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The absence of the absolutely continuous spectrum for δ' Wannier–Stark ladders

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A modification of the Kronig–Penney model consisting of equidistantly spaced δ'–interactions is considered. We prove that absolutely continuous spectrum of such a system disappears under influence of an external electric field. The result extends to periodic lattices of non–identical δ' interactions and potentials whose behavior, up to a bounded term, is powerlike with a power $\mu \le 1$ on the decreasing side.

1 Introduction

One–dimensional Schrödinger operators with potentials composed of a periodic and an aperiodic, most often linear, part have been studied by many authors — see, *e.g.*, [4, 7, 8, 10] and references therein — most attention being paid to the resonance structure of such systems. A basic ingredient of the typical scattering picture is that the spectrum of the corresponding Hamiltonian is absolutely continuous and covers the whole real line provided the aperiodic part is below unbounded; this property can be proven under rather weak differentiability requirements on the potential.

The aim of the present paper is to show that the spectral properties may change substantially if a smooth periodic potential is replaced by an array of singular interactions. The best known system of this type is the Kronig–Penney model, *i.e.*, a sequence of equally spaced δ–interactions; its spectral properties in presence of an external field remain an open problem.

However, in one–dimensional systems there are singular interactions different from δ. Another important class is represented by the δ'–interactions specified by the boundary conditions (1) below; a detailed discussion of their properties can be found in [2]. In distinction to δ, they cannot be approximated by families of Schrödinger operators with scaled short–range potentials, instead one can use families of rank–one operators [12] or velocity–dependent potentials [5, 6]. Moreover, this does not exhaust all possible aspects of the δ'–interaction; elsewhere we have presented a heuristic argument showing that it can be regarded as a paradigm for geometric scatterers [3].

An important distinction between the two types of contact interactions is manifested in Kronig–Penney–type models (without an external force): if we replace δ by δ' we obtain the spectrum in which the band widths are asymptotically constant while the gaps are widening. The heuristic picture of tilted bands then suggests that an unrestricted propagation in such a system under influence of an external force may not be possible.

Our main result confirms this conjecture: using the stability of the absolutely continuous spectrum with respect to trace–class perturbations in a way somewhat analogous to [14], we demonstrate that the spectrum of δ' Wannier–Stark–ladder Hamiltonian $H(\beta,E,a)$ is purely singular for any nonzero "coupling constant" β and external field strength E. Moreover, the validity of this result can be extended to lattices of non–identical δ' interactions and a wide family of external potentials, as we shall discuss in Section 4.

Let me add the following remark. The problem treated here was formulated in collaboration with J.E. Avron and Y. Last. Together we devised the mentioned strategy and wrote a proof for the δ' Kronig–Penney model with a linear potential. The result was announced in Ref.[3]; recently the same conclusion has been reached by a completely different method [9]. Our original proof had a flaw, however; in attempt to rectify it we formulated separately two different arguments. The one published here appeared to be applicable to a considerably wider class of operators, and my coauthors insisted that I publish it in my name. While I respect their decision and appreciate their scrupulous attitude, I want to state that the credit for the main result of the paper is shared equally by all three of us.

2 The main result

Consider the free Stark Hamiltonian $H_E := -\frac{d^2}{dx^2} - Ex$ on $L^2(I\!\!R)$ with $D(H_E) := \{\, f \in H^{2,2}(I\!\!R) :\; H_E f \in L^2 \,\}$, and an equidistant lattice, $\mathcal{L} := \{na\}_{n=-\infty}^{\infty}$ with a spacing $a>0$. Suppose that at each point of $\mathcal{L}$ we introduce the δ'–interaction of a strength β_n, *i.e.*, we define the operator $H(\{\beta_n\},E,a)$ in the following way: it acts as H_E on the intervals $J_n := (na,(n+1)a)$ and its domain differs from $D(H_E)$ by replacing the smoothness requirement at the points of $\mathcal{L}$ by the boundary conditions

$$f'(na+) \,=\, f'(na-) \,=:\, f'(na)\,, \qquad f(na+) - f(na-) \,=\, \beta_n f'(na)\,; \tag{1}$$

the β_n are real numbers or $+\infty$ in which case (1) is replaced by the Neumann condition, $f'(na)=0$. If the interaction strength at each point is the same, $\beta_n=\beta$, we write $H(\{\beta_n\},E,a) =: H(\beta,E,a)$; in particular, $H(0,E,a)=H_E$ and $H(\infty,E,a)$ is the orthogonal sum of the single–interval operators obtained by imposing the Neumann condition at each point of $\mathcal{L}$.

