injective in $\mathscr{A}^{\mathrm{op}}$. For our purposes it is therefore sufficient to deal with projective objects.

11.3. Proposition. *An object P of an exact category is projective if and only if any one of the following conditions holds:*

(i) *For all admissible epics $A \twoheadrightarrow A''$ and all morphisms $P \to A''$ there exists a solution to the lifting problem*

$$\begin{array}{ccc} & P & \\ {\scriptstyle\exists}\swarrow & & \searrow \\ A & \twoheadrightarrow & A'' \end{array}$$

making the diagram above commutative.

(ii) *For every admissible epic $A \twoheadrightarrow A''$ the induced morphism of abelian groups $\mathrm{Hom}_{\mathscr{A}} (P, A) \to \mathrm{Hom}_{\mathscr{A}} (P, A'')$ is surjective.*

(iii) *Every admissible epic $A \twoheadrightarrow P$ splits (has a right inverse).*

Proof. Since $\mathrm{Hom}_{\mathscr{A}} (P, -)$ transforms exact sequences to left exact sequences in $\mathbf{Ab}$ for all objects P (see the proof of Corollary A.6), it is clear that conditions (i) and (ii) are equivalent to the projectivity of P. If P is projective and $A \twoheadrightarrow P$ is an admissible epic then $\mathrm{Hom}_{\mathscr{A}} (P, A) \twoheadrightarrow \mathrm{Hom}_{\mathscr{A}} (P, P)$ is surjective, and every preimage of 1_P is a splitting map of $A \twoheadrightarrow P$. Conversely, let us prove that condition (iii) implies condition (i): given a lifting problem as in (i), form the following pull-back diagram

$$\begin{array}{ccc} D & \xrightarrow{a'} & P \\ {\scriptstyle f'}\downarrow & \text{PB} & \downarrow{\scriptstyle f} \\ A & \xrightarrow{a} & A''. \end{array}$$

By hypothesis, there exists a right inverse b' of a' and $f'b'$ solves the lifting problem because $af'b' = fa'b' = f$. $\square$

11.4. Corollary. *If P is projective and $P \to A$ has a right inverse then A is projective.*

Proof. This is a trivial consequence of condition (i) in Proposition 11.3. □

11.5. Remark. If $\mathscr{A}$ is weakly idempotent complete, the above corollary amounts to the familiar "direct summands of projective objects are projective" in abelian categories.

11.6. Corollary. *A sum $P = P' \oplus P''$ is projective if and only if both P' and P'' are projective.* □

More generally:

11.7. Corollary. *Let $\{P_i\}_{i \in I}$ be a family of objects and assume that the coproduct $P = \coprod_{i \in I} P_i$ exists in $\mathscr{A}$. The object P is projective if and only if each P_i is projective.* □

11.8. Remark. The dual of the previous result is that a product (if it exists) is injective if and only if each of its factors is injective.

12. Resolutions and Classical Derived Functors

12.1. Definition. An exact category $\mathscr{A}$ is said to have *enough projectives* if for every object $A \in \mathscr{A}$ there exists a projective object P and an admissible epic $P \twoheadrightarrow A$.

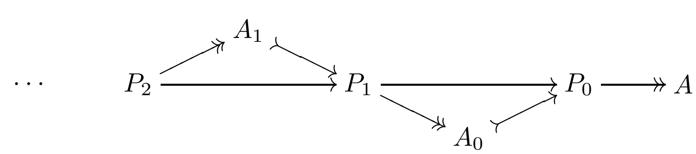
12.2. Definition. A *projective resolution* of the object A is a positive complex with projective components together with a morphism $P_0 \to A$ such that the *augmented complex*

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to A$$

is exact.

12.3. Proposition (Resolution Lemma). *If $\mathscr{A}$ has enough projectives then every object $A \in \mathscr{A}$ has a projective resolution.*

Proof. This is an easy induction. Because $\mathscr{A}$ has enough projectives, there exists a projective object P_0 and an admissible epic $P_0 \twoheadrightarrow A$. Choose an admissible monic $A_0 \rightarrowtail P_0$ such that $A_0 \rightarrowtail P_0 \twoheadrightarrow A$ is exact. Now choose a projective P_1 and an admissible epic $P_1 \twoheadrightarrow A_0$. Continue with an admissible monic $A_1 \rightarrowtail P_1$ such that $A_1 \rightarrowtail P_1 \twoheadrightarrow A_0$ is exact, and so on. One thus obtains a sequence

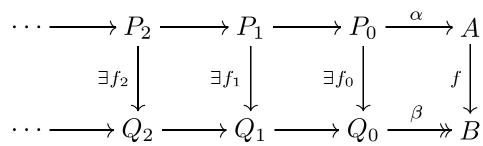


which is exact by construction, so $P_\bullet \to A$ is a projective resolution. □

12.4. Remark. The defining concept of projectivity is not used in the previous proof. That is, we have proved: If $\mathscr{P}$ is a class in $\mathscr{A}$ such that for each object $A \in \mathscr{A}$ there is an admissible epic $P \twoheadrightarrow A$ then each object of $\mathscr{A}$ has a $\mathscr{P}$-resolution $P_\bullet \twoheadrightarrow A$.

Consider a morphism $f : A \to B$ in $\mathscr{A}$. Let $P_\bullet$ be a complex of projectives with $P_n = 0$ for $n < 0$ and let $\alpha : P_0 \to A$ be a morphism such that the composition $P_1 \to P_0 \to A$ is zero [e.g. $P_\bullet \to A$ is a projective resolution of A]. Let $Q_\bullet \xrightarrow{\beta} B$ be a resolution.

12.5. Theorem (Comparison Theorem). *Under the above hypotheses there exists a chain map* $f_\bullet : P_\bullet \to Q_\bullet$ *such that the following diagram commutes:*

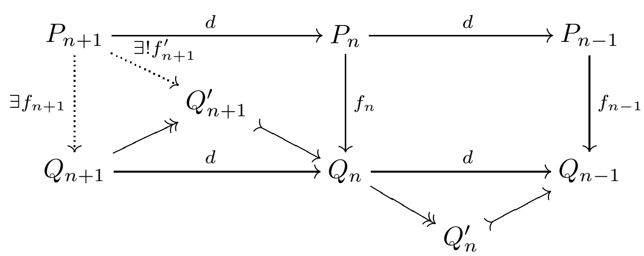


Moreover, the lift $f_\bullet$ *of* f *is unique up to homotopy equivalence.*

Proof. It is convenient to put $P_{-1} = A$, $Q'_0 = Q_{-1} = B$ and $f_{-1} = f$.

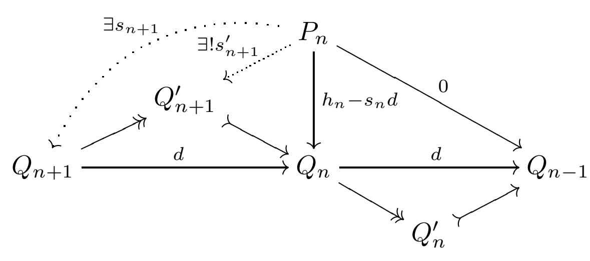
Existence: The question of existence of f_0 is the lifting problem given by the map $f\alpha : P_0 \to B$ and the admissible epic $\beta : Q_0 \twoheadrightarrow B$. This problem has a solution by projectivity of P_0.

Let $n \geq 0$ and suppose by induction that there are morphisms $f_n : P_n \to Q_n$ and $f_{n-1} : P_{n-1} \to Q_{n-1}$ such that $df_n = f_{n-1}d$. Consider the following diagram:



By induction hypothesis, the right hand square is commutative, so the morphism $P_{n+1} \to Q_{n-1}$ is zero because the morphism $P_{n+1} \to P_{n-1}$ is zero. The morphism $P_{n+1} \to Q'_n$ is zero as well because $Q'_n \rightarrowtail Q_{n-1}$ is a mono. Since $Q'_{n+1} \rightarrowtail Q_n \twoheadrightarrow Q'_n$ is exact, there exists a unique morphism $f'_{n+1} : P_{n+1} \to Q'_{n+1}$ making the upper right triangle in the left hand square commute. Because P_{n+1} is projective and $Q_{n+1} \twoheadrightarrow Q'_{n+1}$ is an admissible epi, there is a morphism $f_{n+1} : P_{n+1} \to Q_{n+1}$ such that the left hand square commutes. This settles the existence of $f_\bullet$.

