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MARTINGALE-COBOUNDARY REPRESENTATION FOR A CLASS OF RANDOM FIELDS

MIKHAIL GORDIN

Abstract. It is known that under some conditions a stationary random sequence admits a representation as the sum of two others: one of them is a martingale difference sequence, and another is a so-called coboundary. Such a representation can be used for proving some limit theorems by means of the martingale approximation.

A multivariate version of such a decomposition is presented in the paper for a class of random fields generated by several commuting non-invertible probability preserving transformations. In this representation summands of mixed type appear which behave with respect to some group of directions of the parameter space as reversed multiparameter martingale differences (in the sense of one of several known definitions) while they look as coboundaries relative to the other directions. Applications to limit theorems will be published elsewhere.

1. Introduction

Martingale approximation is one of methods of proving limit theorems for stationary random sequences. The method, in its simplest version, consists of representing the original random sequence as the sum of a martingale difference sequence and a coboundary sequence. In this introduction we give a brief sketch of this approach. The aim of the present paper is to extend the martingale approximation method to a certain class of random fields. This is the topic of the next two sections of the paper.

Let $\xi = (\xi_n)_{n\in\mathbb{Z}}$ be a stationary (in the strict sense) random sequence.

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Under certain assumptions [4] it can be represented in the form

$$\xi_n = \eta_n + \zeta_n,$$

where $\eta = (\eta_n)_{n\in\mathbb{Z}}$ is a stationary sequence of *martingale differences* (this means that $E(\eta_n|\eta_{n-1}, \eta_{n-2}, \dots) = 0$ for all $n \in \mathbb{Z}$) , and $\zeta = (\zeta_n)_{n\in\mathbb{Z}}$ is a so-called *coboundary* (or *coboundary sequence*) which can be written as $\zeta_n = \theta_n - \theta_{n-1}, n \in \mathbb{Z}$, by means of a certain stationary sequence $\theta = (\theta_n)_{n\in\mathbb{Z}}$. It is assumed that the random sequences ξ, η, θ in this representation are stationarily connected, that is the sequence $\big((\xi_n, \eta_n, \theta_n)\big)_{n\in\mathbb{Z}}$ of random vectors is stationary. Let us observe that, while studying the asymptotic distributions of the sums $\sum_{k=0}^{n-1} \xi_k$, $n \geq 1$, in many cases one can neglect by the contribution of the sequence ζ into these sums and extend to $\xi = \eta + \zeta$ limit theorems originally known for the martingale difference η only (notice that the limit theory for martingale differences is well developed). To be negligible in this sense, the sequence ζ needs not be a coboundary: some conditions are known [13, 14] under which approximation of the sums of the sequence ξ by those of the martingale difference sequence η is precise enough to conclude that some limit theorems are applicable to ξ once they hold for η; nevertheless, the difference $\zeta = \xi - \eta$ may not be a coboundary under these conditions. However, we consider here more special situation when the negligible summand does have a form of a coboundary: it is this case which admits the most transparent description and analysis, and seems to be more appropriate for an attempt to extend the martingale approach to random fields.

Conditions of limit theorems which are proved by means of the martingale approximation are usually formulated in terms of a ceratin filtration. This filtration is defined on the basic probability space; it is assumed to be stationarily connected with the sequence ξ (the latter means that the filtration is the sequence of the past $\sigma-$fields of a certain auxiliary stationary sequence stationarily connected with ξ). In general, the martingale approximation is applicable even if ξ is not adapted to this filtration. However, the adapted case deserves a special attention not only by pedagogical reasons. It is this situation when there are more satisfactory answers to some natural questions, such as those about the applicability of the Central Limit Theorem (CLT) and about the variance of the limiting normal distribution. In the adapted case the sequence (ξ_n) can be thought of as a non-anticipating function of a Markov chain. A simple condition in terms of the transition operator (solvability of the so-called Poisson equation) guaranties the desired representation to hold which implies the applicability of the CLT. There is a simple formula expressing the variance of the limiting

normal distribution in terms of the solution of the Poisson equation (see [6, 12] and this Sect. 1). Notice now that the "time reversal" in the stationary case does not hurt the validity of conclusions about convergence in probability or in distribution: such assertions are valid or not simultaneously for both the original and the reversed sequences. Thus, applying to the adapted case the time reversal, one obtains a convenient setup where without loss of generality one can assume that all stationary sequences of interest are given rise by a certain probability preserving transformation (the latter should be non-invertible in nontrivial cases). The decreasing filtration mentioned above arises in this situation as the sequence of $\sigma-$fields of preimages of measurable sets with respect to the degrees of the basic transformation. It is this setting which we have chosen as a framework for a discussion of multivariate generalizations of the martingale approximation method. Notice that various definitions of multivariate arrays of martingale differences are possible (see, for example, [1, 2]). Our assumptions lead us with necessity to one of them (see Remark 2) which is tightly related to one of several definitions in [2]. In the present paper we did not discuss in detail these diverse definitions (though this topic is slightly touched in Remark 2) because such a discussion seems to be more appropriate in the context of limit theorems which will be considered elsewhere.
In the rest of Sect. 1 we remind how in such a setting a simplest result on martingale approximation for a random sequence is formulated. In the next sections of the paper we turn to establishing an analogous representation for random fields generated by a class of measure preserving actions of the additive semigroup of integral $d-$dimensional vectors with nonnegative entries.

Let T be a measure preserving transformation of a probability space $(X, \mathcal{F}, P)$. Stationary sequences we are going to consider are of the form $(f \circ T^n)_{n\geq 0}$, where f is a real-valued measurable function on X. Set for $f \in L_2 = L_2(X, \mathcal{F}, P)$ $\;Uf = f \circ T$, and let $U^* : L_2 \to L_2$ be the conjugate of the operator U. The operators $U \;\; U^*$ are, respectively, an isometry and a coisometry in L_2. Both of them preserve values of constant functions and map nonnegative functions to nonnegative ones. Consider U^* as a transition operator of a Markov chain taking values in X and having P as a stationary distribution. The current state of the chain uniquely determines the previous one by means of the transformation T. Let $E^{\mathcal{G}}$ and I denote the conditional expectation operator with respect to some $\sigma-$field $\mathcal{G} \subset \mathcal{F}$ and the identity operator, correspondingly. The relations

$$(U^*)^n U^n = I,\, U^n (U^*)^n = E^{T^{-n}\mathcal{F}},\, n \geq 0,$$

hold between the operators U U^*.