Our main result is the following.

Theorem 2.1 *Let* $E,\beta \ne 0$ *; then the absolutely continuous spectrum of the* δ' *Wannier–Stark Hamiltonian* $H(\beta,E,a)$ *is empty.*

As mentioned above, we shall demonstrate this property by using known results about the stability of the absolutely continuous spectrum. More specifically, the proof will be based on comparing the resolvent of $H(\beta, E, a)$ to that of a suitable operator with a pure point spectrum.

3 The proof

To simplify the problem, one can cut the line into two halflines and to consider each of them separately. Recall that changing the boundary conditions at a point of $\mathcal{L}$ to the Neumann one means decoupling the Hamiltonian into an orthogonal sum; at the same time it represents a rank–one perturbation in the resolvent which does not change the essential spectrum.

3.1 The resolvent of $H(\{\beta_n\}, E, a)$

We shall start with the more general operator $H(\{\beta_n\}, E, a)$ introduced above and compare it to an operator of the same type with the boundary condition changed to the Neumann one at some points of $\mathcal{L}$. First we have to introduce the notation.

Given $z = k^2$, we have in each interval J_n just one solution $u_n \equiv u_n(\cdot, k)$ of the equation $-u''(x) - [Ex + z]u(x) = 0$ which satisfies the boundary conditions $u_n(na+) = 1$, $u_n'(na+) = 0$. Similarly, there is exactly one solution v_n with $v_n((n+1)a-) = 1$, $v_n'((n+1)a-) = 0$; their Wronskian equals $W_n := W(u_n, v_n) = v_n'(na+) = -u_n'((n+1)a-)$. These solutions are of the form $f(-E^{1/3}(\cdot\, + \frac{z}{E}))$, where f is a combination of Airy functions; we assume for definiteness that $E > 0$.

More generally, let $\{n_\ell\}_{\ell=1}^{\infty}$ be an increasing sequence of integers which specifies the sublattice $\tilde{\mathcal{L}} := \{n_\ell a\}_{\ell=1}^{\infty} \subset \mathcal{L}$. We denote

$$\tilde{\beta}_n := \begin{cases} \infty & \dots \quad n = n_\ell \\ \beta_n & \dots \quad \text{otherwise} \end{cases}$$

and moreover, $\tilde{J}_\ell := (n_\ell a, n_{\ell+1} a)$. Then to any $z = k^2$, there is just one function $\tilde{u}_\ell$ which satisfies in $\tilde{J}_\ell$ the equation $-u''(x) - [Ex + z]u(x) = 0$ together with the b.c. (1) at the points $(n_\ell + 1)a, \dots, (n_{\ell+1} - 1)a$ and $\tilde{u}_\ell(n_\ell a+) = 1$, $\tilde{u}_\ell'(n_\ell a+) = 0$. Similarly, there is a unique $\tilde{v}_\ell$ with $\tilde{v}_\ell(n_{\ell+1}a-) = 1$, $\tilde{v}_\ell'(n_{\ell+1}a-) = 0$, and the corresponding Wronskian equals

$$\tilde{W}_n := W(\tilde{u}_\ell, \tilde{v}_\ell) = \tilde{v}_\ell'(n_\ell a+) = -u_\ell'(n_{\ell+1} a-).$$

The functions $\tilde{u}_\ell$, $\tilde{v}_\ell$ can be again written explicitly in terms of Airy functions but we shall not need it.

Theorem 3.1 *Assume that $|\beta_n| \geq \beta$ for all n and some $\beta > 0$. For any z from the resolvent sets of the two operators, the difference $C := (H(\{\beta_n\}, E, a) - z)^{-1} -$*

