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What are the Quantum Mechanical Lyapunov Exponents?

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Dedicated to Paul Urban, teacher and friend

Abstract

The origin of chaotic behaviour of a dynamical system in the sense of sensitive dependence on initial conditions is the action of a certain group. We are used to the importance of a group from the theory of relativity and in fact this group happens to be a subgroup of the Poincare group. Physical consequences like sensitive dependence on initial conditions or exponential decay of time correlations can be deduced if we add some continuity requirements on the action of this group. This will be illustrated by examples from classical mechanics, quantum mechanics and quantum field theory.

1 Introduction

In the past years it has been possible to widen the scope of ergodic theory so as to contain also quantum systems. It was rather easy to generalize the notions of mixing [1] and topological mixing [2,3]. The dynamical entropy was a long time ago defined for type II_1 systems [4] but now there are three different proposals for the general case [5,6,7]. They agree for commutative systems but show different features in the quantum domain. The topological entropy could also be generalized to noncommutative systems [8]. Recently there have been proposals for a generalization of the Lyapunov exponents [9,10,11] for quantum systems. I shall follow here a definition first given in [12] for the quantum cat and elaborated in a general setting in [13]. It seems that the heart of the matter is a certain group. Its action immediately dictates sensitive dependence on initial conditions and exponential decay of correlations provided the action is sufficiently continuous. We shall find many examples of the action of this group starting from elementary mechanics to wave mechanics and to quantum field theory. They give feeling when the continuity requirements are met so that our general theorems become applicable.

The reader might be suspicious since physical properties depend crucially on continuity properties. It is a sad fact that the key notions in ergodic theory are highly discontinuous objects and this does by no means become better in the quantum domain. As example may serve the cat map on the noncommutative torus which depends on a rotation parameter Θ ($= \hbar$ in appropriate units). This system is random in the sense of algorithmic complexity theory only for $\Theta = 0$ ([14] where this was called failure of the correspondence principle). The Connes–Størmer entropy is zero for almost all Θ's [15] but for rational Θ's it has the classical value and for countably many Θ's it is positive [12]. Also the topological structure of this system considered is quite different for rational and irrational Θ's [16], in the first case it is a fibre bundle, in the latter it is not. On the other hand, the Alicki–Fannes entropy has the classical value irrespective of Θ and the Voiculescu entropy has always at least half the classical value. The Lyapunov exponent does not show this nervous behaviour and all definitions [9,10,11,12] give it its classical value for all Θ's.

2 The Anosov groups

The intuition behind the sensitive dependence on initial conditions is that next to any point on any trajectory there is a point at a distance s such that in the course of time their distance grows exponentially. In formulas it reads

$$\tau^t \circ \sigma^s = \sigma^{se^{-\lambda t}} \circ \tau^t \tag{2.1}$$

which is the multiplication law of the semidirect product $\mathbf{R} \otimes \mathbf{R}$. From this we abstract the

Definition (2.2) The multiplication laws

(i) $(t,s)\circ(t',s') = (t+t', s+s'e^{-\lambda t})$

(ii) $e = (0,0)$

(iii) $(t,s)^{-1} = (-t, -e^{\lambda t}s)$

define for

(a) $(t,s) \in \mathbf{Z}\times\mathbf{R}^+$ the Anosov semigroup

(b) $(t,s) \in \mathbf{Z}\times\mathbf{R}$ the Anosov group

(c) $(t,s) \in \mathbf{R}\times\mathbf{R}$ the continuous Anosov group.

An Anosov dynamical system is a realization of these groups by endomorphisms τ^t, σ^s of a C* algebra $\mathcal{A}$.

Remarks (2.3)

1. We shall consider only the biggest Lyapunov exponent and in the continuous case the groups with different λ's are isomorphic. Then we shall scale λ to 1.

2. Rescaling s does not change λ. In fact σ^s should shift by the distance s which has to be gauged by some metric. In [13] it is shown that λ is independent of the metric.

3. Since we are interested in the long time behaviour $t\in\mathbf{Z}$ or $t\in\mathbf{R}$ are equally good. However we need τ to be invertible so $t\in\mathbf{Z}^+$ would not do.

4. s has to be continuous since on a compact manifold the final distance cannot be arbitrarily big only the initial distance arbitrarily small. We do not only consider the case $s\in\mathbf{R}$ because we shall see that then an invariant state cannot be τ–KMS.

5. The systems we shall consider will be time–reversible, i.e. there is an automorphism (or antiautomorphisms) T with $T^{-1}\circ\tau^t\circ T = \tau^{-t}$. In this case $\sigma_-^s = T^{-1}\circ\sigma^s\circ T$ satisfies (2.1) with $\lambda\to-\lambda$.

6. In the physics we have in mind $\mathcal{A}$ is the algebra of observables and τ^t its true evolution. In spite of the nonlinearity of the dynamics τ^t is represented by linear transformations, namely isometries of the Banach space underlying $\mathcal{A}$. With the aid of invariant states we shall later construct unitary representations in a Hilbert space. There are finite–dimensional non–unitary representations. Since σ^s generates an invariant subgroup $(t,s)\to e^t$ is a representation which represents the factor group τ^t faithfully. $(t,s)\to\begin{pmatrix}1 & 0\\ s & e^{-t}\end{pmatrix}$ represents the whole group faithfully.

