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Double Cosets for SU(2)×···×SU(2) and Outer Automorphisms of Free Groups

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Double cosets for $\mathrm{SU}(2) \times \cdots \times \mathrm{SU}(2)$ and outer automorphisms of free groups

Yuri A. Neretin[1]

Consider the space of double cosets of the product of n copies of SU(2) with respect to the diagonal subgroup. We get a parametrization of this space, the radial part of the Haar measure, and explicit formulas for the actions of the group of outer automorphisms of the free group F_{n-1} and of the braid group of $n-1$ strings.

1 Introduction

1.1. The group SU(2)**.** Denote by SU(2) the group of unitary 2×2-matrices with determinant $= 2$. A matrix $g \in \mathrm{SU}(2)$ has the form

$$g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad \text{where } |a|^2 + |b|^2 = 1.$$

1.2. Double cosets. Denote by SU(2) the group of unitary 2×2 matrices with determinant $= 1$. Denote by $G(n)$ the product of n copies of SU(2). Elements of $G(n)$ are n-tuples

$$(g_1, g_2, \dots, g_n), \qquad \text{where } g_j \in \mathrm{SU}(2). \tag{1.1}$$

Denote by $K = K(n) \simeq \mathrm{SU}(2)$ the diagonal subgroup in $G(n)$; elements of K have the form

$$(h, \dots, h), \qquad \text{where } h \in \mathrm{SU}(2).$$

The object of the present paper is the space of double cosets

$$\Pi(n) := K \setminus G/K.$$

In other words, we consider n-tuples (1.1) up to the equivalence

$$(g_1, g_2, \dots, g_n) \sim (hg_1q, \dots, hg_nq), \qquad \text{where } h,\, q \in \mathrm{SU}(2).$$

1.3. Conjugacy classes.

Observation 1.1 *There is a canonical one-to-one correspondence between $\Pi(n)$ and the space of conjugacy classes of $G(n-1)$ with respect to the subgroup $K(n-1)$.*

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Indeed,

$$(g_1, g_2, \dots, g_n) \sim (1, g_1^{-1} g_2, \dots, g_1^{-1} g_n)$$

Next,

$$(1, r_1, \dots, r_{n-1}) \sim (1, h r_1 h^{-1}, \dots, h r_{n-1} h^{-1}).$$

1.4. Closed polygonal curves on the sphere. Consider the 3-dimensional sphere S^3 endowed with the usual angular distance $d(\cdot, \cdot)$. Fix positive numbers $\theta_1, \dots, \theta_{n-1}$. Consider a closed polygonal curve $A_1 A_2 \dots A_{n-1} A_1$ in S^3 such that $d(A_j, A_{j+1}) = \theta_j$, $d(A_{n-1}, A_1) = \theta_{n-1}$. Denote by $\mathcal{X}(\theta)$ the set of all such spaces defined up to proper rotations of the sphere.

Observation 1.2 *There is one-to-one correspondence between $\mathcal{X}(\theta)$ and the set of $(n-1)$-tuples.*

$$(1, r_1, \dots, r_{n-1}), \qquad \text{where } r_j \in \mathrm{SU}(2) \tag{1.2}$$

defined up to a simultaneous conjugation and satisfying the conditions:

— *the eigenvalues of r_k are $e^{\pm i\theta_k}$;*

— $r_1 r_2 \dots r_{k-1} = 1$.

Indeed, SU(2) can be considered as a 3-dimensional sphere. To a tuple (1.2), we assign the polygonal curve

$$1,\ r_1,\ r_1 r_2,\ \dots,\ r_1 r_2 \dots r_{n-1} = 1.$$

The space $\mathcal{X}(\theta)$ (and its analog for $\mathbb{R}^3$ and the Lobachevsky 3-space) became a subject of investigations after Klyachko's work [5]. Relations of the present work with this literature is not quite clear for the author.

1.5. Spectral forms. For a point of $\Pi(n)$, we write the *spectral form*

$$Q(\lambda) := \det\Bigl(\sum_j \lambda_j g_j\Bigr) =: \sum_{i,j} s_{ij} \lambda_i \lambda_j.$$

We describe the set $\Xi(n)$ of possible spectral forms. Namely, they satisfy the conditions:

— $Q(\lambda) \geqslant 0$;

— $\operatorname{rk} Q \leqslant 4$;

— $s_{jj} = 1$.