Uniqueness: Let $g_\bullet : P_\bullet \to Q_\bullet$ be another lift of f and put $h_\bullet = f_\bullet - g_\bullet$. We will construct by induction a chain contraction $s_n : P_{n-1} \to Q_n$ for h. For $n \leq 0$ we put $s_n = 0$. For $n \geq 0$ assume by induction that there are morphisms s_{n-1}, s_n such that $h_{n-1} = ds_n + s_{n-1}d$. Because of this assumption and the fact that h is a chain map, we have $d(h_n - s_n d) = h_{n-1}d - (h_{n-1} - s_{n-1}d)d = 0$ so the following diagram commutes



and as in the existence proof we get a morphism $s_{n+1} : P_n \to Q_{n+1}$ such that $ds_{n+1} = h_n - s_n d$. □

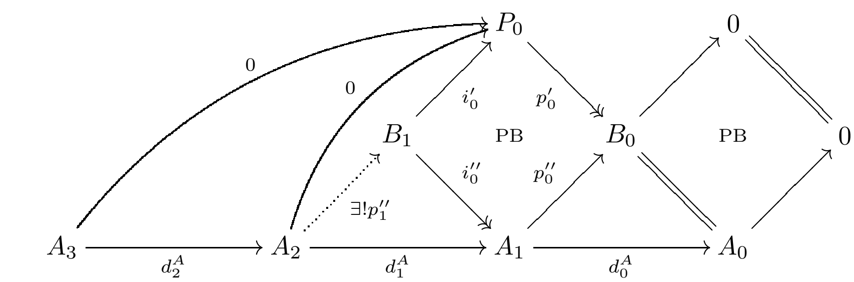
12.6. Corollary. *Any two projective resolutions of an object* A *are chain homotopy equivalent.* □

12.7. Corollary. *Let* $P_\bullet$ *be a right bounded complex of projectives and let* $A_\bullet$ *be an acyclic complex. Then* $\mathrm{Hom}_{\mathbf{K}(\mathscr{A})}(P_\bullet, A_\bullet) = 0$. □

In order to deal with derived functors on the level of the derived category, one needs to sharpen both the resolution lemma and the comparison theorem.

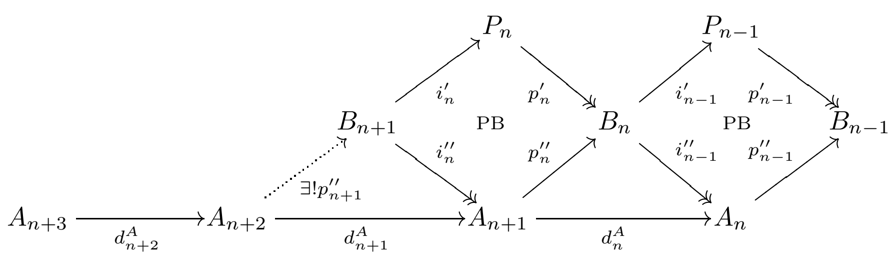
12.8. Theorem ([23, 4.1, Lemma, b)]). *Let $\mathscr{A}$ be an exact category with enough projectives. For every bounded above complex $A \in \mathbf{Ch}^-(\mathscr{A})$ there exists a complex with projective components $P \in \mathbf{Ch}^-(\mathscr{A})$ and a quasi-isomorphism $P \xrightarrow{\alpha} A$.*

Proof. Renumbering if necessary, we may suppose $A_n = 0$ for $n < 0$. The complex P will be constructed by induction. For the inductive formulation it is convenient to define $P_n = B_n = 0$ for $n < 0$. Put $B_0 = A_0$, choose an admissible epi $p'_0 : P_0 \twoheadrightarrow B_0$ from a projective P_0 and define $p''_0 = d^A_0$. Let B_1 be the pull-back over p'_0 and p''_0. Consider the following commutative diagram:



The morphism p''_1 exists by the universal property of the pull-back and moreover $p''_1 d^A_2 = 0$ because $d^A_1 d^A_2 = 0$.

Suppose by induction that in the following diagram everything is constructed except B_{n+1} and the morphisms terminating or issuing from there. Assume further that P_n is projective and that $p''_n d^A_{n+1} = 0$.



As indicated in the diagram, we obtain B_{n+1} by forming the pull-back over p'_n and p''_n. We complete the induction by choosing an admissible epi $p'_{n+1} : P_{n+1} \twoheadrightarrow B_{n+1}$ from a projective P_{n+1}, constructing p''_{n+1} as in the first paragraph and finally noticing that $p''_{n+1} d^A_{n+2} = 0$.

The projective complex is given by the P_n's and the differential $d^P_{n-1} = i'_{n-1} p'_n$, which satisfies $(d^P)^2 = 0$ by construction.

Let α be given by $\alpha_n = i''_{n-1} p'_n$ in degree n, manifestly a chain map. We claim that α is a quasi-isomorphism. The mapping cone of α is seen to be exact using Proposition 2.12: For each n there is an exact sequence

$$B_{n+1} \xrightarrow{i_n = \left[\begin{smallmatrix} -i'_n \\ i''_n \end{smallmatrix}\right]} P_n \oplus A_{n+1} \xrightarrow{p_n = \left[\begin{smallmatrix} p'_n & p''_n \end{smallmatrix}\right]} B_n.$$

We thus obtain an exact complex C with $C_n = P_n \oplus A_{n+1}$ in degree n and differential

$$d^C_{n-1} = i_{n-1} p_n = \begin{bmatrix} -i'_{n-1} p'_n & -i'_{n-1} p''_n \\ i''_{n-1} p'_n & i''_{n-1} p''_n \end{bmatrix} = \begin{bmatrix} -d^P_{n-1} & 0 \\ \alpha_n & d^A_n \end{bmatrix}$$

which shows that $C = \operatorname{cone}(\alpha)$. □

12.9. Theorem (Horseshoe Lemma). *A horseshoe can be filled in: Suppose we are given a horseshoe diagram*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2' & \longrightarrow & P_1' & \longrightarrow & P_0' & \twoheadrightarrow & A' \\
 & & & & & & & & \downarrow \\
 & & & & & & & & A \\
 & & & & & & & & \downarrow \\
\cdots & \longrightarrow & P_2'' & \longrightarrow & P_1'' & \longrightarrow & P_0'' & \twoheadrightarrow & A'',
\end{array}$$

that is to say, the column is short exact and the horizontal rows are projective resolutions of A' and A''. Then the direct sums $P_n = P_n' \oplus P_n''$ assemble to a projective resolution of A in such a way that the horseshoe can be embedded into a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2' & \longrightarrow & P_1' & \longrightarrow & P_0' & \twoheadrightarrow & A' \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \twoheadrightarrow & A \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P_2'' & \longrightarrow & P_1'' & \longrightarrow & P_0'' & \twoheadrightarrow & A''.
\end{array}$$

12.10. Remark. All the columns except the rightmost one are split exact. However, the morphisms $P_{n+1} \to P_n$ are *not* the sums of the morphisms $P_{n+1}' \to P_n'$ and $P_{n+1}'' \to P_n''$. This only happens in the trivial case that the sequence $A' \rightarrowtail A \twoheadrightarrow A''$ is already split exact.