3 Examples

We shall first show that the Anosov groups appear quite frequently in physics. We begin with the simplest situation, namely classical linear systems. Contrary to the folklore sensitive dependence on initial conditions is not a characteristic of nonlinear systems.

3.1 Hamiltonian flows

Abelian C* algebras $\mathcal{A}$ are spaces of continuous functions and any automorphism τ is induced by a homeomorphism τ_* of the underlying space: $\tau(f) = f \circ \tau_*^{-1}\ \forall\ f \in \mathcal{A}$. For the flows corresponding to τ_*^t and σ_*^s we shall take the Hamiltonian flows generated by $H, K \in C(T^*(\mathbf{R}))$ and because of $(\tau^t \circ \sigma^s)_* = \tau_*^t \circ \sigma_*^s$ the differential version of (2.1) imples

$$\{H, K\} = K \quad \text{and} \quad \tau^t K = e^{-t}K, \quad \sigma^s H = H + sK. \tag{3.1}$$

3.1.1 Repulsive harmonic forces

These are the prototype of sensitive dependence on initial conditions since the particle runs with exponentially increasing velocity to infinity.

$$H = \tfrac{1}{2}(p^2 - x^2),\ K = p - x$$

$$\tau^t(x,p) = x \cosh t + p \sinh t, p \cosh t + x \sinh t)$$

$$\sigma^s(x,p) = (x + s, p + s)$$

which satisfy (3.1).

3.1.2 The dilation

It represents the flow near a hyperbolic fixed point and corresponds to 3.1.1 written in the coordinates

$$X = \frac{x+p}{2},\ P = p - x$$

$$H = XP,\ K = P$$

$$\tau^t(X,P) = (Xe^t, Pe^{-t})$$

$$\sigma^s(X,P) = (X + s, P).$$

We see here that restriction to the half plane $(x,p) \in \mathbf{R} \times \mathbf{R}^+$ still gives an action of the continuous Anosov group whereas on $(x,p) \in \mathbf{R}^+ \times (\mathbf{R} \text{ or } \mathbf{R}^+)$ we have an action of the semigroup $(s,t) \in \mathbf{R}^+ \times (\mathbf{R} \text{ or } \mathbf{Z})$. Thus Anosov $\not\Rightarrow$ ergodicity.

3.1.3 The exponential of the age operator

If one wants to use H as a canonical variable its conjugate, the "age operator" is the logarithm of K:

$$H = -p,\ K = e^x$$

$$\tau^t(x,p) = (x-t,p)$$

$$\sigma^s(x,p) = (x, p - se^x).$$

Note that now K is a positive function, in (3.1.1,2) it was not. However , in (3.1.2) we could have used $K = |p|$ which generates the flow $\sigma^s(x,p) = (x + sp/|p|, p)$ which also satisfies the Anosov relation.

3.1.4 Discretized Arnold map on the torus

So far our phase space was $\mathbf{R}^2$ and there was no invariant probability measure. We can also take the compact torus $\mathbf{R}^2/\mathbf{Z}^2$ but then τ^t will no longer be a diffeomorphism. However, for discrete time steps $\tau(x,p) \to (ax+bp, cx+dp)$, $a,b,c,d \in \mathbf{Z}$, $ad-bc=1$, we obtain a symplectic transformation $T \in SL(2,\mathbf{Z})$ that stretches in one direction and contracts in another. If σ^s shifts in the expanding direction we have again a realization of the Anosov group. It is not continuous and

$$\lambda = \frac{t}{2} + \sqrt{\frac{t^2}{4} - 1}, \qquad t = \mathrm{Tr}\ T = a + d > 2.$$

3.2 Quantum mechanics of one particle

The examples (3.1.1,2,3) can be readily translated into quantum mechanics where x and $p = \frac{1}{i}\frac{\partial}{\partial x}$ are operators on $L^2(\mathbf{R})$. In all three cases H and K are essentially selfadjoint on $C_0^\infty(\mathbf{R})$ and therefore define two families of unitaries e^{iHt}, e^{iKs} which satisfy

$$e^{iKse^{-t}}e^{iHt} = e^{iHt}e^{iKs}.$$

Therefore they generate automorphisms of $\mathcal{B}(L^2(\mathbf{R}))$ by

$$\tau^t(A) = e^{iHt}Ae^{-iHt}, \qquad \sigma^s(A) = e^{iKs}Ae^{-iKs}$$

which satisfy (2.1). The quantum analogue of (3.1.4) is the Weyl algebra W generated by

$$e^{i(\alpha x + \beta p)} = W(\alpha,\beta), \qquad \alpha,\beta \in \Theta\mathbf{Z}\times\mathbf{Z},\ \Theta \in (0,2\pi).$$

The "cat map" $\tau(W(\alpha,\beta)) = W(T^t(\alpha,\beta))$ gives an automorphism of W and together with a transversal shift in direction $\vec{e}$

$$\sigma^s(W(\alpha,\beta)) = e^{is(\alpha e_1 + \beta e_2)}W(\alpha,\beta)$$

we have again a realization of the Anosov group with the same Lyapunov exponent λ as in the classical case. Note that λ does not depend on Θ which plays the rôle of Planck's constant.

