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CALABI'S PROBLEM ON BOUNDED MINIMAL SURFACE

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1. Introduction

Let (M, g) be a Riemann surface with a complete Riemannian metric g on M and let

$$\Psi : (M, g) \to \mathbb{R}^3$$

be an isometrical and minimal immersion. Is it possible that the image $\Psi(M)$ is a subset of the unit ball in $\mathbb{R}^3$? This question was raised by Calabi and repeated by Yau in [1]. In [2] this problem is referred as Chern's problem. Jorge and Xavier, [3], proved the existence of a complete minimally immersed surface between two planes. The aim of this paper is to show the existence of a bounded minimal surface.

Theorem. *There exists a complete minimally immersed surface in* $\mathbb{R}^3$ *which is a subset of the unit ball.*

Our example of minimal surface has somewhat similar to the example of Jorge and Xavier: we also use the Weierstrass representation of minimal surfaces and also use the Runge approximation theorem.

2. Preliminaries

Let $\Omega \subset \mathbb{C}$ be a domain and $\varphi : \Omega \to \mathbb{C}^3$ be a conformal map $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, satisfying, $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 \equiv 0$. Then

$$X(z) = Re \int_{z_0}^{z} \varphi \qquad (1)$$

is a minimal surface in $\mathbb{R}^3$. Also any minimal surface $X : \Omega \to \mathbb{R}^3$ can be locally represented in the form (1) and if Ω is simply connected then X is globally represented by (1), (see [4], [5]). In order for Ω to be immersed in $\mathbb{R}^3$ one requires

$$\sum_{i=1}^{3} |\varphi i(z)| \neq 0$$

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for all $z \in \Omega$.

Let us assume that $\varphi_1 - i\varphi_2 \not\equiv 0$ and set

$$f = \varphi_1 - i\varphi_2,$$
$$g = \varphi_3/(\varphi_1 - i\varphi_2)$$

Then f is a holomorphic and g is a meromorphic function on Ω. The surface (1) can be obtained by

$$X(z) = Re \int_{z_0}^{z} (\frac{1}{2}f(1-g^2), \frac{i}{2}f(1+g^2), fg) \tag{2}$$

This is called the Weierstrass representation of a minimal surface. The induced metric g_X on Ω is given by:

$$g_X = (\frac{1}{2}|f|(1+|g|^2)|dz|)^2 \tag{3}$$

The meromorphic map g has an important geometrical meaning: it is the composition of the Gauss map of $X(m)$ with the stereographic projection of the unit sphere to the equatorial plane, from the north pole. Let minimal immersion $X : \Omega \to \mathbb{R}^3$ is given by (2) and h be a holomorphic function on Ω, $h \neq 0$ in Ω. Set $\tilde{f} = fh, \tilde{g} = g/h$,

$$\tilde{X}(z) = Re \int_{z_0}^{z} (\frac{1}{2}\tilde{f}(1-\tilde{g}^2), \frac{i}{2}\tilde{f}(1+\tilde{g}^2), fg) \tag{4}$$

Then $\tilde{X} : \Omega \to \mathbb{R}^3$ is a minimal immersion.

Notation. Let D_r be a disk on $\mathbb{C} : |z| < r$, $S_r := \partial D_r$, $B_r \subset \mathbb{R}^3$ be a ball $|x| < r$. By l_0 we denote a ray in $\mathbb{C}$, $l_\theta = \alpha e^{i\theta}, \alpha > 0$. Let $E \subset \mathbb{C}$ be a set, $\varepsilon > 0$. By $N[\varepsilon](E)$ we denote ε-neighbourhood of the set E.