Proof. This is an easy application of the five lemma 3.2 and the 3×3-lemma 3.6. By lifting the morphism $\varepsilon'' : P_0'' \to A''$ over $A \twoheadrightarrow A''$ we obtain a morphism $\varepsilon : P_0 \to A$ and a commutative diagram

$$\begin{array}{ccccc}
\operatorname{Ker}\varepsilon' & \rightarrowtail & P_0' & \overset{\varepsilon'}{\twoheadrightarrow} & A' \\
\vdots & & \downarrow \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & & \downarrow \\
\operatorname{Ker}\varepsilon & \rightarrowtail & P_0 & \overset{\varepsilon}{\longrightarrow} & A \\
\vdots & & \downarrow [0\ 1] & & \downarrow \\
\operatorname{Ker}\varepsilon'' & \rightarrowtail & P_0'' & \overset{\varepsilon''}{\twoheadrightarrow} & A''.
\end{array}$$

It follows from the five lemma that ε is actually an admissible epic, so its kernel exists. The two vertical dotted morphisms exist since the second and the third row are short exact. Now the 3×3-lemma implies that the dotted column is short exact. Finally note that $P_1' \to P_0'$ and $P_1'' \to P_0''$ factor over admissible epics to $\operatorname{Ker}\varepsilon'$ and $\operatorname{Ker}\varepsilon''$ and proceed by induction. □

12.11. Remark. In concrete situations it may be useful to remember that only the projectivity of P_n'' is used in the proof.

12.12. Remark (Classical Derived Functors). Using the results of this section, the theory of classical derived functors, see e.g. Cartan-Eilenberg [10], Mac Lane [26], Hilton-Stammbach [20] or Weibel [39], is easily adapted to the following situation:

Let $(\mathscr{A}, \mathscr{E})$ be an exact category with enough projectives and let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor to an abelian category. By the resolution lemma 12.3 a projective resolution $P_\bullet \twoheadrightarrow A$ exists for every object $A \in \mathscr{A}$ and is well-defined up to homotopy equivalence by the comparison theorem (Corollary 12.6). It follows that for two projective resolutions $P_\bullet \twoheadrightarrow A$ and $Q_\bullet \twoheadrightarrow A$ the complexes $F(P_\bullet)$ and $F(Q_\bullet)$ are chain homotopy equivalent. Therefore it makes sense to define the *left derived functors* of F as

$$L_iF(A) := H_i(F(P_\bullet)).$$

Let us indicate why $L_iF(A)$ is a functor. First observe that a morphism $f : A \to A'$ extends uniquely up to chain homotopy equivalence to a chain map $f_\bullet : P_\bullet \to P'_\bullet$ if $P_\bullet \twoheadrightarrow A$ and $P'_\bullet \twoheadrightarrow A'$ are projective resolutions of A and A'. From this uniqueness it follows easily that $L_iF(fg) = L_iF(f)L_iF(g)$ and $L_iF(1_A) = 1_{L_iF(A)}$ as desired. Using the horseshoe lemma 12.9 one proves that a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ yields a long exact sequence

$$\cdots \to L_{i+1}F(A'') \to L_iF(A') \to L_iF(A) \to L_iF(A'') \to L_{i-1}F(A') \to \cdots$$

and that L_0F sends exact sequences to right exact sequences in $\mathscr{B}$ so that the L_iF are a universal δ-functor. Moreover, L_0F is characterized by being the best left exact approximation to F and the L_iF measure the failure of L_0F to be exact. In particular, if F sends exact sequences to right exact sequences then $L_0F \cong F$ and if F is exact, then in addition $L_iF = 0$ if $i > 0$.

13. Examples

It is of course impossible to give an exhaustive list of examples. We simply list some of the popular ones.

13.1. **Additive Categories.** Every additive category $\mathscr{A}$ is exact with respect to the class $\mathscr{E}_{\min}$ of split exact sequences, i.e., the sequences isomorphic to

$$A \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} A \oplus B \xrightarrow{[0\ 1]} B$$

for $A, B \in \mathscr{A}$. Every object $A \in \mathscr{A}$ is both projective and injective with respect to this exact structure.

13.2. **Quasi-Abelian Categories.** We have seen in Section 4 that quasi-abelian categories are exact with respect to the class $\mathscr{E}_{\max}$ of all kernel-cokernel pairs. Evidently, this class of examples includes in particular all abelian categories. There is an abundance of non-abelian quasi-abelian categories arising in functional analysis:

13.1. Example. Let **Ban** be the category of Banach spaces and bounded linear maps over the field k of real or complex numbers. It has kernels and cokernels—the cokernel of a morphism $f : A \to B$ is given by $B/\overline{\operatorname{im} f}$. It is an easy consequence of the open mapping theorem that **Ban** is quasi-abelian. Notice that the forgetful functor **Ban** $\to$ **Ab** is exact and reflects exactness, it preserves monics but fails to preserve epics (morphisms with dense image). The ground field k is projective and by Hahn-Banach it is also is injective. More generally, it is easy to see that for each set S the space $\ell^1(S)$ is projective and $\ell^\infty(S)$ is injective. Since every Banach space A isometrically isomorphic to a quotient of $\ell^1(B_{\leq 1}A)$ and to a subspace of $\ell^\infty(B_{\leq 1}A^*)$ there are enough of both, projective and injective objects in **Ban**.

13.2. Example. Let **Fre** be the category of completely metrizable topological vector spaces (Fréchet spaces) and continuous linear maps. Again, **Fre** is quasi-abelian by the open mapping theorem, and there are exact functors **Ban** $\to$ **Fre** and **Fre** $\to$ **Ab**. It is still true that k is projective, but k fails to be injective (Hahn-Banach breaks down). There are neither enough injectives nor enough projectives in **Fre**.

13.3. Example. The category **Pol** of polish abelian groups (i.e., second countable and completely metrizable topological groups) and continuous homomorphisms. Again by the open mapping theorem, **Pol** is quasi-abelian.

Further functional analytic examples are discussed in detail e.g. in Rump [31] and Schneiders [35]. Rump [34] gives a rather long list of examples at the beginning of the introduction.

13.3. **Fully Exact Subcategories.** The proof of the following lemma is an easy exercise left to the reader:

13.4. Lemma. *Let $\mathscr{A}$ be an exact category and suppose that $\mathscr{B}$ is a full additive subcategory of $\mathscr{A}$ which is closed under extensions in the sense that the existence of a short exact sequence $B' \rightarrowtail A \twoheadrightarrow B''$ with $B', B'' \in \mathscr{B}$ implies that $A \in \mathscr{B}$. The restriction of the exact structure of $\mathscr{A}$ to $\mathscr{B}$ is an exact structure on $\mathscr{B}$.* □

13.5. Definition. A *fully exact subcategory* of an exact category $\mathscr{A}$ is a full additive subcategory $\mathscr{B}$ which is closed under extensions and equipped with the exact structure from the previous lemma.

13.6. Example. By the embedding theorem A.1, every small exact category is a fully exact subcategory of an abelian category.

13.7. Example. The full subcategories of projective or injective objects of an exact category $\mathscr{A}$ are fully exact. The induced exact structures are the split exact structures.

13.8. Example. Let $\widehat{\otimes}$ be the projective tensor product of Banach spaces. A Banach space F is flat if $F \,\widehat{\otimes}\, -$ is exact. It is well-known that the flat Banach spaces are precisely the $\mathscr{L}_1$-spaces of Lindenstrauss-Pelczyński. The full subcategory of flat Banach spaces is fully exact. The exact structure is the pure exact structure consisting of the short sequences whose Banach dual sequences are split exact, see [8, Ch. IV.2] for further information and references.

13.4. **Further Examples.**

13.9. Example. Let X be a scheme. The category of algebraic vector bundles over X, i.e., the category of locally free and coherent (sheaves of) $\mathscr{O}_X$-modules, is an exact category with the usual notion of exact sequences.

13.10. Example. If $(\mathscr{A}, \mathscr{E})$ is an exact category then the category of chain complexes $\mathbf{Ch}(\mathscr{A})$ is an exact category with respect to the exact structure $\mathbf{Ch}(\mathscr{E})$ of short sequences of complexes which are exact in each degree, see Lemma 9.1.

