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ON THE LIEB-THIRRING ESTIMATES FOR THE PAULI OPERATOR

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Abstract. We establish Lieb-Thirring type estimates for the sums $\sum_k |\lambda_k|^\gamma$ of the negative eigenvalues λ_k of the two-dimensional Pauli operator with a non-homogeneous magnetic field perturbed by a decreasing electric potential.

1. Introduction

The aim of the paper is to establish some spectral properties of the Pauli operator, that is of the operator describing the motion of a particle with spin in a magnetic field. It acts in $L^2(\mathbb{R}^d)\otimes\mathbb{C}^2$ with $d=2$ or $d=3$ and has the form

$$H^{(d)}_{Pauli} = H^{(d)}_{\mathbf{a}}\mathbb{I} - \Sigma\cdot\mathbf{B},\ \ \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $H^{(d)}_{\mathbf{a}} = (-i\nabla - \mathbf{a})^2$ is the usual spinless Schrödinger operator with the magnetic vector-potential $\mathbf{a} = \{a_1,\dots,a_d\}$, $\mathbf{B} = \nabla\times\mathbf{a}$ is the field and Σ stands for the vector $\sigma_1,\sigma_2,\sigma_3$ of 2×2 Pauli matrices (see [4]). Suppose that the field $\mathbf{B}$ is pointed along the x_3-axis, i.e. $\mathbf{a} = (a_1,a_2,0)$ with $a_k = a_k(x_1,x_2)$ (which is always true for $d=2$). In this case $\mathbf{B} = (0,0,B)$, $B = \partial_1 a_2 - \partial_2 a_1$ and $H^{(d)}_{Pauli}$ looks especially simple:

$$H^{(2)}_{Pauli} = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix},\ \ A_\pm = H^{(2)}_{\mathbf{a}} \mp B, \tag{1.1}$$

$$H^{(3)}_{Pauli} = H^{(2)}_{Pauli} + \begin{pmatrix} -\partial_3^2 & 0 \\ 0 & -\partial_3^2 \end{pmatrix}. \tag{1.2}$$

Though this operator does not seem to be non-negative, the entries $A_\pm$ can be rewritten as $A_\pm = Q_\pm^* Q_\pm$ with the operators

$$Q_\pm = \Pi_1 \pm i\Pi_2,\quad \Pi_k = -i\partial_k - a_k,\ k=1,2, \tag{1.3}$$

which allows one to define $H^{(2)}_{Pauli}$ as a non-negative self-adjoint operator (see Sect. 2 below). A remarkable property of $H^{(2)}_{Pauli}$ is that the point $\lambda = 0$ belongs to its

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spectrum. This assertion was proved under fairly broad conditions on the magnetic field B (see [1], [6] and also [4],[8]). If the operator (1.1) or (1.2) is perturbed by a real-valued function (electric potential) $V = \bar{V}$, decaying at infinity, then it can have some discrete spectrum below $\lambda = 0$. Denote by $\lambda_k = \lambda_k(\mathbf{a}, V)$, $k \in \mathbb{N}$, the negative eigenvalues of $H^{(d)}_{Pauli} + V\mathbb{I}$ enumerated in the non-decreasing order counting multiplicity. We study the sums

$$\mathcal{M}_\gamma(\mathbf{a}, V) = \sum |\lambda_k|^\gamma, \ \gamma \geq 0.$$

It is well-known that, without any magnetic field, $\mathcal{M}_\gamma$ satisfies the following estimate[2]:

$$\mathcal{M}_\gamma(\mathbf{0}, V) \leq C_{d,\gamma} \int V_-(x)^{\gamma + \frac{d}{2}} dx, \quad \gamma + d/2 > 1, \tag{1.4}$$

which is usually referred to as the *Lieb–Thirring inequality* if $\gamma > 0$ and the *Rosenblum–Lieb–Cwickel inequality* if $\gamma = 0$. The same estimate holds for the negative spectrum of the spinless operator $H^{(d)}_{\mathbf{a}} + V$ with $\mathbf{a} \neq 0$. The crucial technical reason for that is the so-called diamagnetic inequality (see [2]). It means, loosely speaking, that the magnetic field "pushes the spectrum upwards", which leads to (1.4) for $H^{(d)}_{\mathbf{a}} + V$ as well. This type of argument does not work for the Pauli operator and as a consequence, the standard Lieb-Thirring estimate (1.4) no longer holds. A suitable replacement for (1.4) for a homogeneous field $B(x_1, x_2) = const$ and $\gamma = 1$ was found in [10] (see also short communication [11]) and [12]:

$$\mathcal{M}_1(\mathbf{a}, V) \leq C'_d \int V_-(x)^{1 + \frac{d}{2}} dx + C''_d |B| \int V_-(x)^{\frac{d}{2}} dx, \ d = 2, 3.$$

A natural generalization of this result for non-homogeneous fields and arbitrary γ would be as follows:

$$\mathcal{M}_\gamma(\mathbf{a}, V) \leq C'_{\gamma,d} \int V_-(x)^{\gamma + \frac{d}{2}} dx + C''_{\gamma,d} \int |B(x_1, x_2)| V_-(x)^{\gamma + \frac{d}{2} - 1} dx, \ \gamma + d/2 > 2. \tag{1.5}$$

The validity of this conjecture was thoroughly investigated in [7] (see also [8] for more details and further references) for $d = 3$. In particular, (1.5) was established for the magnetic fields satisfying the lower bound $B \geq B_0 > 0$ under some additional constraints on the behaviour at infinity. Besides, the author constructed a counterexample showing that the bound (1.5) cannot hold for arbitrary B. In this connection it was conjectured in [7] on the grounds of physical considerations, that in order to save (1.5), one has to replace the magnetic field in (1.5) with a suitable "smeared" modification of B.

In the present paper we give a simple proof of (1.5) in the case $d = 2, \gamma > 1$ under fairly general conditions on B. The results on $d = 2, \gamma = 1$ and $d = 3$ will be published elsewhere. We find out that (1.5) holds with B replaced by an "effective" magnetic field $b(x)$ (see Sect. 2 for precise definition). The field b coincides with the

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[2] Here and in what follows we denote by C and c (with or without indices) various positive constants whose precise value is of no importance.

initial field B, if the latter obeys some regularity condition, which, in particular, does not allow B to decay at infinity too rapidly. This is quite consistent with restrictions on B under which (1.5) was obtained in [7]. On the contrary, if the decay of B is too fast, the effective field b will be considerably different from B. The distinction is especially spectacular for a compactly supported B. This point is discussed in detail in Sect. 2. One should stress that the method of the proof is essentially different from that used in [7]. Instead of the technically involved path integration approach adopted in [7], we apply only elementary methods of the spectral theory for Schrödinger operators: the Birman-Schwinger principle, the diamagnetic inequality and Cwickel type estimates.

We point out that until recently there have been only a few mathematically rigorous results on the Pauli operator in a non-homogeneous magnetic field. In the physical literature most of its properties were assumed without proof. For instance, it was unclear under which conditions on the potential V the operator $H^{(2)}_{Pauli} + V\mathbb{I}$ can be defined as a self-adjoint operator. A partial answer to this question was given in [8]. In the present paper we obtain a more general criterion, which guarantees self-adjointness of the perturbed operator (see Theorem 2.3).