If $\operatorname{rk} Q = 4$, its preimage $\in \Pi(n)$ is a two-point set; we have a branching along the surface $\operatorname{rk} Q = 3$.

Note, that points of the surface $\operatorname{rk} Q = 3$ corresponds to smooth points of the quotient space $\Pi(n) = K \setminus G/K$; the singular locus of $\Pi(n)$ corresponds to the surface $\operatorname{rk} Q \leqslant 2$.

1.6. Radial part of Haar measure. The group $G(n)$ is endowed with the Haar measure. We consider its pushforward on the space $\Xi(n)$. For $n = 3$ we get the usual Lebesgue measure $ds = ds_{12}\, ds_{13}\, ds_{23}$ on $\Xi(3)$, see [7]. For $n = 4$ the measure is given by

$$\det(Q)^{-1/2}\, ds,$$

where ds is the Lebesgue measure. For $n \geqslant 5$ the description of the measure is given in Theorem 3.5.

1.7. The group $\mathrm{Out}(F_k)$, (see [2], [1]). Consider the free group F_k with k generators $c_1, \dots, c_k$. Denote by $\mathrm{Aut}(F_k)$ the group of automorphisms of F_k. Each automorphism $\varkappa$ is determined by images of the generators:

$$c_j \mapsto \varkappa(c_j) = c_{j_1}^{\varepsilon_{j_1}} c_{j_2}^{\varepsilon_{j_1}} \dots, \tag{1.3}$$

where $\varepsilon_j = \pm 1$. Certainly, the collections $\{\varkappa(c_j)\}$ are not arbitrary (generally, formula of the type (1.3) determines a non-surjective and non-injective map $F_k \to F_k$).

By the Nielsen theorem (see [6]), the group $\mathrm{Aut}(F_k)$ is generated by the following transformations of the set of generators:

a) permutations of generators;

b) the map

$$c_1 \mapsto c_1^{-1}, \quad c_2 \mapsto c_2, \quad c_3 \mapsto c_3, \dots;$$

c) the map

$$c_1 \mapsto c_1, \quad c_2 \mapsto c_1 c_2, \quad c_3 \mapsto c_3, \quad c_4 \mapsto c_4, \dots$$

The group F_k acts on itself by interior automorphisms, it is a normal subgroup in $\mathrm{Aut}(F_k)$. We denote by

$$\mathrm{Out}(F_k) := \mathrm{Aut}(F_k)/F_k$$

the *group of outer automorphisms of the free group.*

1.8. The action of $\mathrm{Out}(F_{n-1})$ on $\Pi(n)$. For a transformation (1.3) we write the following transformation of $\widetilde{\Pi}_n$:

$$\widetilde{r}_j = r_{j_1}^{\varepsilon_{r_1}} c_{r_2}^{\varepsilon_{j_1}} \dots$$

In Section 4 we obtain explicit formulas for the Nielsen generators.

1.9. Braid group. Denote by Br_k the Artin *braid group.* It has generators $\sigma_1, \dots \sigma_{k-1}$ and relations

$$\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}$$
$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if } |i-j| > 1.$$

There is the following Artin embedding $\mathrm{Br}_k \mapsto \mathrm{Aut}(F_k)$ (see [4]). The element h_j corresponds to the transformation

$$c_j \mapsto c_{j+1}, \quad c_{j+1} \mapsto c_{j+1}^{-1} c_j c_{j+1},$$

other generators are fixed.

There is a characterization of the image of Br_k in $\mathrm{Aut}(F_k)$. Namely, $\varkappa \in \mathrm{Out}(F_k)$ is contained in Br_k if

1) $\varkappa$ sends each generator c_j to

$$A_j^{-1} c_{\xi(j)} A_j$$

where $A_j \in F_k$ and ξ is a permutation of generators.

2) $\varkappa$ sends $c_1 \dots c_k$ to itself.

In particular, we get the map $\mathrm{Br}_k \to \mathrm{Out}(F_k)$. It is not injective (see, e.g., [3]), the kernel is generated by

$$\Bigl((\sigma_1\sigma_2\dots\sigma_{k-1})(\sigma_1\sigma_2\dots\sigma_{k-2})\dots\sigma_1\Bigr)^2.$$

In Section 4 we get explicit formulae for the action of generators of the braid group in the terms of spectral forms.

1.10. The structure of the paper. In Section 2, we get the characterization of spectral forms. Section 3 contains evaluation of the radial part of the Haar measure. In Section 4, we write out actions of the the groups $\mathrm{Out}(F_{n-1})$ and Br_{n-1}.