Let us now assume that, for some function $f \in L_2$, a function $g \in L_2$ solves the *Poisson equation*

$$f = g - U^* g. \tag{1.1}$$

Then, setting $h_1 = U^* g$, we have

$$U^* f = h_1 - U^* h_1,$$

which implies

$$E^{T^{-1}\mathcal{F}}(f - Uh_1 + h_1) = UU^*(f - Uh_1 + h_1) = U(U^* f - h_1 + U^* h_1) = 0.$$

Since

$$f - Uh_1 + h_1 = g - U^* g - UU^* g + U^* g = g - UU^* g,$$

we obtain the representation

$$f = h + (U - I)h_1, \tag{1.2}$$

where

$$h = g - UU^* g,\ h_1 = U^* g. \tag{1.3}$$

We observe that the summands of the right hand side of (1.2) give rise to the stationary sequences $(U^n h)_{n \geq 0}$ and $(U^{n+1} h_1 - U^n h_1)_{n \geq 0}$ of the reversed martingale differences and the coboundaries, respectively. Representation (1.2) is the basis for applying the martingale approximation method for proving the Central Limit Theorem and other probabilistic limit results. Also certain conditions for solvability of the equation (1.1) are known which are based on the statistical ergodic theorem for the operator U^*.

Remark 1. There exist expressions in terms of the solution of the Poisson equation for the conditional and the unconditional variances of the martingale difference appearing in (1.3). Indeed, taking into account the first of relations (1.3), we obtain (cp. [6, 12])

$$E^{T^{-1}\mathcal{F}}|h|^2 = UU^*|g - UU^* g|^2 = UU^*|g|^2 - U|U^* g|^2, \tag{1.4}$$

or

$$E^{T^{-1}\mathcal{F}}|h|^2 = E^{T^{-1}\mathcal{F}}|g|^2 - |E^{T^{-1}\mathcal{F}} g|^2. \tag{1.5}$$

It follows from (1.4) that

$$E|h|^2 = E|g|^2 - E|U^* g|^2. \tag{1.6}$$

The latter quantity equals the limiting variance in the Central Limit Theorem for the sequence (ξ). □

In the present paper a multivariate analogue of the above situation is considered. Some conditions are investigated which ensure the validity of a representation and relations similar to (1.2) and (1.3). The unicity issue of such a representation is also examined. However, we do not touch applications to limit theorems. Though the case of square-integrable variables is of main interest, our considerations concern the L_p spaces where $1 \le p \le \infty$ or $1 \le p < \infty$. A multivariate generalization of the representation (1.2) is presented in Proposition 1. The main assumption here is the solvability of the equation (2.2), a higher analogue of the Poisson equation (1.1). Solvability conditions for the equation (2.2) are given in Propositions 2 and 3. A discussion of the definition of multivariate martingale differences used in the present paper and comments on the structure of the representation (2.3) and its role in the investigation of the asymptotics of sums over the random field can be found in Remarks 2 and 3, respectively.
Applications to limit theorems will be presented in separate publications which are in preparation. In one of them, by M. Weber and the author [11], a particular form of the representation from the present paper is applied to a problem considered in [10] and related to the so-called Baker sequences. Application of the martingale approach allows us to give a complete analysis of possible degenerations of the limit in this problem. The second paper, joint with H. Dehling and M. Denker [9], introduces a concept of $U-$ and $V-$statistics of a measure preserving transformation and treats asymptotic results for them by means of a formalism parallel to that of the present paper; however, it is applied to some functional spaces, distinct from the L_p spaces and chosen in accordance with the situation considered there. A part of this paper was written during the authors stays

2. Notation and statements of results

Let $T_1, \ldots, T_d$ be commuting measure preserving transformations of a probability space $(X, \mathcal{F}, P)$. Denote by $\mathbb{Z}^d_+$ the additive semigroup of $d-$dimensional coordinate vectors with non-negative integral entries. Then the relation $\mathbf{n} = (n_1, \ldots, n_d) \mapsto T^{\mathbf{n}} = T_1^{n_1} \cdots T_d^{n_d}$, $\mathbf{n} \in \mathbb{Z}^d_+$, defines a measure preserving action of the semigroup $\mathbb{Z}^d_+$ on the space $(X, \mathcal{F}, P)$.

Let $\mathcal{S}_d$ ($\mathcal{S}_{r,d}$) be the set of all subsets (correspondingly, of all sets of cardinality $r \in [0, d]$) of the set $\mathbb{N}(d) = \{1, \ldots, d\}$. Define for every $S \in \mathcal{S}_d$ a subsemigroup $\mathbb{Z}^{d,S}_+ \subseteq \mathbb{Z}^d_+$ by the relation

$$\mathbb{Z}^{d,S}_+ = \{(n_1, \ldots, n_d) \in \mathbb{Z}^d_+ : n_k = 0 \text{ for all } k \notin S\}.$$

