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EQUIVARIANT QUANTIZATIONS FOR AHS–STRUCTURES

ANDREAS ČAP AND JOSEF ŠILHAN

Dedicated to Peter W. Michor at the occasion of his 60th birthday

Abstract. We construct an explicit scheme to associate to any potential symbol an operator acting between sections of natural bundles (associated to irreducible representations) for a so–called AHS–structure. Outside of a finite set of critical (or resonant) weights, this procedure gives rise to a quantization, which is intrinsic to this geometric structure. In particular, this provides projectively and conformally equivariant quantizations for arbitrary symbols on general (curved) projective and conformal structures.

1. Introduction

Consider a smooth manifold M, two vector bundles E and F over M and a linear differential operator $D : \Gamma(E) \to \Gamma(F)$, where $\Gamma(\ )$ indicates the space of smooth sections. If D is of order at most k, then it has a well defined (kth order) *principal symbol* σ_D, which can be viewed as a vector bundle map $S^kT^*M \otimes E \to F$ or as a smooth section of the vector bundle $S^kTM \otimes E^* \otimes F$. Here TM and T^*M are the tangent respectively cotangent bundle of M, E^* is the bundle dual to E, and S^k denotes the kth symmetric power.

A *quantization* on M is a right inverse to the principal symbol map. This means that to each smooth section τ of the bundle $S^kTM \otimes E^* \otimes F$, one has to associate a differential operator $A_\tau : \Gamma(E) \to \Gamma(F)$ of order k with principal symbol τ. Note that operators of order 0 coincide with their principal symbols, so there a unique possible quantization in order 0. Given any kth order operator D with principal symbol τ, the difference $D - A_\tau$ is of order $k - 1$. Iterating this, we conclude that, having a quantization in each order $\leq k$, one actually obtains an isomorphism between the space $Diff^k(E, F)$ of differential operators $\Gamma(E) \to \Gamma(F)$ of order at most k and the space of smooth sections of the bundle $\oplus_{i=0}^k S^iTM \otimes E^* \otimes F$.

A classical example of a quantization is provided by the Fourier transform for smooth functions on $\mathbb{R}^n$. However, it is well known that (even for $E = F = M \times \mathbb{R}$) there is no canonical quantization on a general manifold M, but one has to make additional choices. For our purposes, the most relevant example is to choose linear connections on the vector bundles E and TM. Having done this, one obtains induced linear connections on duals and tensor products of these bundles, and we

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will denote all these connections by ∇. For a smooth section s of E, one can then form the k–fold covariant derivative $\nabla^k s$, which is a section of $\otimes^k T^*M \otimes E$. Symmetrizing in the T^*M entries, we obtain a section $\nabla^{(k)}s$ of $S^kT^*M\otimes E$. Viewing a symbol τ as a bundle map $S^kT^*M\otimes E \to F$, we can simply put $A_\tau(s) := \tau(\nabla^{(k)}s)$. Clearly this defines a differential operator A_τ of order k and it is well known that its principal symbol is τ, so we have obtained a quantization in this way.

This provides a link to geometry. Suppose that M is endowed with some geometric structure which admits a canonical connection. Then one obtains quantizations for all natural bundles associated to this structure. The classical example of this situation is the case when (M, g) is a Riemannian manifold. Then the natural bundles are tensor and spinor bundles, and on each such bundle one has the Levi–Civita connection. Hence the above procedure leads to a natural quantization (in the sense that it is intrinsic to the Riemannian structure) for any pair E and F of natural vector bundles.

At this point there arises the question whether weaker geometric structures, which do not admit canonical connections, still do admit natural quantizations. This problem has been originally posed in [15] and has been intensively studied since then. The examples above naturally lead to the two geometric structures for which this problem has been mainly considered. On the one hand, one may replace a single linear connection on TM by a projective equivalence class of such connections. Here two connections are considered as equivalent if they have the same geodesics up to parametrization. On the other hand, the most natural weakening of Riemannian metrics is provided by conformal structures. Here one takes an equivalence class of (pseudo–)Riemannian metrics which are obtained from each other by multiplication by positive smooth functions.

Projective and conformal structures fit into the general scheme of so–called AHS–structures. These are geometric structures which admit an equivalent description by a canonical Cartan connection modelled on a compact Hermitian symmetric space G/P, where G is semisimple and $P \subset G$ is an appropriate parabolic subgroup. These geometries and the more general class of parabolic geometries have been studied intensively during the last years, and several striking results have been obtained, see e.g. [8]. In particular, an efficient differential calculus for these structures based on so–called tractor bundles has been worked out in [4].

This general point of view has shown up in the theory of equivariant quantizations already. Namely, it turns out that the homogeneous space G/P always contains a dense open subset (the big Schubert–cell) which is naturally diffeomorphic to $\mathbb{R}^n$. While the G–action on G/P cannot be restricted to this subspace, one obtains a realization of the Lie algebra $\mathfrak{g}$ of G as a Lie algebra of vector fields on $\mathbb{R}^n$. For the homogeneous model G/P and geometries locally isomorphic to it, naturality of a quantization is then equivalent to equivariancy for the action of this Lie algebra of vector fields. In many articles, the question of quantizations naturally associated to a projective and/or conformal structure is posed in this setting. Also, the algebras corresponding to general AHS–structures have been studied in this setting under the name "IFFT–equivariant quantizations", see [1]. It should be pointed out however, that these methods only apply to geometries locally isomorphic to G/P (e.g. to locally conformally flat conformal structures). As it is well known from the theory of linear invariant differential operators, passing from the locally flat category to general structures is a very difficult problem.

Most of the work on natural quantizations only applies to operators on sections of line bundles (density bundles). It was only recently that the methods for projective structures have been extended to general natural vector bundles in [12]. The construction there uses the Thomas–Whitehead (or ambient) description of projective structures, which is an equivalent encoding of the canonical Cartan connection for projective structures. This approach is only available in the projective case, though. As mentioned in [12], there is hope to use the Fefferman–Graham ambient metric for conformal structures to find conformally invariant quantizations, but there several immediate problems with this approach. For the other AHS–structures, there is no clear analog of the ambient description.

It should be also mentioned that the results for projective structures have been obtained using the canonical Cartan connection, see [16]. After this article was essentially completed, we learned about the recent preprint [17], in which the Cartan approach is extended to prove existence of a natural quantization for conformal structures and it is claimed that the method further extends to all AHS–structures.

In this article, we use the recent advances on invariant calculi for parabolic geometries to develop a scheme for constructing equivariant quantizations. This scheme is explicit and uniform, it applies to all AHS–structures and to all (irreducible) natural bundles for such structures. As it is known from the special cases studied so far, equivariant quantizations do not always exist, so our scheme does not always lead to an equivariant quantization.

To formulate the result more precisely, we need a bit more background. It turns out that for any AHS–structure there is a family of natural line bundles $\mathcal{E}[w]$ parametrized by a real number w, the so–called density bundles. Any natural bundle E can be twisted by forming tensor products with density bundles to obtain bundles $E[w] := E \otimes \mathcal{E}[w]$. (For conformal structures, this free parameter is known as "conformal weight".) Doing this to the target bundle of differential operators, we can view a section $\tau \in \Gamma(S^kTM \otimes E^* \otimes F \otimes \mathcal{E}[\delta])$ as the potential symbol of an operator $\Gamma(E) \to \Gamma(F[\delta])$. We first universally decompose the bundle of symbols into a finite direct sum of subbundles. On the level of sections, we write this decomposition as $\tau = \sum_i \tau_i$. Given such a section, our scheme constructs a differential operator $A_\tau : \Gamma(E) \to \Gamma(F[\delta])$ for any choice of weight δ. The principal symbol of A_τ is $\sum_i \gamma_i \tau_i$ for real numbers γ_i which only depend on i, and δ (and not on τ or on the manifold in question). We prove that each γ_i is nonzero except for finitely many values of δ. Whenever all γ_i are non–zero, we obtain a natural quantization by mapping τ to $A_{\sum_i \gamma_i^{-1}\tau_i}$.

Our method does not only lead to an abstract proof that the set of critical weights (i.e. of weights δ for which some γ_i vanishes) is finite. We also get general information on the number and size of critical weights. In each concrete example, one can determine the set of critical weights explicitly, and this needs only finite dimensional representation theory.

We should mention that the developments in this article are closely related to the results in the recent thesis [14] of J. Kroeske, in which the author systematically constructs bilinear natural differential operators for AHS–structures and, more generally, for parabolic geometries.

2. AHS–structures and invariant calculus

In this section we review basics facts on AHS–structures and invariant differential calculus for these geometries. Our basic references are [18], [6], and [7].