3.3 Remarks

1. In the version (3.1.3) $\ln|K|$ and H appear as canonically conjugate variables and if $t \to e^{iHt}$ is strongly continuous the von Neumann uniqueness theorem of the representations of the CCR applies. This means that the spectrum of H and $\ln|K|$ has to be zero or absolutely continuous and fill all of $\mathbf{R}$ (resp. $\mathbf{R}^+$), only the multiplicity is free. Hence there is no chance to find the Anosov properties in finite quantum systems where H has a discrete spectrum. If the representation is not faithful or not unitary $\ln|K|$ may not exist as operator. In the first case we have the one–dimensional representations $(t,s) \to e^{i\gamma t}$, $\gamma \in \mathbf{R}$, so $K = 0$ and $\ln|K| = -\infty$. In the faithful two–dimensional representation given in (2.3.6) we have

$$H = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad K = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

   They do satisfy $[K,H] = K$, but $[\ln|K|, H] = -i$ cannot be represented by matrices and indeed

$$|K| = |K^*K|^{1/2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{so} \qquad \ln|K| = \begin{pmatrix} -\infty & 0 \\ 0 & 0 \end{pmatrix}.$$

2. In these cases the dynamics is reversible and there exists also a contracting direction. In the form (3.1.1) time reversal is simply $T(x,p) = (x,-p)$ and σ_-^s is generated by $-p+x$ and gives $(x,p) \to (x-s, p-s)$.

3.4 One–dimensional quantum fields

The CCR and CAR algebras $\mathcal{A}$ are characterized by

$$[\psi_f, \psi_g^*]_\mp = \langle f|g\rangle, \qquad [\psi_f, \psi_g]_\mp = 0, \qquad f,g \in L^2(\mathbf{R}). \tag{3.4.1}$$

For fermions $\|\psi_f\| = \|f\|_2$ but for bosons ψ_f is not bounded and $\mathcal{A}$ is generated by the Weyl operators $e^{i(z\psi_f + z^*\psi_f^*)}$, $z \in C$, $f \in L^2(\mathbf{R})$. Since the unitary actions $f \to f_t = e^{-iHt}f$, $f \to f_s = e^{-iKs}f$ leave (3.4.1) invariant they generate automorphisms $\tau^t(\psi_f) = \psi_{f_t}$, $\sigma^s(\psi_f) = \psi_{f_s}$ which again satisfy (2.1).

3.5 Classical motion in extended phase space

Here the time becomes a dynamical variable x_0 and the energy p_0 is conjugate to it. The Poisson brackets are

$$\{x_i, p_k\} = \eta_{ik}, \qquad \eta_{ik} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and physics requires the constraint $p^i p^k \eta_{ik} + m^2 = 0$.

3.5.1 Free motion

The Lorentz transformation ("boost") generated by $L = x_1 p_0 - x_0 p_1$ and the shift along the light cone $K = (p_0 + p_1)/2$ satisfy $\{L, K\} = K$. Therefore their flows realize the Anosov group. This was to be expected because the flow lines of the boost in configuration space are hyperbolas as the time evolution in phase space in (3.1.1). A physical realization of such a dynamics is given by a constant acceleration to be considered next.

3.5.2 The constant electric field

A constant field is generated by a vector potential $A = \frac{1}{2}(-Ex_1, Ex_0)$ and the dynamics (with respect to proper time) is generated by the Hamiltonian

$$H = (p_i + eA_i)(p_k + eA_k)\frac{\eta^{ik}}{2m} = \frac{1}{2m}\left(\left(p_1 + \frac{eE}{2}x_0\right)^2 - \left(p_0 - \frac{eE}{2}x_1\right)^2\right).$$

The mass shell is given by $H = -m/2$. The two velocities

$$\dot{x}_0 = -\frac{1}{m}\left(p_0 - \frac{eE}{2}x_1\right) =: \frac{P}{m^{1/2}} \qquad \text{and} \qquad \dot{x}_1 = \left(p_1 + \frac{eE}{2}x_0\right)\frac{1}{m} =: \frac{eE}{m^{3/2}}Q$$

are up to constants conjugate, $\{Q, P\} = 1$ and H obtains the form (3.1.1)

$$H = -\frac{1}{2}(P^2 - \lambda^2 Q^2), \qquad \lambda = \frac{eE}{m}.$$

Therefore H and the shift generated by $K = Q + P$ satisfy $\{H, K\} = K$. Here is another pair of canonical coordinates [17] which do not appear in H. They describe the foci of the hyperbolas and are constant. Thus the 4 Lyapunov exponents are $\pm\lambda$, ± 0.

3.5.3 The de Sitter universe

The inflationaary universe is the prototype of an exponentially expanding space and we shall now see how the Anosov group is at work. In two space–time dimensions the de Sitter space can be identified with the hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 = 1 =: x_i x_k \eta^{ik} = (x|x) \tag{3.5.1}$$

in the three-dimensional Minkowski space where

$$\eta = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}.$$

The geodesic flow follows from an Lagrangian

$$L = \frac{1}{2}(\dot{x}|\dot{x}) \tag{3.5.2}$$

and taking (3.5.1) into account by a Lagrangian multiplier c we get as Euler equations

$$\ddot{x} = cx. \tag{3.5.3}$$

(3.5.1) requires

$$0 = (x|\dot{x}) = (\ddot{x}|\dot{x}). \tag{3.5.4}$$

Thus $(\dot{x}|\dot{x})$ is constant and we shall take the proper time as flow parameter which normalizes this to