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2 Spectral forms

2.1. Spectral forms and the map ζ. For any element of $\Pi(n)$ we write out the quadratic form

$$Q(\lambda_1,\dots,\lambda_n) = \det\bigl(\sum \lambda_j g_j\bigr) = \det\left(\sum \lambda_j \begin{pmatrix} a_j & b_j \\ -\overline{b}_j & \overline{a}_j \end{pmatrix}\right)$$

We denote by ζ the map from $\Pi(n)$ to the space of quadratic forms.

Proposition 2.1 *a) We get a well-defined map from the space $\Pi(n)$ to the space of quadratic forms.*

b) Coefficients of Q are real, coefficients in the front of λ_j^2 are 1.

c) Q is positive semidefinite.

d) The rank of Q is $\leqslant 4$.

Proof. Indeed, for real λ,

$$Q(\lambda) = \Bigl(\sum \lambda_j a_j\Bigr)\Bigl(\sum \lambda_j \overline{a}_j\Bigr) + \Bigl(\sum \lambda_j b_j\Bigr)\Bigl(\sum \lambda_j \overline{b}_j\Bigr) =$$
$$= \Bigl|\sum \lambda_j a_j\Bigr|^2 + \Bigl|\sum \lambda_j b_j\Bigr|^2 = \sum \lambda_j^2 + 2\sum_{i<j} \mathrm{Re}(a_j\overline{a}_j + b_i\overline{b}_j)$$

and all the statements become obvious. □

2.2. A description of $\Pi(n)$**.** Denote by $\Xi = \Xi(n)$ the set of all quadratic forms Q satisfying the conditions of the previous statement.

Obviously,

$$\zeta(g_1^t, \dots, g_n^t) = \zeta(g_1, \dots, g_n),$$

where ^{t} denotes the transposed matrix.

Theorem 2.2 *a) The map* $\zeta : \Pi(n) \to \Xi(n)$ *is surjective.*

b) The ζ*-preimage of a point* $Q \in \Xi$ *consists of two points if* $\operatorname{rk} Q = 4$ *and of one point if* $\operatorname{rk} Q \leqslant 3$*.*

c) Moreover, $\operatorname{rk} Q \leqslant 3$ *iff* $(g_1^t, \dots, g_n^t)$ *and* $(g_1, \dots, g_n)$ *represent one point of* Π*.*

Proof. For a positive semi-definite quadratic form

$$Q(\lambda) = \sum_{kl} s_{kl} \lambda_k \lambda_l$$

on $\mathbb{R}^n$ there is a collection (configuration) of vectors v_j in a Euclidean space such that

$$\langle v_k, v_j \rangle = s_{ij}$$

Since $\operatorname{rk} Q \leqslant 4$, this configuration can be realized in $\mathbb{R}^4$. Since $s_{jj} = 1$, these vectors lie on the unit sphere.

Moreover such a configuration $v_j \in \mathbb{R}^4$ is unique up to the action of the orthogonal group $\mathrm{O}(4)$.

Recall that $\mathrm{SU}(2)$ is the 3-dimensional sphere, and s_{ij} are inner products of points of the sphere. Recall that $\mathrm{SO}(4) \simeq \mathrm{SU}(2) \times \mathrm{SU}(2)/\{\pm 1\}$. In other words, proper isometries of the sphere S^3 correspond to the left-right action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathrm{SU}(2)$ (see, e.g. [8]).

If $\operatorname{rk} Q = 4$, then v_j are not contained in a 3-dimensional hyperplane. Therefore an improper isometry of the sphere gives a non-equivalent configuration in $\Pi(n)$.

If $\operatorname{rk} Q < 3$, then we the point configuration v_j is contained in a hyperplane. The reflection with respect to the hyperplane fix this configuration. □

3 The radial part of the Haar measure

3.1. A reduction. As we noted above any element of the double coset space $\Pi(n)$ can reduced to the form $(1, g_2, \dots, g_n)$ and g_j are determined up to a simultaneous conjugation. Next, we can assume that

$$g_2 = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}, \quad \text{where } 0 \leqslant \varphi \leqslant \pi.$$

Proposition 3.1 *Each element of* $\Pi(n)$ *has a representative of the form*

$$\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}e^{i\varphi}&0\\0&e^{-i\varphi}\end{pmatrix}, \begin{pmatrix}a_2&b_2\\-\overline{b}_2&\overline{a}_2\end{pmatrix}, \dots, \begin{pmatrix}a_{n-1}&b_{n-1}\\-\overline{b}_{n-1}&\overline{a}_{n-1}\end{pmatrix}\right\},$$
$$\text{where } b_2 \geqslant 0 \text{ and } 0 \leqslant \varphi \leqslant \pi. \quad (3.1)$$

For elements in general position such a representative is unique.