Another important property which required justification is the coincidence of the non-zero spectra of the entries $A_\pm$ in (1.1) (see [4]):

$$\sigma(A_+) \setminus \{0\} = \sigma(A_-) \setminus \{0\}.$$

This fact follows from the formal relations $A_+ = Q_+^* Q_+$ and $A_- = Q_+ Q_+^*$, which, strictly speaking, are fulfilled only if $Q_\pm^* = Q_\mp$. The latter equality is not true in general. In the appendix we find out that under some conditions on the magnetic potential $\mathbf{a}$ and the field B this equality does take place.

Notation. For any measurable real-valued function f we denote by f_+ and f_- its positive and negative parts respectively: $f_\pm = (|f| \pm f)/2$. This convention does not apply to operators (cf. (1.1)). The notation $\|f\|_{L^q}$ stands for the norm of f in the space $L^q(\mathbb{R}^2)$, $q \geq 1$. If $q = 2$ we simply write $\|f\|$.

For a self-adjoint operator T, $R(z, T) = (H - z)^{-1}$ denotes its resolvent. If T is semi-bounded, then $T[\cdot, \cdot]$ denotes the closed form associated to T, with the domain $D[T]$. Notation $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\mathbb{R}^2)$.

2. RESULT, DISCUSSION

1. Basic definitions and notation. For the operator (1.1) is diagonal, we can study its entries A_+, A_- individually. To state results simultaneously for "+" and "-" we use, as a rule, the double subscript: "±". In this case each statement must be undestood separately for the upper subscript and the lower one.

Let $\mathbf{a} = (a_1, a_2) \in L^2_{loc}(\mathbb{R}^2)$ be a magnetic vector-potential with real-valued components. The operators $Q_\pm, \Pi_k$ defined in (1.3) are closable on $C_0^\infty(\mathbb{R}^2)$, since Π_k are symmetric and $Q_\pm \subset Q_\mp^*$. We use the same letters Q_+, Q_-, Π_k for their closures. Define $A_\pm = Q_\pm^* Q_\pm$. This operator can be also interpreted as that associated with the closed quadratic form $A_\pm[u, u] = \|Q_\pm u\|^2$ with the domain $D[A_\pm] = D(Q_\pm)$. In the same way we define the usual Schrödinger operator $H_\mathbf{a}$ with the magnetic field $\mathbf{a}$: as an operator associated with the form $H_\mathbf{a}[u, v] =$

$\langle \Pi_k u, \Pi_k v\rangle$ (summation over repeating indices is assumed). We omit the superscript "(2)" from notation (cf. (1.1)) without any risk of confusion, for we study only the case $d = 2$. It follows from the inequality

$$A_\pm[u,u] = H_{\mathbf{a}}[u,u] \pm i\big(\langle \Pi_2 u, \Pi_1 u\rangle - \langle \Pi_1 u, \Pi_2 u\rangle\big) \le 2H_{\mathbf{a}}[u,u], \ \forall u \in C_0^\infty(\mathbb{R}^2), \quad (2.1)$$

that $D[H_{\mathbf{a}}] \subset D[A_\pm]$. As a rule below we impose

Assumption 2.1. *The magnetic potential* $\mathbf{a} \in L^2_{loc}(\mathbb{R}^2)$ *is such that the magnetic field*

$$B(x) = -i[\Pi_1, \Pi_2] = \operatorname{rot} \mathbf{a}(x) = \partial_1 a_2(x) - \partial_2 a_1(x),$$

defined in the sense of distributions, belongs to $L^\infty_{loc}(\mathbb{R}^2)$.

It is easy to check that for any $u, v \in C_0^\infty(\mathbb{R}^2)$

$$A_\pm[u,v] = H_{\mathbf{a}}[u,v] \mp \langle Bu, v\rangle, \quad (2.2)$$

which implies that

$$A_+ - A_- = -2B(x), \quad (2.3)$$

in the sense of sesqui-linear forms on $C_0^\infty(\mathbb{R}^2)$. Observe that if $B_\pm \in L^\infty(\mathbb{R}^2)$, then (2.2) entails the estimate

$$A_\pm[u,u] \ge H_{\mathbf{a}}[u,u] - \|B_\pm\|_{L^\infty}\|u\|^2,$$

which, along with (2.1), implies that $D[A_\pm] = D[H_{\mathbf{a}}]$. Without the condition $B_\pm \in L^\infty(\mathbb{R}^2)$, we can only claim that the form domains of the operators $A_+, A_-, H_{\mathbf{a}}$ coincide locally. Precisely, the following lemma holds:

Lemma 2.2. *Let the magnetic potential* $\mathbf{a}$ *obey Assumption 2.1. If a function* u *belongs to one of the domains* $D[A_+], D[A_-]$ *or* $D[H_{\mathbf{a}}]$, *then the function* χu, $\chi \in C_0^\infty(\mathbb{R}^2)$, *belongs to all of them.*

This lemma follows from the obvious commutator relation for the operators (1.3):

$$[Q_\pm, \chi] = [-i\partial_1 \pm \partial_2, \chi] = (-i\partial_1 \pm \partial_2)\chi = \chi_\pm, \quad \forall \chi \in C_0^\infty(\mathbb{R}^2), \quad (2.4)$$

and Assumption 2.1.

¿From now on we impose the following conditions on the magnetic field. Let $\ell \in C(\mathbb{R}^2)$ be a positive function such that

$$|\ell(x) - \ell(y)| \le \varrho|x-y|, \ 0 \le \varrho < 1, \ \forall x, y \in \mathbb{R}^2. \quad (2.5)$$

Sometimes we call this function *slowly varying.* Denote

$$\mathcal{D}(x) = \{y \in \mathbb{R}^2 : |x-y| < \ell(x)\}. \quad (2.6)$$

We assume that there exists a positive function $b \in L^\infty_{loc}(\mathbb{R}^2)$ such that

$$|B_\pm(x)| \le b(x), \ \text{a.a. } x \in \mathbb{R}^2; \quad (2.7)$$

$$C_1 b(x) \le b(y) \le C_2 b(x), \ \text{a.a. } y \in \mathcal{D}(x), \ \text{a.a. } x \in \mathbb{R}^2; \quad (2.8)$$

$$b(x)\ell(x)^2 \ge c, \ \text{a.a. } x \in \mathbb{R}^2. \quad (2.9)$$

The next theorem specifies conditions on V under which the operator $P_\pm = A_\pm + V$ can be defined as a form sum on $D[A_\pm]$.

Theorem 2.3. *Let B obey the conditions (2.7) – (2.9) with some functions $\ell(x)$ and $b(x)$. Let V satisfy for some $p > 1$ the estimate*

$$\text{v-sup}_x \int_{\mathcal{D}(x)} |V(y)|^p \big(b(y) + 1\big) dy < \infty.$$

Then the form $P_\pm[\cdot,\cdot] = A_\pm[\cdot,\cdot] + V[\cdot,\cdot]$ is closed on $D[P_\pm] = D[A_\pm]$.

We point out that if $B_\pm \in L^\infty(\mathbb{R}^2)$, then Theorem 2.3 with $b(x) = \text{v-sup}\, B_\pm(x)$, $\ell(x) = 1$ guarantees that $D[P_\pm] = D[A_\pm]$ under the condition

$$\text{v-sup}_x \int_{|x-y|\le 1} |V(y)|^p dy < \infty.$$

Note that this condition is known to be sufficient for the equality $D[H_{\mathbf{a}}+V] = D[H_{\mathbf{a}}]$ (see [4]). This fact agrees with the observation made above, that $D[A_\pm] = D[H_{\mathbf{a}}]$, if $B_\pm \in L^\infty(\mathbb{R}^2)$.