2.1. **$|1|$–graded Lie algebras and first order structures.** The starting point for defining an AHS–structure is a simple Lie algebra $\mathfrak{g}$ endowed with a so called $|1|$–grading, i.e. a decomposition $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, where we agree that $\mathfrak{g}_\ell = 0$ for $\ell \notin \{-1, 0, 1\}$. The classification of such gradings is well known, since it is equivalent to the classification of Hermitian symmetric spaces. We put $\mathfrak{p} := \mathfrak{g}_0 \oplus \mathfrak{g}_1 \subset \mathfrak{g}$. By the grading property, $\mathfrak{p}$ is a subalgebra of $\mathfrak{p}$ and $\mathfrak{g}_1$ is a nilpotent ideal in $\mathfrak{p}$.

Given a Lie group G with Lie algebra $\mathfrak{g}$, there are natural subgroups $G_0 \subset P \subset G$ corresponding to the Lie subalgebras $\mathfrak{g}_0 \subset \mathfrak{p} \subset \mathfrak{g}$. For P one may take a subgroup lying between the normalizer $N_G(\mathfrak{p})$ of $\mathfrak{p}$ in G and its connected component of the identity. Then $G_0 \subset P$ is defined as the subgroup of all elements whose adjoint action preserves the grading of $\mathfrak{g}$. In particular, restricting the adjoint action to $\mathfrak{g}_{-1}$, one obtains a representation $G_0 \to GL(\mathfrak{g}_{-1})$. This representation is infinitesimally injective, so it makes sense to talk about first order G–structures with structure group G_0 on smooth manifolds of dimension $\dim(\mathfrak{g}_{-1})$.

By definition, such a structure is given by a smooth principal bundle $p : \mathcal{G}_0 \to M$ with structure group G_0, such that the associated bundle $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_{-1}$ is isomorphic to the tangent bundle TM. It turns out that the Killing form on $\mathfrak{g}$ induces a G_0–equivariant duality between $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$, so $\mathcal{G}_0 \times_{G_0} \mathfrak{g}_1 \cong T^*M$. Using this, one can realize arbitrary tensor bundles on M as associated bundles to $\mathcal{G}_0$. More generally, any representation of G_0, via forming associated bundles, gives rise to a natural vector bundle on manifolds endowed with such a structure. It turns out that G_0 is always reductive with one–dimensional center. Hence finite dimensional representations of G_0 on which the center acts diagonalizably (which we will always assume in the sequel) are completely reducible, i.e. they split into direct sums of irreducible representations.

The one–dimensional center of G_0 leads to a family of natural line bundles. For $w \in \mathbb{R}$, we can define a homomorphism $G_0 \to \mathbb{R}_+$ by mapping $g \in G_0$ to $|\det(\mathrm{Ad}_-(g))|^{\frac{w}{n}}$, where $n = \dim(\mathfrak{g}_{-1})$ and $\mathrm{Ad}_-(g) : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is the restriction of the adjoint action of g. This evidently is a smooth homomorphism, thus giving rise to a one–dimensional representation $\mathbb{R}[w]$ of G_0. It is easy to see that $\mathbb{R}[w]$ is non–trivial for $w \neq 0$. (The factor $\frac{1}{n}$ is included to get the usual normalization in the case of conformal structures.) The corresponding associated bundle will be denoted by $\mathcal{E}[w]$, and adding the symbol $[w]$ to the name of a natural bundle will always indicate a tensor product with $\mathcal{E}[w]$. Using the convention that 1–densities are the objects which can be naturally integrated on non–orientable manifolds, $\mathcal{E}[w]$ is by construction the bundle of $(-\frac{w}{n})$–densities. In particular, all the bundles $\mathcal{E}[w]$ are trivial line bundles, but there is no canonical trivialization for $w \neq 0$.

2.2. **Canonical Cartan connections and AHS–structures.** The exponential mapping restricts to a diffeomorphism from $\mathfrak{g}_1$ onto a closed normal Abelian subgroup $P_+ \subset P$ such that P is the semidirect product of G_0 and P_+. Hence G_0 can also naturally be viewed as a quotient of P. In particular, given a principal P–bundle $\mathcal{G} \to M$, the subgroup P_+ acts freely on $\mathcal{G}$, and the quotient $\mathcal{G}/P_+$ is naturally a principal bundle with structure group G_0. Next, suppose that there is a Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ on the principal bundle $\mathcal{G}$. Then the $\mathfrak{g}_{-1}$–component of ω descends to a well defined one–form $\theta \in \Omega^1(\mathcal{G}/P_+, \mathfrak{g}_{-1})$, which is G_0–equivariant and strictly horizontal. This means that $(\mathcal{G}/P_+ \to M, \theta)$ is a first order structure with structure group G_0. In this sense, any Cartan geometry $(p : \mathcal{G} \to M, \omega)$ of type (G, P) has an underlying first order structure with structure group G_0. Conversely, one can talk about extending a first order structure to a Cartan geometry.

It turns out (see e.g. [7]) that, for almost all choices of (G, P), for any given first order structure with structure group G_0 there is a unique (up to isomorphism) extension to a Cartan geometry of type (G, P), for which the Cartan connection ω satisfies a certain normalization condition. This is usually phrased as saying that such structures admit a canonical Cartan connection. The main exception is $\mathfrak{g} = \mathfrak{gl}(n+1, \mathbb{R})$ with a $|1|$–grading such that $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{g}_{\pm 1} \cong \mathbb{R}^n$. For an appropriate choice of G, the adjoint action identifies G_0 with $GL(\mathfrak{g}_{-1}) = GL(n, \mathbb{R})$. A first order structure for this group on a manifold M is just the full linear frame bundle of M and hence contains no information. In this case, an extension to a normal Cartan geometry of type (G, P) is equivalent to the choice of a projective equivalence class of torsion free connections on the tangent bundle TM, i.e. to a classical projective structure.

Normal Cartan geometries of type (G, P) as well as the equivalent underlying structures (i.e. classical projective structures respectively first order structures with structure group G_0) are often referred to as *AHS–structures*. AHS is short for “almost Hermitian symmetric”. To explain this name, recall that the basic example of a Cartan geometry of type (G, P) is provided by the natural projection $G \to G/P$ and the left Maurer–Cartan form as the Cartan connection. This is called the *homogeneous model* of geometries of type (G, P). Now the the homogeneous spaces G/P for pairs (G, P) coming from $|1|$–gradings as described above, are exactly the compact irreducible Hermitian symmetric spaces.

2.3. **Natural bundles and the fundamental derivative.** Via forming associated bundles, any representation of the group P gives rise to a natural bundle for Cartan geometries of type (G, P). As we have seen above, P is the semi–direct product of the reductive subgroup G_0 and the normal vector subgroup P_+, so its representation theory is fairly complicated. Via the quotient homomorphism $P \to G_0$, any representation of G_0 gives rise to a representation of P. It turns out that the representations of P obtained in this way are exactly the completely reducible representations, i.e. the direct sums of irreducible representations. Correspondingly, we will talk about completely reducible and irreducible natural bundles on Cartan geometries of type (G, P). If we have a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ with underlying structure $(p_0 : \mathcal{G}_0 \to M, \theta)$ and V is a representation of G_0, which we also view as a representation of P, then we can naturally identify $\mathcal{G} \times_P V$ with $\mathcal{G}_0 \times_{G_0} V$. Hence completely reducible bundles can be easily described in terms of the underlying structure.

There is a second simple source of representations of P, which leads to an important class of natural bundles. Namely, one may restrict any representation of G to the subgroup P. The corresponding natural vector bundles are called *tractor bundles*, their general theory is developed in [4]. The most important tractor bundle is the *adjoint tractor bundle*. For a Cartan geometry $(p : \mathcal{G} \to M, \omega)$ it is defined by $\mathcal{A}M := \mathcal{G} \times_P \mathfrak{g}$, so it is the associated bundle with respect to the restriction of the adjoint representation of G to P. Now the P–invariant subspaces $\mathfrak{g}_1 \subset \mathfrak{p} \subset \mathfrak{g}$ give rise to a filtration $\mathcal{A}^1M \subset \mathcal{A}^0M \subset \mathcal{A}M$ of the adjoint tractor bundle by smooth subbundles. By construction, $\mathcal{A}^1M \cong T^*M$ and since $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$ we see that $\mathcal{A}M/\mathcal{A}^0M \cong TM$. We will write $\Pi : \mathcal{A}M \to TM$ for the resulting natural projection. Hence the adjoint tractor bundle has the cotangent bundle as a natural subbundle and the tangent bundle as a natural quotient.

The Killing form defines a G–invariant, non–degenerate bilinear form on $\mathfrak{g}$. It turns out that $\mathfrak{g}_1$ is the annihilator of $\mathfrak{p}$ with respect to the Killing form, which leads to duality with $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-1}$ observed above. On the level of associated bundles, we obtain a natural non–degenerate bilinear form on the adjoint tractor bundle $\mathcal{A}M$, which thus can be identified with the dual bundle $\mathcal{A}^*M$. Under this pairing, the subbundle $\mathcal{A}^1M$ is the annihilator of $\mathcal{A}^0M$. The resulting duality between $\mathcal{A}^1M$ and $\mathcal{A}M/\mathcal{A}^0M$ is exactly the duality between T^*M and TM.