For every $p \in [1, \infty]$ let $q = q(p) = p/(1-p) \in [1, \infty]$. Set

$$U_k f = f \circ T_k$$

for every $f \in L_p = L_p(X, \mathcal{F}, P)$ and $k \in \mathbb{N}(d)$. For every $p \in [1, \infty)$ and $k \in \mathbb{N}(d)$ let U_k^* denote the conjugate of the operator U_k. The operator U_k^* is acting on $L_q, 1 < q \leq \infty$. The same symbol U_k^* denotes an operator on L_1 such that its conjugate is $U_k : L_\infty \to L_\infty$ (the existence of such an operator follows easily from the measure-preserving character of T_k). Every operator U_k is acting on every space L_p as an isometry which preserves values of constant functions and the cone of nonnegative functions. Therefore, U_k^* is acting on every such space as a contraction which preserves nonnegativity and values of constants. Furthermore, as was noticed in Sect. 1, for every $k \in \mathbb{N}(d)$ $n \geq 0$ we have

$$U_k^{*n} U_k^n = I, U_k^n U_k^{*n} = E^{T_k^{-n}\mathcal{F}}. \tag{2.1}$$

If for every $i, j \in \mathbb{N}(d)$, $i \neq j$, we also have

$$U_i U_j^* = U_j^* U_i,$$

then the transformations $T_1, \ldots, T_d$ are said to be *completely commuting*. This property, unlike commutativity, depends on the probability measure P. It implies that the conditional expectations

$$\left(E^{T_k^{-n}\mathcal{F}}\right)_{n \geq 0,\, k \in \mathbb{N}(d)}$$

mutually commute as well. Let us set for $n \geq 0$ and $k \in \mathbb{N}(d)$

$$\mathcal{F}_k^n = T_k^{-n}\mathcal{F}, E_k^n = E^{\mathcal{F}_k^n},$$

and

$$\mathcal{F}_k^\infty = \cap_{n=0}^\infty \mathcal{F}_k^n, E_k^\infty = E^{\mathcal{F}_k^\infty}.$$

The above commutativity of conditional expectations extends, by passing to the limit, to the family

$$\left(E_k^n\right)_{0 \leq n \leq \infty,\, k=1,\ldots,d}.$$

Further, let $\overline{\mathbb{Z}_+^d}$ be a completion of $\mathbb{Z}_+^d$, whose elements $\mathbf{n} = (n_1, \ldots, n_d)$ have entries $n_1, \ldots, n_d$ with possible values $0, 1, \ldots, \infty$. Endow $\overline{\mathbb{Z}_+^d}$ with a natural partial order extending that of $\mathbb{Z}_+^d$. For every $\mathbf{n} = (n_1, \ldots, n_d) \in \overline{\mathbb{Z}_+^d}$ we set

$$\mathcal{F}^{\mathbf{n}} = \bigcap_{k=1}^d \mathcal{F}_k^{n_k},\ E^{\mathbf{n}} = E^{\mathcal{F}^{\mathbf{n}}},$$

and obtain

$$E^{\mathbf{n}} = \prod_{k=1}^{d} E_k^{n_k}.$$

Remark 2. Let for $\mathbf{m}=(m_1,\ldots,m_d)$, $\mathbf{n}=(n_1,\ldots,n_d)$, $\mathbf{m},\mathbf{n} \in \overline{\mathbb{Z}_+^d}$, the relation $\mathbf{m} \leq \mathbf{n}$ means, by definition, that $m_1 \leq n_1, \ldots, m_d \leq n_d$. It is clear that $\mathcal{F}^{\mathbf{n}} \subseteq \mathcal{F}^{\mathbf{m}}$ whenever $\mathbf{m} \leq \mathbf{n}$. Thus,

$$\left(\mathcal{F}^{\mathbf{n}}\right)_{\mathbf{n}\in\overline{\mathbb{Z}_+^d}}$$

is a decreasing filtration parametrized by a partially ordered set $\overline{\mathbb{Z}_+^d}$. Let $\bigvee$ be the binary operation of taking the coordinatewise maximum in $\overline{\mathbb{Z}_+^d}$. The commutation relations between conditional expectations observed above have the following probabilistic meaning: assume that $\mathbf{l},\mathbf{m},\mathbf{n} \in \overline{\mathbb{Z}_+^d}$ and $\mathbf{n} = \mathbf{l}\bigvee\mathbf{m}$, then the $\sigma-$fields $\mathcal{F}^{\mathbf{l}}$ $\mathcal{F}^{\mathbf{m}}$ are conditionally independent given $\mathcal{F}^{\mathbf{n}}$. Such a property of a filtration (rather for the increasing case than for the decreasing one as in our setup) is well-known in the literature (see, for example, [2]).
We will discuss now the definition of reversed martingale differences we choose in this paper. We are led to this definition by Proposition 1 below. A family $\left(\xi_{\mathbf{n}}, \mathcal{F}^{\mathbf{n}}\right)_{\mathbf{n}\in\mathbb{Z}_+^d}$ of random variables defined on $(X,\mathcal{F},P)$, and sub-σ-fields of $\mathcal{F}$, is said to be a family of *reversed martingale differences* if we have

(1) for every $\mathbf{n} \in \mathbb{Z}_+^d$ the random variable $\xi_{\mathbf{n}}$ is measurable with respect to $\mathcal{F}^{\mathbf{n}}$;
(2) $E^{\mathcal{F}^{\mathbf{m}}}\xi_{\mathbf{n}} = 0$ whenever $\mathbf{m} \not\leq \mathbf{n}$.

This definition without changes applies to any partially ordered set instead of $\mathbb{Z}_+^d$. Like the above conditional independence condition, it also can be found in the literature. Indeed, in the paper [2], which is devoted to stochastic integrals and martingales in $\mathbb{R}^2$, concepts of 1-2-martingales, among several others, are introduced . The definition given above in the case $d = 2$ is an analogue (for discrete and reversed "time") of the property of a random field to be a 1- and a 2-martingale simultaneously. Comparing the requirements imposed by the definition given above, we see, for example, that it is less restrictive than the one given in [1], and more restrictive than the definition in [7]. Conditions imposed on filtration is a separate question. As was noticed above, in the setup of the present paper a rather special property of conditional independence holds. □

From now on we assume in this paper that the transformations $T_1, \dots, T_d$ are completely commuting.