$$(\dot{x}|\dot{x}) = -1. \tag{3.5.5}$$

By (3.5.1, 4 and 5) the originally six–dimensional phase space $(x, \dot{x})$ is reduced to a three–dimensional mass shell $\mathcal{M}$. In configuration space the geodesics are simply the intersections of the hyperboloid (3.5.1) with planes through the origin and steeper than 45^0. This follows from the constancy of the generators of $SO(2,1)$

$$\ell_i = \varepsilon_{ik\ell} x^k \dot{x}^\ell \tag{3.5.6}$$

since $(\ell|x) = 0$ any point $x(t)$ on the trajectory remains confined to the plane $-\ \ell$. Any point on $\mathcal{M}$ can be reached by a Lorentz transformation $L \in SO(2,1)$ from the point $x(0) = (0,1,0)$, $\dot{x}(0) = (1,0,0)$. This induces a diffeomorphism of $\mathcal{M}$ and $SO(2,1)$. The geodesic through this point is according to the above $x(t) = M(t)x(0)$, $\dot{x}(t) = \dot{M}(t)x(0)$ with

$$M(t) = \begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(2,1). \tag{3.5.7}$$

Since Lorentz transformations L are isometries they transform geodesics into geodesics. Thus the geodesic through the point $(Lx(0), L\dot{x}(0))$ will be $(LM(t)x(0), L\dot{M}(t)x(0))$ or with the identification of $\mathcal{M}$ and $SO(2,1)$ we have

$$\tau^t(L) = LM(t). \tag{3.5.8}$$

The tranasversal shift is easily found. With the Lie algebra of $SO(2,1)$

$$(\ell_0, \ell_1, \ell_2) = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right). \tag{3.5.9}$$

$M(t)$ can be written as $e^{t\ell_2}$, and since one calculates $[\ell_0 - \ell_1, \ell_2] = \ell_1 - \ell_0$ we see that τ^t and $\sigma^s(L) = Le^{s(\ell_1 - \ell_0)}$ represent the continuous Anosov group.

To get the flow explicitly it is convenient to use the isomorphism $SO(2,1) = SL(2,\mathbf{R})/\{1,-1\}$ which gave the two–dimensional representation of section 2. A point on (3.5.1) is given by a 2×2 matrix

$$X = \begin{pmatrix} x_2 & x_1 + x_0 \\ x_1 - x_0 & -x_2 \end{pmatrix}$$

which is characterized by Tr $X = 0$, $\det X = -1$. $SO(2,1)$ is represented by the action $X \to L^{-1}XL$, $L \in S(2,\mathbf{R})$ which leaves Tr and det invariant. The geodesic through $X(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with $\dot{X}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is given by $X(t) = m^{-1}(t)X(0)m(t)$, $m(t) = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix}$. Thus the geodesic through $\widetilde{X} = L^{-1}X(0)L$ is given by $\widetilde{X}(t) = L^{-1}m^{-1}(t)X(0)m(t)L$. A family of geodesics with $\dot{X}(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ which covers every point on (3.5.1) is obtained with $L = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$ and one calculates $(x_0, x_1, x_2) = (\sinh t, \cos 2\varphi \cosh t, \sin 2\varphi \sinh t)$. In this representation the transversal shift has the action

$$\sigma^s(L^{-1}X(0)L) = L^{-1}\gamma_s^{-1}X(0)\gamma_s L \qquad \text{with} \qquad \gamma_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}.$$

Since $\gamma_s m_t = m_t \gamma_{se^t}$ we also have

$$\sigma^s(\tau^t(L^{-1}X(0)L)) = \tau^t(\sigma^{se^t}(L^{-1}X(0)L)).$$

3.6 Remarks

1. The spacelike hyperboloid $-x_0^2 + x_1^2 + x_2^2 = -1$ is locally isometric to $\mathbf{R} \times \mathbf{R}^+$ with $g = (dx^2 + dy^2)/y^2$ ("Poincaré half plane") and exactly the same arguments show that there the geodesic flow also furnishes an Anosov dynamics. This still holds classically if one compactifies $\mathcal{M}$ by going to the coset space $SL(2,\mathbf{R})/SL(2,\mathbf{Z})$. However quantum mechanically this is no longer the case since the Laplacian on this space of finite volume has a discrete spectrum [13].

2. In section 3.5.1 the wedge $x_1 > |x_0|$ is invariant under τ^t but σ^s maps the wedge into the wedge only for $s > 0$. Thus the algebra of functions supported on the wedge give an example of an Anosov semigroup.

3. We shall not go into the quantum mechanical generalizations of the examples 3.5. Since we have identified the Anosov group as a subgroup of both the Poincaré and the de Sitter groups any quantum field theory with an action of these groups gives a realization of an Anosov dynamics.