The adjoint tractor bundle gives rise to a basic family of natural differential operators for AHS–structures (and more generally for parabolic geometries). These have been introduced in [4] under the name "fundamental D–operators", more recently, the name *fundamental derivative* is commonly used. Let us start with an arbitrary representation V of P and consider the corresponding natural bundle $E := \mathcal{G} \times_P V \to M$ for a geometry $(p : \mathcal{G} \to M, \omega)$. Then smooth sections of this bundle are in bijective correspondence with smooth maps $f : \mathcal{G} \to V$, which are P–equivariant. In the special case $V = \mathfrak{g}$ of the adjoint tractor bundle, we can then use the trivialization of $T\mathcal{G}$ provided by the Cartan connection ω to identify P–equivariant functions $\mathcal{G} \to \mathfrak{g}$ with P–invariant vector fields on $\mathcal{G}$. For a section $s \in \Gamma(\mathcal{A}M)$, we can form the corresponding vector field $\xi \in \mathfrak{X}(\mathcal{G})$ and use it to differentiate the equivariant function $f : \mathcal{G} \to V$ corresponding to a section $\sigma \in \Gamma(E)$. The result will again be equivariant, thus defining a smooth section $D_s\sigma \in \Gamma(E)$. Hence we can view the fundamental derivative as an operator $D = D^E : \Gamma(\mathcal{A}M) \times \Gamma(E) \to \Gamma(E)$. The basic properties of this operator as proved in section 3 of [4] are:

Proposition 1. *Let V be a representation of P and let $E = \mathcal{G} \times_P V$ be the corresponding natural bundle for an AHS–structure $(p : \mathcal{G} \to M, \omega)$. Then we have:*
(1) $D : \Gamma(\mathcal{A}M) \times \Gamma(E) \to \Gamma(E)$ is a first order differential operator which is natural, i.e. intrisic to the AHS–structure on M.
(2) D is linear over smooth functions in the $\mathcal{A}M$–entry, so we can also view $\sigma \mapsto D\sigma$ as an operator $\Gamma(E) \to \Gamma(\mathcal{A}^*M \otimes E)$.
(3) For $s \in \Gamma(\mathcal{A}M)$, $\sigma \in \Gamma(E)$, and $f \in C^\infty(M, \mathbb{R})$, we have the Leibniz rule $D_s(f\sigma) = (\Pi(s)\cdot f)\sigma + fD_s\sigma$, where $\Pi : \Gamma(\mathcal{A}M) \to \Gamma(TM)$ is the natural tensorial projection.
(4) For a second natural bundle $F = \mathcal{G} \times_P W$, a P–equivariant map $V \to W$, and the corresponding bundle map $\Phi : E \to F$, the fundamental derivatives on E and F are related by $D^F_s(\Phi \circ \sigma) = \Phi \circ D^E_s\sigma$ for all $s \in \Gamma(\mathcal{A}M)$ and $\sigma \in \Gamma(E)$.

The naturality statement in (4) justifies denoting the fundamental derivatives on all natural bundles by the same letter. Since there is no restriction on the bundle E, the fundamental derivative in the form of part (2) can evidently be iterated. For $\sigma \in \Gamma(E)$ we can form $D\sigma$, $D^2\sigma = D(D\sigma)$ and inductively $D^k\sigma \in \Gamma(\otimes^k\mathcal{A}^*M \otimes E)$.

2.4. **Curved Casimir operators.** Curved Casimir operators form another basic set of natural differential operators defined on AHS–structures. They have been introduced in [9] in the general context of parabolic geometries. That article contains all the facts about curved Casimir operators we will need, as well as the general construction for splitting operators that we will use below.

As above, we start with a representation V of P and consider the corresponding natural vector bundle $E = \mathcal{G} \times_P V$ for an AHS–structure $(p : \mathcal{G} \to M, \omega)$. As noticed above, the composition of two fundamental derivatives defines an operator $D^2 : \Gamma(E) \to \Gamma(\otimes^2 \mathcal{A}^* M \otimes E)$. From 2.3 we know that the Killing form on $\mathfrak{g}$ induces a non–degenerate bilinear form on $\mathcal{A}M$. Using this to identify $\mathcal{A}M$ with $\mathcal{A}^*M$, we also get a natural bilinear form B on $\mathcal{A}^*M$. This can be used to define a bundle map $B \otimes \mathrm{id} : \otimes^2 \mathcal{A}^* M \otimes E \to E$. Now one defines the *curved Casimir operator* $\mathcal{C} = \mathcal{C}^E : \Gamma(E) \to \Gamma(E)$ by $\mathcal{C}(\sigma) := (B \otimes \mathrm{id}) \circ D^2 \sigma$.

Part (4) of proposition 1 easily implies (compare with proposition 2 of [9]) that for another natural vector bundle F and a bundle map $\Phi : E \to F$ coming form a P–equivariant map between the inducing representations, one gets $\mathcal{C}^F(\Phi \circ \sigma) = \Phi \circ \mathcal{C}^E(\sigma)$. This is the justification for denoting all curved Casimir operators by the same symbol.

From the construction it is clear that $\mathcal{C}$ is a natural differential operator of order at most 2. However, it turns out that $\mathcal{C}$ actually always is of order at most one. Moreover, on sections of bundles induced by irreducible representations, the operator $\mathcal{C}$ acts by a scalar which can be computed from representation theory data. One can associate to any irreducible representation of $\mathfrak{g}_0$ a highest and a lowest weight by passing to complexifications, see section 3.4 of [9]. The weights are functionals on the Cartan subalgebra $\mathfrak{h}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, which at the same time is a Cartan subalgebra for $(\mathfrak{g}_0)_{\mathbb{C}}$. Recall that the Killing form of $\mathfrak{g}$ induces a positive definite inner product on the real space of functionals on $\mathfrak{h}$ spanned by possible weights for finite dimensional representations. Denoting this inner product by $\langle\ ,\ \rangle$ and the corresponding norm by $\|\ \|$, the following result is proved as theorem 1 in [9].

Proposition 2. *Let V be a representation of P and let $E = \mathcal{G} \times_P V$ be the corresponding natural vector bundle for an AHS–structure $(p : \mathcal{G} \to M, \omega)$. Then*
(1) $\mathcal{C} : \Gamma(E) \to \Gamma(E)$ is a natural differential operator of order at most one.
(2) If the representation V is irreducible of lowest weight $-\lambda$, then $\mathcal{C}$ acts on $\Gamma(E)$ by multiplication by $\|\lambda\|^2 + 2\langle\lambda, \rho\rangle$, where ρ is half the sum of all positive roots of $\mathfrak{g}_{\mathbb{C}}$.

3. The quantization scheme

Throughout this section, we fix a pair (G, P), two irreducible representations V and W of G_0 with corresponding natural bundles E and F, as well as an order $k > 0$. Given these data, we try to construct a quantization for kth order symbols of operators mapping sections of E to sections of $F[\delta]$ for $\delta \in \mathbb{R}$.

The basic idea for the construction is very simple. The bundle of symbols in this situation is $S^k TM \otimes E^* \otimes F[\delta]$. We know from 2.3 that TM naturally is a quotient of the adjoint tractor bundle $\mathcal{A}M$, so the bundle of symbols is a quotient of $S^k \mathcal{A}M \otimes E^* \otimes F[\delta]$. Using the general machinery of splitting operators, we can associate to a symbol a section of the latter bundle. But such a section can be interpreted as a bundle map $S^k\mathcal{A}^*M\otimes E\to F[\delta]$, so we can apply it to the values of the symmetrized k–fold fundamental derivative of sections of E.

3.1. **Some properties of the fundamental derivative.** To carry out this idea, we first have to derive some properties of the iterated fundamental derivative D^k and its symmetrization $D^{(k)}:\Gamma(E)\to\Gamma(\mathcal{W}M)$, where $\mathcal{W}M:=S^k\mathcal{A}^*M\otimes E$. Recall from 2.3 that $\mathcal{A}M$ admits a natural filtration of the form $\mathcal{A}^1M\subset\mathcal{A}^0M\subset\mathcal{A}^{-1}M:=\mathcal{A}M$. Since elements of $\mathcal{W}M$ can be interpreted as k–linear, symmetric maps $(\mathcal{A}M)^k\to E$, we get an induced filtration of the bundle $\mathcal{W}M$. We first take the natural filtration of $S^k\mathcal{A}M$, with components indexed from $-k$ to k, and then define $\mathcal{W}^\ell M$ to be the annihilator of the filtration component with index $-\ell+1$. Explicitly, this means that $\mathcal{W}^\ell M$ to consist of all maps $\Psi\in\mathcal{W}M$ such that $\Psi(s_1,\dots,s_k)=0$ for arbitrary elements $s_j\in\mathcal{A}^{i_j}M$, provided that $i_1+\dots+i_k>-\ell$. Then by definition, we get $\mathcal{W}^{\ell+1}M\subset\mathcal{W}^\ell M$ for each ℓ, $\mathcal{W}^{k+1}M=0$, and $\mathcal{W}^{-k}M=\mathcal{W}M$. Moreover, a map $\Phi\in\mathcal{W}^kM$ by definition vanishes if at least one of its entries is from $\mathcal{A}^0M\subset\mathcal{A}M$. Hence this factors to a k–linear symmetric map on copies of $\mathcal{A}M/\mathcal{A}^0M\cong TM$, and we get an isomorphism $\mathcal{W}^kM\cong S^kT^*M\otimes E$. We will denote by $\iota:S^kT^*M\otimes E\to\mathcal{W}M$ the corresponding natural inclusion.