Indeed, after a reduction of g_2 to a diagonal form, we can conjugate our tuple by diagonal matrices.

3.2. The Haar measure on SU(2)**.** We can regard the group SU(2) as the unit sphere in the Euclidean space $\mathbb{C}^2$. The Haar measure on SU(2) is the usual surface Lebesgue measure on the sphere. We denote this measure by dg.

Recall the following simple facts.

Proposition 3.2 *a) The image of the Haar measure under the map* $(a, b) \mapsto a$ *is the Lebesgue measure* $da\, d\overline{a}$ *on the circle* $|a| \leqslant 1$*.*

b) Represent b *in the form* $b = \rho e^{i\theta}$*. Then the image of the Lebesgue measure under the map* $(a, b) \mapsto (a, \theta)$ *is*

$$d\theta\, da\, d\overline{a}.$$

c) Consider the map taking a matrix g *to its collection of eigenvalues* $e^{i\varphi}$*,* $e^{-i\varphi}$*, where* $0 \leqslant \varphi \leqslant \pi$*. The image of the Haar measure under the map* $g \mapsto \varphi$ *is* $\sin^2 \varphi d\varphi$*.*

Corollary 3.3 *The pushforward of the Haar measure in the coordinates (3.1) is*

$$\sin^2 \varphi\, d\varphi\, da_2\, d\overline{a}_2\, dg_3 \dots dg_n$$

3.3. Coordinates. n-tuples of matrices. To be definite, take $n = 5$,

$$(g_1, g_2, g_3, g_4, g_5) :=$$
$$:= \left(\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}e^{i\varphi}&0\\0&e^{-i\varphi}\end{pmatrix}, \begin{pmatrix}a_1&b_1\\-\overline{b}_1&\overline{a}_1\end{pmatrix}, \begin{pmatrix}a_2&b_2\\-\overline{b}_2&\overline{a}_2\end{pmatrix}, \begin{pmatrix}a_3&b_3\\-\overline{b}_3&\overline{a}_3\end{pmatrix}\right),$$

where $0 \leqslant \varphi \leqslant \pi$. We also denote

$$a_1 =: x_1 + iy_1 \qquad a_2 =: x_1 + iy_2 \qquad a_3 = x_3 + iy_3, \qquad (3.2)$$
$$b_1 =: e^{i\theta_1}\sqrt{1 - x_1^2 - y_1^2} \quad b_2 =: e^{i\theta_2}\sqrt{1 - x_2^2 - y_2^2} \quad b_3 =: e^{i\theta_3}\sqrt{1 - x_3^2 - y_3^2}. \qquad (3.3)$$

Here

$$\theta_1 - \theta_2, \quad \theta_2 - \theta_3, \quad \theta_1 - \theta_3$$

make sense (but not θ_1, θ_2, θ_3 themselves).

3.4. Coordinates. Spectral forms. Consider the spectral form and denote its coefficients in the following way:

$$\det\left(\sum \lambda g_1 + \mu g_2 + \nu_1 g_3 + \nu_2 g_4 + \nu_3 g_5\right) =:$$
$$=: \lambda^2 + \mu^2 + \nu^2 + 2p\lambda\mu \ + 2q_1\lambda\nu_1 + 2q_2\lambda\nu_2 + 2q_3\lambda\nu_3$$
$$+ 2r_1\mu\nu_1 + 2r_2\mu\nu_2 + 2r_3\mu\nu_3 \ + 2t_{12}\nu_1\nu_2 + 2t_{13}\nu_1\nu_3 + 2t_{23}\nu_2\nu_3.$$

The matrix of the form is

$$\Delta = \begin{pmatrix} 1 & p & q_1 & q_2 & q_3 \\ p & 1 & r_1 & r_2 & r_3 \\ q_1 & r_1 & 1 & t_{12} & t_{13} \\ q_2 & r_2 & t_{12} & 1 & t_{23} \\ q_3 & r_3 & t_{13} & t_{23} & 1 \end{pmatrix}. \tag{3.4}$$