$(H(\{\tilde\beta_n\},E,a)-z)^{-1}$ is an integral operator with the kernel

$$
\begin{aligned}
C(x,y) \;&=\; \sum_{\ell,m}\left\{\frac{\tilde u_\ell(x)}{\sqrt{\beta_{n_\ell}}\tilde W_\ell \tilde W_m}\left(\frac{M_{\ell,m-1}\tilde v_m(y)}{\sqrt{\beta_{n_{m-1}}}}-\frac{M_{\ell m}\tilde u_m(y)}{\sqrt{\beta_{n_m}}}\right)\right. \\
&+\; \left.\frac{\tilde v_\ell(x)}{\sqrt{\beta_{n_{\ell-1}}}\tilde W_\ell \tilde W_m}\left(\frac{M_{\ell-1,m}\tilde u_m(y)}{\sqrt{\beta_{n_m}}}-\frac{M_{\ell-1,m-1}\tilde v_m(y)}{\sqrt{\beta_{n_{m-1}}}}\right)\right\},
\end{aligned} \tag{2}
$$

where M is a symmetric operator on ℓ^2 which is the inverse to the tridiagonal Γ given by

$$
\begin{aligned}
\Gamma_{\ell\ell} \;&:=\; \frac{1}{\beta_{n_\ell}}\left(\beta_{n_{\ell+1}}-\frac{\tilde v_{\ell+1}(n_{\ell+1}a+)}{\tilde W_{\ell+1}}-\frac{\tilde u_\ell(n_{\ell+1}a-)}{\tilde W_\ell}\right), \\
\Gamma_{\ell,\ell+1} \;&=\; \Gamma_{\ell+1,\ell} \;:=\; \frac{1}{\sqrt{\beta_{n_\ell}\beta_{n_{\ell+1}}}\tilde W_{\ell+1}}\,.
\end{aligned} \tag{3}
$$

The series in (2) converges in the strong sense.

Proof: For notational simplicity, let us prove the theorem for the situation when $H_\beta := H(\{\beta_n\},E,a)$ is compared to $H_N := H(\infty,E,a)$, *i.e.*, $\tilde{\mathcal L}=\mathcal L$; the argument for a general $\tilde{\mathcal L}$ is obtained simply by replacing $u_n,\,v_n$ by $\tilde u_n,\,\tilde v_n$, *etc.* The operator H_N has the following kernel:

$$
G_N(x,y) \;=\; \begin{cases} -W_n^{-1}u_n(x_<)v_n(x_>) & \dots \quad x,y\in J_n \\ 0 & \dots \quad x,y \text{ belong to different } J_n \end{cases}
$$

where $x_< := \min(x,y)$, $x_> := \max(x,y)$. Suppose first that $\{\tilde\beta_n\}$ differs from Neumann at a finite number N of points only; then the kernel of H_β can be expressed by Krein's formula in the form

$$
\begin{aligned}
G_\beta(x,y) \;=\; G_N(x,y) \;&+\; \sum_{n,m}\Big[\lambda^{11}_{nm}u_n(x)u_m(y)+\lambda^{12}_{nm}u_n(x)v_m(y) \\
&+\; \lambda^{21}_{nm}v_n(x)u_m(y)+\lambda^{22}_{nm}v_n(x)v_m(y)\Big],
\end{aligned} \tag{4}
$$

where λ^{jk}_{nm} are coefficients to be found; they fulfill the obvious symmetry requirements

$$
\lambda^{jj}_{nm}=\lambda^{jj}_{mn}\,, \quad \lambda^{12}_{nm}=\lambda^{21}_{mn}\,.
$$

By definition, H_β maps $L^2(I\!R^+)$ into $D(H_\beta)$; hence applying the *rhs* of (4) to an arbitrary $g\in L^2(I\!R^+)$, we obtain a vector which belongs to the domain of H_β, in particular, it must satisfy the b.c. (1) at each point of $\mathcal L$. This yields a system of $4N(N+1)$ linear equations; choosing

$$
\begin{aligned}
\lambda^{11}_{nm} \;&=\; -\frac{M_{nm}}{\sqrt{\beta_n\beta_m}W_nW_m}\,, \qquad & \lambda^{21}_{nm} \;&=\; \frac{M_{n-1,m}}{\sqrt{\beta_{n-1}\beta_m}W_nW_m}\,, \\
\lambda^{12}_{nm} \;&=\; \frac{M_{n,m-1}}{\sqrt{\beta_n\beta_{m-1}}W_nW_m}\,, \qquad & \lambda^{22}_{nm} \;&=\; -\frac{M_{n-1,m-1}}{\sqrt{\beta_{n-1}\beta_{m-1}}W_nW_m}\,,
\end{aligned}
$$