Appendix A. The Embedding Theorem

For abelian categories, one has the Freyd-Mitchell embedding theorem, see [13] and [28], allowing one to prove diagram lemmas in abelian categories "by chasing elements". In order to prove diagram lemmas in exact categories, a similar technique works. More precisely, one has:

A.1. Theorem ([37, A.7.1, A.7.16]). *Let $(\mathscr{A}, \mathscr{E})$ be a small exact category.*

(i) There is an abelian category $\mathscr{B}$ and a fully faithful exact functor $i : \mathscr{A} \to \mathscr{B}$ that reflects exactness. Moreover, $\mathscr{A}$ is closed under extensions in $\mathscr{B}$.
(ii) The category $\mathscr{B}$ may canonically be chosen to be the category of left exact functors $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ and i to be the Yoneda embedding $i(A) = \mathrm{Hom}_{\mathscr{A}}(-, A)$.
(iii) Assume moreover that $\mathscr{A}$ is weakly idempotent complete. If f is a morphism in $\mathscr{A}$ and $i(f)$ is epic in $\mathscr{B}$ then f is an admissible epic.

A.2. Remark. Quillen states in [32, p. "92/16/100"]:

> Now suppose given an exact category $\mathscr{M}$. Let $\mathscr{A}$ be the additive category of additive contravariant functors from $\mathscr{M}$ to abelian groups which are left exact, i.e. carry [an exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$] to an exact sequence
>
> $$0 \to F(M'') \to F(M) \to F(M').$$
>
> (Precisely, choose a universe containing $\mathscr{M}$, and let $\mathscr{A}$ be the category of left exact functors whose values are abelian groups in the universe.) Following well-known ideas (e.g. [16]), one can prove $\mathscr{A}$ is an abelian category, that the Yoneda functor h embeds $\mathscr{M}$ as a full subcategory of $\mathscr{A}$ closed under extensions, and finally that a [short] sequence [...] is in $\mathscr{E}$ if and only if h carries it into an exact sequence in $\mathscr{A}$. The details will be omitted, as they are not really important for the sequel.

Freyd stated a similar theorem in [12], again without proof, and with the additional assumption that idempotents split, since he uses Heller's axioms. The first proof published is in Laumon [25, 1.0.3], relying on the Grothendieck-Verdier theory of sheafification [36]. However, Laumon's proof that the embedding reflects exactness and its image is closed under extensions seems to be flawed by the confusion of epics in $\mathscr{B}$ and epics in the Yoneda category $\mathscr{Y} = \mathbf{Ab}^{\mathscr{A}^{\mathrm{op}}}$. The sheafification approach was also used and further refined by Thomason [37, Appendix A]. A quite detailed sketch of the proof alluded to by Quillen is given in Keller [23, A.3].

The proof given here is the one in Thomason [37, A.7] amalgamated with the one given by Laumon [25, 1.0.3]. We also take the opportunity to fix a slight gap in Thomason's argument (our Lemma A.8, compare with the first sentence after [37, (A.7.10)]). Since Thomason fails to spell out the nice sheaf-theoretic interpretations of his construction and since referring to SGA 4 seems rather brutal, we use the terminology of the more lightweight Mac Lane-Moerdijk [27, Chapter III]. Other good introductions to the theory of sheaves may be found in Artin [1] or Borceux [5], for example.

A.1. **Separated Presheaves and Sheaves.** Let $(\mathscr{A}, \mathscr{E})$ be a small exact category. For each object $A \in \mathscr{A}$, let

$$\mathscr{C}_A = \{(p' : A' \twoheadrightarrow A) \, : \, A' \in \mathscr{A}\}$$

be the set of admissible epics onto A. The elements of $\mathscr{C}_A$ are the *coverings* of A.

A.3. Lemma. *The family $\{\mathscr{C}_A\}_{A \in \mathscr{A}}$ is a* basis for a Grothendieck topology J *on $\mathscr{A}$, that is:*

(i) If $f : A \to B$ is an isomorphism then $f \in \mathscr{C}_B$.
(ii) If $g : A \to B$ is arbitrary and $(q' : B' \twoheadrightarrow B) \in \mathscr{C}_B$ then the pull-back

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ {\scriptstyle p'}\downarrow & PB & \downarrow{\scriptstyle q'} \\ A & \xrightarrow{g} & B \end{array}$$

yields a morphism $p' \in \mathscr{C}_A$. ("Stability under base-change")
(iii) If $(p : B \twoheadrightarrow A) \in \mathscr{C}_A$ and $(q : C \twoheadrightarrow B) \in \mathscr{C}_B$ then $pq \in \mathscr{C}_A$. ("Transitivity")

In particular, $(\mathscr{A}, J)$ is a site.

Proof. This is obvious from the definition, see [27, Definition 2, p. 111]. □

The Yoneda functor $y : \mathscr{A} \to \mathbf{Ab}^{\mathscr{A}^{\mathrm{op}}}$ associates to each object $A \in \mathscr{A}$ the presheaf (of abelian groups) $y(A) = \mathrm{Hom}_{\mathscr{A}}(-, A)$. In general, a *presheaf* is just a functor $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$, which we will assume to be additive except in the next lemma. We will see shortly that $y(A)$ is in fact a *sheaf* on the site $(\mathscr{A}, J)$.

A.4. Lemma. *Consider the site $(\mathscr{A}, J)$ and let $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ be a functor.*

(i) The presheaf G is separated *if and only if for each admissible epic p the morphism $G(p)$ is monic.*

(ii) The presheaf G is a sheaf *if and only if for each admissible epic $p : A \twoheadrightarrow B$ the diagram*

$$G(B) \xrightarrow{G(p)} G(A) \underset{d^0 = G(p_0)}{\overset{d^1 = G(p_1)}{\rightrightarrows}} G(A \times_B A)$$

is a difference kernel *(equalizer), where $p_0, p_1 : A \times_B A \twoheadrightarrow A$ denote the two projections. In other words, the presheaf G is a sheaf if and only if for all admissible epics $p : A \twoheadrightarrow B$ the diagram*

$$\begin{array}{ccc} G(B) & \xrightarrow{G(p)} & G(A) \\ \big\downarrow{\scriptstyle G(p)} & & \big\downarrow{\scriptstyle d^1} \\ G(A) & \xrightarrow{d^0} & G(A \times_B A) \end{array}$$

is a pull-back.

Proof. Again, this is obtained by making the definitions explicit. Point (i) is the definition, [27, p. 129], and point (ii) is [27, Proposition 1[bis], p. 123]. □

The following lemma shows that the sheaves on the site $(\mathscr{A}, J)$ are quite familiar gadgets.

A.5. Lemma. *Let $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ be an additive functor. The following are equivalent:*

(i) The presheaf G is a sheaf on the site $(\mathscr{A}, J)$.

(ii) For each admissible epic $p : B \twoheadrightarrow C$ the sequence

$$0 \to G(C) \xrightarrow{G(p)} G(B) \xrightarrow{d^0 - d^1} G(B \times_C B)$$

is exact.

(iii) For each short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ in $\mathscr{A}$ the sequence

$$0 \to G(C) \to G(B) \to G(A)$$

is exact, i.e., G is left exact.