Then

$$p = \cos\varphi \tag{3.5}$$
$$q_j = x_j \tag{3.6}$$
$$r_j = x_j \cos\varphi + y_j \sin\varphi \tag{3.7}$$
$$t_{ij} = x_i x_j + y_i y_j + \sqrt{1 - x_i^2 - y_i^2}\sqrt{1 - x_j^2 - y_j^2}\cos(\theta_i - \theta_j) \tag{3.8}$$

It is easy to write the inverse map:

$$\varphi = \arccos p \tag{3.9}$$
$$x_j = q_j \tag{3.10}$$
$$y_j = \frac{r_j - q_j p}{\sqrt{1 - p^2}} \tag{3.11}$$

$$\theta_i - \theta_j = \pm \arccos \frac{\det\begin{pmatrix} 1 & p & q_i \\ p & 1 & r_i \\ q_j & r_j & t \end{pmatrix}}{\det^{1/2}\begin{pmatrix} 1 & p & q_i \\ p & 1 & r_i \\ q_i & r_i & 1 \end{pmatrix} \det^{1/2}\begin{pmatrix} 1 & p & q_j \\ p & 1 & r_j \\ q_j & r_j & 1 \end{pmatrix}} \tag{3.12}$$

The last formula requires some calculations. For this reason, we present some intermediate formulas:

$$1 - x_1^2 - y_1^2 = \frac{\det\begin{pmatrix} 1 & p & q_1 \\ p & 1 & r_1 \\ q_1 & r_1 & 1 \end{pmatrix}}{1 - p^2}, \tag{3.13}$$

$$t - x_1 x_2 - y_1 y_2 = \frac{\det\begin{pmatrix} 1 & p & q_1 \\ p & 1 & r_1 \\ q_2 & r_2 & t \end{pmatrix}}{1 - p^2}.$$

Note, that we can not reconstruct the sign of $\theta_i - \theta_j$ from the formula (3.12). Recall that the substitution

$$\theta_1 \mapsto -\theta_1, \quad \theta_2 \mapsto -\theta_2, \quad \theta_3 \mapsto -\theta_3$$

corresponds to the simultaneous transposition

$$(g_1, g_2, g_3, g_4, g_5) \mapsto (g_1^t, g_2^t, g_3^t, g_4^t, g_5^t).$$

3.5. What happens if we forget t_{23}? Next, let we know p, all q_j, all r_j, and t_{12}, t_{13}. Then we can reconstruct φ, x_j, y_j and

$$\cos(\theta_1 - \theta_2), \quad \cos(\theta_1 - \theta_3).$$

Without loss of a generality, we can assume $\theta_1 = 0$. Then we know $\pm\theta_2$ and $\pm\theta_3$ and there are two possible variants for $|\theta_2 - \theta_3|$.

Chose $h \in \mathrm{SU}(2)$ such that

$$h^{-1} g_2 h = g_2^t, \quad h^{-1} g_3 h = g_3^t$$

(recall that g_1 is the unit matrix). Then without t_{23} we can not distinguish

$$(g_1, g_2, g_3, g_4, g_5) \quad \text{and} \quad (g_1, g_2, g_3, g_4, h g_5^t h^{-1}) \tag{3.14}$$

3.6. The radial part of the Haar measure. The cases $n = 3$, $n = 4$.

Theorem 3.4 *a) Let $n = 3$. The pushforward of the Haar measure under the map $\zeta : \Pi(3) \to \Xi(3)$ is*

$$dp\, dq_1\, dr_1.$$

b) Let $n = 4$. Then the image of the Haar measure under the map $\zeta : \Pi(4) \to \Xi(4)$ is

$$\det \begin{pmatrix} 1 & p & q_1 & q_2 \\ p & 1 & r_1 & r_2 \\ q_1 & r_1 & 1 & t_{12} \\ q_2 & r_2 & t_{12} & 1 \end{pmatrix}^{-1/2} dp\, dq_1\, dq_2\, dr_1\, dr_2\, dt_{12}.$$

Proof. Consider the case $n = 4$. The radial part of the Haar measure in the coordinates φ, x_1, y_1, x_2, y_2, θ is given by

$$\sin^2 \varphi\, d\varphi\, dx_1\, dy_1\, dx_2\, dy_2\, d\theta$$

Next, we must write the Jacobian of the map (3.9)–(3.12). Evidently, the Jacobian is

$$\frac{\partial \varphi}{\partial p} \cdot \frac{\partial \theta}{\partial t},$$

this can be easily evaluated.