3. PROOF OF THE THEOREM

Lemma. *Let* $X \in C^\infty(\bar{D}_1; \mathbb{R}^3)$ *and*

$$X : D_1 \to B_r \subset \mathbb{R}^3, \tag{5}$$

$r > 0$, *be a minimal immersion,* $X(0) = 0$. *Assume that* (D_1, g_X) *is a geodesic disk of radius* ρ *centred in 0. Then for any* $\varepsilon, \rho > 0$ *there exists a minimal immersion*

$$Y : D_1 \to B_R \subset \mathbb{R}^3,$$

$R = \sqrt{r^2 + s^2} + \varepsilon$, *such that* (D_1, g_Y) *is a geodesic disk of radius* $\rho + s$ *and*

$$|X - Y| < \varepsilon \text{ on } D_{1-\varepsilon}$$

This Lemma will be proved in Section 4. We show now that the Theorem is a consequence of the Lemma.

We define a sequence of minimal immersions

$$X_n : D_1 \to \mathbb{R}^3$$

by induction over $n = 1, 2, \dots$. Let $X_1 : (D_1, |dz|) \to \mathbb{R}^3$ be a linear isometry, $X_1(0) = 0$. Let $\varepsilon_n, n = 1, 2, \dots$, be a sequence of positive numbers which will be specified later. Assume that a minimal immersion $X_{n-1} = X$ is already defined. Set $\varepsilon = \varepsilon_n, s = 1/n$ and let the minimal immersion Y be defined by the lemma. Define $X_n = Y$. If the ε_k tend sufficiently fast to zero as $k \to \infty$ then the following holds:

(a) $X_k \to X$ as $k \to \infty$ in the open disk D_1 and

$$X : D_1 \to \mathbb{R}^3$$

is a minimal immersion;

(b)

$$X_k : D_1 \to B_{r_k} \subset \mathbb{R}^3$$

where $r_k \leq r_{k-1} + 1/k^2$ and hence $r_k \leq 2$ for all k;

(c) since (D_1, g_{X_k}) is a geodesic disk of radius ρ_k, where ρ_k is given by

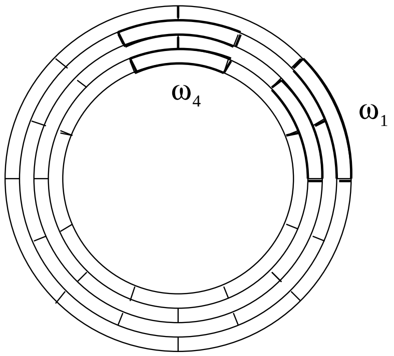
$$\rho_k = \sum_{j=1}^{k} 1/j$$

the metric (D_1, g_X) is complete provided that the ε_k tend to zero sufficiently fast.

The theorem is proved.

4. PROOF OF THE LEMMA

(4.1) Consider the following labyrinth, see the picture below.



It's not difficult to find a way from the inner circle to the outside but any such way is fairly long although the Euclidean distance is short.

Now we give a formal description of a partition of the unit disk, which is illustrated by the picture above.

N-partition of a disk. Let $N \in \mathbb{N}$. Denote $r_i = 1 - i/N^3$, $i = 0, \dots, 2N^2$, $\mathfrak{A} = D_1 \backslash D_{r_{2N^2}+1}$, $A_i = D_{r_{2i}} \backslash D_{r_{2i+1}}$, $\tilde{A}_i = D_{r_{2i-1}} \backslash D_{r_{2i}}$,

$$A = \cup_{i=0}^{N^2} A_i,$$
$$\tilde{A} = \cup_{i=1}^{N^2} \tilde{A}_i,$$
$$S = \cup_{i=0}^{2N^2} S_{r_i},$$

$$l = \cup_{i=0}^{N} l_{i2\pi/N},$$
$$\tilde{l} = \cup_{i=0}^{N} l_{(2i+1)\pi/N},$$

$L = l \cap A$, $\tilde{L} = \tilde{l} \cap \tilde{A}$, $H = S \cup L \cap \tilde{L}$, $P = N[1/4N^3](H)$, $\Omega = \mathfrak{A} \backslash P$, $s_i = l_{i\pi/N} \cap \mathfrak{A}$.

Let us denote by $\omega_j, j = 1, \dots, 2N$, the union of the segment s_j and those components of the set Ω which have nonempty intersection with s_j.

