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Diffeomorphism covariant star products and noncommutative gravity

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Abstract. The use of a diffeomorphism covariant star product enables us to construct diffeomorphism invariant gravities on noncommutative symplectic manifolds without twisting the symmetries. As an example, we construct noncommutative deformations of all two-dimensional dilaton gravity models thus overcoming some difficulties of earlier approaches. One of such models appears to be integrable. We find all classical solutions of this model and discuss their properties.

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1. Introduction

The diffeomorphism invariance is one of the most important features of any gravity theory. Despite recent advances in noncommutative gravity [1] there is still no unique and totally satisfactory way to realize the full diffeomorphism group on a noncommutative manifold. One can use the Seiberg-Witten approach [2] which reduces all symmetries, including the diffeomorphisms, to the commutative ones at the expense of a non-linear field redefinition. However, calculations beyond the leading order in the noncommutativity parameter are hardly possible in this approach, see, e.g, [3]. Another way to extend the diffeomorphism transformations to noncommutative spaces is to make their action twisted [4]. One can construct a full twisted-invariant noncommutative gravity action having just the right number of symmetries. However, the twisted symmetries are not *bona fide* physical symmetries. One cannot use them to gauge away any degrees of freedom.

Here we develop a different approach to the diffeomorphism invariance on noncommutative spaces. The star product we use is a particular case of the geometric construction by Fedosov [5] suggested in [6]. This star product is diffeomorphism covariant. As we show, the tensor algebra built up with this star product has many nice properties and is suitable for the construction of gravity theories on noncommutative manifolds. As an example, we consider dilaton gravities in two dimensions and show that all of them have fully diffeomorphism invariant noncommutative counterparts. One of these models (a conformally transformed Witten black hole model) appears to be classically integrable in the noncommutative case. We construct all solutions of this model and discuss briefly their properties.

2. The star product

Let us suppose that the space-time $\mathcal{M}$ is a symplectic manifold. That is, $\mathcal{M}$ is equipped with a closed non-degenerate two-form ω. In a local coordinate system this implies that

$$\partial_\mu \omega_{\nu\rho} + \partial_\rho \omega_{\mu\nu} + \partial_\nu \omega_{\rho\mu} = 0 \,. \tag{1}$$

The inverse of $\omega_{\mu\nu}$, $\omega^{\nu\rho}$, is defined through the equation

$$\omega_{\mu\nu}\omega^{\nu\rho} = \delta_\mu^\rho \,. \tag{2}$$

$\omega^{\nu\rho}$ is a Poisson bivector. It satisfies the Jacobi identities as a consequence of (1).

Let us choose a Christoffel symbol on $\mathcal{M}$ such that the symplectic form is covariantly constant,

$$\nabla_\mu \omega_{\nu\rho} = \partial_\mu \omega_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\omega_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\omega_{\nu\sigma} = 0. \tag{3}$$

Thus $\mathcal{M}$ is a Fedosov manifold [7]. Let us suppose that the connection Γ is flat, i.e. the curvature tensor‡

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\sigma\nu} - \partial_\sigma \Gamma^\mu_{\rho\nu} + \Gamma^\lambda_{\sigma\nu}\Gamma^\mu_{\rho\lambda} - \Gamma^\lambda_{\rho\nu}\Gamma^\mu_{\sigma\lambda} \tag{4}$$

vanishes. We also suppose that the connection is symmetric, $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$, which implies that the torsion vanishes. Due to these two assumptions, the covariant derivatives commute.

We can now define a star product

$$\begin{aligned} f \star g &= f \exp\left(\overleftarrow{\nabla}_\mu \frac{i}{2} \omega^{\mu\nu} \overrightarrow{\nabla}_\nu \right) g \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \omega^{\mu_1\nu_1} \dots \omega^{\mu_n\nu_n} (\nabla_{\mu_1} \dots \nabla_{\mu_n} f)(\nabla_{\nu_1} \dots \nabla_{\nu_n} g) \,. \end{aligned} \tag{5}$$

The algebra of smooth functions on $\mathcal{M}$ with this multiplication will be denoted $\mathcal{A}_\omega$. This is an associative noncommutative algebra. The product (5) solves the deformation quantization problem [8] in the sense that for any two scalar functions f and g on $\mathcal{M}$

$$f \star g = f \cdot g + \frac{i}{2} \omega^{\mu\nu} \partial_\mu f \cdot \partial_\nu g + O\left((\omega^{\mu\nu})^2 \right), \tag{6}$$

i.e., the linear in $\omega^{\mu\nu}$ term reproduces the Poisson bracket between f and g. For another diffeomorphism covariant star product introduced in [9] only e weaker property holds. Namely, the Poisson bracket is given by the linear order of the star-commutator.