3.7. The Haar measure, general case. For an n-tuple $(g_1, \dots, g_n)$ consider its spectral form

$$\det\Bigl(\sum_j \lambda_j g_j\Bigr) =: \sum_j \lambda_j^2 + 2 \sum_{i<j} s_{ij} \lambda_i \lambda_j$$

Theorem 3.5 *a) The coefficients s_{12}, s_{13}, s_{23} are distributed as*

$$ds_{12}\, ds_{13}\, ds_{23}$$

in the domain $\begin{pmatrix} 1 & s_{12} & s_{13} \\ s_{12} & 1 & s_{23} \\ s_{13} & s_{23} & 1 \end{pmatrix} \geqslant 0.$

b) For fixed s_{12}, s_{13}, s_{23} in a general position, a vector $v_j := \begin{pmatrix} s_{1j} & s_{2j} & s_{3j} \end{pmatrix}$ is distributed as

$$\det(\Delta_j)^{-1/2} ds_{j1}\, ds_{j2}\, ds_{j3},$$

where

$$\Delta_j = \begin{pmatrix} 1 & s_{12} & s_{13} & s_{1j} \\ s_{12} & 1 & s_{23} & s_{2j} \\ s_{13} & s_{23} & 1 & s_{3j} \\ s_{1j} & s_{2j} & s_{3j} & 1 \end{pmatrix}.$$

A vector $\begin{pmatrix} s_{1j} & s_{2j} & s_{3j} \end{pmatrix}$ ranges in the domain $\Delta_j \geqslant 0$.

c) The random variables v_4, v_5, ..., v_n are independent.

d) Let us fix s_{1j}, s_{2j}, s_{3j} for all j. For such a collection in a general position there are 2 equiprobable variants of a choice of s_{4j}. These samplings are independent for $j = 5, 6, \dots$.

e) Let us fix s_{1j}, s_{2j}, s_{3j} for all j and fix s_{4j}. Then a.s. all other variables s_{ij} are uniquely determined.

PROOF. The statements a)-b) are a rephrasing of Theorem (3.4).

Next, for fixed g_1, g_2, g_3 the matrices ('random variables') g_3, g_4, ... are independent. A matrix g_j determines a vector v_j, and g_j is uniquely determined by a vector v_j. This proves c).

Now we write the matrix of a spectral form

$$\begin{pmatrix} 1 & s_{12} & s_{13} & s_{14} & s_{15} & s_{16} & \dots \\ s_{12} & 1 & s_{23} & s_{24} & s_{25} & s_{26} & \dots \\ s_{13} & s_{23} & 1 & s_{34} & s_{35} & s_{36} & \dots \\ s_{14} & s_{24} & s_{34} & 1 & ?_1 & ?_2 & \dots \\ s_{15} & s_{25} & s_{35} & ?_1 & 1 & * & \dots \\ s_{16} & s_{26} & s_{36} & ?_2 & * & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The left upper 5×5 minor is zero. This gives a quadratic equation for s_{45}. Both the solutions are admissible, they correspond to collections of matrices given by (3.14). We repeat the same operation for s_{46} etc.

In the notation of Subsections 3.4–3.5, the knowledge of the first three rows gives us $\pm\theta_1$, $\pm\theta_2$, $\pm\theta_3$ etc. All these choices are equiprobable.

Having all s_{4j}, we get a unique way to complete the matrix to the matrix of rank 4.

4 Actions of $\mathrm{Out}(F_{n-1})$ and braid group

4.1. The action of $\mathrm{Out}(F_{n-1})$ on $\Pi(n)$. We realize $\Pi(n)$ as set of collections $(1, g_2, \dots, g_n)$ defined up to a simultaneous conjugation.

The Nielsen transformations act in the obvious way. We get

a) Permutations of g_j.

b) The transformation $g_2 \mapsto g_2^{-1}$

c) The map $(g_2, g_3, g_4, \dots) \to (g_2, g_2 g_3, g_4, \dots)$.

To be definite, take $n = 5$. The action of permutations is obvious.