By definition the operator $A_\pm$ is non-negative. Below we shall impose on the potential V the constraints, which will guarantee that the negative spectrum of the operator $P_\pm$ associated with the form $P_\pm[\cdot,\cdot]$ is discrete. Let $\lambda_k^{(\pm)}$, $k \in \mathbb{N}$, be negative eigenvalues of $P_\pm$, enumerated in the non-decreasing order counting multiplicity. We study the following quantities:

$$\mathcal{M}_\gamma^{(\pm)} = \sum_k |\lambda_k^{(\pm)}|^\gamma, \quad \mathcal{N}_\gamma^{(\pm)} = \sup_k |\lambda_k^{(\pm)}|^\gamma k, \gamma \ge 0. \tag{2.10}$$

Note that $\mathcal{N}_\gamma^{(\pm)} \le \mathcal{M}_\gamma^{(\pm)}$. When necessary we reflect the dependence of various objects on the fields $\mathbf{a}, V$: for example, $\lambda_k^{(\pm)} = \lambda_k^{(\pm)}(\mathbf{a}, V)$, $\mathcal{M}_\gamma^{(\pm)} = \mathcal{M}_\gamma^{(\pm)}(\mathbf{a}, V)$. Next Theorem establishes the main result of the paper.

Theorem 2.4. *Let the conditions (2.7) – (2.9) be fulfilled. Suppose that V obeys the conditions of Theorem 2.3 and $V_- \in L^{\gamma+1}(\mathbb{R}^2)$, $V_-^\gamma b \in L^1(\mathbb{R}^2)$ for some $\gamma \ge 1$. Then the negative spectrum of $P_\pm$ is discrete and*

$$\mathcal{M}_\gamma^{(\pm)} \le C_{1,\gamma} \int V_-(x)^{\gamma+1} dx + C_{2,\gamma} \int V_-(x)^\gamma b(x) dx, \;\; \gamma > 1, \tag{2.11}$$

$$\mathcal{N}_1^{(\pm)} \le C_{1,1} \int V_-(x)^2 dx + C_{2,1} \int V_-(x) b(x) dx, \;\; \gamma = 1, \tag{2.12}$$

The constants $C_{1,\gamma}, C_{2,\gamma}$ can be calculated explicitly, but apparently their values are far from being optimal and for this reason we do not give them.

2. Discussion of Theorem 2.4. Let us replace the vector-potential $\mathbf{a}$ with $\mu\mathbf{a}$ where $\mu > 0$ is a parameter measuring intensity of the magnetic field. It is clear that the conditions (2.7)–(2.9) for B and b imply (2.7)–(2.9) for the fields μB and μb, if $\mu \ge 1$. Hence by Theorem 2.4 the sum $\mathcal{M}_\gamma^{(\pm)}(\mu\mathbf{a}, V)$ obeys the bound

$$\mathcal{M}_\gamma^{(\pm)}(\mu\mathbf{a}, V) \le C_{1,\gamma} \int V_-(x)^{\gamma+1} dx + C_{2,\gamma}\mu \int V_-(x)^\gamma b(x) dx, \;\; \gamma > 1, \;\; \mu \ge 1. \tag{2.13}$$

On the contrary, if $\mu \to 0$, then the condition (2.9) for μb may be violated and Theorem 2.4 will be no longer applicable.

Theorem 2.4 allows one to estimate $\mathcal{M}_\gamma^{(\pm)}$ for the Pauli operator containing the Planck constant $\hbar$:

$$\tilde{P}_\pm(\hbar) = (-i\hbar\nabla - \mathbf{a})^2 \mp \hbar B + V, \ \hbar \in (0, 1].$$

To that end observe that

$$\tilde{P}_\pm(\hbar) = \hbar^2 P_\pm(\hbar^{-1}\mathbf{a}, \hbar^{-2}V),$$

which means that the sum of negative eigenvalues $\lambda_k(\tilde{P}_\pm)$ raised to the power γ equals

$$\hbar^{2\gamma}\mathcal{M}_\gamma^{(\pm)}(\hbar^{-1}\mathbf{a}, \hbar^{-2}V).$$

Using for the r.h.s. the estimate (2.13) with $\mu = \hbar^{-1} \geq 1$, one obtains that

$$\sum_k |\lambda_k(\tilde{P}_\pm)|^\gamma \leq C_{1,\gamma}\hbar^{-2}\int V_-(x)^{\gamma+1}dx + C_{2,\gamma}\hbar^{-1}\int V_-(x)^\gamma b(x)dx, \ \gamma > 1.$$

Let us consider two examples of Theorem 2.4.

Example 1. Suppose that $B(x) > 0$, $B \in C^1(\mathbb{R}^2)$ and

$$|\nabla B(x)| \leq CB(x)^{\frac{3}{2}}, \ \forall x \in \mathbb{R}^2. \tag{2.14}$$

For $B_- = 0$, it is clear that

$$\mathcal{M}_\gamma^{(-)} \leq C_1 \int V_-(x)^{\gamma+1}dx, \ \gamma > 1.$$

We claim that $\mathcal{M}_\gamma^{(+)}$ satisfies (2.11) with $b = B$. Indeed, define $\ell(x) = \varsigma B(x)^{-1/2}$, $b(x) = B(x)$ with some $\varsigma > 0$. Clearly, (2.7) and (2.9) are fulfilled. Let us verify the conditions (2.5) and (2.8). Due to (2.14)

$$|\nabla\ell(x)| \leq \frac{\varsigma}{2}B(x)^{-\frac{3}{2}}|\nabla B(x)| \leq C\frac{\varsigma}{2},$$

and consequently, for sufficiently small ς the function $\ell(x)$ obeys (2.5) with $\varrho = C\varsigma/2 < 1$. Furthermore, (2.5) provides the bound

$$(1+\varrho)^{-2}B(x) \leq B(y) \leq (1-\varrho)^{-2}B(x), \quad \forall y \in \mathcal{D}(x),$$

which ensures (2.8). Applying Theorem 2.4, we obtain (2.11) with $b = B$.

Note that the condition (2.14) admits a quick growth of B at infinity. For instance, any positive function $B \in C^1(\mathbb{R}^2)$ which equals $\exp(|x|^m), m > 0$, for $|x| \geq R > 0$, obeys (2.14). On the contrary, (2.14) does not allow $B(x)$ to decrease as $|x| \to \infty$ too rapidly. In fact, if $B(x) = |x|^{-\alpha}$ for $|x| \geq r > 0$, then the condition (2.14) is not fulfilled if $\alpha > 2$.

Example 2. Compactly supported magnetic field. Suppose that $B(x) = B,\ |x| \leq R$ and $B(x) = 0,\ |x| > R$ for some $B > 0, R > 0$. As in example 1, we look only at $\mathcal{M}_\gamma^{(+)}$. It is easy to see that the functions

$$\ell(x) = \varrho\sqrt{1+|x|^2},\ 0 < \varrho < 1, \qquad b(x) = \begin{cases} B & , |x| \leq R, \\ \dfrac{2B}{1+|x|^2R^{-2}}, & |x| > R, \end{cases} \tag{2.15}$$

satisfy the conditions (2.5), (2.7) – (2.9), which leads to (2.11) and (2.12). Note that the effective field b is not only non-compactly supported, but even non-integrable! Let us explain why one cannot find another function b satisfying all the conditions of Theorem 2.4 and decreasing quicker than the function in (2.15). The thing is that the choice of $b(x)$ should be made simultaneously with that of the slowly varying function $\ell(x)$. In view of (2.9), in order to minimize $b(x)$, one should maximize $\ell(x)$. Clearly, $\ell(x) \sim \varrho|x|$ is the fastest growing (as $|x| \to \infty$) function which obeys (2.5). Now the condition (2.9) ensures that $b(x) \geq C|x|^{-2}$.