Proof. By Lemma A.4 (ii) we have that G is a sheaf if and only if the sequence

$$0 \to G(C) \xrightarrow{\left[\begin{smallmatrix} G(p) \\ G(p) \end{smallmatrix}\right]} G(B) \oplus G(B) \xrightarrow{\left[\begin{smallmatrix} G(p_0) & -G(p_1) \end{smallmatrix}\right]} G(B \times_C B)$$

is exact. Since $p_1 : B \times_C B \twoheadrightarrow B$ is a split epic with kernel A, there is an isomorphism $B \times_C B \to A \oplus B$ and it is easy to check that the above sequence is isomorphic to

$$0 \to G(C) \to G(B) \oplus G(B) \to G(A) \oplus G(B).$$

Because left exact sequences are stable under taking direct sums and passing to direct summands, (i) is equivalent to (iii). That (i) is equivalent to (ii) is obvious by Lemma A.4 (ii). □

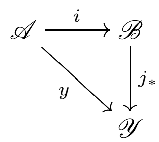
A.6. Corollary ([37, A.7.6]). *For every object $A \in \mathscr{A}$ the represented functor $y(A) = \mathrm{Hom}_{\mathscr{A}}(-, A)$ is a sheaf.*

Proof. Given an exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ we need to prove that

$$0 \to \mathrm{Hom}_{\mathscr{A}}(B'', A) \to \mathrm{Hom}_{\mathscr{A}}(B, A) \to \mathrm{Hom}_{\mathscr{A}}(B', A)$$

is exact. That the sequence is exact at $\mathrm{Hom}_{\mathscr{A}}(B, A)$ follows from the fact that $B \twoheadrightarrow B''$ is a cokernel of $B' \rightarrowtail B$. That the sequence is exact at $\mathrm{Hom}_{\mathscr{A}}(B'', A)$ follows from the fact that $B \twoheadrightarrow B''$ is epic. □

A.2. **Outline of the Proof.** Let now $\mathscr{Y}$ be the category of additive functors $\mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$ and let $\mathscr{B}$ be the category of (additive) sheaves on the site $(\mathscr{A}, J)$. Let $j_* : \mathscr{B} \to \mathscr{A}$ be the inclusion. By Corollary A.6, the Yoneda functor y factors as



via a functor $i : \mathscr{A} \to \mathscr{B}$. We will prove that the category $\mathscr{B} = \mathrm{Sheaves}(A, J)$ is abelian and we will check that the functor i has the properties asserted in the embedding theorem.

The category $\mathscr{Y}$ is a Grothendieck abelian category (small products and coproducts exist and filtered colimits are exact)—as a functor category, these properties are inherited from **Ab**, as limits and colimits are taken pointwise. The crux of the proof of the embedding theorem is to show that j_* has a left adjoint j^* such that $j^* j_* = \mathrm{id}_{\mathscr{B}}$, namely sheafification. As soon as this is established, the rest will be relatively painless.

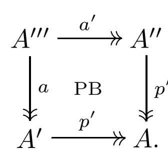
A.3. **Sheafification.** The goal of this section is to construct the sheafification functor on the site $(\mathscr{A}, J)$ and to prove its basic properties. We will construct an endofunctor $L : \mathscr{Y} \to \mathscr{Y}$ which associates to each presheaf a separated presheaf and to each separated presheaf a sheaf. The sheafification functor will then be given by $j^* = LL$.

We need one more concept from the theory of sites:

A.7. Lemma. *Let $A \in \mathscr{A}$. A covering $p'' : A'' \twoheadrightarrow A$ is a* refinement *of the covering $p' : A' \twoheadrightarrow A$ if and only if there exists a morphism $a : A'' \to A'$ such that $p'a = p''$.*

Proof. This is just a translation of the definition of a *matching family* as given in [27, p. 121] into the present setting. □

By definition, refinement gives the structure of a filtered category on $\mathscr{C}_A$ for each $A \in \mathscr{A}$. More precisely, let $\mathscr{D}_A$ be the following category: the objects are the coverings $(p' : A' \twoheadrightarrow A)$ and there exists at most one morphism between any two objects of $\mathscr{D}_A$: there exists a morphism $(p' : A' \twoheadrightarrow A) \to (p'' : A'' \twoheadrightarrow A)$ in $\mathscr{D}_A$ if and only if there exists $a : A'' \to A'$ such that $p'a = p''$. To see that $\mathscr{D}_A$ is filtered, let $(p' : A' \twoheadrightarrow A)$ and $(p'' : A'' \twoheadrightarrow A)$ be two objects and put $A''' = A' \times_A A''$, so there is a pull-back diagram



Put $p''' = p'a = p''a'$, so the object $(p''' : A''' \twoheadrightarrow A)$ of $\mathscr{D}_A$ is a common refinement of $(p' : A' \twoheadrightarrow A)$ and $(p'' : A'' \twoheadrightarrow A)$.

A.8. Lemma. *Let $A_1, A_2 \in \mathscr{A}$ be any two objects.*

(i) *There is a functor $Q : \mathscr{D}_{A_1} \times \mathscr{D}_{A_2} \to \mathscr{D}_{A_1 \oplus A_2}$, $(p'_1, p'_2) \mapsto (p'_1 \oplus p'_2)$.*

(ii) Let $(p' : A' \twoheadrightarrow A_1 \oplus A_2)$ be an object of $\mathscr{D}_{A_1 \oplus A_2}$ and for $i = 1, 2$ let

$$\begin{array}{ccc} A_i' & \longrightarrow & A' \\ \Big\downarrow{\scriptstyle p_i'} & \text{PB} & \Big\downarrow{\scriptstyle p'} \\ A_i & \longrightarrow & A_1 \oplus A_2 \end{array}$$

be a pull-back diagram in which the bottom arrow is the inclusion. This construction defines a functor

$$P : \mathscr{D}_{A_1 \oplus A_2} \longrightarrow \mathscr{D}_{A_1} \times \mathscr{D}_{A_2}, \quad p' \longmapsto (p_1', p_2').$$

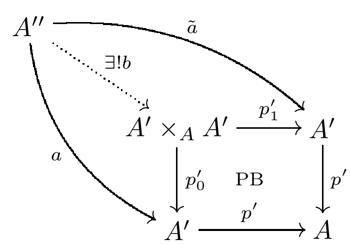
(iii) There are a natural transformation $\mathrm{id}_{\mathscr{D}_{A_1 \oplus A_2}} \Rightarrow PQ$ and a natural isomorphism $QP \cong \mathrm{id}_{\mathscr{D}_{A_1} \times \mathscr{D}_{A_2}}$. In particular, the images of P and Q are cofinal.

Proof. That P is a functor follows from its construction and the universal property of pull-back diagrams in conjunction with axiom [E2$^{\mathrm{op}}$]. That Q is well-defined follows from Proposition 2.9 and that $PQ \cong \mathrm{id}_{\mathscr{D}_{A_1} \times \mathscr{D}_{A_2}}$ is easy to check. That there is a natural transformation $\mathrm{id}_{\mathscr{D}_{A_1 \oplus A_2}} \Rightarrow QP$ follows from the universal property of products. □

Let $(p'' : A'' \twoheadrightarrow A)$ be a refinement of $(p' : A' \twoheadrightarrow A)$, and let $a : A'' \to A'$ be such that $p'a = p''$. By the universal property of pull-backs, a yields a unique morphism $A'' \times_A A'' \to A' \times_A A'$ which we denote by $a \times_A a$. Hence, for every additive functor $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$, we obtain a commutative diagram in $\mathbf{Ab}$:

$$\begin{array}{ccccc} \mathrm{Ker}\,(d^0 - d^1) & \longrightarrow & G(A') & \xrightarrow{d^0 - d^1} & G(A' \times_A A') \\ \vdots{\scriptstyle \exists!} & & \Big\downarrow{\scriptstyle G(a)} & & \Big\downarrow{\scriptstyle G(a \times_A a)} \\ \mathrm{Ker}\,(d^0 - d^1) & \longrightarrow & G(A'') & \xrightarrow{d^0 - d^1} & G(A'' \times_A A''). \end{array}$$

The next thing to observe is that the dotted morphism does not depend on the choice of a. Indeed, if $\tilde{a}$ is another morphism such that $p'\tilde{a} = p''$, consider the diagram



and $b : A'' \to A' \times_A A'$ is such that

$$G(b)(d^0 - d^1) = G(b)G(p_0') - G(b)G(p_1') = G(a) - G(\tilde{a}),$$

so $G(a) - G(\tilde{a}) = 0$ on $\mathrm{Ker}\,(d^0 - d^1)$.

For $G : \mathscr{A}^{\mathrm{op}} \to \mathbf{Ab}$, we put $\ell G(p' : A' \twoheadrightarrow A) := \mathrm{Ker}\,(G(A') \xrightarrow{d^0 - d^1} G(A' \times_A A'))$ and we have just seen that this defines a functor $\ell G : \mathscr{D}_A \to \mathbf{Ab}$.