we can reduce it by an elementary algebra to the system of N^2 equations

$$\frac{M_{n+1,m}}{\sqrt{\beta_{n+1}\beta_m}W_{n+1}} - \frac{M_{nm}}{\sqrt{\beta_n\beta_m}}\left(\beta_{n+1} - \frac{v_{n+1}((n+1)a+)}{W_{n+1}} - \frac{u_n((n+1)a-)}{W_n}\right) + \frac{M_{n-1,m}}{\sqrt{\beta_{n-1}\beta_m}W_{n-1}} = \delta_{nm}\,,$$

which is solved by $M = \Gamma^{-1}$. The limit $N \to \infty$ then yields the result; the strong convergence of the series can be verified first on functions of $C_0^\infty(\mathbb{R}^+)$, and then extending the result by density using the uniform boundedness of the resolvent away of the spectrum. ∎

3.2 A trace–class estimate

As we have said, the proof of Theorem 2.1 can be split into two halfline problems. The growing–potential case is easy. The Kronig–Penney operator without an electric field is below bounded, $H(\beta,0,a) \ge -c$ for some $c \ge 0$. Changing the boundary condition to Neumann at a point $-na$ does not change the essential spectrum and a finite interval does not contribute to it, so

$$\inf \sigma_{ess}(H(\beta,E,a)) \ge -c + Ena\,.$$

Since n can be chosen arbitrarily, the spectrum is pure point.

Let us turn to the more difficult case when the potential decreases along the halfline. If the b.c. (1) are changed to the Neumann one at infinitely many points, the corresponding operator $H(\{\tilde\beta_n\},E,a)$ has certainly a pure point spectrum. Our aim is to show that for a suitably chosen sequence $\{n_\ell\}$ and some $z \in \mathbb{C}$, the resolvent difference of Theorem 3.1 is a trace–class operator, in which case the sought result follows from the Birman–Kuroda theorem [11, Sec.XI.3].

We denote $z = \rho + i\nu$, where both $\rho,\,\nu$ will be chosen large positive. If ρ is large, the asymptotic properties of the Airy functions [1, Chap.10] show that the elementary solutions introduced above are in the intervals J_n of the form

$$u_n(x) = \cos k_n(x-na)\,\mathrm{Err}_{1/2}(k_n)\,, \quad v_n(x) = \cos k_n(x-(n+1)a)\,\mathrm{Err}_{1/2}(k_n)\,, \tag{5}$$

where $k_n := \sqrt{z + E\overline{x}_n}$ for some $\overline{x}_n \in J_n$, and $\mathrm{Err}_\alpha(y)$ stands as a shorthand for $1 + \mathcal{O}(y^{-\alpha})$. The momentum values behave asymptotically as

$$k_n = \left(An^{1/2} + Bn^{-1/2}\right)\mathrm{Err}_1(n) \tag{6}$$

with $A := \sqrt{ca}$ and $B := \frac{z}{2A}$ as $n \to \infty$. The Wronskian is then $W_n = k_n \sin(k_n a)\,\mathrm{Err}_{1/4}(n)$; since z is complex, it is never zero and one can estimate it by

$$|W_n| \ge \frac{\nu a}{2}\,\mathrm{Err}_{1/4}(n)\,. \tag{7}$$

We also need an upper bound for

$$\frac{W_{n-1}}{W_n} = \left(1 + \frac{\sin k_{n-1}a - \sin k_n a}{\sin k_n a}\right) \mathrm{Err}_1(n)\,;$$

it is clear that one should pay attention only to the vicinity of the points where $\mathrm{Re}\,(k_n a) \approx \pi\ell$. There we have

$$\left|\frac{\sin k_{n-1}a - \sin k_n a}{\sin k_n a}\right| \le 2\left(\frac{A^2}{\nu} + \frac{1}{n}\right) \mathrm{Err}_1(n)\,,$$

which yields the bound

$$\left|\frac{W_{n-1}}{W_n} - 1\right| \le \frac{c'}{\nu} + \mathcal{O}(n^{-1}) \tag{8}$$

with a constant independent of n.