Explicit calculations for this example were carried out in [8]. They show that the compactly supported magnetic field creates a non-integrable "tail" behaviour of a relevant quantity (ground state density), whith decay at infinity as $B|x|^{-2}(\log|x|)^{-1}$. The function $b(x)$ in (2.15) provides quite a precise estimate for such a tail.

3. Auxiliary information. Here we provide well-known facts to be used in the sequel.

(a)Compact operators (see [3]). Let T be a compact operator. We use the notation $s_n(T)$, $n \in \mathbb{N}$, for its singular values ($s-$values) and denote by $n(s,T) = \#\{s_n > s\}, s > 0$ their distribution function. Recall that $s_n(T)$ are defined as eigenvalues of the selfadjoint operator $(T^*T)^{1/2}$, so that

$$n(s^2, T^*T) = n(s,T). \tag{2.16}$$

Moreover, the *Weyl inequality* holds:

$$n(s_1+s_2, T_1+T_2) \leq n(s_1,T_1) + n(s_2,T_2). \tag{2.17}$$

We denote by $\mathfrak{S}_p, p \geq 1$, the Neumann-Schatten classes of compact operators with the norm

$$\| T \|_p = \left[\sum_n s_n(T)^p\right]^{\frac{1}{p}}.$$

It is easy to see that

$$n(s,T) \leq s^{-p}\| T \|_p^p,\ \forall s > 0. \tag{2.18}$$

(b) Birman-Schwinger principle. Let $N_\pm(\lambda) = \#\{\lambda_k^{(\pm)} < -\lambda\},\ \lambda > 0$ be the number of negative eigenvalues of the operator $P_\pm$. The quantities (2.10) can be represented as follows:

$$\left.\begin{aligned} \mathcal{M}_\gamma^{(\pm)} &= -\int_0^\infty \lambda^\gamma\, dN_\pm(\lambda) = \gamma\int_0^\infty \lambda^{\gamma-1}N_\pm(\lambda)d\lambda, \\ \mathcal{N}_\gamma^{(\pm)} &= \sup_{\lambda>0} N_\pm(\lambda)\lambda^\gamma, \end{aligned}\right\} \tag{2.19}$$

which reduces the problem to the study of the function $N_\pm(\lambda)$. To estimate it we use the following classical argument. For a function Y defined on $D(A_\pm)$ denote

$$K_\pm(\mu; Y) = Y(A_\pm + \mu)^{-1}Y^*. \tag{2.20}$$

Then

$$N_\pm(\lambda) \le n\big(1, K_\pm(\mu; Y)\big),\ Y = (V - \mu + \lambda)_-^{\frac{1}{2}},\ \forall \mu \in (0, \lambda], \tag{2.21}$$

for any $\lambda > 0$. This result is a version of *the Birman-Schwinger principle.*

(c) Diamagnetic inequality. Let $H_{\mathbf{a}}$ be the Schödinger operator with a magnetic field. Then for any $\lambda > 0, \varkappa \ge 0$ one has the following point-wise estimate:

$$|R(-\lambda, H_{\mathbf{a}})^\varkappa u|(x) \le R(-\lambda, H_{\mathbf{0}})^\varkappa |u(x)|,\ \forall u \in L^2(\mathbb{R}^2),\ \text{a.a. } x \in \mathbb{R}^2.$$

This inequality yields (see [2] and references therein)

Proposition 2.5. *Let X be multiplication by a measurable function and $\varkappa > 0$. Then for any $\lambda > 0$*

$$\|XR(-\lambda; H_{\mathbf{a}})^\varkappa\| \le \|XR(-\lambda; H_{\mathbf{0}})^\varkappa\|$$

and for any positive integer n

$$\big| XR(-\lambda; H_{\mathbf{a}})^\varkappa \big|_{2n} \le \big| XR(-\lambda; H_{\mathbf{0}})^\varkappa \big|_{2n}. \tag{2.22}$$

The first part of this proposition with $\varkappa = 1/2$ implies

Corollary 2.6. *Let X be as in Proposition 2.5. Then the inequality*

$$\|Xu\|^2 \le \epsilon H_{\mathbf{0}}[u, u] + M\|u\|^2,\ \forall u \in C_0^\infty(\mathbb{R}^2),$$

with some positive ϵ and M, implies that

$$\|Xu\|^2 \le \epsilon H_{\mathbf{a}}[u, u] + M\|u\|^2,\ \forall u \in C_0^\infty(\mathbb{R}^2).$$

An estimate for the r.h.s. of (2.22) can be proved with the help of a simple Cwickel estimate for the operators of the form $a(x)b(-i\partial)$ (see, e.g. [15]). The latter provides for any $X \in L^p(\mathbb{R}^2), p \ge 2$, and $\lambda > 0$ the bound

$$\big| XR(-\lambda; H_{\mathbf{0}})^\varkappa \big|_p \le C\lambda^{-\varkappa + \frac{1}{p}} \|X\|_{L^p},\ \forall \varkappa > p^{-1}. \tag{2.23}$$

(d) Multiplicative inequality. In the proof of Theorem 2.3 we shall use the following inequality (see [13] and also [5]):

Lemma 2.7. *Let $u \in C_0^\infty(\mathbb{R}^2)$. Then for any $q \in [2, \infty)$ one has*

$$\|u\|_{L^q} \le C_q \|u\|^\beta \|\partial u\|^{1-\beta},\ \beta = 2q^{-1}.$$

The constant C_q does not depend on u.

This Lemma is nothing but a convenient version of the embedding theorem in $\mathbb{R}^2$ with the "critical" exponent (see [5] for details).

3. Proof of Theorem 2.3

1. Partition of unity. The first step in the proof is to construct a partition of unity associated with the function $\ell(x)$ introduced in the beginning of Sect. 2.:

Lemma 3.1. *Let* $\ell(x)$ *be a continuous function satisfying (2.4). Then there exists a set* $x_k \in \mathbb{R}^2$, $k \in \mathbb{N}$ *such that the open disks* $\mathcal{D}_k = \mathcal{D}(x_k)$ *(see (2.6) for definition) form a covering of* $\mathbb{R}^2$ *with "the finite intersection property" (i.e. each disk intersects no more than* $N = N(\varrho) < \infty$ *other disks). Moreover, there exists a set of non-negative functions* $\phi_k \in C_0^\infty(\mathcal{D}_k)$, $k \in \mathbb{N}$, *such that*

$$\sum_k \phi_k(x)^2 = 1, \tag{3.1}$$

and

$$|\partial^m \phi_k(x)| \le C_m \ell(x)^{-|m|}, \ \forall m, \tag{3.2}$$

uniformly in k.

Note that more common definition of the partition of unity requires $\sum_k \phi_k = 1$ instead of (3.1). Nevertheless, for us the square will be convenient. Proof of this Lemma is analogous to that of Theorem 2.1.8 from [9] and we do not reproduce it here.

