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Vienna, Preprint ESI 2156 (2009)

June 29, 2009

Supported by the Austrian Federal Ministry of Education, Science and Culture
Available via http://www.esi.ac.at

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Abstract

Let F be a non-Archimedean locally compact field, let G be a split connected reductive group over F. For a parabolic subgroup $Q \subset G$ and a ring L we consider the G-representation on the L-module

$$(*) \qquad C^{\infty}(G/Q, L)/\sum_{Q' \supsetneq Q} C^{\infty}(G/Q', L).$$

Let $I \subset G$ denote a Iwahori subgroup. We define a certain free finite rank L-module $\mathfrak{M}$ (depending on Q; if Q is a Borel subgroup then $(*)$ is the Steinberg representation and $\mathfrak{M}$ is of rank one) and construct an I-equivariant embedding of $(*)$ into $C^{\infty}(I, \mathfrak{M})$. This allows the computation of the I-invariants in $(*)$. We then prove that if L is a field with characteristic equal to the residue characteristic of F and if G is a classical group, then the G-representation $(*)$ is irreducible. This is the analog of a theorem of Casselman (which says the same for $L = \mathbb{C}$); it had been conjectured by Vignéras.

Introduction

Let F be a non-Archimedean locally compact field with ring of integers $\mathcal{O}_F$ and residue field k_F. Let G be a connected split reductive group over F. Let T be a split maximal torus, $N \subset G$ its normalizer and $W = N/T$, the corresponding Weyl group. Let $\Phi \subset X^*(T)$ be the set of roots, let $\Phi^+ \subset \Phi$ be the set of positive roots with respect to a Borel subgroup P containing T and let $\Delta \subset \Phi^+$ be the corresponding set of simple roots. For a subset $J \subset \Delta$ let $W_J \subset W$ denote the subgroup generated by the simple reflections associated with the elements of J. Let P_J denote the parabolic subgroup generated by P and by representatives (in N) of the elements of W_J. Any parabolic subgroup of G is conjugate to P_J for some J. For a ring L (commutative, with $1 \in L$) we call the G-representation

$$\mathrm{Sp}_J(G, L) = \frac{C^{\infty}(G/P_J, L)}{\sum_{\alpha \in \Delta - J} C^{\infty}(G/P_{J \cup \{\alpha\}}, L)}$$

the J-special representation of G with coefficients in L. For $J = \emptyset$ this is the Steinberg representation of G with coefficients in L. By an old theorem of Casselman, the representations

MSC Classification 22E50, 11S99

Keywords: p-adic reductive group, modular representation, Steinberg representation

$\mathrm{Sp}_J(G,\mathbb{C})$ are irreducible for all J, they form the irreducible constituents, each with multiplicity one, of $C^\infty(G/P,\mathbb{C})$. Published proofs of this irreducibility use techniques specific for the coefficient field $L=\mathbb{C}$, see [4] ch. X, Theorem 4.11 or [10] Theorem 8.1.2. For L a field of characteristic $\ell\neq p=\mathrm{char}(k_F)$ it is known that the irreducibility of say $\mathrm{Sp}_\emptyset(G,L)$ depends on ℓ. See e.g. [16] III, Theorem 2.8 (b).

In this paper we investigate the representation $\mathrm{Sp}_J(G,L)$ for arbitrary coefficient rings L (and on the way obtain results previously unknown even for $L=\mathbb{C}$). We need the L-module

$$\mathfrak{M}_J(L)=\frac{L[W/W_J]}{\sum_{\alpha\in\Delta-J}L[W/W_{J\cup\{\alpha\}}]}.$$

Let $I\subset G$ be an Iwahori subgroup adapted to P, i.e. such that we have an Iwahori decompositon $G=\cup_{w\in W}IwP$. Our first main theorem is the following (Theorem 2.3), which even for $L=\mathbb{C}$ seems to have been unknown before:

Theorem 1: *There exists an I-equivariant embedding*

$$\mathrm{Sp}_J(G,L)\hookrightarrow C^\infty(I,\mathfrak{M}_J(L));$$

its formation commutes with base changes in L.