The functions $\tilde u_\ell$, $\tilde v_\ell$ can be estimated using "transfer matrices" relating the elementary solutions in neighboring intervals. A general solution of the equation $-f''(x) - [Ex+z]f(x) = 0$ in J_n has the form $f_n(x) = \xi_n u_n(x) + \eta_n v_n(x)$; we look for T_n such that

$$\begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = T_n \begin{pmatrix} \xi_{n-1} \\ \eta_{n-1} \end{pmatrix}.$$

Using the b.c. (1), one proves easily

$$T_n = \begin{pmatrix} -\beta_n W_{n-1} + \frac{W_{n-1}}{W_n} v_n(na+) + u_{n-1}(na-) & 1 \\ -\frac{W_{n-1}}{W_n} & 0 \end{pmatrix},$$

$$T_n^{-1} = \begin{pmatrix} 0 & -\frac{W_n}{W_{n-1}} \\ 1 & -\beta_n W_n + v_n(na+) + \frac{W_n}{W_{n-1}} u_{n-1}(na-) \end{pmatrix}.$$

Using now (5)–(8), we get

$$\begin{aligned} |(T_n)_{11}| &\ge |\beta_n|\,|W_{n-1}| - |u_{n-1}(na-) + v_n(na+)| - |v_n(na+)|\,\frac{c'}{\nu} \\ &\ge \left(\frac{\beta a}{2}\,\nu - 2 - \frac{c'}{\nu}\right) \mathrm{Err}_{1/4}(n)\,, \end{aligned}$$

so choosing ν large enough, the *rhs* can be made positive and sufficiently large as $n \to \infty$. At the same time,

$$|(T_n)_{21}| \le 1 + \frac{c'}{\nu} + \mathcal{O}(n^{-1}).$$

Since the coefficients of $\binom{\xi_n}{\eta_n}$ corresponding to $\tilde{u}_\ell(x)$ start from $\binom{1}{0}$ at $x = n_\ell a$, the above bounds tell us that for all sufficiently large n there is a number $d > 1$ independent of n such that

$$|\xi_n| \;\geq\; d^j |\xi_{n-j}|\,, \qquad |\xi_n| \;\geq\; d|\eta_n|\,. \tag{9}$$

The norm of $\tilde{u}_\ell$ can be then estimated as follows,

$$\|\tilde{u}_\ell\|^2 \;\leq\; 2a \sum_{n=n_\ell}^{n_{\ell+1}-1} \left(|\xi_n|^2 + |\eta_n|^2\right) \;\leq\; 2a\,|\xi_N|^2\left(1+d^{-2}\right) \sum_{j=0}^{N'} d^{-2j}\,,$$

where $N := n_{\ell+1} - 1$ and $N' := n_{\ell+1} - n_\ell - 1$. Hence there is a positive d' independent of ℓ such that $\|\tilde{u}_\ell\| \leq d'|\xi_{n_{\ell+1}-1}|$. The coefficients of $\tilde{v}_\ell$ behave similarly if we change the direction and switch the roles of ξ_n, η_n; this yields the inequality $\|\tilde{v}_\ell\| \leq d'|\eta_{n_\ell}|$.

For simplicity, we introduce $U_\ell := \frac{\tilde{u}_\ell(\cdot)}{\tilde{u}'_\ell(n_{\ell+1}a-)}$ and $V_\ell := \frac{\tilde{v}_\ell(\cdot)}{\tilde{v}'_\ell(n_\ell a+)}$. The denominators depend clearly on the choice of the sequence $\{n_\ell\}$. We pick its points roughly in "the middle of the gaps" assuming, *e.g.*,

$$\left| k_{n_\ell} \;-\; \frac{\pi(2\ell+1)}{2a} \right| \;\leq\; \frac{\pi}{4a}\,, \tag{10}$$

so that $n_\ell = \left(\frac{\pi\ell}{Aa}\right)^2 \mathrm{Err}_1(\ell)$. Since $v_{n_{\ell+1}-1}(n_{\ell+1}a-) = 0$ by assumption, we have in view of (5) and (10)

$$|\tilde{u}'_\ell(n_{\ell+1}a-)| \;\geq\; \frac{1}{\sqrt{2}}\, k_{n_{\ell+1}-1}|\xi_{n_{\ell+1}-1}|\; \mathrm{Err}_{1/2}(\ell)\,.$$

In the same way, $|\tilde{v}'_\ell(n_\ell a+)|$ has a bound proportional to $|\eta_{n_\ell}|$; using the above inequalities for the norms of these functions, we get

$$\|U_\ell\| \;\leq\; \frac{\sqrt{2}\,d'}{k_{n_{\ell+1}-1}}\; \mathrm{Err}_{1/2}(\ell)\,, \qquad \|V_\ell\| \;\leq\; \frac{\sqrt{2}\,d'}{k_{n_\ell}}\; \mathrm{Err}_{1/2}(\ell)\,. \tag{11}$$