We single out an important consequence of the finite intersection property for disks $\mathcal{D}_k$. Denote

$$\mathfrak{M}_j = \{k \in \mathbb{N} : \mathcal{D}_j \cap \mathcal{D}_k \ne \emptyset\}. \tag{3.3}$$

and define by induction the sets

$$\mathfrak{M}_j^{(r+1)} = \{n \in \mathbb{N} : \mathcal{D}_n \cap \mathcal{D}_k \ne \emptyset, \ \forall k \in \mathfrak{M}_j^{(r)}\}, \ \ \mathfrak{M}_j^{(0)} = \mathfrak{M}_j, \ r = 0, 1, \dots. \tag{3.4}$$

Then

$$\operatorname{card} \mathfrak{M}_j^{(r)} \le \big(N(\varrho) + 1\big) N(\varrho)^r, \ r = 0, 1, \dots \tag{3.5}$$

with the number $N(\varrho)$ defined in Lemma 3.1.

2. Proof of Theorem 2.3. It suffices to verify that for any $\varepsilon < 1$ there exists a constant $C = C(V, B, \varepsilon)$ such that

$$\| |V|^{1/2} u\|^2 \le \varepsilon A_\pm[u,u] + C\|u\|^2, \ \forall u \in C_0^\infty(\mathbb{R}^2). \tag{3.6}$$

Step 1. We claim that for any $W \in L^p(\mathbb{R}^2)$, $p > 1$, and $u \in C_0^\infty(\mathbb{R}^2)$ one has the bound

$$\| |W|^{1/2} u\|^2 \le C \|W\|_{L^p} \mathcal{H}_\omega(u), \ \forall \omega > 0, \tag{3.7}$$

where

$$\mathcal{H}_\omega(u) = \omega H_{\mathbf{a}}[u,u] + \omega^{\frac{\beta-1}{\beta}} \|u\|^2, \ \beta = 1 - p^{-1}.$$

Indeed, by Hölder inequality and Lemma 2.7, for $q = 2p(p-1)^{-1} > 2$ and $\beta = 2q^{-1}$ we have

$$\| |W|^{\frac12} u\|^2 \le \|W\|_{L^p} \|u\|_{L^q}^2 \le C\|W\|_{L^p} \|u\|^{2\beta} \|\partial u\|^{2(1-\beta)}.$$

By the Young inequality, the r.h.s. does not exceed

$$C\|W\|_{L^p}\left[\omega\|\partial u\|^2+\omega^{\frac{\beta-1}{\beta}}\|u\|^2\right],\ \forall\omega>0.$$

In view of Corollary 2.6 this leads to the estimate (3.7).

Step 2. Let $\mathcal{D}_j$ be the disks forming the covering of $\mathbb{R}^2$ described above and let ϕ_j be the set of functions constructed in Lemma 3.1. Set $b_j=b(x_j),\ \ell_j=\ell(x_j)$. Denote by χ_j the characteristic function of $\mathcal{D}_j$. By (3.1) the l.h.s. of (3.6) equals

$$\sum_j\langle\chi_j|V|u_j,u_j\rangle, u_j=\phi_j u.$$

Applying (3.7) with $W=\chi_j V$ and $u=u_j$, we see that

$$\||V|^{1/2}u_j\|^2\le C\|\chi_j V\|_{L^p}\mathcal{H}_\omega(u_j),\ \forall\omega>0.$$

Adding and subtracting the term $\mp\omega\langle Bu_j,u_j\rangle$, one obtains

$$\mathcal{H}_\omega(u_j)=\omega A_\pm[u_j,u_j]+\omega^{\frac{\beta-1}{\beta}}\|u_j\|^2\pm\omega\langle Bu_j,u_j\rangle.$$

Estimate the first summand. Due to (2.4) and (3.2), one has

$$A_\pm[u_j,u_j]\le 2\|\phi_j Q_\pm u\|^2+2\|[Q_\pm,\phi_j]u\|^2\le 2\|\phi_j Q_\pm u\|^2+C\ell_j^{-2}\|\chi_j u\|^2.$$

Moreover, $\pm\langle Bu_j,u_j\rangle\le\langle B_\pm u_j,u_j\rangle$. Therefore, in view of (2.9) and (2.7),

$$\mathcal{H}_\omega(u_j)\le 2\omega\|\phi_j Q_\pm u\|^2+C\left[\omega^{\frac{\beta-1}{\beta}}\|u_j\|^2+\omega b_j\|\chi_j u\|^2\right]. \tag{3.8}$$

The last term is bounded by

$$C\omega^{\frac{\beta-1}{\beta}}(1+\omega^{\frac{1}{\beta}}b_j)\|\chi_j u\|^2.$$

Let us pick

$$\omega=\delta(1+b_j)^{-\beta},\ \delta\le 1,$$

and plug it in (3.8), taking into account that $1-\beta=1/p$:

$$\begin{aligned}\||V|^{\frac12}u_j\|^2&\le C\|\chi_j V\|_{L^p}\left[2\delta(1+b_j)^{-\beta}\|\phi_j Q_\pm u\|^2+\delta^{\frac{\beta-1}{\beta}}(1+b_j)^{1-\beta}\|\chi_j u\|^2\right]\\&\le C\|\chi_j V\|_{L^p}(1+b_j)^{\frac1p}\left[2\delta\|\phi_j Q_\pm u\|^2+\delta^{\frac{\beta-1}{\beta}}\|\chi_j u\|^2\right].\end{aligned}$$

With the notation

$$L=\text{v-sup}_{x\in\mathbb{R}^2}\left[\int_{\mathcal{D}(x)}|V(y)|^p(1+b(y))\,dy\right]^{\frac1p}$$

this estimate transforms into the bound

$$\||V|^{\frac12}u_j\|^2\le C\delta L\|\phi_j Q_\pm u\|^2+C(\delta)L\|\chi_j u\|^2.$$

Let us sum up the contributions from different disks $\mathcal{D}_j$:

$$\||V|^{\frac12}u\|^2\le C\delta L A_\pm[u,u]+CL\sum_j\|\chi_j u\|^2.$$

Since the number of mutual intersections of $\mathcal{D}_j$ is bounded, the last term does not exceed $C'L\|u\|^2$. Letting $\delta=(CL)^{-1}\varepsilon$, we arrive at (3.6). □

4. Proof of Theorem 2.4

1. Properties of the operator $A_\pm$. In the sequel we shall retrieve spectral properties of the operators $A_\pm$ by comparing it with the operator $H = H_{\mathbf{a}} + W(x), W \in L^\infty_{loc}(\mathbb{R}^2), W \geq 0$. In view of (2.2)

$$H = A_\pm \pm B + W.$$

Our basic tool will be the resolvent identity relating $R(z, H)$ and $R(z, A_\pm)$. Let $\phi \in C_0^\infty(\mathbb{R}^2)$. Then

$$\left.\begin{aligned} \phi R(z, A_\pm) &= R(z, H)\phi + R(z, H) X_\pm(\phi, W \pm B) R(z, A_\pm), \\ X_\pm(\phi, f) &= Q_\mp \phi_\pm + \phi_\mp Q_\pm + f\phi. \end{aligned}\right\} \tag{4.1}$$

To derive (4.1) we use Lemma 2.2.