A.9. Lemma. *Define*

$$LG(A) = \varinjlim_{\mathscr{D}_A} \ell G(p' : A' \twoheadrightarrow A).$$

(i) LG is an additive contravariant functor in A.
(ii) L is a covariant functor in G.

PROOF. This is immediate from going through the definitions:

To prove (i), let $f : A \to B$ be an arbitrary morphism. By taking pull-backs (Lemma A.3 (ii)), we obtain a functor

$$\mathscr{D}_B \xrightarrow{f^*} \mathscr{D}_A$$

which, by passing to the colimit, induces a unique morphism $LG(B) \xrightarrow{LG(f)} LG(A)$ compatible with f^*. From this uniqueness, we deduce $LG(fg) = LG(g)LG(f)$. The additivity of LG is a consequence of Lemma A.8.

To prove (ii), let $\alpha : F \Rightarrow G$ be a natural transformation between two (additive) presheaves. Given an object $A \in \mathscr{A}$, we obtain a morphism between the colimit diagrams defining $LF(A)$ and $LG(A)$ and we denote the unique resulting map by $L(\alpha)_A$. Given a morphism $f : A \to B$, there is a commutative diagram

$$\begin{array}{ccc} LF(B) & \xrightarrow{LF(f)} & LF(A) \\ \downarrow{\scriptstyle L(\alpha)_B} & & \downarrow{\scriptstyle L(\alpha)_A} \\ LG(B) & \xrightarrow{LG(f)} & LG(A), \end{array}$$

as is easily checked. The uniqueness in the definition of $L(\alpha)_A$ implies that for each $A \in \mathscr{A}$ the equation

$$L(\alpha \circ \beta)_A = L(\alpha)_A \circ L(\beta)_A$$

holds. The reader in need of more details may consult [5, p. 206f]. □

A.10. LEMMA ([37, A.7.8]). *The functor $L : \mathscr{Y} \to \mathscr{Y}$ has the following properties:*

(i) It is additive and preserves finite limits.
(ii) There is a natural transformation $\eta : \mathrm{id}_{\mathscr{Y}} \Rightarrow L$.

PROOF. That L preserves finite limits follows from the fact that filtered colimits and kernels in **Ab** commute with finite limits, as limits in $\mathscr{Y}$ are formed pointwise, see also [5, Lemma 3.3.1]. Since L preserves finite limits, it preserves in particular finite products, hence it is additive. This settles point (i).

For each $(p' : A' \twoheadrightarrow A) \in \mathscr{D}_A$ the morphism $G(p') : G(A) \to G(A')$ factors uniquely over

$$\tilde{\eta}_{p'} : G(A) \to \mathrm{Ker}\,(G(A') \to G(A' \times_A A')).$$

By passing to the colimit over $\mathscr{D}_A$, this induces a morphism $\tilde{\eta}_A : G(A) \to LG(A)$ which is clearly natural in A. In other words, the $\tilde{\eta}_A$ yield a natural transformation $\eta_G : G \Rightarrow LG$, i.e., a morphism in $\mathscr{Y}$. We leave it to the reader to check that the construction of η_G is compatible with natural transformations $\alpha : G \Rightarrow F$ so that the η_G assemble to yield a natural transformation $\eta : \mathrm{id}_{\mathscr{Y}} \Rightarrow L$, as claimed in point (ii). □

A.11. LEMMA ([37, A.7.11, (a), (b), (c)]). *Let $G \in \mathscr{Y}$ and let $A \in \mathscr{A}$.*

(i) For all $x \in LG(A)$ there exists an admissible epic $p' : A' \twoheadrightarrow A$ and $y \in G(A')$ such that $\eta(y) = LG(p')(x)$ in $LG(A')$.
(ii) For all $x \in G(A)$, we have $\eta(x) = 0$ in $LG(A)$ if and only if there exists an admissible epic $p' : A' \twoheadrightarrow A$ such that $G(p')(x) = 0$ in $G(A')$.
(iii) We have $LG = 0$ if and only if for all $A \in \mathscr{A}$ and all $x \in G(A)$ there exists an admissible epic $p' : A' \twoheadrightarrow A$ such that $G(p')(x) = 0$.

PROOF. Points (i) and (ii) are immediate from the definitions. Point (iii) follows from (i) and (ii). □

A.12. LEMMA ([27, Lemma 2, p. 131], [37, A.7.11, (d), (e)]). *Let $G \in \mathscr{Y}$.*

(i) The presheaf G is separated if and only if $\eta_G : G \to LG$ is monic.

(ii) The presheaf G is a sheaf if and only if $\eta_G : G \to LG$ is an isomorphism.

Proof. Point (i) follows from Lemma A.11 (ii) and point (ii) follows from the definitions. □

A.13. Proposition ([37, A.7.12]). *Let $G \in \mathscr{Y}$.*

(i) The presheaf LG is separated.
(ii) If G is separated then LG is a sheaf.

Proof. Let us prove (i) by applying Lemma A.4 (i), so let $x \in LG(A)$ and let $b : B \twoheadrightarrow A$ be an admissible epic for which $LG(b)(x) = 0$. We have to prove that then $x = 0$ in $LG(A)$. By the definition of $LG(A)$, we know that x is represented by some $y \in \operatorname{Ker}(G(A') \xrightarrow{d^0 - d^1} G(A' \times_A A'))$ for some admissible epic $(p' : A' \twoheadrightarrow A)$ in $\mathscr{D}_A$. Since $LG(b)(x) = 0$ in $LG(B)$, we know that the image of y in $\operatorname{Ker}(G(A' \times_A B) \xrightarrow{d^0 - d^1} G((A' \times_A B) \times_B (A' \times_A B)))$ is equivalent to zero in the filtered colimit over $\mathscr{D}_B$ defining $LG(B)$. Therefore there exists a morphism $D \to A' \times_A B$ in $\mathscr{E}$ such that its composite with the projection onto B is an admissible epic $D \twoheadrightarrow B$. By Lemma A.11 (ii), it follows that y maps to zero in $G(D)$. Now the composite $D \twoheadrightarrow B \twoheadrightarrow A$ is in $\mathscr{D}_A$ and hence y is equivalent to zero in the filtered colimit over $\mathscr{D}_A$ defining $LG(A)$. Thus, $x = 0$ in $LG(A)$ as required.

Let us prove (ii). If G is a separated presheaf, we have to check that for every admissible epic $B \twoheadrightarrow A$ the diagram

$$LG(A) \longrightarrow LG(B) \underset{d^0 = G(p_0)}{\overset{d^1 = G(p_1)}{\rightrightarrows}} LG(B \times_A B)$$

is a difference kernel. By (i) LG is separated, so $LG(A) \to LG(B)$ is monic, and it remains to prove that every element $x \in LG(B)$ with $(d^0 - d^1)x = 0$ is in the image of $LG(A)$. By Lemma A.11 (i) there is an admissible epic $q : C \twoheadrightarrow B$ and $y \in G(C)$ such that $\eta(y) = LG(q)(x)$. It follows that $\eta G(p_0)(y) = \eta G(p_1)(y)$ in $LG(C \times_A C)$. Now, G is separated, so $\eta : G \Rightarrow LG$ is monic by Lemma A.12, and we conclude from this that $G(p_0)(y) = G(p_1)(y)$ in $G(C \times_A C)$. In other words, $y \in \operatorname{Ker}(G(C) \xrightarrow{d^0 - d^1} G(C \times_A C))$ yields a class in $LG(A)$ representing x. □

The following observation is quite useful:

A.14. Corollary. *For a presheaf $G \in \mathscr{Y}$ we have $LG = 0$ if and only if $LLG = 0$.*

Proof. Obviously $LG = 0$ entails $LLG = 0$ as L is additive by Lemma A.10. Conversely, as LG is separated by Lemma A.12 (i), the morphism $\eta_{LG} : LG \to LLG$ is monic by Lemma A.12 (i), so if $LLG = 0$ we must have $LG = 0$. □

A.15. Definition. The *sheafification functor* is $j^* = LL : \mathscr{Y} \to \mathscr{B}$.