To estimate the difference of the resolvents, we need also a bound for the coefficients $M_{\ell m}$ in (2). Since all the β_n are the same by assumption, the operator (3) can be written as $\Gamma = I + N$ with

$$N_{\ell\ell} \;:=\; -\beta^{-1}\left(U_\ell(n_{\ell+1}a-) + V_{\ell+1}(n_{\ell+1}a+)\right), \qquad N_{\ell,\ell+1} \;=\; N_{\ell+1,\ell} \;:=\; \beta^{-1}\tilde{W}_\ell^{-1}\,.$$

The above estimates then yield

$$\begin{aligned}
|N_{\ell\ell}| \;&\leq\; \frac{2a}{\pi\ell\beta}\left(1+\frac{\sqrt{2}}{d}\right)\; \mathrm{Err}_1(\ell)\,,\\
|N_{\ell,\ell+1}| \;&\leq\; \frac{\sqrt{2}\,a}{\pi\ell\beta}\, d^{-N'}\, \mathrm{Err}_1(\ell) \;\leq\; \frac{2a}{\pi\ell\beta}\, \exp\left(-\,2\ell\,\left(\frac{\pi}{Aa}\right)^2 \ln\, d\right)
\end{aligned}$$

hence if $\{n_\ell\}$ starts with ℓ large enough (which is equivalent to choosing a sufficiently large ρ), we have $\|N\| < 1$. The inverse is then easily computed as a geometric series,

$$M_{\ell m} = \delta_{\ell m} - N_{\ell m} + \sum_{s=2}^{\infty} (-1)^s \sum_{r_2,\dots,r_s} N_{\ell r_2} N_{\ell r_3} \dots N_{r_s m} .$$

Since the elements of the r–th side diagonal contain at least r off–diagonal elements of N (recall that the latter is tridiagonal!), there are positive C, d'' such that

$$|M_{\ell,\ell+r}| \le C\, e^{-d''|r|} . \tag{12}$$

Putting the estimates (11) and (12) together in combination with (6) and (10), we get

$$\begin{aligned}
\mathrm{Tr}\,|C| \;\le\; & \frac{1}{\beta} \sum_{\ell,m} \Big\{ |M_{\ell m}|\, \|U_\ell\|\, \|U_m\| + |M_{\ell,m-1}|\, \|U_\ell\|\, \|V_m\| \\
& + |M_{\ell-1,m}|\, \|V_\ell\|\, \|U_m\| + |M_{\ell-1,m-1}|\, \|V_\ell\|\, \|V_m\| \Big\} \\
\le\; & \frac{C(2d'a)^2}{\pi^2\beta} \sum_\ell \Big\{ \frac{1}{\ell} \sum_{r\ge 0} \left(\frac{1}{\ell+r} + \frac{1}{\ell+1+r} \right) e^{-d''r} \\
& + \frac{1}{\ell-1} \sum_{r\ge 0} \left(\frac{1}{\ell+r} + \frac{1}{\ell-1+r} \right) e^{-d''r} \Big\} < \infty ,
\end{aligned}$$

what we wanted to prove. ∎

4 Generalizations

The presented proof of Theorem 2.1 extends easily at least in two directions:

(i) *the background potential need not be linear:* let $H_V := -\frac{d^2}{dx^2} + V(x)$ be the "free" Schrödinger operator with a potential V to be specified below; then we can define $H(\{\beta_n\}, V, a)$ as in Section 2, *i.e.*, through the b.c.(1).

(ii) one may consider a lattice of *non–identical δ' interactions*, *i.e.*, a general sequence $\{\beta_n\}$.

For simplicity, let us consider again only the halfline problem with a decreasing potential. The key observation is that the asymptotics (5) of the elementary solutions comes in fact from the WKB expansion, and therefore it is valid for any any potential V which is decreasing and regular enough provided we define $k_n := \sqrt{z - V(\overline{x}_n)}$. Assume, *e.g.*, that

(a) $V(x) = -c|x|^\mu + W(x)$, where $c > 0$, $\mu \in (0,1]$, and W is bounded and piecewise C^2 smooth with $\max\{|W'(x)|, |W''(x)|\} \le d|x|^\mu$ for some $d > 0$.