4 Sensitive dependence on initial conditions

The unlimited expansion as it happened in some of our examples can certainly not happen on a compact space. So it is better to look at the Anosov property as saying that something arbitrarily small can become finite rather than that something finite can become arbitrarily big. So we will consider the limit and require the continuity

$$\lim_{s\to 0} D_s(A) = 0 \quad \forall\ A \in \mathcal{A} \quad \text{with} \quad D_s(A) = \|\sigma^s(A) - A\|. \tag{4.1}$$

$D_s(A)$ will not always be differentiable in s but one can construct a norm–dense set $\mathcal{D} \subset \mathcal{A}$ where it is. In this case we get the

Sensitive dependence on initial conditions (4.2)

$$\lim_{s\to 0} \frac{D_s(\tau^t A)}{D_s(A)} = e^{\lambda t} \qquad \forall\ A \in \mathcal{D}.$$

Proof: Since τ^t is norm preserving we see

$$\lim_{s\to 0} \frac{\|\sigma^s\tau^t(A) - \tau^t(A)\|}{D_s(A)} = \lim_{s\to 0} \frac{\|\sigma^{se^{\lambda t}}(A) - A\|}{D_s(A)} = \lim_{s\to 0} \frac{D_{se^{\lambda t}}(A)}{D_s(A)} = e^{\lambda t} \quad \forall\ A \in \mathcal{D}.$$

Remarks (4.3)

1. We do not need continuity in t, $t \in \mathbf{Z}$ is just as good. s has to be continuous but $s \in \mathbf{R}^+$ is also sufficient.

2. Differentiability means that there is a derivation $\delta : \mathcal{D} \to \mathcal{A}$ such that

$$\lim_{s\to 0} \left\| \frac{\sigma^s(A) - A}{s} - \delta A \right\| = 0.$$

   This implies weak differentiability and thus we have for all τ–invariant states ω

$$\lim_{s\to 0} \frac{\omega(\sigma^s\tau^t(A) - \tau^t(A))}{\omega(\sigma^s(A) - A)} = e^{\lambda t} \qquad \forall\ A \in \mathcal{D}.$$

Example (4.4) Going back to section 3 we see that the continuity (4.1) is not realized in the examples (3.1.1,2,3) only in (3.1.4). The reason is that for continuous functions f the translates $f_s(x) = f(x+s)$ are an equicontinuous family on compact sets but not necessarily in an infinite space. Similarly in the quantum version $s \to \sigma^s(A) = e^{iKs}Ae^{-iKs}$ will be strongly continuous but not necessarily norm continuous. Only in the second quantized version for fermions τ^t and σ^s will be norm continuous. The reason is that

$$\|\sigma^s(\psi_f) - \psi_f\| = \|\psi_{e^{-iKs}f} - \psi_f\| = \|e^{-iKs}f - f\| \to 0$$

for $s \to 0$. In this case $\mathcal{D}$ is norm–dense in $\mathcal{A}$ since K has a dense set of analytic vectors and (4.2) is fully applicable.

Continuity can be gained by restriction to the subalgebra of quasiperiodic functions generated

$$W(\alpha,\beta) = e^{i(\alpha x + \beta p)}, \qquad (\alpha,\beta) \in \mathbf{R}^2. \tag{4.5}$$

The shifts of (3.1.1,2,4) are continuous since

$$\|\sigma^s W(\alpha,\beta) - W(\alpha,\beta)\| = \|(e^{i(\alpha+\beta)s}-1)W(\alpha,\beta)\| = |e^{i(\alpha+\beta)s}-1| \to 0 \quad \text{for } s\to 0.$$

Exactly the same holds in quantum mechanics, (4.5) generates the Weyl algebra W and the noncommutativity

$$W(\alpha,\beta)W(\alpha',\beta') = e^{i\hbar(\alpha\beta'-\beta\alpha')}W(\alpha+\alpha',\beta+\beta')$$

does not change the Anosov properties. In fact, λ is independent of $\hbar$.

It is instructive to verify that (4.2) holds on W and not on all of $\mathcal{B}(L^2(\mathbf{R}))$. Consider $e^{i\alpha x}\in W$ and $e^{i\alpha x^2}\in\mathcal{B}(L^2(\mathbf{R}))$. We have

$$\lim_{s\to 0}\frac{\|\sigma^s\tau^t(e^{i\alpha x})-\tau^t(e^{i\alpha x})\|}{\|\sigma^s(e^{i\alpha x})-e^{i\alpha x}\|} = \lim_{s\to 0}\frac{\sup_{x\in\mathbf{R}}|e^{i\alpha e^t(x+s)}-e^{i\alpha e^t x}|}{\sup_{x\in\mathbf{R}}|e^{i\alpha(x+s)}-e^{i\alpha x}|} = \lim_{s\to 0}\frac{|e^{i\alpha e^t s}-1|}{|e^{i\alpha s}-1|} = e^t$$

but

$$\lim_{s\to 0}\frac{\|\sigma^s\tau^t(e^{i\alpha x^2})-\tau^t(e^{i\alpha x^2})\|}{\|\sigma^s(e^{i\alpha x^2})-e^{i\alpha x^2}\|} = \lim_{s\to 0}\frac{\sup_{x\in\mathbf{R}}|e^{i\alpha e^{2t}(x+s)^2}-e^{i\alpha(e^t x)^2}|}{\sup_{x\in\mathbf{R}}|e^{i\alpha(x+s)^2}-e^{i\alpha x^2}|} = \frac{2}{2} = 1 \qquad \forall\, t\in\mathbf{R}.$$