Using the Iwahori decomposition, the proof of Theorem 1 is reduced to the proof of exactness of a certain natural sequence

(1)
$$\bigoplus_{\substack{\alpha\in\Delta-J\\ w\in W/W_{J\cup\{\alpha\}}}} C^\infty(I/I\cap wP_{J\cup\{\alpha\}}w^{-1},L)\longrightarrow\bigoplus_{w\in W/W_J}C^\infty(I/I\cap wP_Jw^{-1},L)\longrightarrow C^\infty(I,\mathfrak{M}_J(L))$$

(Proposition 2.2). This exactness proof proceeds by induction along a certain filtration of (1). The key to defining this filtration is to consider certain subsets of Φ which we call J-quasi-parabolic: a subset $D\subset\Phi$ is called J-quasi-parabolic if $\prod_{\alpha\in D}U_\alpha$ is the intersection of unipotent radicals of parabolic subgroups which are W-conjugate to P_J. Here $U_\alpha\subset G$ denotes the root subgroup associated to α. For such D we define a subset $W^J(D)$ of W/W_J as consisting of those classes wW_J for which $\prod_{\alpha\in D}U_\alpha$ is contained in the unipotent radical of the parabolic subgroup opposite to wP_Jw^{-1}. Fixing a size-increasing enumeration of all J-quasi-parabolic subsets D, the corresponding $W^J(D)$'s give the said filtration of (1). The exactness of (1) is then reduced to the exactness, for any D, of

$$\bigoplus_{\alpha\in\Delta-J}L[W^{J\cup\{\alpha\}}(D)]\longrightarrow L[W^J(D)]\longrightarrow\mathfrak{M}_J(L)$$

(Proposition 1.2), a purely combinatorial fact on finite crystallographic reflection groups. We mention that if L is a complete field extension of F, Theorem 1 holds verbatim, with the same proof, for the corresponding representations on spaces of locally analytic (rather than locally constant) functions.

A vigorously emerging subject in current p-adic number theory is the smooth representation theory of p-adic reductive groups, like G, on $\overline{\mathbb{F}}_p$-vector spaces. So far, the research has focused mostly on the case $G = \mathrm{GL}_2(F)$, for finite extensions F of $\mathbb{Q}_p$, but even for those G the theory turns out to be fairly complicated and is far from being well understood. However, it already becomes quite clear that a good understanding of the theory depends crucially on a good understanding of the functor 'taking invariants under a (pro-p-)Iwahori-subgroup'. At present there is literally no general technique available to compute this functor. For example, although Vignéras had proved the irreducibility of the Steinberg representation of our G's in characteristic p, the space of its (pro-p-)Iwahori invariants was not known (except for $G = \mathrm{GL}_2(F)$); this was the motivating problem for our investigations.

As an immediate consequence of Theorem 1 we obtain that the submodule of I-invariants $\mathrm{Sp}_J(G,L)^I$ is free of rank at most the rank of $\mathfrak{M}_J(L)$, i.e. $\mathrm{rk}_L(\mathrm{Sp}_J(G,L)^I) \le \mathrm{rk}_L(\mathfrak{M}_J(L))$, as was conjectured by Vignéras [15]. The reverse inequality $\mathrm{rk}_L(\mathrm{Sp}_J(G,L)^I) \ge \mathrm{rk}_L(\mathfrak{M}_J(L))$ follows easily by summing over all J, using that $\sum_J \mathrm{rk}_L(\mathfrak{M}_J(L)) = |W|$. In particular, the module of I-invariants in the Steinberg representation is free of rank one, for any L.