For $\mathbf{n} = (n_1, \dots, n_d) \in \mathbb{Z}_+^d$ we set

$$U^{\mathbf{n}} = U_1^{n_1} \cdots U_d^{n_d},\ U^{*\mathbf{n}} = (U^{\mathbf{n}})^*.$$

For every $S \in \mathcal{S}_d$ denote by $\mathcal{I}_S$ the $\sigma-$field of those $A \in \mathcal{F}$ for which the relation $T_k^{-1}A = A$ holds for every $k \in S$, and let $E^{\mathcal{I}_S}$ be the corresponding conditional expectation. Notice that for the empty set $\emptyset$ $\mathcal{I}_\emptyset = \mathcal{F}$ and $E^{\mathcal{I}_\emptyset} = I$. Write $\mathcal{I}_k$ instead of $\mathcal{I}_{\{k\}}$ for $k \in \mathbb{N}(d)$. Set $\mathbf{1}_d = (1, \dots, 1) \in \mathbb{Z}_+^d$. For $\mathbf{N} = (N_1, \dots, N_d) \in \mathbb{Z}_+^d$ set

$$S_{\mathbf{N}} = \sum_{0 \le \mathbf{n} \le \mathbf{N} - \mathbf{1}_d} U^{\mathbf{n}},\ S_{\mathbf{N}}^* = \sum_{0 \le \mathbf{n} \le \mathbf{N} - \mathbf{1}_d} U^{*\mathbf{n}},$$

if $\mathbf{1}_d \le \mathbf{N}$, and

$$S_{\mathbf{N}} = 0,\ S_{\mathbf{N}}^* = 0$$

otherwise.

The following assertion presents a multivariate analogue of the representation in the form of a sum of a martingale difference and a coboundary which was discussed in Sect. 1. Comments on this multivariate representation are given in Remark 3 below.

Proposition 1. *Let $1 \le p \le \infty$, and let for a function $f \in L_p$ a function $g \in L_p$ solves the equation*

$$f = \Big(\prod_{k=1}^{d} (I - U_k^*)\Big) g. \tag{2.2}$$

Then f can be represented in the form

$$f = \sum_{S \in \mathcal{S}_d} \Big(\prod_{k \in S} (U_k - I) \prod_{l \notin S} (I - U_l U_l^*)\Big) h_S, \tag{2.3}$$

where for every $S \in \mathcal{S}_d$ the function $h_S \in L_p$ is defined by the relation

$$h_S = \Big(\prod_{m \in S} U_m^*\Big) g\,. \tag{2.4}$$

Conversely, if for some $g \in L_p$ a function $f \in L_p$ admits the representation (2.3) *with functions h_S defined by the relations* (2.4)*, then g is a solution of the equation* (2.2)*.*

Let a function $f \in L_p$ admits two representations of the form (2.3) *with the function $(h_S)_{S \in \mathcal{S}_d}$ and $(h'_S)_{S \in \mathcal{S}_d}$, correspondingly (now it is not* *a priori assumed that relations of the type of* (2.4) *hold). Then for every* $S \in \mathcal{S}_d$

$$\Big(\prod_{k\in S}(U_k - I)\prod_{l\notin S}(I - U_l U_l^*)\Big)h'_S = \Big(\prod_{k\in S}(U_k - I)\prod_{l\notin S}(I - U_l U_l^*)\Big)h_S.$$

Remark 3. It is clarified here the meaning of decomposition (2.3) the role of its components in the asymptotics of the sums

$$S_{\mathbf{N}}f = \sum_{0\le \mathbf{n}\le \mathbf{N}-\mathbf{1}_d} U^{\mathbf{n}} f. \tag{2.5}$$

Let $S \in \mathcal{S}_{r,d}$. The summand

$$A_S = \Big(\prod_{k\in S}(U_k - I)\prod_{l\notin S}(I - U_l U_l^*)\Big)h_S$$

of the right-hand side of relation (2.3) satisfies the equations

$$E_t^1 A_S = 0, t \notin S. \tag{2.6}$$

To establish this fact, we represent A_S, using the commutation relations, as $A_S = \big(\prod_{l\notin S}(I - U_l U_l^*)\big)B_S$ and then apply the relations $U_t U_t^*(I - U_t U_t^*) = 0$, $t \notin S$. This implies that for every $\mathbf{m} \in \mathbb{Z}_+^d$ the family
$\big(U^{\mathbf{m}+\mathbf{n}}A_S, \mathcal{F}^{\mathbf{m}+\mathbf{n}}\big)_{\mathbf{n}\in\mathbb{Z}_+^{d,\mathbb{N}(d)\setminus S}}$ is a $(d-r)-$dimensional random field of reversed martingale differences (in the sense of Remark 2.) The latter means that for $\mathbf{n} \in \mathbb{Z}_+^{d,\mathbb{N}(d)\setminus S}$ the random variable $U^{\mathbf{m}+\mathbf{n}}A_S$ is $\mathcal{F}^{\mathbf{m}+\mathbf{n}}$-measurable and satisfies the relations

$$E^{\mathbf{m}+\mathbf{n}+\mathbf{e}_l}U^{\mathbf{m}+\mathbf{n}}A_S = 0, l \notin S.$$

Here we set $\mathbf{e}_l = (\delta_{l,1}, \dots, \delta_{l,d})$, where $\delta_{l,m} = 1$ for $l = m$, and $\delta_{l,m} = 0$, if $l \neq m$.
Further, for every $\mathbf{m} \in \mathbb{Z}_+^d$ the family $(U^{\mathbf{m}+\mathbf{n}}A_S)_{\mathbf{n}\in\mathbb{Z}_+^{d,S}}$ is a $r-$dimensional stationary random field of $r-$coboundaries, that is it has the form $\big(U^{\mathbf{n}}\big(\prod_{k\in S}(U_k - I)\big)C_{S,\mathbf{m}}\big)\big)_{\mathbf{n}\in\mathbb{Z}_+^{d,S}}$. In particular, this implies that the sums

$$\sum_{\{\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{Z}_+^{d,S}:\, n_k\in[0,N_k-1] \text{ for every } k\in S\}} U^{\mathbf{m}+\mathbf{n}}A_S$$

are bounded in L_p. Putting off the analysis of distributions of the sums (2.5) to another case, we will describe now at the heuristic level the role played by decomposition (2.3) in this issue. We have, in view of

(2.3),

$$
\begin{aligned}
S_{\mathbf{N}} f &= S_{\mathbf{N}}\Big(\sum_{S\in\mathcal{S}_d}\big(\prod_{k\in S}(U_k - I)\prod_{l\notin S}(I - U_l U_l^*)\big)h_S\Big) \\
&= S_{\mathbf{N}}\sum_{S\in\mathcal{S}_d} A_S = \sum_{S\in\mathcal{S}_d} S_{\mathbf{N}} A_S.
\end{aligned} \tag{2.7}
$$