To see how (4.2) works in relativistic quantum field theory we consider finally a massless Dirac field in two space–time dimensions. It has two components, the right and left movers $\psi_j(x_1+(-)^j x_0)$, $j=1,2$ which satisfy the anticommutation relations

$$\{\psi_j(x),\psi_k^*(x')\} = \delta(x-x')\delta_{jk}. \tag{4.6}$$

The actions of τ^t, σ^s from section 3.5.1 (boost and shift on the light cone) are

$$\tau^t(\psi_j(x)) = e^{(-)^j t/2}\psi_j(xe^{-t}), \qquad \sigma^s(\psi_j(x)) = \begin{matrix}\psi_j(x+s) & \text{for } j=1\\ \psi_j(x) & \text{for } j=2.\end{matrix}$$

The $\psi_j(x)$ do not form an algebra but the smeared $\psi_{j,f}=\int dx f(x)\psi_j(x)$, $f\in L^2(\mathbf{R})$ do. For the $\psi_{j,f}$, $f'\in L^2(\mathbf{R})$ (4.2) holds and they generate a norm dense set in the CAR algebra:

$$\begin{aligned}\lim_{r\to 0}\frac{\|\sigma^s\tau^t(\psi_f)-\tau^t(\psi_f)\|^2}{\|\sigma^s(\psi_f)-\psi_f\|^2} &= \lim_{s\to 0}\frac{\int dx e^t|f(e^t(x+s))-f(e^t x)|^2}{\int dx|f(x+s)-f(x)|^2}\\ &= e^{2t}\frac{\int dx e^t|f'(xe^t)|^2}{\int dx|f'(x)|^2} = e^{2t}.\end{aligned}$$

5 Exponential decay of time correlations

It is sometimes considered as characteristic for chaos that time correlations decay exponentially. To see whether this happens in Anosov systems we have first to verify that there are invariant states $\omega=\omega\circ\tau^t=\omega\circ\sigma^s$. A priori this is assured because the Anosov group is amenable. Rather then proving this assertion we make the following observation. Abelian groups are always amenable so there exists an invariant state $\omega=\omega\circ\tau^t$ and therefore τ^t is unitarily implemented, $\tau^t(a)=V_{-t}aV_t$. V_t can be chosen such that $V_t|\Omega\rangle=|\Omega\rangle$ where $|\Omega\rangle$ is the cyclic vector in the GNS construction with ω. Now we have the simple

Lemma (5.1) If σ^s is unitarily implemented, $\sigma^s(a) = U_{-s} a U_s$ and $s \to U_s$ is strongly continuous then $V_t|\Omega\rangle = |\Omega\rangle$ implies $U_s|\Omega\rangle = |\Omega\rangle$.

Proof: $\forall\, t \in \mathbf{R}$ we have

$$U_s|\Omega\rangle = U_s V_t|\Omega\rangle = V_t U_{e^t s}|\Omega\rangle \stackrel{t\to-\infty}{\longrightarrow} V_t|\Omega\rangle = |\Omega\rangle.$$

This has the sad consequence that ω cannot be an equilibrium (= KMS) state for the time evolution τ^t. This follows from the following well–known

Lemma (5.3) If ω is τ^t–KMS and σ^s–invariant for an automorphism group σ^s then $[\tau^t, \sigma^s] = 0$.

Proof: For any s ω is KMS for the automorphism group $\bar{\tau}^t = \sigma^{-s}\tau^t\sigma^s$ since

$$\omega(b\bar{\tau}^{i\beta}(a)) = \omega(b\sigma^{-s}\tau^{i\beta}\sigma^s(a)) = \omega(\sigma^s(b)\tau^{i\beta}\sigma^s(a)) = \omega(\sigma^s(a)\sigma^s(b)) = \omega(ab).$$

However the modular automorphism of a state is unique, thus $\bar{\tau}^t = \tau^t$ or $\tau^t\sigma^s = \sigma^s\tau^t$. We have given the standard proof because it shows the way out of this dilemma. If σ^s, $s \geq 0$, is only a semigroup of endomorphisms then it is possible that ω is τ^t–KMS.

Example (5.3) (The poor man's Bisognano–Wichmann theorem.) We take the example of the end of section 4, namely a Dirac field in $1+1$ dimensions,

$$\psi = \begin{pmatrix} \psi_1(x_1 - x_0) \\ \psi_2(x_1 + x_0) \end{pmatrix}$$

where τ^t is the boost and σ^s the shift in a light–like direction $u = x_0 + x_1$. We consider only one component:

$$\psi(u) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} a(p) e^{ipu}, \qquad [\psi(u), \psi^*(u')]_+ = \delta(u-u'), \qquad [a(p), a^*(p')]_+ = \frac{1}{2\pi}\delta(p-p').$$