The reductive group underlying G can be defined over $\mathcal{O}_F$; as such we denote it by $\mathcal{G}_{x_0}$. Its group $\mathcal{G}_{x_0}(\mathcal{O}_F)$ of $\mathcal{O}_F$-rational points is a subgroup of G, let $\overline{G} = \mathcal{G}_{x_0}(k_F)$ denote the group of k_F-rational points of the split reductive group over k_F obtained by reduction. Its root system is the same as for G. We may copy the definition of the G-representations $\mathrm{Sp}_J(G,L)$ to define $\overline{G}$-representations $\mathrm{Sp}_J(\overline{G},L)$, for all $J \subset \Delta$ (replace locally constant functions on G by functions on $\overline{G}$). Let $\overline{P} \subset \overline{G}$ denote the Borel subgroup obtained by reduction of $I \subset \mathcal{G}_{x_0}(\mathcal{O}_F)$. Then using Theorem 1 we find a canonical identification (Proposition 3.2):

$$\mathrm{Sp}_J(G,L)^I = \mathrm{Sp}_J(\overline{G},L)^{\overline{P}}. \tag{2}$$

Our second main theorem is the analog of Casselman's theorem for a field L with $p = \mathrm{char}(L) = \mathrm{char}(k_F)$ (of course, this analog implies and gives a purely algebraic proof of Casselman's theorem). Let $I_1 \subset I$ denote the pro-p-Iwahori subgroup inside I. The G-representation $\mathrm{Sp}_J(G,L)$ is generated by $\mathrm{Sp}_J(G,L)^I = \mathrm{Sp}_J(G,L)^{I_1}$ (see [15]). As any smooth representation of a pro-p-group on a non-zero vector space in characteristic p admits a non-zero invariant vector, it is enough to show that $\mathrm{Sp}_J(G,L)^I$ is irreducible as a module under the Iwahori Hecke algebra $\mathcal{H}(G,I)$. We may view $\mathrm{Sp}_J(G,L)^I = \mathrm{Sp}_J(\overline{G},L)^{\overline{P}}$ as a module under the Hecke algebra $\mathcal{H}(\overline{G},\overline{P})$. In a first step we show (Proposition 3.4) that each $\mathcal{H}(\overline{G},\overline{P})$-submodule of $\mathrm{Sp}_J(G,L)^I = \mathrm{Sp}_J(\overline{G},L)^{\overline{P}}$ contains the class of the characteristic function $\chi_{Iw_\Delta P_J}$ of the subset $Iw_\Delta P_J \subset G$; here $w_\Delta \in W$ denotes the longest element. This follows from explicit formulae for the action on $\mathrm{Sp}_J(\overline{G},L)^{\overline{P}}$ of the Hecke operators associated to simple reflections (these formulae boil down to the Bruhat decomposition of $\overline{G}$ and require our assumption $p = \mathrm{char}(L) = \mathrm{char}(k_F)$), together with a combinatorial lemma (Lemma 1.5) on W. In a second step we need to show that the class of $\chi_{Iw_\Delta P_J}$ generates $\mathrm{Sp}_J(G,L)^I$ as a $\mathcal{H}(G,I)$-module. We can prove this if the root system Φ belongs to one of the infinite series $(A_l)_l$, $(B_l)_l$, $(C_l)_l$ or $(D_l)_l$. Our argument uses a combinatorial result (Proposition 1.6) on the weak (left)ordering of W (an ordering weaker than the Bruhat ordering) which we can prove only for such root systems. It may also hold true for the root systems of type E_6 or E_7 (hence we would get the irreducibility result in these cases, too), but certainly fails for the root systems of the types E_8, F_4 and G_2. Thus in these cases another argument (for the generation of $\mathrm{Sp}_J(G,L)^I$ by $\chi_{Iw_\Delta P_J}$) would be needed. In conclusion, what we prove is (Corollary 4.3, Corollary 4.4):

Theorem 2: *If L is a field with $\mathrm{char}(L) = \mathrm{char}(k_F)$ and if the root-system Φ is of type A_l, B_l, C_l or D_l, then the G-representation $\mathrm{Sp}_J(G,L)$ is irreducible. The $\mathrm{Sp}_J(G,L)$ for the various J form the irreducible constituents, each one occuring with multiplicity one, of $C^\infty(G/P,L)$.*

This theorem had been conjectured by Vignéras [15] (without the restriction on Φ), and, as indicated above, she had proven the irreducibility of the Steinberg representation $\mathrm{Sp}_\emptyset(G,L)$.