Let curve $\sum \subset D_1 \backslash \Omega$ connects the point 0 and S_1. Then we have

$$\text{length } (\Sigma) > 10N$$

Let h be a continuous function in $D_1, h \geq 1$ on D_1, $h \geq N^4$ on Ω. Let a smooth curve σ connect 0 and S_1 in D_1. Then

$$\int_\sigma h ds > N \tag{6}$$

where ds is the arc length parameter on σ.

(4.2) Let $G : D_1 \rightarrow S^2$ be the Gauss map of the minimal surface (5). Since the X is smooth in $\bar{D}_1$ the map G is continuous in $\bar{D}_1$. Hence for any $\delta > 0$ there exists $N = N(\delta)$ such that for any domain ω_i of the N-partition of D_1 the following inequality holds

$$diam G(\omega_i) < \delta.$$

(4.3) Let $E_1, E_2 \subset \mathbb{C}$ be compact such that each complement $\complement E_i$ is connected, $i = 1, 2$, and $E_1 \cap E_2 = \emptyset$. Let $T > 1$. By Rynge theorem there exists a holomorphic function w on $\mathbb{C}$ such that

$$|w| < 1/T \text{ on } E_1$$

and

$$|w - T| < 1/T \text{ on } E_2.$$

Let us denote

$$h[T, E_1, E_2](z) := e^{w(z)}$$

(4.4) Let N and T be sufficiently large positive constants which will be specified later. We define a sequence of minimal immersions

$$F_k : D_1 \rightarrow \mathbb{R}^3$$

$k = 0, \ldots, 2N^3 = K$ by induction over k. Set $F_0 = X$. Assume that a map F_{i-1} is already defined. Let us pick a point $q \in S^1$ such that

$$dist(q, G(\omega_i)) = 1/\sqrt{N} \tag{7}$$

We assume that in orthogonal coordinates x_1, x_2, x_3 in $\mathbb{R}^3$ the vector q is directed along x_3. Let (2) be the Weierstrass representation of F_{i-1}. Set

$$h = h[T, \omega_i, D_1 \backslash N[1/4N^3](\omega_i)],$$

$\tilde{f} = fh, \tilde{g} = g/h$ and $\tilde{X}$ is defined by (4). Set $F_i := \tilde{X}$. Then

$$\pi(F_i) = \pi(F_{i-1}), \tag{8}$$

where π is the orthogonal projection $\mathbb{R}^3$ on x_3 axes. Denote $g_{F_i} = a|dz|$. By (3) we have

$$a \geq |f| max(|h|, |g|/|h|)$$

Hence $a \to \infty$ on ω_i as $T \to \infty$, $g_{F_i} \to g_{F_{i-1}}$ on $D_1 \backslash N[1/4N^3](\omega_i)$ as $T \to \infty$. By (7) on the set $N[1/4N^3](\omega_i) \backslash \omega_i$ the following inequality holds:

$$a \geq 1/4\sqrt{N}$$

Thus, by (6) for sufficiently large T the geodesic distance between point 0 and S_1 in the metric (D_1, g_{F_K}) is no less than $\sqrt{N}/4$.

Let d be a geodesic disk in (D_1, g_{F_K}) of radius $\rho + s$ with the centre at 0. Since the Gaussian curvature of g_{F_K} is nonpositive the ∂d is a smooth curve in D_1. For sufficiently large N it follows: $D_{1-\varepsilon} \subset d$ and

$$|X - F_K| < \varepsilon \text{ on } D_{1-\varepsilon}$$

Let $\eta \in \partial d$. From Assertion (4.2) and from (5), (8), it follows that

$$\overline{\lim}_{\substack{\delta \to 0 \\ T \to \infty}} |F_K(\eta)| \leq \sqrt{r^2 + s^2}$$

Let $w : D_1 \to d$ be a biholomorphic map such that $w(0) = 0, w^1(0) = 1$. Set $Y = F_K \circ w$. By the Schwarz lemma for sufficiently large N the following inequality holds:

$$|X - Y| < \varepsilon \text{ on } D_{1-\varepsilon}$$

The lemma is proved.

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