Keeping S fixed, the behavior of the sums $S_{\mathbf{N}}A_S$ by $\mathbf{N} = (N_1,\dots,N_d) \to \infty$ depends on existence of the moments and some other properties of the summands. For $p \geq 2$ the above-mentioned cobounadry properties of A_S guarantee the boundedeness of L_2-norms of the random variables

$$
\big(\prod_{l\notin S} N_l\big)^{-1/2} S_{\mathbf{N}} A_S. \tag{2.8}
$$

If $(T^{\mathbf{n}})_{\mathbf{n}\in\mathbb{Z}_+^d}$ is a mixing action and $\big(\prod_{l\notin S}(I - U_lU_l^*)\big)h_S \neq 0$ (we will call such a function f *S-nondegenerate*), the L_2-norms of such variables have a (finite) positive limit. Moreover, notice (though we do not need it in the present paper) that these variables converge in distribution to a centered Gaussian law whose variance is the square of this limit. In case of $\emptyset-$non-degeneracy of f, the summand

$$
S_{\mathbf{N}} A_\emptyset = S_{\mathbf{N}}\big(\prod_{l=1}^{d}(I - U_l U_l^*)\big)g
$$

dominates in the sum $\sum_{S\in\mathcal{S}_d} S_{\mathbf{N}} A_S$. Indeed, denoting by $|\cdot|_2$ the L_2-norm, we obtain

$$
\sigma_\emptyset^2(f) = \big|\big(\prod_{l=1}^{d}(I - U_lU_l^*)\big)g\big|_2^2 \ \Big(= \sum_{r=0}^{d}(-1)^r \sum_{S\in\mathcal{S}^{r,d}}\big|\prod_{k\in S} U_k^* g\big|_2^2\Big). \tag{2.9}
$$

Since reversed martingale differences $\big(U^{\mathbf{n}}\big(\prod_{l=1}^d (I - U_lU_l^*)\big)g\big)_{\mathbf{n}\in\mathbb{Z}_+^d}$ are mutually orthogonal, we have for $\mathbf{N} = (N_1,\dots,N_d)$

$$
\big|S_{\mathbf{N}} A_\emptyset\big|_2^2 = \big(\prod_{k=1}^{d} N_k\big)\sigma_\emptyset^2(f).
$$

Comparing this amount with (2.8) for $S \neq \emptyset$, it is clear that $S_{\mathbf{N}}A_\emptyset$ dominates in the sums $S_{\mathbf{N}} f$ as $\mathbf{N} \to \infty$ whenever $\sigma_\emptyset^2(f) > 0$. This fact is crucial when one proves limit theorems for sums $S_{\mathbf{N}} f$ by reduction the problem to the case of reversed martingale differences. It also shows that $\sigma_\emptyset^2(f)$ does not depend on the choice of the representation of the type of (2.3), and that the notation introduced above is consistent.

Moreover, it is clear that the random variable $A_\emptyset$ generating a $d-$dimensional field of reversed martingale differences is uniquely determined. This analysis of the asymptotics can be continued to obtain the uniqueness of summands in the representation of the type (2.3) on the way distinct from that taken in the proof of Proposition 1. □

In the rest of the present section conditions for solvability of the equation (2.2) are discussed, and a description of the set of its solutions is given.

The following remark will be needed in the course of the proof of the Proposition 2 to identify the limit in the statistical ergodic theorem for the operators U_k^*.

Remark 4. Here some general properties of the actions under consideration are summarized. Since the transformations $T_1, \dots, T_d$ commute, the conditional expectations $E^{\mathcal{I}_S}, S \in \mathcal{S}_d$, commute as well. Notice that for $k \in \mathbb{N}(d)$

$$\mathcal{I}_k \subseteq \mathcal{F}_k^\infty. \tag{2.10}$$

Let us make clear the interrelation between the invariant elements of the operators U_k and U_k^*. Assume that for some $k \in \mathbb{N}(d)$ a certain $f \in L_p$ satisfies $U_k f = f$. Apply U_k^* to the both parts of the last equation. Then the relation $U_k^* U_k = I$ yields $U_k^* f = f$. Conversely, let $U_k^* f = f$. Then for all $n \geq 0$ $U_k^{*n} f = f$, $U_k^n U_k^{*n} f = U_k^n f$ and, consequently, $E_k^n f = U_k^n f$. Since the operator U_k is an isometry, the conditional expectation in the left-hand side preserves the L_p-norm of f, which is only possible if the expectation acts on f identically. The latter means that f is $\mathcal{F}_k^n-$measurable. Since n is arbitrary, f is $\mathcal{F}_k^\infty-$measurable. Further, it follows from the relations between U_k U_k^* that they act on the space of $\mathcal{F}_k^\infty$-measurable L_p-functions as mutually inverse isometries which implies $U_k f = f$. Therefore, the operators U_k U_k^* have the same invariant elements in the spaces L_p. The same conclusion also holds for every $S \in \mathcal{S}_d$ for jointly invariant elements of every of two sets of operators: $\{U_k : k \in S\}$ and $\{U_k^* : k \in S\}$. Hence, we have for every $S \in \mathcal{S}_d$ and every $p \in [1, \infty]$ that

$$\{f \in L_p : U_k^* f = f, k \in S\} = \{f \in L_p : U_k f = f, k \in S\} =$$
$$= \{f \in L_p : f \text{ is measurable with respect to } \mathcal{I}_S\}.$$

□

Let us call a function $g \in L_p$ *normal*, if

$$g = \Big(\prod_{k \in \mathbb{N}(d)} (I - E^{\mathcal{I}_k}) \Big) g.$$

Normality of $g \in L_p$ is equivalent to

$$E^{\mathcal{I}_k} g = 0, k \in \mathbb{N}(d).$$

Denote by $\mathrm{Ker}(A)$ and $\mathrm{Ran}(A)$ the kernel and the image of a linear operator A, respectively .