In the final section L is arbitrary as before and we consider realizations of $\mathrm{Sp}_J(G,L)$ as modules of harmonic chains on the (semisimple) building X of G. It follows from the results of [3] that if $\mathcal{C} = \mathcal{S}(\emptyset)$ denotes the set of all pointed chambers of X, the Steinberg representation $\mathrm{Sp}_\emptyset(G,L)$ is the quotient of the G-representation $L[\mathcal{C}]$ divided by all sums of pointed chambers which share a common pointed one-codimensional face. For general J it is still easy to see that $\mathrm{Sp}_J(G,L)$ is a quotient of the G-representation $L[\mathcal{S}(J)]$ for a suitable G-stable set $\mathcal{S}(J)$ of pointed $|\Delta - J|$-dimensional simplices in X. Indeed, using the previous notations, it not hard to see that the $\overline{G}$-representation $\mathrm{Sp}_J(\overline{G},L)$ can be realized as a quotient of $L[\overline{\mathcal{S}}(J)]$, where $\overline{\mathcal{S}}(J)$ denotes a certain $\mathcal{G}_{x_0}(\mathcal{O}_F)$-stable set of $|\Delta - J|$-dimensional simplices in X containing the unique special vertex x_0 of X fixed by $\mathcal{G}_{x_0}(\mathcal{O}_F)$. Now we simply endow the elements of $\overline{\mathcal{S}}(J)$ with the pointing by x_0: then $\mathcal{S}(J) = G.\overline{\mathcal{S}}(J)$ works. However, for $J \neq \emptyset$ it is a hard problem to give explicit local generators for the kernel of $L[\mathcal{S}(J)] \to \mathrm{Sp}_J(G,L)$, i.e. the needed 'harmonicity' relations. This problem has been solved in the case $G = \mathrm{GL}_n(F)$ for some J, namely for J consisting of the first $|J|$ simple roots in the Dynkin digram. (See [7] and [1]. The important definitions, as well as the proof in the case $\mathrm{char}(F) = 0$, as a byproduct of another investigation, are due to de Shalit. A later proof for general F is due to Aït Amrane. In fact, the definitions of de Shalit for such J realize $\mathrm{Sp}_J(G,L)$ even as a quotient of the free L-module on the set of *all* pointed $|\Delta - J|$-dimensional simplices, instead of just the set $\mathcal{S}(J)$ considered above. For general J this may be asking for to much.) Here we give local harmonicity relations for all J if $G = \mathrm{GL}_n(F)$ (Theorem 5.1). Finally we give an explicit description of our embedding from Theorem 1 in terms of this realization of $\mathrm{Sp}_J(G,L)$ (if $G = \mathrm{GL}_n(F)$).

We expect that the methods and results of this paper are indispensable for further investigations on the representations $\mathrm{Sp}_J(G,L)$, for L a field of characteristic p. For example, if $L = \mathbb{C}, \mathbb{Q}_\ell$ or if $\mathrm{char}(L) = \ell \neq p$, cohomological results on the representations $\mathrm{Sp}_J(G,L)$ obtained in [5], [11] and [12] have been important for understanding the cohomology of the Drinfel'd symmetric space $\mathcal{X}$ associated with $G = \mathrm{GL}_n(F)$, see [5] and [12]. For $L = k_F$ some of the representations $\mathrm{Sp}_J(G,L)$ occur in the (coherent) cohomology of the natural formal $\mathcal{O}_F$-model of $\mathcal{X}$.