The C* algebra $\mathcal{A}$ is generated by the $\psi_f = \int_{-\infty}^{\infty} du \psi(u) f(u)$, $f \in L^2(\mathbf{R})$. The automorphisms act as $\tau^t(\psi_f) = \psi_{f_t}$, $f_t(u) = e^{-t/2} f(e^{-t}u)$, and $\sigma^s(\psi_f) = \psi_{f_s}$, $f_s(u) = f(u+s)$. We know that the spectrum of the generator H of τ^t is all of $\mathbf{R}$ whereas the generator K of σ^s can have positive (or negative) spectrum. Formally

$$H = \int_{-\infty}^{\infty} \frac{dp}{2\pi} : a^*(p) \frac{1}{2i}(p\frac{\partial}{\partial p} + \frac{\partial}{\partial p} p) a(p) :, \qquad K = \int_{-\infty}^{\infty} \frac{dp}{2\pi} : a^*(p) p a(p) :$$

and to realize the first possibility we let the : : refer to a vacuum where the negative p–states are filled:

$$\omega(a^*(p)a(p')) = \frac{1}{2\pi}\theta(-p)\delta(p-p')$$

and therefore

$$\omega(a(p')a^*(p)) = \frac{1}{2\pi}\theta(p)\delta(p-p')$$

which means

$$: a^*(p)a(p) : = -a(p)a^*(p) \qquad \text{for } p < 0.$$

Next we consider the subalgebra $\mathcal{A}^{\pm}$ generated by ψ_f with supp $f \in \mathbf{R}^{\pm}$. ω is faithful over $\mathcal{A}^+$ since

$$\|\psi_f|\Omega\rangle\|^2 = \langle\Omega|\psi_f^*\psi_f|\Omega\rangle = \int_{-\infty}^0 \frac{dp}{2\pi}|\widetilde{f}(p)|^2$$

and for supp $f \in \mathbf{R}^+$, $\widetilde{f}(p) = \int_0^\infty du e^{ipu} f(u)$ is analytic in the upper half plane. $\widetilde{f}(p) = 0$ $\forall\ p < 0$ implies then $f = 0$ or $\psi_f = 0$. On $\mathcal{A}^+$ τ^t acts as automorphism and σ^s for $s > 0$ as endomorphism. ω is invariant under both since these actions imply $\widetilde{f}(p) \to e^{t/2}\widetilde{f}(e^t p)$ or $\widetilde{f}(p) \to e^{ips}\widetilde{f}(p)$. Furthermore $\mathcal{A}^+|\Omega\rangle$ is dense in $\mathcal{H}_\omega$ the representation space of $\Pi_\omega(\mathcal{A})$ since

$$\langle\Omega|\psi_g^*\psi_f|\Omega\rangle = \int_{-\infty}^0 \frac{dp}{2\pi}\widetilde{g}^*(p)f(p) = 0 \qquad \forall\ \psi_f \in \mathcal{A}^+$$

implies supp $g \subset \mathbf{R}^+$ and thus $\psi_g|\Omega\rangle = 0$. Thus $\Pi_\omega|_{\mathcal{A}^+}(\mathcal{A}^+) = \Pi_\omega(\mathcal{A}^+)$ and we claim that $\omega|_{\mathcal{A}^+}$ is $\tau^{2\pi i}$–KMS and $\Pi_\omega(\mathcal{A}^+)' = \Pi_\omega(\mathcal{A}^-)''$ but ω is not a product state over $\mathcal{A} = \mathcal{A}^+ \otimes \mathcal{A}^-$. To show this note that $\tau^t\psi_f|\Omega\rangle = \psi_{f_t}|\Omega\rangle$ can with $\widetilde{f}_t(p) = e^{t/2}\widetilde{f}(e^t p)$ be continued in the upper half plane to $t = i\pi$ and similarly $\psi_f^*|\Omega\rangle$ in the lower half plane to $-i\pi$ since $\widetilde{f}^*(p) = \int_0^\infty du e^{-ipu} f^*(u)$ is analytic for Im $p < 0$. $\tau^{i\pi}\psi_f = \psi_{f_-}$, $\widetilde{f}_-(p) = if(-p)$ and the KMS condition says

$$\langle\Omega|\psi_g^*\psi_f|\Omega\rangle = \int_{-\infty}^0 \frac{dp}{2\pi}\widetilde{g}^*(p)f(p) = \langle\Omega|\tau^{-i\pi}\psi_f\tau^{i\pi}\psi_g^*|\Omega\rangle = \int_0^\infty \frac{dp}{2\pi}\widetilde{g}^*(-p)f(-p)$$

and is therefore satisfied for the two point function. Since the two point function determines here the n–point function these considerations can be extended to all of $\mathcal{A}^+$.

Remark: Note that $\Pi_\omega(\mathcal{A}^+)''$ is of type III_1 whereas $\Pi_\omega(\mathcal{A})''$ is type I_∞ so this feature appears already in this simple example.

Theorem (5.4) Let ω be τ^t invariant, τ^t and σ^s represented by strongly continuous unitaries $V_t = e^{iHt}$, $U_s = e^{iKs}$, resp. Let $|\Omega\rangle$ be the cyclic vector in Π_ω such that $H|\Omega\rangle = 0$ and therefore $K|\Omega\rangle = 0$. Let χ_Δ be the characteristic function of $\Delta = (-\infty, -a) \cup (a, \infty)$, $a > 0$ and $A, B \in \mathcal{A}$ such that

(i) $A|\Omega\rangle \in D(K^r)$ the domain of K^r, $r > 0$, and

(ii) $\chi_{\Delta^c}(K)B|\Omega\rangle = 0$, then

$$|\omega(A\tau^t B)| \leq e^{-tr}a^{-r}\|B|\Omega\rangle\| \cdot \|K^r A|\Omega\rangle\|.$$

Remarks

1. Without loss of generality we may assume that $K \geq 0$, otherwise K^r may be read as $|K|^r$.

2. Since $K|\Omega\rangle$ is zero (ii) implies $\langle\Omega|B|\Omega\rangle = 0$ which can be achieved by $B \to B - \langle\Omega|B|\Omega\rangle$. Thus the left hand side can be replaced by $|\omega(A\tau^t B) - \omega(A)\omega(B)|$ which shows that the theorem proves decay of correlations.