In the proof of Theorem 2.4 we use (4.1) with the functions ϕ_j and different operators $H_{\pm,j} = H_{\mathbf{a}} + B_\mp + b_j,\ b_j = b(x_j)$. Since $H_{\pm,j} - A_\pm = B_\pm + b_j$, one has

$$\left.\begin{aligned} \phi_j^2 R(z, A_\pm) &= \phi_j R(z, H_{\pm,j})\phi_j + \phi_j R(z, H_{\pm,j}) X_{\pm,j} R(z, A_\pm), \\ X_{\pm,j} &= X_\pm(\phi_j, B_\pm + b_j). \end{aligned}\right\} \tag{4.2}$$

Note that $H_{\pm,j} \geq H_{\mathbf{a}} + b_j$, which implies that

$$R(-\lambda, H_{\pm,j})^\varkappa \leq R\big(-(\lambda + b_j), H_{\mathbf{a}}\big)^\varkappa,\ \forall \lambda > 0,\ \varkappa \in [0, 1]. \tag{4.3}$$

In order to control the second term in (4.2) we need the following elementary properties of the resolvent $R(z, A_\pm)$:

Lemma 4.1. *Let* $\mathbf{a} \in L^2_{loc}(\mathbb{R}^2)$ *and* $k = 0, 1$. *Then*

$$\|Q_\pm^k R(\lambda, A_\pm)^{\frac{1}{2}}\| \leq \lambda^{\frac{k-1}{2}},\ \forall \lambda > 0. \tag{4.4}$$

Suppose that Assumption 2.1 is fulfilled. Let $\psi \in C_0^\infty(\mathbb{R}^2)$ *and*

$$\|f\|_\psi = \text{v-sup}_{x \in \operatorname{supp} \psi} |f(x)|,\ \ f \in L^\infty_{loc}(\mathbb{R}^2).$$

Then

$$\|Q_\mp \psi R(\lambda, A_\pm)^{\frac{1}{2}}\|^2 \leq 2\Big[\lambda^{-1}\big(\|\psi\|^2_{L^\infty} \|B_\pm\|_\psi + \|\psi_\pm\|^2_{L^\infty}\big) + \|\psi\|^2_{L^\infty}\Big]. \tag{4.5}$$

Proof. For $k = 0$ (4.4) is obvious. For $k = 1$ the bound (4.4) follows from the inequality

$$\|Q_\pm u\|^2 \leq \|(A_\pm + \lambda)^{\frac{1}{2}} u\|^2,\quad \forall u \in C_0^\infty(\mathbb{R}^2). \tag{4.6}$$

Let us prove (4.5). For any $u \in C_0^\infty(\mathbb{R}^2)$ the relations (2.3) and (2.4) ensure that

$$\begin{aligned} \|Q_\mp \psi u\|^2 &= \langle A_\mp \psi u, \psi u\rangle = \langle A_\pm \psi u, \psi u\rangle \pm 2\langle B\psi u, \psi u\rangle \\ &\leq \|Q_\pm \psi u\|^2 + 2\|\psi\|^2_{L^\infty} \|B_\pm\|_\psi \|u\|^2 \\ &\leq 2\|\psi Q_\pm\|^2 + 2\|[Q_\pm, \psi]u\|^2 + 2\|\psi\|^2_{L^\infty} \|B_\pm\|_\psi \|u\|^2 \\ &\leq 2\|\psi\|^2_{L^\infty} \|Q_\pm u\|^2 + 2\big[\|\chi_\pm\|^2_{L^\infty} + \|\psi\|^2_{L^\infty} \|B_\pm\|_\psi\big] \|u\|^2. \end{aligned}$$

It remains to use (4.6). $\square$

If B obeys the conditions of Theorem 2.4, then Lemma 4.1 leads to

Lemma 4.2. *Let the vector-potential* $\mathbf{a}$ *and the field* B *be as in Theorem 2.4. Then for any* $\lambda > 0$ *and* $j \in \mathbb{N}$ *one has*

$$\|X_{\pm,j} R(-\lambda, A_\pm)^{\frac{1}{2}}\| \le C\big(b_j^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} b_j\big), b_j = b(x_j).$$

where the constant does not depend on j *or the functions* B, b.

Proof. By (2.7) and definition (4.1)

$$\begin{aligned}\|X_{\pm,j} R(-\lambda, A_\pm)^{\frac{1}{2}}\| \le \|Q_\mp(\phi_j)_\pm R(-\lambda, A_\pm)^{\frac{1}{2}}\| \\ + \|(\phi_j)_\mp Q_\pm R(-\lambda, A_\pm)^{\frac{1}{2}}\| + C b_j \|R(-\lambda, A_\pm)^{\frac{1}{2}}\|.\end{aligned}$$

Denote $\ell_j = \ell(x_j)$. In view of Lemma 4.1 and (3.2) the r.h.s. does not exceed

$$\begin{aligned}&C\left[\lambda^{-\frac{1}{2}}\big(\ell_j^{-2} + b_j^{\frac{1}{2}}\ell_j^{-1}\big) + \ell_j^{-1}\right] + C'\ell_j^{-1} + C b_j \lambda^{-\frac{1}{2}} \\ &\le C\ell_j^{-1} + C'\lambda^{-\frac{1}{2}}\ell_j^{-1}\big(\ell_j^{-1} + b_j^{\frac{1}{2}}\big) + C b_j \lambda^{-\frac{1}{2}} \le C\big(b_j^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} b_j\big).\end{aligned}$$

To obtain the last inequality we used (2.9). □

2. Spectral estimates for the operator $K_\pm(\lambda, Y)$. Here we obtain an estimate for the counting function $n(s, K_\pm)$ where $K_\pm = K_\pm(\lambda, Y)$ is the operator defined in (2.20). For the sake of brevity below we sometimes omit " ± " from the notation and write K, A, Q instead of $K_\pm, A_\pm, Q_\pm$.

For some functions Y, b and an open set $\Omega \subset \mathbb{R}^2$ define the integrals

$$I_1(Y; \Omega) = \int_\Omega |Y(x)|^4 dx, \quad I_2(Y, b; \Omega) = \int_\Omega |Y(x)|^2 b(x) dx. \tag{4.7}$$

Our aim is

Theorem 4.3. *Let Assumption 2.1 be fulfilled and the field* B *obey the conditions (2.7)–(2.9). Let* Y *be a function such that* $Y \in D[A]$. *If* $I_1(Y; \mathbb{R}^2) < \infty$, $I_2(Y, b; \mathbb{R}^2) < \infty$, *then the operator* $K(\lambda, Y)$ *is compact for any* $\lambda > 0$ *and*

$$n\big(s, K(\lambda, Y)\big) \le C(s)\lambda^{-1}\big[I_1(Y; \mathbb{R}^2) + I_2(Y, b; \mathbb{R}^2)\big], \ \forall s > 0. \tag{4.8}$$

To study properties of the operator $K(\lambda; Y)$, we rewrite it as follows:

$$\begin{aligned}K(\lambda; Y) &= Y(A+\lambda)^{-1}Y^* = Y(A+\lambda)^{-1}(A+\lambda)(A+\lambda)^{-1}Y^* \\ &= Y(A+\lambda)^{-1}Q^*Q(A+\lambda)^{-1}Y^* + Y(A+\lambda)^{-1}\lambda(A+\lambda)^{-1}Y^*.\end{aligned}$$