It is a great pleasure to express my deep gratitude to Marie-France Vignéras. She invited me to spend June 2006 at Université Paris 7. During that visit she suggested the problem of computing the Iwahori invariants in p-modular Steinberg representations: this was the origin of the present work. Later she gave helpful comments on a preliminary version of this paper. I am extremely grateful to Peter Schneider. Having explained to him an unnecessarily complicated proof of Theorem 1, valid only in a restricted setting, he insisted on getting a better conceptual understanding. His numerous suggestions were decisive for approaching Theorem 1 in the correct context and for discovering the proof in its full generality. He also outlined some possible further developments. Thanks go to Yacine Aït Amrane for an enlightning discussion on harmonic (co)chains and special representations. Thanks also to Guy Henniart for his interest in this work, and to Alain Genestier for a suggestion concerning the remaining cases of exceptional groups. I thank the universities Paris 7 and Paris 13 for giving me the opportunity to present my results in their seminars. I thank the Deutsche Forschungs Gemeinschaft (DFG) as this work was done while I was supported by the DFG as a Heisenberg fellow.

Contents

1 Reflection groups

In this section we collect some results on finite crystallographic reflection groups. Proposition 1.2 will be needed for Theorem 2.3, the embedding of $\mathrm{Sp}_J(G,L)$ into $C^\infty(I,\mathfrak{M}_J(L))$. Lemma 1.5 will be needed for Proposition 3.4 which concerns the $\mathcal{H}(\overline{G},\overline{P};L)$-module structure of $\mathrm{Sp}_J(G,L)^I$, and Corollary 1.7 will be needed for the proof of Theorem 4.2 on the irreducibility of $\mathrm{Sp}_J(G,L)^I$ as a $\mathcal{H}(G,I;L)$-module.

Consider a reduced crystallographic root system Φ and let W be its corresponding Weyl group. Fix a system $\Delta\subset\Phi$ of simple roots and denote by $\Phi^+\subset\Phi$ the corresponding set of positive roots. Let $\Phi^-=\Phi-\Phi^+$. For $\alpha\in\Phi$ let $s_\alpha\in W$ denote the associated reflection. Let $\ell(.):W\to\mathbb{Z}_{\geq 0}$ be the length function with respect to Δ. For a subset $J\subset\Delta$ let $W_J\subset W$ be the subgroup generated by all s_α for $\alpha\in J$. Let

$$\Phi_J(1)=\Phi^- - (\Phi^-\cap W_J.J)$$

where $W_J.J = \{w\alpha \,|\, w \in W_J, \alpha \in J\} \subset \Phi$ is the sub-root system generated by J. For $w \in W$ we then define the subset

$$\Phi_J(w) = w\Phi_J(1)$$

of Φ. It depends only on the class of w in W/W_J. Observe $\Phi_{J'}(w) \subset \Phi_J(w)$ for $J \subset J'$. We say that a subset $D \subset \Phi$ is *J-quasi-parabolic* if it is the intersection of subsets $\Phi_J(w)$ for some (at least one) $w \in W$. Let

$$W^J = \{w \in W \mid w(J) \subset \Phi^+\}.$$

It is well known (cf. e.g. [15], remark after definition 6) that this is a set of representatives for W/W_J and can alternatively be described as

$$W^J = \{w \in W \mid \ell(ws_\alpha) > \ell(w) \text{ for all } \alpha \in J\}. \tag{3}$$

For a subset $D \subset \Phi$ let

$$W^J(D) = \{w \in W^J \mid D \subset \Phi_J(w)\}.$$

Let

$$V^J = W^J - \bigcup_{\alpha \in \Delta - J} W^{J \cup \{\alpha\}}.$$

Then $W = \cup_{J \subset \Phi} V^J$ (disjoint union). We have

$$V^J = \{w \in W^J \mid w(\Delta - J) \subset \Phi^-\}.$$

Lemma 1.1. *For $J \subset J'$ and $w \in W^{J'}$ we have $\Phi_J(w) - \Phi_{J'}(w) \subset \Phi^-$.*

PROOF: Each element in $\Phi_J(w) - \Phi_{J'}(w) = w(\Phi_J(1) - \Phi_{J'}(1))$ can be written as $w(\sum_\nu -\alpha_\nu)$ with certain $\alpha_\nu \in J'$. As $w \in W^{J'}$ the claim follows. $\square$

Let L be a ring. For a set S let $L[S]$ denote the free L-module with basis S.