Proposition 4.1 *The transformation $g_2 \mapsto g_2^{-1}$ corresponds to the map $\Xi(5) \to \Xi(5)$ given by*

$$\begin{aligned} \widetilde{p} &= p; \\ \widetilde{q}_j &= q_j; \\ \widetilde{t}_{ij} &= t_{ij} \\ \widetilde{r}_j &= -r_j + 2pq_j \end{aligned}$$

PROOF. In the notation (3.2)–(3.3), 3.5–3.8), e have

$$\widetilde{r}_1 = x_1 \cos\varphi - x_1 \sin\varphi = r_1 - 2y_1 \sin\varphi.$$

On the other hand

$$y_1 \sin\varphi = r_1 - x_1 \cos\varphi = r_1 - pq_1.$$

This completes the calculation. □

Theorem 4.2 *The transformation*

$$(1, g_2, g_3, g_4, g_5) \mapsto (1, g_2, g_2 g_3, g_4, g_5)$$

corresponds to the map $\widetilde{\Xi}(5) \to \widetilde{\Xi}(5)$ given by

$$\widetilde{p} = p;$$
$$\widetilde{q}_2 = q_2, \quad \widetilde{q}_3 = q_3;$$
$$\widetilde{q}_1 = -r_1 + 2pq_1;$$
$$\widetilde{r}_2 = r_2, \quad \widetilde{r}_3 = q_3;$$
$$\widetilde{t}_{1j} = p\, t_{1j} - q_j r_1 + q_1 r_j. \mp \det \begin{pmatrix} 1 & p & q_1 & q_j \\ p & 1 & r_1 & r_j \\ q_1 & r_1 & 1 & t_{1j} \\ q_j & r_j & t_{1j} & 1 \end{pmatrix}^{1/2}, \quad \textit{where } j = 2,\, 3. \tag{4.1}$$
$$\widetilde{t}_{23} = t_{23}$$

The group $\mathrm{Out}(F_n)$ acts on $\widetilde{\Xi}(5)$ and not on $\Xi(5)$ and a choice of signs $\mp$ requires explanations. They are given below.

PROOF. First, we write the coefficients of the spectral form for the transformed collection. Only the variables q_1, r_1, t_{12}, t_{13} change. We have

$$\widetilde{r}_1 = \widetilde{x}_1 \cos\varphi + \widetilde{y}_1 \sin\varphi = \\ = (x_1\cos\varphi - y_1\sin\varphi)\cos\varphi + (y_1\cos\varphi + x_1\sin\varphi)\sin\varphi = x_1 = q_1.$$

and

$$\widetilde{q}_1 = \widetilde{x}_1 = x_1\cos\varphi - y_1\sin\varphi = q_1 p - (r_1 - q_1 p) = -r_1 + 2q_1 p. \tag{4.2}$$

An evaluation of $\widetilde{t}_{1j}$ is heavier,

$$\widetilde{t}_{12} = \widetilde{x}_1\widetilde{x}_2 + \widetilde{y}_1\widetilde{y}_2 + \sqrt{1-\widetilde{x}_1^2-\widetilde{y}_1^2}\sqrt{1-\widetilde{x}_2^2-\widetilde{y}_2^2}\cos(\widetilde{\theta}_1-\widetilde{\theta}_2).$$

By construction, $\widetilde{x}_2 = x_2$, $\widetilde{y}_2 = y_2$, $\widetilde{\theta}_2 = \theta_2$. Next,

$$\widetilde{a}_1 = \widetilde{x}_1 + i\widetilde{y}_1 = (x_1 + iy_1)\,e^{i\varphi}$$

and therefore

$$1-\widetilde{x}_1^2-\widetilde{y}_1^2 = 1 - x_1^2 - y_1^2.$$

Also, $\widetilde{\theta}_1 = \theta_1 + \varphi$. We also denote $\theta := \theta_1 - \theta_2$. Thus,

$$\widetilde{t}_{12} = \widetilde{x}_1 x_2 + \widetilde{y}_1 y_2 + \sqrt{1-x_1^2-y_1^2}\,\sqrt{1-x_2^2-y_2^2}\big(\cos\theta\cos\varphi - \sin\theta\sin\varphi\big).$$

The variable $\widetilde{x}_1$ is evaluated in (4.2),

$$\widetilde{y}_1 = x_1\sin\varphi + y_1\cos\varphi = \frac{r_1 - q_1 p}{\sqrt{1-p^2}}\cdot p + q\sqrt{1-p^2}.$$

We use formula (3.13) for square roots and formula (3.12) for $\cos\theta$. After this, we can evaluate $\sin\theta$,

$$\sin^2\theta = \frac{(1-p^2)\cdot\det\begin{pmatrix}1 & p & q_1 & q_2\\ p & 1 & r_1 & r_2\\ q_1 & r_1 & 1 & t_{12}\\ q_2 & r_2 & t_{12} & 1\end{pmatrix}}{\det\begin{pmatrix}1 & p & q_1\\ p & 1 & r_1\\ q_1 & r_1 & 1\end{pmatrix}\det\begin{pmatrix}1 & p & q_2\\ p & 1 & r_2\\ q_2 & r_2 & 1\end{pmatrix}}.$$

After this, we get a unexpectedly long chain of cancelations and get the desired formula. □.