Proposition 2. *Let $p \in [1, \infty]$.*

(1) *Every $f \in L_p$, for which equation (2.2) has a solution $g \in L^p$, is normal.*

(2) *A function $g \in L_p$ is a solution of equation (2.2) if and only if it can be represented in the form $g = g' + e$, where g' is a normal solution of equation (2.2) and $e \in \mathrm{Ker}\big(\prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k})\big)$. Equation (2.2) has at most one normal solution.*

(3) *Let $p < \infty$ and $f \in L_p$ is a normal function. Equation (2.2) has a solution in L^p if and only if the limit*

$$\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty} (N_1 \cdots N_d)^{-1} \sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d} S^*_{\mathbf{M}} f \tag{2.11}$$

exists in the L_p-norm. This limit represents a normal solution of equation (2.2).

Substituting the normality assumption by a stronger condition, one can simplify the solvability criterion of (2.2) and the procedure of constructing its solution. Let us call a function $f \in L_1$ *strictly normal* whenever $f = \prod_{k\in\mathbb{N}_d}(I - E_k^{\infty})f$ (an equivalent property says that for all $k \in \mathbb{N}(d)$ $E_k^{\infty} f = 0$.) The strict normality is stronger than the normality because $\mathcal{I}_k \subset \mathcal{F}_k^{\infty}$ (recall that $E_k^{\infty} = E^{\mathcal{F}_k^{\infty}}$).

The strict normality of $f \in L_p$ can be characterized by any of the following properties (where convergence is assumed in the sense of the L_p−norm):

$$E^{(n_1,\dots,n_d)} f \underset{\max(n_1,\dots,n_d)\to\infty}{\longrightarrow} 0, \tag{2.12}$$

or

$$U^{*(n_1,\dots,n_d)} f \underset{\max(n_1,\dots,n_d)\to\infty}{\longrightarrow} 0. \tag{2.13}$$

Proposition 3. *Let $p \in [1, \infty)$.*

(1) *If a function $f \in L_p$ can be represented as (2.2), where $g \in L_p$ is strictly normal, then f is strictly normal, and g can be represented in the form*

$$g = \sum_{\mathbf{n}\in\mathbb{Z}_+^d} U^{*\mathbf{n}} f \ \Big(\overset{def}{=} \lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty} S^*_{\mathbf{N}} f\Big), \tag{2.14}$$

where the series converges in the L_p-norm. Equation (2.2) *has at most one strictly normal solution in L_p.*

(2) *Conversely, let for some strictly normal function $f \in L_p$ series* (2.14) *converges in the L_p-norm. Then the sum of series* (2.14) *presents a strictly normal solution of equation* (2.2).

(3) *For every strictly normal function $f \in L_p$ it follows from the convergence of series* (2.14) *that for every $S \in S_d$ the series*

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^{d,S}} U^{*\mathbf{n}} f \ \left(\overset{def}{=} \lim_{N_k\to\infty,\, k\in S} \sum_{\mathbf{n}\in\mathbb{Z}_+^{d,S}} U^{*\mathbf{n}} f \right) \tag{2.15}$$

converges in the $L_p - norm$.

Remark 5. For $d \geq 2$ the convergence of series (2.14) seemingly does not imply that (2.13) holds (that is that f is strictly normal). However, one can omit in assertions (2) and (3) of Proposition 3 the assumption that f is strictly normal, if one assumes instead, in addition to the convergence of (2.14), the convergence of (2.15) for every set S of cardinality 1. □

Remark 6. Propositions 2 and 3 can be extended to the case of the space L_∞, if one considers the convergence of the series in the L_1-topology of the space L_∞ instead of their convergence in the L_∞-norm. Further, for $1 < p < \infty$ the requirement of the existence of the limit (2.11) in Proposition 2 as a sufficient condition of solvability of equation (2.2) can be weakened to that of boundedness in L_p of the corresponding sequence of partial sums.

Example 2.1. *Let $d = 2$. If for a strictly normal function $f \in L_p$ the series $\sum_{n_1,n_2=0}^{\infty} U_1^{*n_1} U_2^{*n_2} f$ converges in the L_p-norm, then f admits a representation in the form*

$$f = C_\emptyset + (U_1 - I)C_1 + (U_2 - I)C_2 + (U_1 - I)(U_2 - I)C_{1,2},$$

where the functions $C_\emptyset, C_1, C_2, C_{1,2} \in L_p$ are strictly normal and

$$E^{T_i^{-1}\mathcal{F}} C_\emptyset = 0 \, (i = 1, 2),$$

$$E^{T_2^{-1}\mathcal{F}} C_1 = 0, E^{T_1^{-1}\mathcal{F}} C_2 = 0.$$

If the transformations T_1 and T_2 are ergodic, then the strictly normal functions $C_\emptyset, C_1, C_2, C_{1,2}$ are uniquely determined. □

3. Proofs

In the course of proofs of Propositions 1, 2 and 3 the following assertion will be needed.

Lemma 1. *For every $S \in \mathcal{S}_d$ the following relations hold:*

$$\Big(\prod_{k\in S}(I - E^{\mathcal{I}_k})\Big)\Big(\prod_{k\in S}(I - U_k^*)\Big) = \Big(\prod_{k\in S}(I - U_k^*)\Big)\Big(\prod_{k\in S}(I - E^{\mathcal{I}_k})\Big) = \prod_{k\in S}(I - U_k^*), \tag{3.1}$$

$$\Big(I - \prod_{k\in S}(I - E^{\mathcal{I}_k})\Big)\Big(\prod_{k\in S}(I - U_k^*)\Big) = \Big(\prod_{k\in S}(I - U_k^*)\Big)\Big(I - \prod_{k\in S}(I - E^{\mathcal{I}_k})\Big) = 0, \tag{3.2}$$

$$\operatorname{Ker}\Big(\prod_{k\in S}(I - U_k^*)\Big) = \operatorname{Ker}\Big(\prod_{k\in S}(I - U_k)\Big) = \operatorname{Ker}\Big(\prod_{k\in S}(I - E^{\mathcal{I}_k})\Big). \tag{3.3}$$