Definition: We define the L-module $\mathfrak{M}_J(L)$ and the L-linear map ∇ by the exact sequence of L-modules

$$\bigoplus_{\alpha \in \Delta - J} L[W^{J \cup \{\alpha\}}] \xrightarrow{\partial} L[W^J] \xrightarrow{\nabla} \mathfrak{M}_J(L) \longrightarrow 0 \tag{4}$$

where for $w \in W^{J \cup \{\alpha\}}$ we set

$$\partial(w) = \sum_{\substack{w' \in W^J \\ w'W_J \subset wW_{J \cup \{\alpha\}}}} w'.$$

Proposition 1.2. *(a) ∇ induces a bijection between V^J and an L-basis of $\mathfrak{M}_J(L)$; in particular, $\mathfrak{M}_J(L)$ is L-free of rank $|V^J|$, and $\mathfrak{M}_J(L') = \mathfrak{M}_J(L) \otimes_L L'$ for any ring morphism $L \to L'$.*

(b) Let $D \subset \Phi$ be a J-quasi-parabolic subset. We have $\partial(\oplus_{\alpha \in \Delta - J} L[W^{J \cup \{\alpha\}}(D)]) \subset L[W^J(D)]$, and the sequence

$$\bigoplus_{\alpha \in \Delta - J} L[W^{J \cup \{\alpha\}}(D)] \xrightarrow{\partial^D} L[W^J(D)] \xrightarrow{\nabla^D} \mathfrak{M}_J(L)$$

obtained by restricting (4) is exact.

PROOF: For $w \in W^{J\cup\{\alpha\}}$ and $w' \in W^J$ with $w'W_J \subset wW_{J\cup\{\alpha\}}$ we have $\Phi_{J\cup\{\alpha\}}(w) = \Phi_{J\cup\{\alpha\}}(w') \subset \Phi_J(w')$. This shows $\partial(\oplus_{\alpha\in\Delta-J}L[W^{J\cup\{\alpha\}}(D)]) \subset L[W^J(D)]$, for any subset D of Φ.

First Step: Let $D \subset \Phi^+$ be a subset. Define $\mathfrak{M}_{J,D}(L)$ and $\tilde{\nabla}^D$ by the exact sequence

$$\bigoplus_{\alpha\in\Delta-J} L[W^{J\cup\{\alpha\}}(D)] \xrightarrow{\partial^D} L[W^J(D)] \xrightarrow{\tilde{\nabla}^D} \mathfrak{M}_{J,D}(L) \longrightarrow 0.$$

Let $V^J(D) = V^J \cap W^J(D)$.

Claim: For all ℓ and all $w \in W^J(D)$ with $\ell(w) \geq \ell$ we have $\tilde{\nabla}^D(w) \in \tilde{\nabla}^D(L[V^J(D)])$.

We prove this by descending induction on ℓ. Suppose we are given such a $w \in W^J(D)$ with $\ell(w) \geq \ell$. If $w \in V^J$ we are done. Otherwise there is some $\alpha \in \Delta - J$ with $w \in W^{J\cup\{\alpha\}}$. By Lemma 1.1 we have $\Phi_J(w) - \Phi_{J\cup\{\alpha\}}(w) \subset \Phi^-$, thus our assumption $D \subset \Phi^+$ implies even $w \in W^{J\cup\{\alpha\}}(D)$. For all $w' \in W^J - \{w\}$ with $w'W_J \subset wW_{J\cup\{\alpha\}}$ we have $\ell(w') > \ell(w)$ (because $w'W_J \subset wW_{J\cup\{\alpha\}}$ implies $w'W_{J\cup\{\alpha\}} = wW_{J\cup\{\alpha\}}$, but in view of (3) we know that w is the unique element of $wW_{J\cup\{\alpha\}}$ of minimal length). Moreover we have $w' \in W^J(D)$ (as noted at the beginning of this proof), thus by induction hypothesis we get $\tilde{\nabla}^D(w') \in \tilde{\nabla}^D(L[V^J(D)])$ for all such w'. Now

$$w = \partial^D(w) - \sum_{\substack{w'\in W^J-\{w\} \\ w'W_J\subset wW_{J\cup\{\alpha\}}}} w'$$