Proposition 3. *(1) The symmetrized k–fold fundamental derivative $D^{(k)}:\Gamma(E)\to\Gamma(\mathcal{W}M)$ has values in the space of sections of the subbundle $\mathcal{W}^0M$.*
(2) Consider any principal connection on the bundle $\mathcal{G}_0\to M$, denote by ∇ all the induced connections on associated vector bundles, by ∇^k the k–fold covariant derivative, and by $\nabla^{(k)}$ its symmetrization.

Then the operator $\Gamma(E)\to\Gamma(\tilde{\mathcal{W}}M)$ given by $\varphi\mapsto D^k\varphi-i(\nabla^k\varphi)$ has order at most $k-1$. In particular, $D^{(k)}\varphi$ is the sum of $i(\nabla^{(k)}\varphi)$ and terms of order at most $k-1$ in φ.

Proof. We will proceed by induction on k. Recall that there is a family of preferred connections on the bundle $\mathcal{G}_0$ which is intrinsic to the AHS–structure, see [4, 6]. Any such connection also determines determines a splitting of the filtration of the adjoint tractor bundle, i.e. an isomorphism $\mathcal{A}M\to T^*M\oplus\mathrm{End}_0(TM)\oplus TM$, where $\mathrm{End}_0(TM)=\mathcal{G}_0\times_{G_0}\mathfrak{g}_0$, which behaves well with respect to the filtration. In particular, the last component is given by the natural projection $\Pi:\mathcal{A}M\to TM$, while the first component restricts to the natural isomorphism $\mathcal{A}^1M\to T^*M$. Fixing one preferred connection, the difference to any other principal connection on $\mathcal{G}_0$ is given by a tensorial operator, so it suffices to prove part (2) for the chosen preferred connection.

A formula for the action of the fundamental derivative on tensor bundles in terms of ∇ and this splitting is derived in section 4.14 of [4]. The argument used there applies to all bundles constructed from completely reducible subquotients of tractor bundles, and hence to all bundles associated to $\mathcal{G}_0$. If $s\in\Gamma(\mathcal{A}M)$ corresponds to (ψ,Φ,ξ) in the splitting determined by ∇, then $D_s\varphi=\nabla_\xi\varphi-\Phi\bullet\varphi$, where $\bullet:\mathrm{End}_0(TM)\times E\to E$ is the tensorial operation induced by the infinitesimal action $\mathfrak{g}_0\times V\to V$. Now $s\in\Gamma(\mathcal{A}^1M)$ if and only if $\xi=0$ and $\Phi=0$, so $D_s\varphi=0$ in this case. On the other hand, $\xi=\Pi(s)$ so $D_s\varphi-\nabla_{\Pi(s)}\varphi=\Phi\bullet\varphi$ is tensorial. Hence we have proved (1) and (2) for $k=1$.

Next observe that naturality of the fundamental derivative implies that for $s_0,\dots,s_k\in\Gamma(\mathcal{A}M)$ we obtain the Leibniz rule

$$(*)\qquad \begin{aligned}&(D^{k+1}\varphi)(s_0,\dots,s_k)=\\ &\qquad D_{s_0}(D^k\varphi(s_1,\dots,s_k))-\textstyle\sum_{i=1}^k(D^k\varphi)(s_1,\dots,D_{s_0}s_i,\dots,s_k),\end{aligned}$$

compare with proposition 3.1 of [4]. Assuming inductively that part (2) holds for k, the second summand is evidently of order at most k in φ. Moreover, the first summand is given by $\nabla_{\Pi(s_0)}(\nabla^k\varphi(\Pi(s_1),\dots,\Pi(s_k)))$ plus terms of order at most $k-1$ in φ which immediately implies (2).

To prove (1), observe $D^k\varphi\in\Gamma(\mathcal{W}^0M)$ if and only if $D^k\varphi(s_1,\dots,s_k)=0$ provided that at least r of the sections s_i have values in $\mathcal{A}^0M$ and at least $k-r+1$ of them even have values in $\mathcal{A}^1M$. We assume this inductively and prove the corresponding property of $D^{k+1}\varphi$. Hence we take sections $s_0,\dots,s_k$, and assume that r' of them have values in $\mathcal{A}^0M$ and $k-r'+2$ even have values in $\mathcal{A}^1M$.

If s_0 has values in $\mathcal{A}^1M$, then D_{s_0} acts trivially on $\Gamma(E)$ as well as on sections of $\mathcal{A}^1M$, it maps sections of $\mathcal{A}M$ to sections of $\mathcal{A}^0M$ and sections of $\mathcal{A}^0M$ to sections of $\mathcal{A}^1M$. Hence the first summand of the right hand side of $(*)$ vanishes. In the second term of this right hand side, only summands in which s_i does not have values in $\mathcal{A}^1M$ can provide a non–zero contribution. If $s_i\in\Gamma(\mathcal{A}^0M)$, then in the corresponding summand we have $r'-1$ sections of $\mathcal{A}^0M$, and $k-r'+2=k-(r'-1)+1$ of them have values in $\mathcal{A}^1M$, so the corresponding summand vanishes by inductive hypothesis. If s_i is not a section of $\mathcal{A}^0M$, then in the corresponding summand we have r' sections of $\mathcal{A}^0M$, and $k-r'+1$ of them have values in $\mathcal{A}^1M$, so again vanishing follows by induction.

If s_0 has values in $\mathcal{A}^0M$ but not in $\mathcal{A}^1M$, then we only need to take into account that, acting on sections of $\mathcal{A}M$, D_{s_0} preserves sections of each filtration component. This shows that in each of the summands in the right hand side of $(*)$, there are $r'-1$ sections of $\mathcal{A}^0M$ inserted into $D^k\varphi$, and $k-r'+2=k-(r'-1)+1$ of them have values in $\mathcal{A}^1M$. Hence again vanishing of each summand follows by induction.

Finally, if s_0 does not have values in $\mathcal{A}^0M$, then we again need only that D_{s_0} preserves sections of each of the filtration components of $\mathcal{A}M$. This shows that in each summand of the right hand side of $(*)$, we have r' sections of $\mathcal{A}^0M$ and $k-r'+2$ of them have values in $\mathcal{A}^1M$. Thus vanishing of each summand again follows by induction, and the proof of (1) follows by symmetrization. □

3.2. **The splitting operators.** According to the the idea described in the beginning of section 3, we should next consider the bundle $S^kTM\otimes E^*\otimes F[\delta]$ of symbols as a quotient of the bundle $\tilde{\mathcal{V}}M:=S^k\mathcal{A}M\otimes E^*\otimes F[\delta]$. However, in view of proposition 3, we can already improve the basic idea. As we have noted in 3.1, the bundle $S^k\mathcal{A}M$ carries a natural filtration. Taking the tensor product with E^* and $F[\delta]$, we obtain a filtration of the bundle $\tilde{\mathcal{V}}M$ of the form

$$\tilde{\mathcal{V}}^kM\subset\cdots\subset\tilde{\mathcal{V}}^0M\subset\cdots\subset\tilde{\mathcal{V}}^{-k}M=\tilde{\mathcal{V}}M.$$

As we have observed in the beginning of section 3, there is a well defined bilinear pairing $\tilde{\mathcal{V}}M\times\mathcal{W}M\to F[\delta]$. By definition of the filtration on $\mathcal{W}M$, this factorizes to a bilinear pairing of $\mathcal{V}M\times\mathcal{W}^0M\to F[\delta]$, where $\mathcal{V}M:=\tilde{\mathcal{V}}M/\tilde{\mathcal{V}}^1M$. We denote all these pairings by $\langle\ ,\ \rangle$. As we shall see below, replacing the bundle $\tilde{\mathcal{V}}M$ by its quotient $\mathcal{V}M$ leads to a smaller set of critical weights δ.