Then $k_n = \left(An^{\mu/2} + Bn^{-\mu/2}\right) \mathrm{Err}_\mu(n)$ as $n \to \infty$ for appropriate A, B and the argument of Section 3.2 modifies easily. In particular, we have

$$\left| \frac{\sin k_{n-1}a - \sin k_n a}{\sin k_n a} \right| \le \frac{2\mu A^2}{\nu} \left(\frac{\pi \ell}{Aa}\right)^{2-2/\mu} + \frac{4\|W\|_\infty}{\nu} + \mathcal{O}(n^{-\mu}),$$

so (8) holds again as long as $\mu \le 1$, and the "transfer–matrix" analysis may be repeated. As for the "coupling constants", we adopt the following assumptions:

(b) $|\beta_n| \ge \beta > 0$ for all n,

(c) there is a monotonic sequence $\{n_\ell\} \subset \mathbb{Z}$ such that $n_\ell = \frac{1}{a} \left[\frac{\pi}{a\sqrt{c}}(\ell + \epsilon_\ell)\right]^{2/\mu}$ with $\epsilon_\ell \in (\frac{1}{4}, \frac{3}{4})$, and $\beta_{n_\ell}\beta^{-1}_{n_{\ell+1}}$ remains bounded as $n_\ell \to \infty$,

The matrix elements (3) of the operator Γ can be now written as

$$\Gamma_{\ell m} = \frac{1}{\sqrt{\beta_{n_\ell}}} (B+S)_{\ell m} \frac{1}{\sqrt{\beta_{n_m}}},$$

where $B := \mathrm{diag}\,\{\ldots, \beta_{n_{\ell+1}}, \ldots\}$, and $S_{\ell\ell} := -U_\ell(n_{\ell+1}a-) - V_{\ell+1}(n_{\ell+1}a+)$, $S_{\ell,\ell+1} = S_{\ell+1,\ell} := \tilde{W}_\ell^{-1}$. If the sequence $\{n_\ell\}$ starts with ℓ large enough, $\|B^{-1}N\| < 1$ and the inverse is given by

$$M_{\ell m} = \beta_{n_\ell}\beta^{-1}_{n_{\ell+1}}\delta_{\ell m} - \sqrt{\beta_{n_\ell}}\beta^{-1}_{n_{\ell+1}} S_{\ell m} \beta^{-1}_{n_{m+1}} \sqrt{\beta_{n_m}}$$

$$+ \sum_{s=2}^{\infty} (-1)^s \sum_{r_2,\ldots,r_s} \sqrt{\beta_{n_\ell}}\beta^{-1}_{n_{\ell+1}} S_{\ell r_2} \beta^{-1}_{n_{r_2+1}} \ldots S_{r_s m} \beta^{-1}_{n_{m+1}} \sqrt{\beta_{n_m}}.$$

Since the numbers $\beta_{n_\ell}\beta^{-1}_{n_{\ell+1}}$ remain bounded at the cutting points in view of (c), the estimate (12) holds again.

In this way, we arrive at the following result.

Theorem 4.1 *Let $H(\{\beta_n\}, V, a)$ be the operator on $L^2(\mathbb{R}^+)$ defined above, with V and $\{\beta_n\} \subset \mathbb{R}$ satisfying the assumptions (a)–(c). Then $\sigma_{ac}(H(\{\beta_n\}, V, a) = \emptyset$.*

The result extends easily to operators on the whole real line with potentials growing in the other direction, because for the other halfline we have used just the fact that the spectrum is purely discrete when the potential tends to infinity. A more involved argument is required, however, if the sequence $\{\beta_n\}$ may approach zero, since then the point–interaction Hamiltonian without the presence of an external field need no be below bounded.

5 Concluding remarks

The assumptions used here are clearly not optimal, but we are not going to push the argument further. Let us remark instead that the mentioned splitting of the problem on the line into two halflines together with Proposition 4.1 shows that the absolutely continuous spectrum is void not only for δ' Wannier–ladder "slopes" but for "hills" with the potential decreasing in both directions as well.