A.16. Lemma. *The sheafification functor j^* is left adjoint to the inclusion functor $j_* : \mathscr{B} \to \mathscr{Y}$ and satisfies $j^* j_* \cong \mathrm{id}_{\mathscr{B}}$. Moreover, sheafification is exact.*

Proof. Since $\eta_G : G \to LG$ is an isomorphism if and only if G is a sheaf by Lemma A.12 (ii), it follows that $j^* j_* \cong \mathrm{id}_{\mathscr{B}}$.

Let $Y \in \mathscr{Y}$ be a presheaf and let $B \in \mathscr{B}$ be a sheaf. The natural transformation $\eta : \mathrm{id}_{\mathscr{Y}} \Rightarrow L$ gives us on the one hand a natural transformation

$$\varrho_Y = \eta_{LY} \eta_Y : Y \longrightarrow LLY = j_* j^* Y$$

and on the other hand a natural isomorphism

$$\lambda_B = (\eta_{LB} \eta_B)^{-1} : j^* j_* B = LLB \longrightarrow B.$$

Now the compositions

$$j_*B \xrightarrow{\varrho_{j_*B}} j_*j^*j_*B \xrightarrow{j_*\lambda_B} j_*B \qquad \text{and} \qquad j^*Y \xrightarrow{j^*\varrho_Y} j^*j_*j^*Y \xrightarrow{\lambda_{j^*Y}} j^*Y$$

are manifestly equal to id_{j_*B} and id_{j^*Y} so that j^* is indeed left adjoint to j_*. In particular j^* preserves cokernels. That j^* preserves kernels follows from the fact that $L : \mathscr{Y} \to \mathscr{Y}$ has this property by Lemma A.10 (i) and the fact that $\mathscr{B}$ is a full subcategory of $\mathscr{Y}$. Therefore j^* is exact. ☐

A.17. Remark. It is an illuminating exercise to prove exactness of j^* directly by going through the definitions.

A.18. Lemma. *The category $\mathscr{B}$ is abelian.*

Proof. It is clear that $\mathscr{B}$ is additive. The sheafification functor $j^* = LL$ preserves kernels by Lemma A.10 (i) and as a left adjoint it preserves cokernels. To prove $\mathscr{B}$ abelian, it suffices to check that every morphism $f : A \to B$ has an analysis

$$\begin{array}{ccccccc} & & A & \xrightarrow{\quad\quad f \quad\quad} & B & & \\ & \nearrow\!\!\!\!\rightarrowtail & & \searrow\!\!\!\!\twoheadrightarrow \qquad \nearrow\!\!\!\!\rightarrowtail & & \searrow\!\!\!\!\twoheadrightarrow & \\ \mathrm{Ker}\,(f) & & \mathrm{Coim}\,(f) & \xrightarrow{\cong} & \mathrm{Im}\,(f) & & \mathrm{Coker}\,(f). \end{array}$$

Since j^* preserves kernels and cokernels and $j^*j_* \cong \mathrm{id}_{\mathscr{B}}$ such an analysis can be obtained by applying j^* to an analysis of j_*f in $\mathscr{Y}$. ☐

A.4. **Proof of the Embedding Theorem.** Let us recapitulate: one half of the axioms of an exact structure yields that a small exact category $\mathscr{A}$ becomes a *site* $(\mathscr{A}, J)$. We denoted the Yoneda category of contravariant functors $\mathscr{A} \to \mathbf{Ab}$ by $\mathscr{Y}$ and the Yoneda embedding $A \mapsto \mathrm{Hom}\,(-, A)$ by $y : \mathscr{A} \to \mathscr{Y}$. We have shown that the category $\mathscr{B}$ of sheaves on the site (A, J) is abelian, being a full reflective subcategory of $\mathscr{Y}$ with sheafification $j^* : \mathscr{Y} \to \mathscr{B}$ as reflector (left adjoint). Following Thomason, we denoted the inclusion $\mathscr{B} \to \mathscr{Y}$ by j_*. Moreover, we have shown that the Yoneda embedding takes its image in $\mathscr{B}$, so we obtained a commutative diagram of categories

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{i} & \mathscr{B} \\ & {\scriptstyle y}\searrow & \downarrow{\scriptstyle j_*} \\ & & \mathscr{Y}, \end{array}$$

in other words $y = j_*i$. By the Yoneda lemma, y is fully faithful and j_* is fully faithful, hence i is fully faithful as well. This settles the first part of the following lemma:

A.19. Lemma. *The functor $i : \mathscr{A} \to \mathscr{B}$ is fully faithful and exact.*

Proof. By the above discussion, it remains to prove exactness.

Clearly, the Yoneda embedding sends exact sequences in $\mathscr{A}$ to left exact sequences in $\mathscr{Y}$. Sheafification j^* is exact and since $j^*j_* \cong \mathrm{id}_{\mathscr{B}}$, we have that $j^*y = j^*j_*i \cong i$ is left exact as well. It remains to prove that for each admissible epic $p : B \twoheadrightarrow C$ the morphism $i(p)$ is epic. By Corollary A.14, it suffices to prove that $G = \mathrm{Coker}\, y(p)$ satisfies $LG = 0$, because $\mathrm{Coker}\, i(p) = j^*\,\mathrm{Coker}\, y(p) = LLG = 0$ then implies that $i(p)$ is epic. To this end we use the criterion in Lemma A.11 (iii), so let $A \in \mathscr{A}$ be any object and $x \in G(A)$. We have an exact sequence of abelian groups

$$\mathrm{Hom}\,(A, B) \xrightarrow{y(p)_A} \mathrm{Hom}\,(A, C) \xrightarrow{q_A} G(A) \to 0$$

so $x = q_A(f)$ for some morphism $f : A \to C$. Now form the pull-back

$$\begin{array}{ccc} A' & \xrightarrow{p'} & A \\ {\scriptstyle f'}\big\downarrow & \text{PB} & \big\downarrow{\scriptstyle f} \\ B & \xrightarrow{p} & C \end{array}$$

and observe that $G(p')(x) = G(p')(q_A(f)) = q_{A'}(fp') = q_{A'}(pf') = 0$. $\square$

A.20. Lemma ([37, A.7.15]). *Let $A \in \mathscr{A}$ and $B \in \mathscr{B}$ and suppose there is an epic $e : B \twoheadrightarrow i(A)$. There exist $A' \in \mathscr{A}$ and $k : i(A') \to B$ such that $ek : A' \to A$ is an admissible epic.*

Proof. Let G be the cokernel of $j_* e$ in $\mathscr{Y}$. Then $0 = j^* G = LLG$ because $j^* j_* e \cong e$ is epic. By Proposition A.13 (i) we know that LG is separated, hence by Lemma A.12 (i) the morphism $\eta_{LG} : LG \to LLG$ is monic. It follows that $LG = 0$. Now $G(A) \cong \mathrm{Hom}\,(A, A)/\,\mathrm{Hom}\,(i(A), B)$ and let $x \in G(A)$ be the class of 1_A. From Lemma A.11 (iii) we conclude that there is an admissible epic $p' : A' \twoheadrightarrow A$ such that $G(p')(x) = 0$ in $G(A') \cong \mathrm{Hom}\,(A', A)/\,\mathrm{Hom}\,(i(A'), B)$. But this means that the admissible epic p' factors as ek for some $k \in \mathrm{Hom}\,(i(A'), B)$ as claimed. $\square$