For the same reason, it is preferable to take a further decomposition according to irreducible components of the bundle of symbols as follows. By construction, the filtration on $S^k\mathcal{A}M$ is induced by P–invariant subspaces of the representation $S^k\mathfrak{g}$, so the filtration of $\tilde{\mathcal{V}}M$ comes from a P–invariant filtration of $S^k\mathfrak{g}\otimes V^*\otimes W[\delta]$. The quotient of this space by the largest proper filtration component by construction is $S^k(\mathfrak{g}/\mathfrak{p})\otimes V^*\otimes W[\delta]$, which induces the bundle of symbols. Now if we restrict to the subgroup $G_0\subset P$, then $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, and the filtration components are just $\mathfrak{g}_0\oplus\mathfrak{g}_1$ and $\mathfrak{g}_1$. Correspondingly, the filtrations on $S^k\mathfrak{g}$ and $S^k\mathfrak{g}\otimes V^*\otimes W[\delta]$, viewed as G_0–representations, are induced from direct sum decompositions.

Since we have assumed that V and W are irreducible representations of P (and hence of G_0), the tensor product $S^k(\mathfrak{g}/\mathfrak{p})\otimes V^*\otimes W[\delta]$ splits into a direct sum $\oplus_i R_i$ of irreducible representations of G_0. Identifying $\mathfrak{g}/\mathfrak{p}$ with $\mathfrak{g}_{-1}$, we can view each R_i as a subspace in the quotient of $S^k\mathfrak{g}\otimes V^*\otimes W$ by the P–invariant filtration component with index 1. Then for each i, we can look at the P–module S_i generated by R_i. Each S_i has a P–invariant filtration with completely reducible subquotients, and the quotient of S_i by the largest proper filtration component is R_i.

Passing to associated bundles, we see that for each i, we can consider $\mathcal{G}\times_P R_i$ as a subbundle of the bundle $S^kTM\otimes E^*\otimes F[\delta]$ of symbols, and these subbundles form a decomposition into a direct sum. In particular, any section τ of the bundle of symbols can be uniquely written as $\tau=\sum_i\tau_i$ of sections $\tau_i\in\Gamma(\mathcal{G}\times_P R_i)$. Likewise, for each i, we can view $\mathcal{G}\times_P S_i$ as a subbundle of $\mathcal{V}M$, so in particular, sections of $\mathcal{G}\times_P S_i$ can be viewed as sections of $\mathcal{V}M$.

Now for each i, we denote by β_i^0 the eigenvalue by which the curved Casimir operator acts on sections of the irreducible bundle $\mathcal{G}\times_P R_i$, see proposition 2. Further, by $\beta_i^1,\dots,\beta_i^{n_i}$ we denote the different Casimir eigenvalues occurring for irreducible components in the other quotients of consecutive filtration components of S_i. Using this, we can now formulate:

Proposition 4. *Let $\Pi:\mathcal{V}M\to S^kTM\otimes E^*\otimes F[\delta]$ be the natural projection and denote the induced tensorial operator on sections by the same symbol. For each i define $\gamma_i:=\prod_{j=1}^{n_i}(\beta_i^0-\beta_i^j)$.*

Then there is a natural operator $L:\Gamma(S^kTM\otimes E^*\otimes F[\delta])\to\Gamma(\mathcal{V}M)$ such that

$$\Pi(L(\tau))=\textstyle\sum_i\gamma_i\tau_i$$

for any section $\tau=\sum_i\tau_i$ of the bundle of symbols.

Proof. Of course for each i, mapping τ to $\tau_i\in\Gamma(\mathcal{G}\times_P R_i)$ defines a tensorial natural operator. The construction of splitting operators in theorem 2 of [9] gives us, for each i, a natural differential operator $L_i:\Gamma(\mathcal{G}\times_P R_i)\to\Gamma(\mathcal{G}\times_P S_i)$. This has the property that denoting by Π_i the tensorial projection in the other direction, we obtain $\Pi_i(L(\tau_i))=\gamma_i\tau_i$ for the number γ_i defined in the proposition. As we have noted above, we can naturally view sections of $\mathcal{G}\times_P S_i$ as sections of $\mathcal{V}M$, so we can simply define $L(\tau):=\sum_i L_i(\tau_i)$. □

It is easy to give an explicit description of L, since the construction of splitting operators in [9] is explicit. Given τ, we have to choose sections $s_i\in\Gamma(\mathcal{G}\times_P S_i)\subset$

$\Gamma(\mathcal{V}M)$ such that $\Pi(s_i)=\tau_i$ for all i. Then we claim that

$$L(\tau)=\sum_i\prod_{j=1}^{n_i}(\mathcal{C}-\beta_i^j)(s_i).$$

The product for fixed i exactly corresponds to the definition of the splitting operator from [9]. Naturality of the curved Casimir operator thus implies that each of the summands equals $L_i(\tau_i)$, viewed as a section of $\mathcal{V}M$, and the claim follows.

3.3. **The quantization scheme.** We are now ready to formulate our first main result.

Theorem 5. *The map* $(\tau,\varphi)\mapsto\langle L(\tau),D^{(k)}\varphi\rangle$ *defines a natural bilinear operator* $\Gamma(S^kTM\otimes E^*\otimes F[\delta])\times\Gamma(E)\to\Gamma(F[\delta])$.

For $\tau=\sum_i\tau_i\in\Gamma(S^kTM\otimes E^*\otimes F[\delta])$, *the operator* $A_\tau:\Gamma(E)\to\Gamma(F[\delta])$ *defined by* $A_\tau(\varphi):=\langle L(\tau),D^{(k)}\varphi\rangle$ *is of order at most* k *and has principal symbol* $\sum_i\gamma_i\tau_i$.

Proof. Naturality of L, $D^{(k)}$, and the pairing $\langle\ ,\ \rangle$ implies naturality of the bilinear operator. Now fix τ and consider the operator A_τ. Choose any principal connection on $\mathcal{G}_0$ and denote by ∇ all the induced linear connections on associated vector bundles. Using proposition 3 we see that $A_\tau(\varphi)=\langle L(\tau),i(\nabla^{(k)}\varphi)\rangle$ up to terms of order at mots $k-1$ in φ. Hence A_τ is of order at most k and by the properties of the pairing $\langle\ ,\ \rangle$, the principal symbol is obatined as the result of pairing $\Pi(L(\tau))\in\Gamma(S^kTM\otimes E^*\otimes F[\delta])$ with $\nabla^{(k)}\varphi\in\Gamma(S^kT^*M\otimes E)$. Thus the result follows from proposition 4. □

Now we define a weight $\delta\in\mathbb{R}$ to be *critical* if at least one of the γ_i is zero for the chosen value of δ. For non–critical weights, our theorem immediately leads to a natural quantization:

Corollary 6. *If the weight* δ *is not critical, then the map* $\tau\mapsto A_{\sum_i\gamma_i^{-1}\tau_i}$ *defines a natural quantization for the bundles* E *and* $F[\delta]$.

We want to emphasize that the naturality result in the corollary in particular implies that in the case of the homogeneous model G/P of the AHS–structure in question the quantization is equivariant (as a bilinear map) under the natural G–action on the spaces of sections of the bundles in question (which are homogeneous vector bundles in this case). We can restrict the quantization to the big Schubert cell in G/P, which is diffeomorphic to $\mathbb{R}^n$, $n=\dim(G/P)$. The G–equivariancy on G/P immediately implies that the result is equivariant for the Lie subalgebra of vector fields on $\mathbb{R}^n$ formed by the fundamental vector fields for this G–action. Hence our quantization will specialize to an equivariant quantization in the usual sense.

3.4. **The set of critical weights.** To complete our results, we have to prove that for any choice of bundles E and F and any order k, the set of critical weights is finite. Verifying this is a question of finite dimensional representation theory. In fact, we not only get an abstract proof of finiteness of the set of critical weights, but a method to determine the set of critical weights for any given example.

In view of proposition 4 and theorem 5, it is clear that we have to understand the dependence of the Casimir eigenvalues, or more precisely of the differences $\beta_i^0-\beta_i^j$, on δ. To get a complete understanding of the set of critical weights, one has to determine the composition series (i.e. the structure of the quotients of iterated filtration components), of the P–modules S_i. Recall from 3.2 that, as a representation of G_0, S_i is simply the direct sum of all the composition factors, so essentially we have to determine the decomposition of S_i into irreducible components as a G_0–module. From proposition 2 we know how to determine the numbers β from the lowest weights of these irreducible components. Notice that changing the weight δ corresponds to taking a tensor product with a one–dimensional representation. In particular, this does not influence the basic decompositions into irreducible components, apart from the fact that each of these components is tensorized with that one–dimensional representation. As we shall see, we can get quite a bit of information without detailed knowledge of the decomposition into irreducibles, using only structural information on the possible irreducible components. We start by proving a basic finiteness result.

Theorem 7. *Fix an irreducible component $R_i \subset S^k\mathfrak{g}_{-1} \otimes V^* \otimes W[\delta]$, consider the corresponding Casimir eigenvalue β_i^0, and one of the other Casimir eigenvalues β_i^j. Then there is exactly one value of δ for which $\beta_i^0 = \beta_i^j$. Hence there are at most n_i many values for δ for which $\gamma_i = 0$, and at most $\sum_i n_i$ critical weights.*

Proof. Let us first make a few comments. The Casimir eigenvalues can be computed from lowest weights, which are defined via complexification of non–complex representations and of the Lie algebra in question. Since these complexifications do not change the decomposition into irreducible components, we may work in the setting of complex $|1|$–graded Lie algebras throughout the proof. Second, recall that for an irreducible representation of a complex semisimple Lie algebra, the negative of the lowest weight coincides with the highest weight of the dual representation. In this way, standard results on highest weights have analogs for the negatives of lowest weights.