Let S denote either the operator Q or the multiplication operator $\sqrt{\lambda}$. Define

$$T = T(\lambda; Y, S) = Y(A+\lambda)^{-1}S^*.$$

It is clear that

$$K(\lambda; Y) = T(\lambda; Y, Q)T(\lambda; Y, Q)^* + T(\lambda; Y, \sqrt{\lambda})T(\lambda; Y, \sqrt{\lambda})^*,$$

which ensures, in view of (2.17) and (2.16), that

$$n_+\big(2s^2, K(\lambda; Y)\big) \le n\big(s, T(\lambda; Y, Q)\big) + n\big(s, T(\lambda; Y, \sqrt{\lambda})\big). \tag{4.9}$$

Thus it is sufficient obtain (4.8) for the operator T. Using the partition of unity ϕ_j constructed in Lemma 3.1, one can represent T as

$$T = \sum_j F_j, \quad F_j = F_j(\lambda; Y, S) = \phi_j^2 T(\lambda; Y, S).$$

According to (4.2), each F_j breaks up into three operators:

$$\begin{aligned} F_j &= F_j^{(1)} + F_j^{(2)} + F_j^{(3)}, \\ F_j^{(1)} &= \phi_j Y R(-\lambda, H_j) S^* \phi_j, \\ F_j^{(2)} &= -\phi_j Y R(-\lambda, H_j)[S^*, \phi_j], \\ F_j^{(3)} &= \phi_j Y R(-\lambda, H_j) X_j R(-\lambda, A) S^*. \end{aligned} \tag{4.10}$$

In the next lemma the constants in all the bounds do not depend on j. They may depend only on the constants in (2.8), (2.9),(3.2) and the parameter ϱ from (2.5). As before, we use the notation $b_j = b(x_j)$, $\ell_j = \ell(x_j)$.

Lemma 4.4. *Let the magnetic potential* $\mathbf{a}$ *and the field* B *be as in Theorem 2.4 and* $Y \in L^4_{loc}(\mathbb{R}^2)$. *Let* I_1, I_2 *be defined in (4.7). Then*

$$\| F_j^{(1)} \|_4^4 + \| F_j^{(2)} \|_4^4 \le C\lambda^{-1} I_1(Y; \mathcal{D}_j), \tag{4.11}$$

$$\| F_j^{(3)} \|_2^2 \le C\lambda^{-1} I_2(Y, b; \mathcal{D}_j). \tag{4.12}$$

Proof. Observe first of all that the inequalities (4.3), (2.22) and (2.23) lead to the bounds

$$\begin{aligned} \| \phi_j Y R(-\lambda, H_j)^{\varkappa} \|_4 &\le \| \phi_j Y \ R\big(-(\lambda + b_j), H_{\mathbf{0}}\big)^{\varkappa} \|_4 \\ &\le C(\lambda + b_j)^{-\varkappa + \frac{1}{4}} \big[I_1(Y; \mathcal{D}_j)\big]^{\frac{1}{4}}, \ \forall \varkappa > 1/4; \end{aligned} \tag{4.13}$$

$$\begin{aligned} \| \phi_j Y R(-\lambda, H_j)^{\varkappa} \|_2 &\le \| \phi_j Y \ R\big(-(\lambda + b_j), H_{\mathbf{0}}\big)^{\varkappa} \|_2 \\ &\le C(\lambda + b_j)^{-\varkappa + \frac{1}{2}} \big[I_2(Y, 1; \mathcal{D}_j)\big]^{\frac{1}{2}}, \ \forall \varkappa > 1/2. \end{aligned} \tag{4.14}$$

Consequently,

$$\| F_j^{(1)} \|_4 \le \| \phi_j Y R(-\lambda, H_j)^{\frac{1}{2}} \|_4 \| R(-\lambda, H_j)^{\frac{1}{2}} S^* \| \le C\lambda^{-\frac{1}{4}} \big[I_1(Y; \mathcal{D}_j)\big]^{\frac{1}{4}}.$$

If $S=\sqrt{\lambda}$, then $F_j^{(2)}=0$. If $S=Q$, then in view of (2.4) and (3.2),

$$\|[S^*,\phi_j]\| \le C\ell_j^{-1},$$

and hence (4.13) yields that

$$\| F_j^{(2)} \|_4 \le C \| \phi_j Y R(-\lambda, H_j) \|_4 \ell_j^{-1} \le C(\lambda+b_j)^{-\frac{3}{4}} \ell_j^{-1} \big[I_1(Y;\mathcal{D}_j)\big]^{\frac{1}{4}}.$$

By (2.9) $b_j^{-1/2}\ell_j^{-1} \le C$. This proves (4.11) for $F_j^{(1)}$ and $F_j^{(2)}$.

Proof of (4.12). By Lemma 4.2

$$\|X_j R(-\lambda,A)S^*\| \le \|X_j R(-\lambda,A)^{\frac{1}{2}}\| \|R(-\lambda,A)^{\frac{1}{2}}S^*\| \le C\big(b_j^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} b_j\big).$$

Consequently, using (4.14) we get

$$\begin{aligned} \| F_j^{(3)} \|_2 &\le C\big(b_j^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} b_j\big) \| \phi_j Y R(-\lambda, H_j) \|_2 \\ &\le C\big(b_j^{\frac{1}{2}} + \lambda^{-\frac{1}{2}} b_j\big)(\lambda+b_j)^{-\frac{1}{2}} \big[I_2(Y,1;\mathcal{D}_j)\big]^{\frac{1}{2}} \le C\lambda^{-\frac{1}{2}} b_j^{\frac{1}{2}} \big[I_2(Y,1;\mathcal{D}_j)\big]^{\frac{1}{2}}, \end{aligned}$$

which provides (4.12) by virtue of (2.8). $\square$

In accordance with (4.10) the operator T can be presented as the sum

$$T = T^{(1)} + T^{(2)} + T^{(3)}, \quad T^{(l)} = \sum_j F_j^{(l)}, \ l=1,2,3.$$

Next lemma puts together contributions from different F_j's.

Lemma 4.5. *Under the conditions of Theorem 4.3 one has* $T^{(1)}, T^{(2)} \in \mathfrak{S}_4$, $T^{(3)} \in \mathfrak{S}_2$ *and*

$$\| T^{(1)} \|_4^4 + \| T^{(2)} \|_4^4 \le C\lambda^{-1} I_1(Y;\mathbb{R}^2), \tag{4.15}$$

$$\| T^{(3)} \|_2^2 \le C\lambda^{-1} I_2(Y,b;\mathbb{R}^2). \tag{4.16}$$

Proof. We prove (4.15) first. Let $\mathfrak{M}_j, \mathfrak{M}_j^{(r)}$ be the sets defined in (3.3), (3.4). For

$$F_k^{(l)}\big(F_j^{(l)}\big)^* = \big(F_j^{(l)}\big)^* F_k^{(l)} = 0, \ k \notin \mathfrak{M}_j, l=1,2,$$

one has

$$\begin{aligned} \| T^{(l)} \|_4^4 &= \| (T^{(l)})^* T^{(l)} (T^{(l)})^* T^{(l)} \|_1 \\ &\le \sum_j \sum_{k\in\mathfrak{M}_j} \sum_{m\in\mathfrak{M}_k} \sum_{n\in\mathfrak{M}_m} \| F_j^{(l)} \|_4 \| F_k^{(l)} \|_4 \| F_m^{(l)} \|_4 \| F_n^{(l)} \|_4 \le \sum_j \Big[\sum_{k\in\mathfrak{M}_j^{(2)}} \| F_k^{(l)} \|_4 \Big]^4. \end{aligned}$$

For the number $\operatorname{card}\mathfrak{M}_j^{(2)}$ is bounded uniformly in j (see (3.5)), one can estimate

$$\Big[\sum_{k\in\mathfrak{M}_j^{(2)}} \| F_k^{(l)} \|_4\Big]^4 \le C \sum_{k\in\mathfrak{M}_j^{(2)}} \| F_k^{(l)} \|_4^4.$$

Furthermore, by (3.5)

$$\| T^{(l)} \|_4^4 \le C \sum_j \operatorname{card}\mathfrak{M}_j^{(2)} \| F_j^{(l)} \|_4^4 \le C' \sum_j \| F_j^{(l)} \|_4^4.$$

In combination with (4.11) this leads to

$$\| T_1^{(l)} \|_4^4 \le C\lambda^{-1} \sum_j I_1(Y; \mathcal{D}_j).$$

In view of (3.5) with $r = 0$ this implies (4.15).