Note, that the use of non-flat connections leads to non-associative star products [10, 11].

‡ In [6] the tensor (4) is called the Riemannian curvature. We shall avoid this terminology since no Riemannian structure (metric) is assumed on $\mathcal{M}$.

The star product (5) is a particular case of the Fedosov construction [5], which was proposed in [6]. Let us formulate some basic properties of the product (5). Obviously, this product may be extended from functions to arbitrary tensors. Therefore, in the formulae below f and g are tensor fields. First we observe that due to (3)

$$\omega^{\mu\nu} \star f = \omega^{\mu\nu} \cdot f \,, \tag{7}$$

i.e., $\omega^{\mu\nu}$ belongs to the center of corresponding commutator algebra. One can also see that ∇_μ is a derivation on $\mathcal{A}_\omega$,

$$\nabla_\mu (f \star g) = (\nabla_\mu f) \star g + f \star (\nabla_\mu g). \tag{8}$$

The product (5) is hermitian,

$$\overline{(f \star g)} = \bar{g} \star \bar{f} \,, \tag{9}$$

where the bar denotes complex conjugation.

In the context of this work, the most important property of the star product (5) is the *diffeomorphism covariance*. Let $f \to f'$ be a diffeomorphism transformation, then

$$(f \star g)' = f' \star' g' \,, \tag{10}$$

where $\star'$ is given by the formula (5) where $\omega^{\mu\nu}$ and the connection are transformed under the diffeomorphism in the standard way (as in the commutative geometry). Again, f and g in (10) may be tensors of any rank.

The star multiplication commutes with lowering/raising indices with $\omega_{\mu\nu}$ and $\omega^{\mu\nu}$, respectively. For example, $(f_\nu \omega^{\nu\mu}) \star g = f^\mu \star g = (f_\nu \star g)\omega^{\nu\mu}$.

There is a natural integration measure [5, 6, 12]

$$d\mu(x) = (\det(\omega^{\mu\nu}))^{-\frac{1}{2}} \, dx \,. \tag{11}$$

It is easy to check that with respect to this measure the star product of tensors is closed provided all indices are contracted in pairs,

$$\int_{\mathcal{M}} d\mu(x) f_{\mu\nu\dots\rho} \star g^{\mu\nu\dots\rho} = \int_{\mathcal{M}} d\mu(x) f_{\mu\nu\dots\rho} \cdot g^{\mu\nu\dots\rho} \,. \tag{12}$$

This equation also implies that contraction of all indices and integration with the measure (11) is a trace on the star-tensor algebra over $\mathcal{M}$. Given nice properties of the tensor algebra one can expect that this approach will also shed a new light onto the problem of construction of Poisson structures and star products on the exterior algebras (differential forms), see [13].

In the Riemannian geometry a torsionless connection is uniquely fixed by the condition of covariant constancy of the metric. In the symplectic geometry, the condition (3) does not fix the connection uniquely, even if one requires that the torsion and the curvature vanish. As noted e.g. in [6], a flat torsionless symplectic connection trivializes, $\nabla_\mu = \partial_\mu$, in a Darboux coordinate system. The Darboux coordinates are defined up to a symplectomorphysm, which is a diffeomorphism preserving $\omega_{\mu\nu}$. Therefore, to fix a star product one has to fix a symplectic form $\omega_{\mu\nu}$ *and* a symplectomorphism.