3. (i) can be achieved by replacing A by $A_f = \int ds f(s)\sigma^s(A)$. In fact by our continuity assumptions A can be approximated in norm by A_f's which qualify for the theorem.

4. Because of $\omega(A\tau^t B) = \omega(\tau^{-t}(A)B) = \omega(B^*\tau^{-t}A^*)^*$ the decay is in both time directions.

Proof: We use $\chi_\Delta(x) \leq \left(\frac{x}{a}\right)^r \ \forall\, r > 0$ to conclude

$$\begin{aligned} |\langle\Omega|A\tau^{-t}B|\Omega\rangle| &= |\langle\Omega|AV_{-t}\chi_\Delta B|\Omega\rangle| = |\langle\Omega|A\chi_{e^t\Delta}V_{-t}B|\Omega\rangle| \\ &\leq \langle\Omega|A\chi_{e^t\Delta}A^*|\omega\rangle^{1/2}\langle\Omega|B^*B|\Omega\rangle^{1/2} \leq (ae^t)^{-r}\|B|\Omega\rangle\| \cdot \|K^r A|\Omega\rangle\|. \end{aligned}$$

Examples

1. The Weyl algebra

   We consider the algebra $\mathcal{W}$ which consists of polynomials of $W(z) = e^{i(\alpha x + \beta p)}$, $z = (\alpha, \beta) \in \mathbf{R}^2$ with the multiplication law

   $$W(z)W(z') = e^{i\sigma(z,z')}W(z + z'),$$

   with the symplctic form $\sigma(z, z') = \frac{1}{2}(\alpha\beta' - \beta\alpha')$. The unitaries $V(t) = \exp[i\frac{1}{2}(px + xp)t] \notin \mathcal{A}$ and $U(s) = e^{ips} \in \mathcal{A}$ generate

   $$\tau^t(W(\alpha, \beta)) = W(e^t\alpha, e^{-t}\beta), \qquad \sigma^s(W(\alpha, \beta)) = e^{i\alpha s}W(\alpha, \beta).$$

   The action τ^t is not norm–continuous. A state ω is characterized by $E(z) = \omega(W(z))$ and has to satisfy $E(0) = 1$, and $e^{-i(\sigma(z,z')}f(z - z')$ has to be a positive integral kernel. (This implies $|E(z)| \leq 1$.) If ω is to be invariant under σ^s we need $E(\alpha, \beta) = 0 \ \forall\ \alpha \neq 0$ and then invariance and τ^t requires that $E(0, \beta)$ is constant on $\mathbf{R}^+$ or $\mathbf{R}^-$ resp. The positivity requirement leaves two possibilities

   a) The tracial state $E_0 = 0 \ \forall\ (\alpha, \beta) \neq (0, 0)$. This state is well–known from the Hilbert space of quasiperiodic functions which is not separable. Here the actions of τ^t and σ^s on $W(z)$ are not weakly continuous since the typical matrix elements of $W(z)$ are

   $$\langle z'|W(z)|z''\rangle := \langle\Omega|W^*(z')W(z)W(z'')|\Omega\rangle = e^{i\sigma(z,z''-z')-i\sigma(z',z'')}E(z + z'' - z').$$

   Thus neither H nor K exists. The representation is type II_1 but not hyperfinite.

b) The other state is $E(\alpha,\beta) = 0 \; \forall \; \alpha \neq 0, = 1$ for $\alpha = 0$ and again τ is not weakly continuously represented so Theorem (5.4) does not apply. Nevertheless in both cases there is some extreme clustering since

$$\langle \Omega | W^*(z) \tau^t W(z') | \Omega \rangle = 0$$

for all t except if $\alpha = e^t \alpha'$.

2. Here we consider the subalgebra $\mathcal{W}_c = \{W(z), z \in \Theta\mathbf{Z} \times \mathbf{Z}\}$. As in the classical case the τ^t of (2.1) is not an automorphism of $\mathcal{W}_c$ but if we choose other directions for expansion and contraction it can happen that for a discrete set of times we get the following automorphism of $\mathcal{W}_c$, $\tau W(z) = W(Tz)$, $T \in SL(2,\mathbf{Z})$. Of course, for $\mathcal{W}$ there exists a τ^t such that $\tau^1 = \tau$ but $\tau^t(\mathcal{W}_c) \subset \mathcal{W}_c$ only for $t \in \mathbf{Z}$. In this case the Lyapunov exponent cannot be scaled to 1 but turns out to have the classical value [11,12,13]

$$\lambda = \frac{c}{2} + \sqrt{\frac{c^2}{4} - 1}, \qquad c = \mathrm{Tr}\; T > 2.$$

E_0 gives the only invariant state and a regular representation Π_0. It acts in a separable Hilbert space and $\Pi_0(\mathcal{W}_c)''$ is type II_1 hyperfinite. If σ is the shift in transversal direction X such that $\sigma^s(W(z)) = e^{is\langle x|z\rangle} W(z)$ then σ^s acts even norm continuously and the theorem applies.

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