Exploiting the same argument, one proves that

$$\| T^{(3)} \|_2^2 = \| (T^{(3)})^* T^{(3)} \|_1 \le \sum_j \| F_j^{(3)} \|_2 \sum_{k\in\mathfrak{M}_j} \| F_k^{(3)} \|_2 \le C\lambda^{-1} \sum_j I_2(Y, b; \mathcal{D}_j).$$

This provides (4.16) due to (3.5). $\square$

Proof of Theorem 4.3. According to (2.17), (2.18) and (4.15),

$$n(s, T^{(1)} + T^{(2)}) \le Cs^{-4}\lambda^{-1} I_1(Y; \mathbb{R}^2),$$
$$n(s, T^{(3)}) \le Cs^{-2}\lambda^{-1} I_2(Y, b; \mathbb{R}^2).$$

Now (4.8) follows from (4.9). $\square$

3. Proof of Theorem 2.4. According to the Birman-Schwinger principle (2.21) with $\mu = \lambda/2$,

$$N_\pm(\lambda) \le n(1, K_\pm), \ \ K_\pm = K_\pm\big(\lambda/2, (V + \lambda/2)_-^{1/2}\big).$$

Consequently, plugging the bound (4.8) into (2.19), one obtains that

$$\mathcal{M}_\gamma^{(\pm)} \le C \int_0^\infty \lambda^{\gamma-2} \int \Big[(V(x) + \lambda)_-^2 + (V(x) + \lambda)_- b(x) \Big] dx d\lambda$$
$$= C \int V_-(x)^{\gamma+1} dx \int_0^1 t^{\gamma-2}(1-t)^2 dt + C \int V_-(x)^\gamma b(x) dx \int_0^1 t^{\gamma-2}(1-t) dt.$$

For $\gamma > 1$ the r.h.s. is finite and therefore (2.11) follows.

To prove (2.12) we use (2.21) with $\mu = 0$, which provides the desired result due to (4.8) and (2.19).

Appendix. On the operators $Q_\pm$

Let $Q_\pm$ be the operators defined in (1.3). As was mentioned in Sect. 2, $Q_\pm$ are closable on $C_0^\infty(\mathbb{R}^2)$, since $Q_\pm \subset Q_\mp^*$. In this appendix we find conditions on the magnetic field, which guarantee $Q_\pm^* = Q_\mp$.

Suppose that there exists a function $\mathcal{B} \in C^1(\mathbb{R}^3)$ such that

$$|B(x)| \le \mathcal{B}(x), \tag{A1}$$

$$\mathcal{B}(x) \ge 1, \ |\nabla \mathcal{B}(x)| \le C\mathcal{B}(x), \ \forall x \in \mathbb{R}^3. \tag{A2}$$

Note that unlike (2.7), the inequality (A1) is required for both positive and negative parts of the magnetic field.

Theorem A1. *Let* $\mathbf{a} \in C^1(\mathbb{R}^2)$ *and the field* B *obeys the conditions (A1), (A2). Then* $Q_\pm^* = Q_\mp$.

To prove Theorem A1 it suffices to establish essential self-adjointness of the symmetric operator

$$\tau = \begin{pmatrix} 0 & Q_- \\ Q_+ & 0 \end{pmatrix} \tag{A3}$$

on $\mathfrak{D} = C_0^\infty(\mathbb{R}^2) \oplus C_0^\infty(\mathbb{R}^2)$. Observe that $\bar{\tau}^*\bar{\tau} = H$, where H is the Pauli operator (1.1):

$$H = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}.$$

It is clear that on $\mathfrak{D}$ one has $H = \tau^2$.

Theorem A1 will result from the following abstract commutator lemma (see [3], [16]):

Lemma A2. *Let* M *be an essentially self-adjoint positive-definite operator. Suppose that a symmetric operator* S *satisfies the conditions* $D(S) = D(M)$ *and*

$$\|Sf\| \le C\|Mf\|, \ \forall f \in D(M), \tag{A4}$$

$$|(Sf, Mf) - (Mf, Sf)| \le C'(Mf, f), \ \forall f \in D(M). \tag{A5}$$

Then S *is essentially self-adjoint.*

Proof. We shall use Lemma A2 with

$$M = H + \mathcal{B}\mathbb{I}, \ \ S = \tau, \ D(S) = D(M) = \mathfrak{D}.$$

Since $\mathcal{B} \ge 1$ and $H \ge 0$, the operator M is positive-definite. In view of (2.2) and (A1),

$$M = H_{\mathbf{a}}\mathbb{I} + X, \ \ X = \begin{pmatrix} \mathcal{B} - B & 0 \\ 0 & \mathcal{B} + B \end{pmatrix} \ge 0.$$

Therefore M is essentially self-adjoint on $\mathfrak{D}$ (see [14]). Since $\mathcal{B} \ge 1$, for any $f \in \mathfrak{D}$ one has

$$\|\tau f\|^2 = \langle \tau^2 f, f\rangle \le \|Mf\|\|\mathcal{B}f\| \le \|Mf\|^2,$$

which guarantees (A4). Now,

$$L[f] = \langle \tau f, (\tau^2 + \mathcal{B}\mathbb{I})f\rangle - \langle(\tau^2 + \mathcal{B}\mathbb{I})f, \tau f\rangle = -\langle[\tau, \mathcal{B}\mathbb{I}]f, f\rangle.$$

It follows from the definition of τ that

$$[\tau, \mathcal{B}\mathbb{I}] = -i \begin{pmatrix} 0 & \partial_1\mathcal{B} - i\partial_2\mathcal{B} \\ \partial_1\mathcal{B} + i\partial_2\mathcal{B} & 0 \end{pmatrix},$$

so that

$$-|\nabla\mathcal{B}|\mathbb{I} \le i[\tau, \mathcal{B}\mathbb{I}] \le |\nabla\mathcal{B}|\mathbb{I}.$$

Therefore, by (A2)

$$|L[f]| \le \langle|\nabla\mathcal{B}|\mathbb{I}f, f\rangle \le C\langle\mathcal{B}\mathbb{I}f, f\rangle \le C\langle Mf, f\rangle.$$

Consequently, (A5) is fulfilled. Now Lemma A2 yields essential self-adjointness of the operator τ on $\mathfrak{D}$, which in its turn provides the equality $Q_\pm^* = Q_\mp$. $\square$

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