As we have noted in proposition 2, for a representation with lowest weight $-\lambda$, the Casimir eigenvalue on sections of the corresponding induced bundle is given by $\|\lambda\|^2 + 2\langle\lambda,\rho\rangle = \langle\lambda,\lambda+2\rho\rangle$. Writing c_λ for this number, the last expression immediately shows that for two weights λ and λ', we have

$$c_{\lambda'} - c_\lambda = 2\langle\lambda'-\lambda,\lambda+\rho\rangle + \|\lambda'-\lambda\|^2. \tag{1}$$

We have to understand, how this is influenced by changing δ. Denoting by μ the highest weight associated to the representation $\mathbb{R}[1]$, which induces the bundle $\mathcal{E}[1]$, the bundle $\mathcal{E}[w]$ corresponds to the weight $w\mu$. Moving from δ to $\delta+w$ corresponds to forming a tensor product with $\mathcal{E}[w]$, and hence replacing λ by $\lambda+w\mu$ and λ' by $\lambda'+w\mu$. This means that the difference of the two weights remains unchanged, and equation (1) shows that

$$c_{\lambda'+w\mu} - c_{\lambda+w\mu} = c_{\lambda'} - c_\lambda + 2w\langle\lambda'-\lambda,\mu\rangle. \tag{2}$$

Now by definition, the weights of the representation $\mathfrak{g}$ are exactly the roots of $\mathfrak{g}$. Consequently, any weight of $S^k\mathfrak{g}$ is a sum of k roots. Further, it is well know that the highest weight of any irreducible component in a tensor product of two irreducible representations can be written as a sum of the highest weight of one of the two factors and some weight of the other factor. Passing to duals, we see that the same statement holds for the negatives of lowest weights. Thus, the negative of the lowest weight of any irreducible component of $S^k\mathfrak{g}\otimes V^*\otimes W$ can be written as a linear combination of the negative of the lowest weight of an irreducible component of $V^*\otimes W$ and at most k roots.

Now recall (see [18]) that for a complex $|1|$–graded Lie algebra, one can choose a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ and positive roots in such a way that there is a unique simple root α_0 for which the corresponding root space is contained in $\mathfrak{g}_1$. More precisely, for a root α, the corresponding root space sits in $\mathfrak{g}_i$ for $i=-1,0,1$, where i is the coefficient of α_0 in the expansion of α as a sum of simple roots. Then the center of $\mathfrak{g}_0$ is generated by the unique element $H_0\in\mathfrak{h}$ for which that $\alpha_0(H_0)=1$, while all other simple roots vanish on H_0. The orthocomplement of H_0 in $\mathfrak{h}$ is a Cartan subalgebra of the semisimple part of $\mathfrak{g}_0$.

Since the semisimple part of $\mathfrak{g}_0$ acts trivially on $\mathbb{R}[1]$, we conclude that $\mu(H)=aB(H,H_0)$ for some nonzero number a and all $H\in\mathfrak{h}$, where B denotes the Killing form of $\mathfrak{g}$. Going through the conventions, it is easy to see that actually $a<0$. By definition of the inner product, this means that for any weight ν, we have $\langle\mu,\nu\rangle=a\nu(H_0)$. Since H_0 acts by a scalar on any irreducible representation, it also acts by a scalar on all of $S^k\mathfrak{g}_{-1}\otimes V^*\otimes W$. But this implies that if $-\nu$ is the lowest weight of an irreducible component of $V^*\otimes W$, then $\nu(H_0)=a_0$ for a fixed number a_0. Consequently, if $-\nu$ is the lowest weight of an irreducible component of the quotient of two consecutive filtrations components in $\mathcal{V}M$, say the one with index ℓ by the one with index $\ell+1$, $\nu(H_0)=a_0+\ell$. In particular, if $-\lambda$ is the lowest weight of R_i, then $\lambda(H_0)=a_0-k$. Likewise if $-\lambda'$ is the lowest weight giving rise to β_i^j then $\lambda'(H_0)=a_0+\ell$ for some $\ell>-k$. Thus we conclude that $\langle\lambda'-\lambda,\mu\rangle=a(k+\ell)<0$, and formula (2) shows that λ and λ' give rise to exactly one critical weight. □

Note that the proof actually leads to an explicit formulae for the critical weights. Suppose that $-\lambda$ and $-\lambda'$ are the lowest weights of irreducible components giving rise to β_i^0 and β_i^j, and that the irreducible component corresponding to $-\lambda'$ sits in the quotient of the ℓth by the $(\ell+1)$st filtration component. Then formulae (1) and (2) from the proof show that the critical weight caused by these two components is given by

$$\delta=\frac{2\langle\lambda'-\lambda,\lambda+\rho\rangle+\|\lambda'-\lambda\|^2}{2\langle\lambda'-\lambda,\mu\rangle}\tag{3}$$

where μ is the highest weight of the representation $\mathbb{R}[1]$. In particular, we can use this formula to completely determine the set of all critical weights if we know all the P–representations S_i together with their composition structure.

3.5. **Restrictions on critical weights.** We can also get some information on the set of critical weights without this detailed knowledge. For any P–module, we can look at the restriction of the P–action to G_0 and the restriction of the infinitesimal action of $\mathfrak{p}$ to the abelian subalgebra $\mathfrak{g}_1$. Since P is the semidirect product of G_0 and $\exp(\mathfrak{g}_1)$, one immediately concludes that any subspace in a representation of P, which is G_0–invariant and closed under the infinitesimal action of $\mathfrak{g}_1$ is actually P–invariant. By construction, the actions of elements of $\mathfrak{g}_1$ on any P–module commute. Hence the iterated action of elements of $\mathfrak{g}_1$ (in the P–module $S^k\mathfrak{g}\otimes V^*\otimes W[\delta]$) on R_i define maps $S^\ell\mathfrak{g}_1\otimes R_i\to S^k\mathfrak{g}\otimes V^*\otimes W[\delta]$. By construction, the image sits in the filtration component with index $\ell-k$ as well as in S_i. Hence we actually obtain a map $\oplus_{\ell=0}^k S^\ell\mathfrak{g}_1\otimes R_i\to S_i$, which is evidently G_0–equivariant.

In particular, the image is a G_0–invariant subspace of S_i and from the construction it follows immediately that it is also closed under the infinitesimal action of $\mathfrak{g}_1$.

The upshot of this is that any G_0–irreducible component of S_i also occurs in $\oplus_{\ell=0}^k S^\ell\mathfrak{g}_1\otimes R_i$. If we determine the set of all weights δ for which an irreducible component of $\oplus_{\ell=1}^k S^\ell\mathfrak{g}_1\otimes R_i$ corresponds to the same Casimir eigenvalue as R_i, then the union of these sets for all i contains the set of all critical weights.

We next work out more details on the set of critical weights for some examples in the case of even dimensional conformal structures of arbitrary signature (p,q). (This is significantly more complicated than the case of projective structures, which is mainly considered in the literature). Hence G_0 is the conformal group $CO(p,q)$ and $\mathfrak{g}_{-1}$ is the standard representation $\mathbb{R}^n$, $n=p+q$ of this group, and we assume that n is even. As above, we may work in the complexified setting, and we will use the notation, conventions and results from [5] for weights. We will fix representations V and W and determine critical weights starting from $S^k\mathfrak{g}_{-1}\otimes V^*\otimes W$ (i.e. with $\delta=0$).

Let us assume that $S^k\mathfrak{g}_{-1}\otimes V^*\otimes W$ contains an irreducible component $R_i\cong\mathbb{R}[w]$ for some $w\in\mathbb{R}$. The decomposition of $S^\ell\mathbb{R}^{n*}$ into irreducible components is given by $S^\ell_0\mathbb{R}^{n*}\oplus S^{\ell-2}_0\mathbb{R}^{n*}[-2]\oplus S^{\ell-4}_0\mathbb{R}^{n*}[-4]\oplus\dots$, where the subscript 0 indicates the totally tracefree part. From 3.5 we thus conclude that in any case all the irreducible components of P–module S_i generated by R_i must be of the form $S^\ell\mathbb{R}^{n*}[w-2m]$ for non–negative integers ℓ and m such that $\ell+2m\le k$.

In particular, for $k=1$, the only possibility is $\mathbb{R}^n[w]$. In the notation from section 2.4 of [5], $\mathbb{R}[w]$ corresponds to the weight $(w|0,\dots)$ while $\mathbb{R}^{n*}[w]$ corresponds to $(w-1|1,0,\dots)$, which immediately shows that the corresponding critical weight is $\delta=-w$. For $k=2$, we get $S^2_0\mathbb{R}^{n*}[w]$ and $\mathbb{R}[w-2]$, which correspond to $(w-2|2,0,\dots)$ and $(w-2|0,\dots)$ and the critical weights $1-w$ and $1-w-\frac{n}{2}$.