3. Noncommutative gravity in two dimensions

Let us first consider generic two-dimensional dilaton gravity on a commutative space [14]. The Euclidean first-order action reads [15]

$$S_{\rm c} = \int d^2x\, \epsilon^{\mu\nu} \left[\bar{Y}(\partial_\mu e_\nu - i\rho_\mu e_\nu) + Y(\partial_\mu \bar{e}_\nu + i\rho_\mu \bar{e}_\nu) \right. \\ \left. + \Phi \partial_\mu \rho_\nu + iV(\Phi)\bar{e}_\mu e_\nu \right] . \tag{13}$$

We use complex fields

$$Y = \frac{1}{\sqrt{2}}(Y^1 + iY^2), \qquad \bar{Y} = \frac{1}{\sqrt{2}}(Y^1 - iY^2), \\ e = \frac{1}{\sqrt{2}}(e^1 + ie^2), \qquad \bar{e} = \frac{1}{\sqrt{2}}(e^1 - ie^2), \tag{14}$$

where the superscript is a $U(1)$ (Euclidean Lorentz) index. $\epsilon^{\mu\nu}$ is the Levi-Civita symbol, $\epsilon^{12} = -\epsilon^{21} = 1$. e_μ is the zweibein, and ρ_μ is the spin connection. Φ is the dilaton, and Y and $\bar{Y}$ are auxiliary fields which generate the torsion constraint. $V(\Phi)$ is an arbitrary function of the dilaton, which defines a particular model within the family. Most general dilaton gravity actions contain also a term $U(\Phi)\bar{Y}Y$. Such a term can be removed by a conformal redefinition of the metric§. An extensive list of physically relevant potentials V and U can be found in [14, 16].

A noncommutative extension of the action (13) reads

$$S_{\rm nc} = \int d\mu(x)\, \omega^{\mu\nu} \star \left[\bar{Y} \star (\nabla_\mu e_\nu - i\rho_\mu \star e_\nu) + (\nabla_\mu \bar{e}_\nu - i\bar{e}_\mu \star \rho_\nu) \star Y \right. \\ \left. + \Phi \star (\nabla_\mu \rho_\nu - i\rho_\mu \star \rho_\nu) + i\bar{e}_\mu \star V_\star(\Phi) \star e_\nu \right] . \tag{15}$$

This action is invariant under the following noncommutative $U(1)_\star$ gauge transformations

$$\delta Y = i\lambda \star Y, \qquad \delta \bar{Y} = -i\bar{Y} \star \lambda, \\ \delta e_\mu = i\lambda \star e_\mu, \qquad \delta \bar{e}_\mu = -i\bar{e}_\mu \star \lambda, \\ \delta \rho_\mu = \partial_\mu \lambda - i[\rho_\mu, \lambda]_\star, \qquad \delta \Phi = i[\lambda, \Phi]_\star , \tag{16}$$

which will be treated as a noncommutative extension of the Euclidean Lorentz symmetry. In the equations above, $[\cdot,\cdot]_\star$ denotes the star-commutator, $[\lambda, \Phi]_\star \equiv \lambda \star \Phi - \Phi \star \lambda$. The invariance of the action (15) with respect to diffeomorphisms is ensured by the diffeomorphism covariance of the star product.

In two dimensions any antisymmetric tensor is proportional to the Levi-Civita symbol,

$$\omega^{\mu\nu} = B(x)\epsilon^{\mu\nu}. \tag{17}$$

Therefore, $\det(\omega^{\mu\nu}) = B^2$, $d\mu(x)\omega^{\mu\nu} = d^2x\epsilon^{\mu\nu}$, and the action (15) indeed reproduces (13) in the commutative limit‖.

§ This conformal redefinition does not provide a full equivalence between models even classically since the corresponding conformal transformation is in general valid only locally. Here we ignore this subtlety and consider the theories with vanishing $U(\Phi)$.

‖ Strictly speaking, the limit $\omega^{\mu\nu} \to 0$ does not exist since $\omega_{\mu\nu}$ diverges. Instead of taking $\omega^{\mu\nu} \to 0$, one has to replace $\omega^{\mu\nu}$ by $\alpha\omega^{\mu\nu}$ in the star product (5) and then take $\alpha \to 0$.

Note, that the deformation (15) of (13) is "fairly unique". This means the following. Of course, one has to make a choice, whether the noncommutative gauge transformations act on Y from the left, or from the right. If they act from the left, as in (16), the gauge transformation for $\bar{e}_\mu$ is defined uniquely since we are going to couple $\bar{e}_\mu$ to Y. The transformations for $\bar{Y}$ and e_μ then follow by complex conjugation. The transformation rules of ρ_μ and Φ and the action (15) are then fixed uniquely by requiring that the commutative limit is (13), and that the action is real and gauge invariant provided we do not include any terms containing products of Y and $\bar{Y}$. This is in contrast to the term $U(\Phi)\bar{Y}Y$ which we discussed briefly below eq. (14). Any interaction of the form $i\sum_a \bar{e}_\mu \star W_\star^{[a]}(\Phi) \star e_\nu \star \bar{Y} \star \tilde{W}_\star^{[a]}(\Phi) \star Y$ with the only restriction $\sum_a W^{[a]}(\Phi)\tilde{W}^{[a]}(\Phi) = U(\Phi)$ will (after the integration over $\mathcal{M}$) be real, gauge-invariant, and possess a correct commutative limit. A physical interpretation of this enormous ambiguity remains unclear. To avoid this ambiguity we shall not consider any interactions containing both $\bar{Y}$ and Y.