Proof of Lemma 1. For every $k \in \mathbb{N}(d)$ we have $E^{\mathcal{I}_k}U_k^* = E^{\mathcal{I}_k}$ (this follows, for example, from the obvious identity $U_kE^{\mathcal{I}_k} = E^{\mathcal{I}_k}$ applied to the dual space). This implies $(I-E^{\mathcal{I}_k})(I-U_k^*) = I-U_k^*$. Taking the product of these relations over all $k \in S$ (the order of the multipliers, in view of their commutativity, is of no importance here) gives (3.1). Subtracting the both parts of (3.1) from the operator $\prod_{k\in S}(I - U_k^*)$, (3.2) follows.
Let us prove (3.3). According to Remark 4 for all $k \in \mathbb{N}(d)$ the relations $\operatorname{Ker}(I - U_k^*) = \operatorname{Ker}(I - U_k) = \operatorname{Ker}(I - E^{\mathcal{I}_k})$ hold. Equalities (3.3) are consequences of these relations for $k \in S$ and the fact that the kernel of the product of two commuting bounded operators is the closure of the sum of their kernels. □

Proof of Proposition 1. Since for g holds (2.2), we have

$$\begin{aligned} f &= \Big(\prod_{k=1}^{d}(I - U_k^*)\Big)g = \Big(\prod_{k=1}^{d}[(I - U_kU_k^*) + (U_k - I)U_k^*)]\Big)g \\ &= \sum_{S\in\mathcal{S}_d}\Big(\prod_{k\in S}(U_k - I)\prod_{l\notin S}(I - U_lU_l^*)\prod_{m\in S}U_m^*\Big)g \\ &= \sum_{S\in\mathcal{S}_d}\Big(\prod_{k\in S}(U_k - I)\prod_{l\notin S}(I - U_lU_l^*)\Big)h_S, \end{aligned} \tag{3.4}$$

where $h_S = \big(\prod_{m\in S}U_m^*\big)g$.
Reversing this chain of equalities, we see that it follows from (2.3) and (2.4) that g satisfies the relation (2.2).
Let us prove now the assertion about the uniqueness of the representation (2.3). We set $H_S = h'_S - h_S$. Subtracting one of the representations

from another, the relation

$$\sum_{S\in\mathcal{S}_d}\big(\prod_{k\in S}(U_k-I)\prod_{l\notin S}(I-U_lU_l^*)\big)H_S=0, \tag{3.5}$$

is obtained. It suffices now to deduce from (3.5) that for every $S\in\mathcal{S}_d$

$$\big(\prod_{k\in S}(U_k-I)\prod_{l\notin S}(I-U_lU_l^*)\big)H_S=0. \tag{3.6}$$

At the initial step (we give number zero to it) we apply the operator $\prod_{k\in\mathbb{Z}_d}U_k^*$ to the both parts of equation (3.5), keeping in mind that operators with different indices commute while for operators with the same index the relations $U_l^*(I-U_lU_l^*)=0, U_l^*(U_l-I)=I-U_l^*, l\in\mathbb{Z}(d)$, hold. It is clear that the operator $\prod_{k\in\mathbb{Z}_d}U_k^*$ vanishes on all summands in the left-hand side of (3.5), except for that for which $S=\mathbb{Z}(d)$. This implies

$$\big(\prod_{k\in\mathbb{Z}_d}(I-U_k^*)\big)H_{\mathbb{Z}(d)}=0,$$

which follows, in view of the first of relations (3.3), that

$$\big(\prod_{k\in\mathbb{Z}_d}(U_k-I)\big)H_{\mathbb{Z}(d)}=0.$$

Therefore, the relation (3.6) for $S=\mathbb{Z}(d)$ is obtained. Subtracting this relation "of level d" from (3.5), the relation

$$\sum_{r=0}^{d-1}\sum_{S\in\mathcal{S}_{r,d}}\big(\prod_{k\in S}(U_k-I)\prod_{l\notin S}(I-U_lU_l^*)\big)H_S=0 \tag{3.7}$$

is established. At the step one each of d products

$$U_2^*\cdots U_d^*,\ U_1^*U_3^*\cdots U_d^*,\ldots,\ U_1^*\cdots U_{d-1}^*$$

of $d-1$ operators is consequently applied to the equation obtained before. This gives the equalities

$$\big(\prod_{k\in S}(I-U_k^*)\prod_{l\notin S}(I-U_lU_l^*)\big)H_S=0, S\in\mathcal{S}_{d-1,d}.$$

Using again the relation (3.3), we see that

$$\big(\prod_{k\in S}(U_k-I)\prod_{l\notin S}(I-U_lU_l^*)\big)H_S=0, S\in\mathcal{S}_{d-1,d}.$$

Therefore, we obtained such way every of d equalities of level $d-1$ from (3.6). Subtract them from (3.7) and continue the process. At the r-th step all those $\binom{d}{r}$ relations from (3.6) will be obtained, which correspond to $S \in \mathcal{S}_{d-r,d}$. The process finishes at the d-th step by obtaining the relation (3.6) for $S = \emptyset$. $\square$

Proof of Proposition 2. It follows from relations (2.2) (3.1) that

$$
\begin{aligned}
f =& \Big(\prod_{l=1}^{d}(I - U_l^*)\Big) g = \Big(\prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k})\Big)\Big(\prod_{k\in\mathbb{N}(d)}(I - U_k^*)\Big) g \\
=& \Big(\prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k})\Big) f,
\end{aligned}
$$

which proves (1).
Let now $g \in L_p$ be a certain solution of (2.2). Set $g' = \big(\prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k})\big)g$ and $e = \big(I - \prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k})\big)g$. It follows from (3.1) by $S = \mathbb{N}(d)$ that g' is a solution of (2.2). This solution is normal, since $\prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k})g' = g'$. Let g_1 and g_2 be two normal solutions of equation (2.2). Then $g_2 - g_1 \in \mathrm{Ker}\big(\prod_{k\in\mathbb{N}(d)}(I - U_k^*)\big)$, which, combined with (3.3), implies $g_2 - g_1 \in \mathrm{Ker}(\prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k}))$. On the other hand, in view of normality of the solutions g_1 g_2 we have $g_2 - g_1 \in \mathrm{Ran}\big(\prod_{k\in\mathbb{N}(d)}(I - E^{\mathcal{I}_k})\big)$. This implies $g_2 - g_1 = 0$, which proves (2).
Let f admit representation (2.2). In view of just established item (2) of Proposition 2, the function g in this representation can be (and will be) chosen to be normal. Set $\mathbf{M} = (M_1, \dots, M_d)$ and $\mathbf{N} = (N_1, \dots, N_d)$. Then we have