For a general order k, the possible representations are $(w-\ell|\ell-2m,0,\dots)$ for $\ell\le k$ and $\ell-2m\ge 0$ and one easily verifies directly:

Proposition 8. *The possible critical weights caused by an irreducible component $\mathbb{R}[w]\subset S^k\mathfrak{g}_{-1}\otimes V^*\otimes W$ are contained in the set*

$$\left\{-w-1+\ell-2m+\frac{m(2+2m-n)}{\ell}:0\le\ell\le k,0\le 2m\le\ell\right\}.$$

We can derive an effective upper bound, above which there are no critical weights for quantization in *any* order. This can be viewed as a vast generalization of the results in section 3.1 of [11] on quantization of operators on functions. Observe first that it may happen that for the representations V and W inducing E and F, the tensor product $V^*\otimes W$ itself splits into several irreducible components. For example, if $V=W$, then one always has the one dimensional invariant subspace spanned by the identity. Given an irreducible component $U\subset V^*\otimes W$ and $\delta\in\mathbb{R}$, we have $S^k\mathfrak{g}_{-1}\otimes U[\delta]\subset S^k\mathfrak{g}_{-1}\otimes V^*\otimes W[\delta]$, so one may talk about symbols of type U of any order and any weight. Of course, one may apply the constructions from 3.1–3.3 directly to this subspace. As an irreducible representation of $\mathfrak{g}_0$, $U[\delta]$ has an associated lowest weight. Using this, we can now formulate

Theorem 9. *Let $-\lambda$ be the lowest weight of $U[\delta]$ and assume that δ is chosen in such a way that λ is $\mathfrak{g}$–dominant. Then for any order k, the weight δ is non–critical for symbols of type U. In particular, this always holds for sufficiently large values of δ.*

Proof. Let us first assume that λ is $\mathfrak{g}$–dominant and integral. Then there is a finite dimensional irreducible representation $\tilde{U}$ of $\mathfrak{g}$ with lowest weight $-\lambda$. We can pass to the dual $\tilde{U}^*$, and look at the $\mathfrak{p}$–submodule generated by a highest weight vector. It is well known that this realizes the irreducible representation of $\mathfrak{p}$ with highest weight λ. Passing back, we see that $U[\delta]$ can be naturally viewed as a quotient of $\tilde{U}$. Consequently, for any $k \geq 0$, we can naturally view $S^k\mathfrak{g}_{-1} \otimes U[\delta]$ as quotient of the representation $S^k\mathfrak{g} \otimes \tilde{U}$ of $\mathfrak{g}$. In particular, for any irreducible component $R_i \subset S^k\mathfrak{g}_{-1} \otimes U[\delta]$ we obtain a corresponding $\mathfrak{g}$–invariant subset $\tilde{S}_i \subset S^k\mathfrak{g} \otimes \tilde{U}$ (which can be taken to be $\mathfrak{g}$–irreducible) with $\mathfrak{p}$–irreducible quotient R_i. It is also evident that applying the natural map $S^k\mathfrak{g} \otimes \tilde{U} \to S^k\mathfrak{g} \otimes U[\delta]$ to $\tilde{S}_i$ and then factoring by the filtration component of degree zero, the image has to contain the $\mathfrak{p}$–submodule S_i generated by R_i. In particular, any $\mathfrak{g}_0$–irreducible component of S_i also has to occur in $\tilde{S}_i$.

But for the bundles corresponding to irreducible representations of $\mathfrak{g}$, the critical weights are described in lemma 2 of [9] in terms of the Kostant Laplacian $\square$ and the value c_0 by which the (algebraic) Casimir operator of $\mathfrak{g}$ acts on the irreducible representation $\tilde{S}_i$. Now c_0 coincides with the Casimir eigenvalue β_i^0 in our sense and hence lemma 2 of [9] shows that $\beta_i^j - \beta_i^0$ can be computed as twice the eigenvalue of $\square$ on the irreducible component giving rise to β_j^i. Now Kostant's theorem from [13] in particular implies that the kernel of $\square$ on $\tilde{S}_i$ consists of R_i (viewed as a $\mathfrak{g}_0$–invariant subspace) only. This implies the result if λ is $\mathfrak{g}$–dominant and integral.

More is known about the eigenvalues of $\square$, however. The lemma in Cartier's remarks ([10]) to Kostant's article shows that all eigenvalues of square are non–positive. In the terminology of the proof of theorem 7 this means that $c_{\lambda'} - c_\lambda < 0$. There we have also seen that $\langle \lambda' - \lambda, \mu \rangle < 0$, so formula (2) from that proof shows that $c_{\lambda'+w\mu} - c_{\lambda+w\mu} < 0$ for $w \geq 0$. Now if $-\lambda$ is the lowest weight of a finite dimensional irreducible representation of $\mathfrak{p}$, then λ is $\mathfrak{p}$–dominant and $\mathfrak{p}$–integral. But this means that $\lambda + w\mu$ is $\mathfrak{g}$–dominant for sufficiently large values of w and $\mathfrak{g}$–integral for all integral values of w, which implies all the remaining claims. $\square$

3.6. **Low order quantizations for even–dimensional conformal structures.** Let us move to more complete examples in the setting from above. We will restrict to the cases that $V^* \otimes W \cong \mathbb{R}$ and $V^* \otimes W \cong \mathbb{R}^n$, and to orders at most three in the first case and at most two in the second case. For $V^* \otimes W \cong \mathbb{R}$, we get quantizations on density bundles, which can be compared to available results in the literature. The case $V^* \otimes W \cong \mathbb{R}^n$ can be used to understand operators mapping weighted one–forms to densities and, vice versa, mapping densities to weighted one–forms.

We have already noted in 3.5 that the decomposition of $S^k\mathfrak{g}_{-1}$ is given by $\oplus_{\ell \leq k/2} S^\ell \mathfrak{g}_{-1}[2\ell]$.

First order operators on densities. Here the symbol representation is $\mathfrak{g}_{-1} \cong \mathbb{R}^n$, so this is irreducible and corresponds to the weight $(1|1, 0\dots)$. Likewise, $\mathfrak{g}$ is an irreducible representation of $\mathfrak{g}$, and there is only one relevant level which may produce critical weights, namely $\mathfrak{g}_0 \cong \Lambda^2\mathbb{R}^n[2] \oplus \mathbb{R}$, which is the quotient of the filtration components of degrees 0 and 1. The summands correspond to the weights $(0|1, 1, 0, \dots) \oplus (0|0, \dots)$ and we obtain the critical weights $-n$ and -2.

Second order operators on densities. The symbol representation splits into two irreducible components R_1 and R_2 corresponding to the weights $(2|2, 0, \dots)$ (tracefree symbols) and $(2|0, \dots)$ (symbols which are pure trace, i.e. of Laplace type). Also, the representation $S^2\mathfrak{g}$ of $\mathfrak{g}$ is not irreducible any more, but splits into four irreducible components. One of them is a trivial representation (corresponding to the Killing form) and one is isomorphic to $\Lambda^4\mathbb{R}^{n+2}$. These two components are entirely contained in the filtration component of degree -1, so they do not contribute to the quotient by the largest filtration component. One of the remaining two irreducible components is isomorphic to $S^2_0\mathbb{R}^{n+2}$. The quotient of this component by its intersection with the largest filtration component is exactly R_2, so all of S_2 must be contained in this part. Finally, there is the highest weight component $\circledcirc^2\mathfrak{g} \subset S^2\mathfrak{g}$ (the Cartan product of two copies of $\mathfrak{g}$), whose quotient by the largest filtration component is R_1. Hence S_1 is contained in this component.

To determine the possible critical weights it thus suffices to analyze the composition structure of the representations $\circledcirc^2\mathfrak{g}$ and $S^2_0\mathbb{R}^{n+2}$. This can be done fairly easily using the description of representations of $\mathfrak{g}$ in terms of their $\mathfrak{p}$–irreducible quotients from section 3 of [3], in particular the result in lemma 3.1 of this article. One has to use the fact that the Lie algebra cohomology groups that occur are algorithmically computable using Kostant's version of the Bott–Borel–Weil theorem.

This shows that in the language of weights, the two relevant levels of $\circledcirc^2\mathfrak{g}$ decompose as

$$\begin{aligned}&(1|2,1,0,\dots)\oplus(1|1,0,\dots)\\&(0|2,2,0,\dots)\oplus(0|2,0,\dots)\oplus(0|1,1,0,\dots)\oplus(0|0,\dots),\end{aligned}$$

and consequently, one obtains the critical weights -3, -2, $-2-n$, $-1-n$, $(-2-n)/2$, and $(-4-n)/2$.