We like to stress, that in this approach one can construct a deformation of *any* $2D$ dilaton gravity model in such a way that the deformed model is invariant under diffeomorphisms and deformed Lorentz transformations. Previously, a noncommutative deformation with untwisted symmetries was constructed only for the Jackiw-Teitelboim [17] model (linear $V(\Phi)$, $U(\Phi) = 0$) by using its equivalence to a BF model with Yang-Mills type symmetries [18]. (This model appeared to be even quantum integrable [19]). Later it was demonstrated, that one cannot add higher order terms to the linear potential of the model [18] and preserve the number of symmetries in a noncommutative gravity theory in two dimensions [20].

There is an interesting relation to the twisted-symmetric models. By taking a constant $\omega^{\mu\nu}$ and "gauge fixing" the connection in ∇_μ to zero one arrives at twisted diffeomorphism invariant gravity in $2D$ (cf. [21]). This is in parallel to the observation made in [22] in the context of the Yang-Mills symmetries. By fixing a gauge in the gauge covariant star product one obtains a twisted-symmetric Yang-Mills theory [23].

Let us consider a noncommutative version of the Witten black hole [24]. After a conformal redefinition of the metric [25] in the commutative case one obtains the action (13) with a constant potential

$$V(\Phi) = \Lambda\,. \tag{18}$$

This model is almost trivial since it describes the flat metric only. In the noncommutative case, the equations of motion following from the action (15) with the potential (18) read

$$\epsilon^{\mu\nu}(\nabla_\mu \rho_\nu - i\rho_\mu \star \rho_\nu) = 0, \tag{19}$$

$$\epsilon^{\mu\nu}(\nabla_\mu e_\nu - i\rho_\mu \star e_\nu) = 0, \tag{20}$$

$$\epsilon^{\mu\nu}(\nabla_\mu \bar{e}_\nu - i\bar{e}_\mu \star \rho_\nu) = 0, \tag{21}$$

$$\nabla_\nu \Phi + i[\Phi, \rho_\nu]_\star - ie_\nu \star \bar{Y} + iY \star \bar{e}_\nu = 0, \tag{22}$$

$$\nabla_\nu \bar{Y} + i\bar{Y} \star \rho_\nu - i\Lambda \bar{e}_\nu = 0, \tag{23}$$

$$\nabla_\nu Y - i\rho_\nu \star Y + i\Lambda e_\nu = 0, \tag{24}$$

Note, that in the equations (19) - (24) one can replace the covariant derivatives ∇_μ by the partial derivatives ∂_μ (except for the covariant derivatives hidden in the star product). The reason is that these derivatives either act on scalars or appear contracted with the Levi-Civita symbol, as, for example, $\epsilon^{\mu\nu}\nabla_\mu\rho_\nu$ in (19).

In all commutative $2D$ dilaton gravity theories there is a quantity $\mathcal{C}(\Phi,\bar{Y},Y)$ which is absolutely conserved, $\partial_\mu\mathcal{C}(\Phi,\bar{Y},Y)=0$, due to the equations of motion. The existence of this quantity is essential for the classical integrability of dilaton gravities. For example, for a constant dilaton potential V given by (18) the conserved quantity reads $\mathcal{C} = Y\bar{Y} + \Lambda\Phi$.