$$
S^*_{\mathbf{M}} f = \sum_{0\le \mathbf{n} \le \mathbf{M}-\mathbf{1}_d} U^{*\mathbf{n}} f = \Big(\prod_{k=1}^{d}\Big(\sum_{n=0}^{M_k-1} U_k^{*n}\Big)\Big) f = \Big(\prod_{k=1}^{d}(I - U_k^{*M_k})\Big) g
$$

and

$$
\begin{aligned}
&(N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d} S^*_{\mathbf{M}}\, f\\
&= (N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d}\Big(\prod_{k=1}^{d}\big(I-U_k^{*M_k}\big)\Big)\, g\\
&= (N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d}\sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}\Big(\prod_{l\in S}U_l^{*M_l}\Big)\, g\\
&= \sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}(N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d}\Big(\prod_{l\in S}U_l^{*M_l}\Big)\, g\\
&= \sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}\Big(\prod_{l\in S}\big(N_l^{-1}\sum_{M_l=0}^{N_l-1}U_l^{*M_l}\big)\Big)\, g\\
&\underset{\mathbf{N}\to\infty}{\longrightarrow}\sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}E^{\mathcal{I}_S}g,
\end{aligned}
\tag{3.8}
$$

where on the last stage the multiparameter statistical ergodic theorem was applied. Since g is normal, $E^{\mathcal{I}_S}g=0$ for every non-empty S, while $E^{\mathcal{I}_\emptyset}g=g$.

Conversely, let for a normal function $f\in L_p$ in the L_p-norm there exists the limit

$$
\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}(N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d}S^*_{\mathbf{M}}\, f=g. \tag{3.9}
$$

The operators $U_1^*,\dots,U_d^*$ send normal functions to normal ones. Hence, g is normal, as a limit of normal functions. Further, acting as in (3.8), we obtain

(3.10)

$$
\begin{aligned}
&\Big(\prod_{k\in\mathbb{N}(d)}(I-U_k^*)\Big)g\\
&=\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}(N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d}S^*_{\mathbf{M}}\Big(\prod_{k\in\mathbb{N}(d)}(I-U_k^*)\Big)f\\
&=\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}(N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d}\Big(\prod_{k=1}^{d}\big(I-U_k^{*M_k}\big)\Big)f\\
&=\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}(N_1\cdots N_d)^{-1}\sum_{0\le\mathbf{M}\le\mathbf{N}-\mathbf{1}_d}\sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}\Big(\prod_{l\in S}U_l^{*M_l}\Big)f\\
&=\sum_{r=0}^{d}(-1)^r\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}\sum_{S\in\mathcal{S}_{r,d}}\Big(\prod_{l\in S}\Big(N_l^{-1}\sum_{M_l=0}^{N_l-1}U_l^{*M_l}\Big)\Big)f\\
&=\sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}E^{\mathcal{I}_S}f=f.
\end{aligned}
$$

□

Proof of Proposition 3. The space of strictly normal L_p-function is invariant with respect to the operators $U_k^*, k=1,\dots,d$. This implies that f is strictly normal. Further,

$$
\begin{aligned}
\sum_{0\le\mathbf{n}\le\mathbf{N}-\mathbf{1}_d}U^{*\mathbf{n}}f&=\sum_{0\le\mathbf{n}\le\mathbf{N}-\mathbf{1}_d}U^{*\mathbf{n}}\Big(\prod_{k=1}^{d}(I-U_k^*)\Big)g\\
&=\Big(\prod_{k=1}^{d}\Big((I-U_k^*)\sum_{n_k=0}^{N_k-1}U_k^{*n_k}\Big)\Big)g=\Big(\prod_{k=1}^{d}\big(I-U_k^{*N_k}\big)\Big)g \qquad (3.11)\\
&=\sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}\prod_{k\in S}U_k^{*N_k}g\underset{\mathbf{N}\to\infty}{\longrightarrow}\sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}\prod_{k\in S}E_k^{\infty}g=g,
\end{aligned}
$$

where the last equation follows from the strict normality of g. It follows from this representation that a normal solution is unique (this follows also from Proposition (3)), and (1) is proved.

Starting to prove (2), let g denote the sum of the series (2.14). The strict normality of g is a consequence of the strict normality of f and the fact that the subspace of strictly normal L_p-functions is closed and invariant with respect to the operators $U_k^*, k=1,\dots,d$. Further, analogously to (3.11),

$$
\begin{aligned}
&\Big(\prod_{k=1}^{d}(I-U_k^*)\Big)g \\
&= \Big(\prod_{k=1}^{d}(I-U_k^*)\Big)\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}\sum_{0\le\mathbf{n}\le\mathbf{N}-\mathbf{1}_d}U^{*\mathbf{n}}f \\
&= \lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}\Big(\prod_{k=1}^{d}\big(I-U_k^{*N_k}\big)\Big)f \qquad (3.12)\\
&= \sum_{r=0}^{d}(-1)^r\sum_{S\in\mathcal{S}_{r,d}}\lim_{\mathbf{N}=(N_1,\dots,N_d)\to\infty}\Big(\prod_{k\in S}U_k^{*N_k}\Big)f \\
&= f.
\end{aligned}
$$

Assertion of item (3) follows from items (1) and (2), if one first uses (2), then notices that $f=\big(\prod_{k\in\mathbb{N}(d)}(I-U^{*k})\big)g$ implies $f=\big(\prod_{k\in S}(I-U^{*k})\big)g'$ with $g'=\big(\prod_{k\notin S}(I-U^{*k})\big)g$, and, finally, applies item (1) to the semigroup $\mathbb{Z}_+^{d,S}$. □

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