In addition to finding weaker restrictions on the potential decay, other generalizations are possible. For instance, one can treat similarly arrays of δ'–interactions, where the spacing assumes a finite number of different values. Furthermore, δ and δ' are extreme cases in the general four–parameter class of one–dimensional point interactions [5, 6, 13]. The behavior of bands and gaps in the absence of the external force suggests that the result will remain valid at least as long as there is a nonzero δ' component in such an interaction, *i.e.*, a discontinuity of the wavefunction at the lattice points which depends on the one–sided derivatives.

Having excluded the absolutely continuous spectrum, one asks naturally how the other parts of the spectrum look like. Consider again our basic model, *i.e.*, the operator $H(\beta,E,a)$ for nonzero $E,\,\beta$. The above proof shows, in particular, that the essential spectrum of $H(\beta,E,a)$ does not change if the system is "chopped" by imposing the Neumann condition at a properly chosen sequence $\{n_\ell a\}$. The discrete spectra of the corresponding "finite sections" of $H(\beta,E,a)$ can be found numerically [15]; in this way we arrive at the following

Conjecture 5.1 *For nonzero* $E,\,\beta$*,*

$$\sigma_{ess}(H(\beta,E,a)) \,=\, \left\{ \frac{4}{\beta a} + \left(\frac{m\pi}{a}\right)^2 - E\left(n+\frac{1}{2}\right)a \;:\; m,n\in \mathbb{Z} \right\}.$$

In other words, the essential–spectrum points are sums of three terms: the energy step of the ladder, the eigenvalues of $H(\infty,E,a)$, anf the asymptotic *halfwidth* of a band. Recall that $\sigma_{ess}((H(\beta,E,a))$ consists of all accumulation points of the spectrum, since a second–order differential operator cannot have eigenvalues of infinite multiplicity. If the conjecture is true, the spectrum exhibits an intriguing dependence on the number–theoretic properties of the external field, namely

(a) for $\gamma := \left(\frac{a}{\pi}\right)^2 Ea$ is rational, the spectrum is nowhere dense, and therefore automatically pure point.

(b) on the other hand, if γ is irrational, $\sigma((H(\beta,E,a)) = \sigma_{ess}((H(\beta,E,a)) = \mathbb{R}$.

In this way, δ' Wannier–Stark systems still represent a challenge.

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References

[1] M. Abramowitz, I.A. Stegun, *eds.*: *Handbook of Mathematical Functions*, Dover, New York 1965.

[2] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden: *Solvable Models in Quantum Mechanics*, Springer, Heidelberg 1988.

[3] J.E. Avron, P. Exner, Y. Last: Periodic Schrödinger operators with large gaps and Wannier–Stark ladders, *Phys.Rev.Lett.* **72** (1994), 896–899.

[4] F. Bentosela, V. Grecchi: Stark–Wannier ladders, *Commun.Math.Phys.* **142** (1991), 169–192.

[5] M. Carreau: Four–parameter point–interactions in 1D quantum systems, J.Phys. **A26** (1993), 427–432.

[6] P.R. Chernoff, R. Hughes: A new class of point interactions in one dimension *J.Funct.Anal.* **111** (1993), 92–117.

[7] H.I. Cycon, R.G. Froese, W. Kirsch, B. Simon: *Schrödinger Operators*, Springer, Berlin 1987.

[8] V. Grecchi, M. Maioli, A. Sacchetti: Wannier ladders and perturbation theory, *J.Phys.* **A26** (1993), L379–384.

[9] M. Maioli, A. Sacchetti: Absence of absolutely continuous spectrum for Stark–Bloch operators with strongly singular periodic potentials, *J. Phys.* **A**, to appear.

[10] G. Nenciu: Dynamics of band electrons in electric and magnetic fields: rigorous justification of the effective Hamiltonians, *Rev.Mod.Phys.* **63** (1993), 91–127.

[11] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, III. Scattering Theory*, Academic Press, New York 1979.

[12] P. Šeba: Some remarks on the δ'–interaction in one dimension, *Rep.Math. Phys.* **24** (1986), 111–120.

[13] P. Šeba: The generalized point interaction in one dimension, *Czech.J.Phys.* **B36** (1986), 667–673.

[14] B. Simon, T. Spencer: Trace class perturbations and the absence of absolutely continuous spectrum, *Commun.Math.Phys.* **125** (1989), 113–125.

[15] There is also a formal argument supporting the conjecture which comes from rephrasing the problem in terms of discrete Schrödinger operators (J.E. Avron, private communication).