CHOICE OF SIGNS. We use formula (3.12) and find

$$\pm(\theta_1-\theta_2), \quad \pm(\theta_1-\theta_3), \quad \pm(\theta_2-\theta_3)$$

These numbers must be consistent, in fact only two variants are possible (this corresponds to a choice of a sheet of the covering map $\widetilde{\Xi} \to \Xi$). Now let we have chosen the signs. Then we take 'minus' in (4.1) if $(\theta_1 - \theta_j) \geqslant 0$. Otherwise, we take 'plus'.

4.2. The action of the braid group.

Lemma 4.3 *The transformation*

$$(1, g_1, g_2, g_3, g_4, g_5) \mapsto (1, g_1, g_1 g_2 g_1^{-1}, g_3, g_4, g_5)$$

corresponds to the map $\widetilde{\Xi}(5) \to \widetilde{\Xi}(5)$ *given by*

$$\begin{aligned}
\widetilde{p} &= p \\
\widetilde{q}_k &= q_k, \quad \textit{where } k = 1,\ 2,\ 3; \\
\widetilde{r}_k &= r_k, \quad \textit{where } k = 1,\ 2,\ 3; \\
\widetilde{t}_{1j} &= t_{1j} - 2 \cdot \det \begin{pmatrix} 1 & p & q_1 \\ p & 1 & r_1 \\ q_j & r_j & t_{1j} \end{pmatrix} \mp 2p \cdot \det \begin{pmatrix} 1 & p & q_1 & q_j \\ p & 1 & r_1 & r_j \\ q_1 & r_1 & 1 & t_{1j} \\ q_j & r_j & t_{1j} & 1 \end{pmatrix}^{1/2}, \\
\widetilde{t}_{23} &= t_{23}.
\end{aligned}$$

Proof. We evaluate

$$\begin{aligned}
\widetilde{t}_{12} &= \operatorname{Re}(a_1 \overline{a}_2 + b_1 \overline{b}_2 e^{2i\varphi}) = \\
&= x_1 x_2 + y_1 y_2 + \sqrt{1 - x_1^2 - y_1^2}\sqrt{1 - x_2^2 - y_2^2}(\cos\theta \cos 2\varphi - \sin\theta \sin 2\varphi)
\end{aligned}$$

as in the previous proof. □

Now we can write the action of the braid group. To write formulas for generators, we represent the matrix of the spectral form as

$$\begin{pmatrix} 1 & p_1 & p_2 & p_3 & \dots \\ p_1 & 1 & h_{12} & h_{13} & \dots \\ p_2 & h_{12} & 1 & h_{23} & \dots \\ p_3 & h_{13} & h_{23} & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We also set $h_{ij} := h_{ji}$. Then the formula for a generator σ_k of the braid group is

$$\begin{aligned}
\widetilde{h}_{ij} &= h_{ij}, \quad \text{if } i,\ j \neq k,\ k+1 \\
\widetilde{p}_i &= p_i, \quad \text{if } i \neq k,\ k+1 \\
\widetilde{p}_k &= p_{k+1} \\
\widetilde{p}_{k+1} &= p_k \\
\widetilde{h}_{(k+1)j} &= h_{kj},
\end{aligned}$$

and

$$\widetilde{h}_{kj} = h_{(k+1)j} - 2\cdot \det \begin{pmatrix} 1 & p_k & p_{k+1} \\ p_k & 1 & h_{k(k+1)} \\ p_j & h_{kj} & h_{(k+1)j} \end{pmatrix} - \\ - 2p_k \cdot \det \begin{pmatrix} 1 & p_k & p_{k+1} & p_j \\ p_k & 1 & h_{k(k+1)} & h_{kj} \\ p_{k+1} & h_{k(k+1)} & 1 & h_{(k+1)j} \\ p_j & h_{kj} & h_{(k+1)j} & 1 \end{pmatrix}^{1/2} \qquad (4.3)$$

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