For the case of symbols which are pure trace, the decompositions of the level for the index -1 is irreducible corresponding to the weight $(1|1,0\dots)$, while the level for index zero decomposes as $(0|2,0,\dots)\oplus(0|0,\dots)$. This gives rise to the critical weights -2, -1 and $(-2-n)/2$.

Third order operators on densities. The analysis is closely analogous to the second order case, we mainly include the results for comparison to [2]. The symbol representation splits into two irreducible components and again these two components correspond to two of the seven irreducible components in $S^3\mathfrak{g}$. Namely, tracefree symbols ($S^3_0\mathbb{R}^n$) correspond to the highest weight component $\circledcirc^3\mathfrak{g}$, while trace–symbols ($\mathbb{R}^n[2]$) correspond to the Cartan product $\mathfrak{g}\circledcirc S^2_0\mathbb{R}^{n+2}$. The relevant parts of the composition series for these two representations of $\mathfrak{g}$ can be determined as in the second order case. From these, one computes the critical weights. In the tracefree case, one obtains -4, -3, -2, $-4-n$, $-3-n$, $-2-n$, $(-7-n)/2$, $(-4-n)/2$, $(-8-n)/3$, $(-8-2n)/3$, $(-6-n)/3$, and $(-6-2n)/3$. For trace–type symbols, we get the critical weights -1, -2, -4, $-5/2$, $-4/3$, $(-4-n)/2$, $(-4-n)/3$, $(-6-n)/3$, and $(-4-2n)/3$. These are the critical weights from [2], plus quite a few additional ones. We'll comment on that in 3.7 below.

First order operators for $V^*\otimes W \cong \mathbb{R}^n$. Here the symbol representation decomposes as

$$\mathbb{R}^n\otimes\mathbb{R}^n = R_1\oplus R_2\oplus R_3 = S^2_0\mathbb{R}^n\oplus\Lambda^2\mathbb{R}^n\oplus\mathbb{R}[2],$$

or in weights $(2|2,0,\dots)\oplus(2|1,1,0,\dots)\oplus(2|0,\dots)$. There is only one relevant level in the composition series of $\mathfrak{g}\otimes\mathbb{R}^n$, which can be determined by decomposing the tensor product $\mathfrak{g}_0\otimes\mathbb{R}^n$ into irreducibles. In terms of weights, the result is

$(1|2,1,0,\dots)\oplus(1|1,1,1,0,\dots)\oplus 2(1|1,0\dots)$, so the last irreducible component occurs with multiplicity two. Decomposing the tensor products $R_i\otimes\mathbb{R}^n$, one concludes that S_1 can only contain the first and a copy of the last irreducible components, while S_3 can only contain one copy of the last irreducible component. Consequently, there are three critical weights for skew symmetric symbols (which turn out to be -1, -4, and $-n$) but only two (namely -3 and $-2-n$) for symmetric symbols. For trace type symbols we obtain only one critical weight, namely -2, which agrees with the result from 3.5.

Second order operators for $V^*\otimes W\cong\mathbb{R}^n$. Here the symbol representation $S^2\mathbb{R}^n\otimes\mathbb{R}^n$ decomposes into four irreducible components, in weight notation, it is given by

$$(3|3,0,\dots)\oplus(3|2,1,0,\dots)\oplus 2(3|1,0,\dots).$$

Here one of the two copies of $\mathbb{R}^n[2]$ is contained in $S^2_0\mathbb{R}^n\otimes\mathbb{R}^n$, while the other comes from the trace part. Let us write this decomposition as $R_1\oplus\dots\oplus R_4$, with R_4 coming from the trace part. From above, we know that $S^2\mathfrak{g}$ contains the irreducible components $\circledcirc^2\mathfrak{g}$ and $S^2_0\mathbb{R}^{n+2}$, which correspond to $S^2_0\mathbb{R}^n$ and $\mathbb{R}[2]\subset S^2\mathbb{R}^n$, respectively. Consequently, we can determine the relevant composition factors for S_1, S_2, and S_3 by decomposing the tensor products of the composition factors of $\circledcirc^2\mathfrak{g}$ as listed above with $\mathbb{R}^n$, and then checking with of the components may be contained in each S_i. For S_4, we proceed similarly with $S^2_0\mathbb{R}^{n+2}$ replacing $\circledcirc^2\mathfrak{g}$.

For the first relevant level (corresponding to filtration index -1), we first have to decompose $(1|2,1,0,\dots)\otimes(1|1,0,\dots)$ which gives

$$(2|3,1,0,\dots)\oplus(2|2,2,0,\dots)\oplus(2|2,1,1,0,\dots)\oplus(2|2,0,\dots)\oplus(2|1,1,0,\dots).$$

Second, $(1|1,0,\dots)\otimes(1|1,0,\dots)\cong(2|2,0,\dots)\oplus(2|1,1,0,\dots)\oplus(2|0,\dots)$.

Looking at the tensor products $R_i\otimes\mathbb{R}^n$, we conclude that S_1 can only contain $(2|3,1,0,\dots)$ and $(2|2,0,\dots)$, S_3 can only contain $(2|2,0,\dots)$ and $(2|1,1,0,\dots)$, while all components of the first sum may occur in S_2. Hence from this level, we get the critical weights -4 and $-4-n$ for R_1. For R_2, we obtain the critical weights -1, -3, -5, $-1-n$, and $-3-n$, while for R_3, the critical weights are -2, -4, and $-2-n$.

The second relevant level is dealt with in an analogous way. The result is that for R_1, we get the additional critical weights -3, $-3-n$, $(-4-n)/2$, and $(-7-n)/2$. For R_2, we obtain $-3/2$, $-7/2$, $(-1-n)/2$, $(-4-n)/2$, $(-7-n)/2$, $(-3-2n)/2$. Finally, for R_3, we get the additional critical weights -1, $-5/2$, and $(-4-n)/2$. A direct evaluation shows that for R_4 we exactly the same critical weights as for R_3 (although the bundle involved is different).

3.7. **Discussion and Remarks.** (1) Note that the results in the examples from 3.6 are consistent with theorem 9, which implies that in all the cases discussed in 3.6 all critical weights have to be negative.

(2) From the examples of operators on densities discussed in 3.6 it is evident that the sets of critical weights we obtain with our general procedure are far from being optimal. It is actually easy to see why this happens, and even to partly improve the procedure, to get smaller sets of critical weights. The point here is that part (1) of proposition 3 can be heavily improved in special cases, and in particular for the fundamental derivative on densities. In the case of densities, already the values of a single fundamental derivative do not exhaust $\mathcal{A}^0M[w]$. On the contrary, projecting to $(\mathcal{A}^0M/\mathcal{A}^1M)[w]\cong\Lambda^2TM[w-2]\oplus\mathcal{E}[w]$, the values always lie in the

density summand only. By naturality of the fundamental derivative, this implies that higher order fundamental derivatives always will lie in subbundles which are much smaller than the bundle $\mathcal{W}^0M$ from proposition 3.

Knowing this, one can run the analog of the procedure from 3.2 and 3.3 on the quotient by the annihilator of this subbundle, which will be significantly smaller than the bundle $\mathcal{V}M$ we have used. For this smaller quotient, there will be less irreducible components in the individual subquotients and hence less critical weights. In fact, it is easy to see directly that in the examples discussed in 3.6 most (but not all of) the superfluous critical weights will disappear.

(3) In the case $V \otimes W \cong \mathbb{R}^n$ the set of critical weights we have obtained in 3.6 will be closer to the optimum than in the case of densities. As we have noted, this case can be used to study both quantizations for operators mapping sections of $\mathcal{E}[w]$ to sections of $TM[w+\delta]$ and for operators mapping sections of $T^*M[w] \cong TM[w-2]$ to sections of $\mathcal{E}[w+\delta]$. While these two cases are completely symmetric from our point of view, this is no more true if one looks at the best possible sets of critical weights. The point is that in the first case, the value of the splitting operator will be paired with $D^{(k)}f \in \Gamma(S^k\mathcal{A}^*M[w])$ for $f \in \Gamma(\mathcal{E}[w])$, and as discussed above, this has values in a much smaller subbundle than just the filtration component of degree zero. In the second interpretation, we will have to pair it with $D^{(k)}\alpha \in \Gamma(S^k\mathcal{A}^*M \otimes T^*M[w])$ for $\alpha \in \Gamma(T^*M[w])$, and the values of this operator fill a more substantial part of the filtration component of degree zero. Hence in the first case, we can remove more superfluous critical weights than in the second one.

(4) There is a systematic way to derive explicit formulae for the procedures we have developed in terms of distinguished connections (e.g. the Levi–Civita connections of the metrics in a conformal class), but this becomes quickly rather tedious. In view of the construction, the main point is to obtain an explicit formula for the curved Casimir operator on irreducible components of $S^k\mathcal{A}M$. This can be done along the lines of proposition 2.2 of [5] which holds (with obvious modifications) for general AHS–structures.

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