A.21. Lemma. *The functor i reflects exactness.*

Proof. Suppose $A \xrightarrow{m} B \xrightarrow{e} C$ is a sequence in $\mathscr{A}$ such that

$$i(A) \xrightarrow{i(m)} i(B) \xrightarrow{i(e)} i(C)$$

is short exact in $\mathscr{B}$. In particular, $i(m)$ is a kernel of $i(e)$. Since i is fully faithful, it follows that m is a kernel of e in $\mathscr{A}$, hence we are done as soon as we can show that e is an admissible epic. Because $i(e)$ is epic, Lemma A.20 allows us to find $A' \in \mathscr{A}$ and $k : i(A') \to i(B)$ such that ek is an admissible epic and since e has a kernel we conclude by the dual of Proposition 2.15. $\square$

A.22. Lemma. *The essential image of $i : \mathscr{A} \to \mathscr{B}$ is closed under extensions.*

Proof. Consider a short exact sequence $i(A) \rightarrowtail G \twoheadrightarrow i(B)$ in $\mathscr{B}$, where $A, B \in \mathscr{A}$. By Lemma A.20 we find an admissible epic $p : C \twoheadrightarrow B$ such that $i(p)$ factors over G. Now consider the pull-back diagram

$$\begin{array}{ccc} D & \twoheadrightarrow & G \\ \big\downarrow & \text{PB} & \big\downarrow \\ i(C) & \xrightarrow{i(p)} & i(B) \end{array}$$

and observe that $D \twoheadrightarrow i(C)$ is a split epic because $i(p)$ factors over G. Therefore $D \cong i(A) \oplus i(C) \cong i(A \oplus C)$. If K is a kernel of p then $i(K)$ is a kernel of $D \twoheadrightarrow G$, so we obtain an exact sequence

$$i(K) \xrightarrow{\left[\begin{smallmatrix} i(a) \\ i(c) \end{smallmatrix}\right]} i(A) \oplus i(C) \twoheadrightarrow G,$$

where $c = \ker p$, which shows that G is the push-out

$$\begin{array}{ccc} i(K) & \xrightarrow{i(c)} & i(C) \\ {\scriptstyle i(a)}\big\downarrow & \text{PO} & \big\downarrow \\ i(A) & \rightarrowtail & G. \end{array}$$

Now i is exact by Lemma A.19 and hence preserves push-outs along admissible monics by Proposition 5.2, so i preserves the push-out $G' = A \cup_K C$ of a along the admissible monic c and thus G is isomorphic to $i(G')$. □

Proof of the Embedding Theorem A.1. Let us summarize what we know: the embedding $i : \mathscr{A} \to \mathscr{B}$ is fully faithful and exact by Lemma A.19. It reflects exactness by Lemma A.21 and its image is closed under extensions in $\mathscr{B}$ by Lemma A.22. This settles point (i) of the theorem.

Point (ii) is taken care of by Lemma A.5 and Corollary A.6.

It remains to prove (iii). Assume that $\mathscr{A}$ has weakly split idempotents. We claim that every morphism $f : B \to C$ such that $i(f)$ is epic is in fact an admissible epic. Indeed, by Lemma A.20 we find a morphism $k : A \to B$ such that $fk : A \twoheadrightarrow C$ is an admissible epic and we conclude by Proposition 7.5. □

Appendix B. Heller's Axioms

B.1. Proposition (Quillen). *Let $\mathscr{A}$ be a weakly idempotent complete additive category and let $\mathscr{E}$ be a class of kernel-cokernel pairs in $\mathscr{A}$. The pair $(\mathscr{A}, \mathscr{E})$ is an exact category if and only if $\mathscr{E}$ satisfies Heller's axioms:*

(i) Identity morphisms are both admissible monics and admissible epics;
(ii) The class of admissible monics and the class of admissible epics are closed under composition;
(iii) Let f and g be composable morphisms. If gf is an admissible monic then so is f and if gf is an admissible epic then so is g;
(iv) Assume that all columns and the second two rows of the commutative diagram

$$\begin{array}{ccccc} A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle b'} & & \downarrow{\scriptstyle c'} \\ A'' & \overset{f''}{\rightarrowtail} & B'' & \overset{g''}{\twoheadrightarrow} & C'' \end{array}$$

are in $\mathscr{E}$ then the first row is also in $\mathscr{E}$.

Proof. Note that (i) and (ii) are just axioms [E0], [E1] and their duals.

For an exact category $(\mathscr{A}, \mathscr{E})$, point (iii) is proved in Proposition 7.5 and point (iv) follows from the 3×3-lemma 3.6.

Conversely, assume that $\mathscr{E}$ has properties (i)-(iv) and let us check that $\mathscr{E}$ is an exact structure.

By properties (i) and (iii) an isomorphism is both an admissible monic and an admissible epic since by definition $f^{-1}f = 1$ and $ff^{-1} = 1$. If the short sequence $\sigma = (A' \to A \to A'')$ is isomorphic to the short exact sequence $B' \rightarrowtail B \twoheadrightarrow B''$ then property (iv) tells us that σ is short exact. Thus, $\mathscr{E}$ is closed under isomorphisms.

Heller proves [19, Proposition 4.1] that (iv) implies its dual, that is: if the commutative diagram in (iv) has exact rows and both (a, a') and (b, b') belong to $\mathscr{E}$ then so does (c, c').[3] It follows that Heller's axioms are self-dual.

[3]Indeed, by (iii) c' is an admissible epic and so it has a kernel D. Because $c'gb = 0$, there is a morphism $B' \to D$ and replacing C' by D in the diagram of (iv) we see that $A' \rightarrowtail B' \twoheadrightarrow D$ is short exact. Therefore $C' \cong D$ and we conclude by the fact that $\mathscr{E}$ is closed under isomorphisms.

Let us prove that [E2] holds—the remaining axiom [E2$^{\mathrm{op}}$] will follow from the dual argument. Given the diagram

$$\begin{array}{ccc} A' & \overset{f'}{\rightarrowtail} & B' \\ {\scriptstyle a}\downarrow & & \\ A & & \end{array}$$

we want to construct its push-out B and prove that the morphism $A \to B$ is an admissible monic. Observe that $\left[\begin{smallmatrix} a \\ f' \end{smallmatrix}\right] : A' \to A \oplus B'$ is the composition

$$A' \xrightarrow{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} A \oplus A' \underset{\cong}{\xrightarrow{\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]}} A \oplus A' \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & f' \end{smallmatrix}\right]} A \oplus B'.$$

By (iii) split exact sequences belong to $\mathscr{E}$, and the proof of Proposition 2.9 shows that the direct sum of two sequences in $\mathscr{E}$ also belongs to $\mathscr{E}$. Therefore $\left[\begin{smallmatrix} a \\ f' \end{smallmatrix}\right]$ is an admissible monic and it has a cokernel $\left[\begin{smallmatrix} -f & b \end{smallmatrix}\right] : A \oplus B' \twoheadrightarrow B$. It follows that the left hand square in the diagram

$$\begin{array}{ccccc} A' & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ {\scriptstyle a}\downarrow & \mathrm{BC} & \downarrow{\scriptstyle b} & & \Vert \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C' \end{array}$$

is bicartesian. Let $g' : B' \twoheadrightarrow C'$ be a cokernel of f' and let g be the morphism $B \to C'$ such that $gf = 0$ and $gb = g'$. Now consider the commutative diagram

$$\begin{array}{ccccc} A' & \overset{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]}{\rightarrowtail} & A \oplus A' & \overset{\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]}{\twoheadrightarrow} & A \\ \Vert & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 1 & a \\ 0 & f' \end{smallmatrix}\right]} & & \downarrow{\scriptstyle f} \\ A' & \overset{\left[\begin{smallmatrix} a \\ f \end{smallmatrix}\right]}{\rightarrowtail} & A \oplus B' & \overset{\left[\begin{smallmatrix} f & b \end{smallmatrix}\right]}{\twoheadrightarrow} & B \\ & & \downarrow{\scriptstyle \left[\begin{smallmatrix} 0 & g' \end{smallmatrix}\right]} & & \downarrow{\scriptstyle g} \\ & & C' & = & C' \end{array}$$

in which the rows are exact and the first two columns are exact. It follows that the third column is exact and hence f is an admissible monic. □

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