Let us try to define a similar quantity in the noncommutative case. This can be done in the same way as in commutative Euclidean theories [15]. One only has to fix properly the order of multipliers. Let us multiply eq. (23) by Y from the left and add to the equation (24) multiplied by $\bar{Y}$ fro the right. Then, use eq. (22) to get rid of the terms containing e and $\bar{e}$. We have

$$(\nabla_\mu - i\,\mathrm{ad}_\star\,\rho_\mu)(Y\star\bar{Y} + \Lambda\Phi) = 0\,, \tag{25}$$

where $(\mathrm{ad}_\star\,a)b = [a,b]_\star$ is the adjoint action. In contrast to the commutative case, equation (25) contains a $U(1)_\star$ covariant derivative. The model is nevertheless classically integrable. The equation (19) yields that ρ_μ is a trivial $U(1)_\star$ connection at least locally, i.e.,

$$\rho_\mu = iu\star\nabla_\mu u^{-1}\,, \tag{26}$$

where $\bar{u}=u^{-1}$ and $u\star u^{-1}=1$. Let us introduce gauge transformed fields

$$\begin{aligned} Y &= u\star Y^u, & e_\mu &= u\star e^u_\mu,\\ \bar{Y} &= \bar{Y}^u\star u^{-1}, & \bar{e}_\mu &= \bar{e}^u_\mu\star u^{-1},\\ \Phi &= u\star\Phi^u\star u^{-1}. \end{aligned} \tag{27}$$

Next, let us substitute the fields (27) into eqs. (20) - (24). The equations still have the same form in terms of transformed the fields $\{Y^u,\bar{Y}^u,e^u_\mu,\bar{e}^u_\mu,\Phi^u\}$, except that ρ_μ disappears. One then easily finds a general solution

$$e^u_\mu = \nabla_\mu E, \qquad Y^u = -i\Lambda E, \qquad \Phi^u = b - \frac{1}{\Lambda}Y^u\star\bar{Y}^u\,, \tag{28}$$

where E is an arbitrary complex function, b is an arbitrary real constant. The solutions for $\bar{e}^u_\mu$ and $\bar{Y}^u$ are given by complex conjugation.

The solution depends on three arbitrary real functions (one parametrizes u, and the other two are the real and imaginary parts of E, respectively). This corresponds to the presence of three local symmetries of the action (15) (two diffeomorphisms and one $U(1)_\star$).

One can define an $U(1)_\star$ (Lorentz) invariant tensor

$$g_{\mu\nu} = \frac{1}{2}(\bar{e}_\mu\star e_\nu + \bar{e}_\nu\star e_\mu), \tag{29}$$

which may be identified with the Riemannian metric on $\mathcal{M}$. The line element $(ds)^2 = g_{\mu\nu}dx^\mu dx^\nu$ is diffeomorphism invariant in the standard sense, i.e., it does not change under the coordinate transformations.

The solution (26), (28) is flat. The connection ρ_μ is a gauge-trivial one, and the zweibein e_μ^u can be reduced, at least locally, to the unit one. This can be done, e.g., by choosing the coordinates $x^1 = \mathrm{Re}\, E$, $x^2 = \mathrm{Im}\, E$. However, the metric (29) *need not be trivial* since it is constructed by using the star product. In general, $g_{\mu\nu}$ cannot be reduced to the unit one by choosing a suitable coordinate system even locally.

4. Conclusions

In this paper we considered a diffeomorphism covariant star product on a symplectic manifold and studied the properties of corresponding tensor algebra. We constructed noncommutative diffeomorphism invariant deformations of all dilaton gravities in two dimensions thus overcoming some difficulties of earlier approaches. For the simplest model with a constant dilaton potential we were able to find all classical solutions. Although the solutions correspond to a flat zweibein e_μ and to a flat spin-connection ρ_μ, the metric need not be flat.

There are many possible extensions of the results reported above. The most immediate one is to consider the dilaton potentials other than the constant one (18). It is not obvious whether corresponding noncommutative gravities will be integrable. An extension to four-dimensional gravities also looks rather straightforward. Although $SO(1,3)$ and $SO(4)$ do not close on noncommutative spaces, one can either work in the metric formalism thus avoiding Lorentz transformations, or add a trivial Lorentz or $SO(4)$ connection to the covariant derivative ∇ in the star product.

The restriction to symplectic manifolds may be weakened. One can consider instead of symplectic manifolds regular Poisson manifolds where the Fedosov construction [5] also works well.

In the model we considered in this paper the symplectic geometry plays the role of "external conditions" which were not restricted by any equations of motion. It would be interesting to make dynamical fields out of the symplectic structure $\omega_{\mu\nu}$ and the symplectomorphism which define the star product (see the discussion at the end of sec. 2). In this respect we like to mention the approach of Pinzul and Stern [26].

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