(inside $L[W^J(D)]$) which shows $\tilde{\nabla}^D(w) \in \tilde{\nabla}^D(L[V^J(D)])$, as desired.

The claim is proved. In particular, setting $\ell = 0$, we get $\tilde{\nabla}^D(L[V^J(D)]) = \mathfrak{M}_{J,D}(L)$.

Second Step: Here we prove (a). That the image of V^J generates the L-module $\mathfrak{M}_J(L)$ follows from the first step (with $D = \emptyset$ there). To see that it remains linearly independent we may assume $L = \mathbb{Z}$ (because the situation for general L arises by base change $\mathbb{Z} \to L$ from the one with $L = \mathbb{Z}$). But then, to prove the linear independence we may just as well assume $L = \mathbb{Q}$ and our task is to show $\dim_{\mathbb{Q}} \mathfrak{M}_J(\mathbb{Q}) = |V^J|$.

By definition, the $\mathbb{Q}$-vector spaces $\mathbb{Q}[W^J]$ and $\mathbb{Q}[W^{J\cup\{\alpha\}}]$ come with the distinguished bases W^J and $W^{J\cup\{\alpha\}}$, hence with isomorphisms with their duals $\mathbb{Q}[W^J] \cong \mathbb{Q}[W^J]^*$ and $\mathbb{Q}[W^{J\cup\{\alpha\}}] \cong \mathbb{Q}[W^{J\cup\{\alpha\}}]^*$. One easily checks that under these identifications, the map

$$L[W^J] \xrightarrow{\partial^*} \bigoplus_{\alpha\in\Delta-J} L[W^{J\cup\{\alpha\}}]$$

dual to ∂ is given as follows: for $w' \in W^J$ the α-component of $\partial^*(w')$ is the unique $w \in W^{J\cup\{\alpha\}}$ with $w'W_{J\cup\{\alpha\}} = wW_{J\cup\{\alpha\}}$. Therefore the kernel of ∂^* is the $\mathbb{Q}$-vector space generated by $\cap_\alpha(W^J \cap W_{J\cup\{\alpha\}}) = \cap_\alpha(W^J - W^{J\cup\{\alpha\}}) = W^J - \cup_\alpha W^{J\cup\{\alpha\}} = V^J$. Thus $\dim_{\mathbb{Q}} \mathfrak{M}_J(L) = \dim_{\mathbb{Q}}\mathrm{coker}(\partial) = \dim_{\mathbb{Q}}\ker(\partial^*) = |V^J|$.

Third Step: Here we prove (b). As D is J-quasi-parabolic we find some $w \in W$ with $wD \subset \Phi^+$. We have a commutative diagram

$$\begin{tikzcd}
\bigoplus_{\alpha\in\Delta-J} L[W^{J\cup\{\alpha\}}(D)] \arrow[r, "\partial^D"] \arrow[d, "\cong"] & L[W^J(D)] \arrow[r, "\nabla^D"] \arrow[d, "\cong"] & \mathfrak{M}_J(L) \arrow[d, "\cong"] \\
\bigoplus_{\substack{\alpha\in\Delta-J \\ w\in W^{J\cup\{\alpha\}}}} L[W^{J\cup\{\alpha\}}(wD)] \arrow[r, "\partial^{wD}"] & L[W^J(wD)] \arrow[r, "\nabla^{wD}"] & \mathfrak{M}_J(L)
\end{tikzcd}$$

where the second and the third (resp. the first) vertical isomorphism is induced by the bijection $W^J \to W^J$, $w' \mapsto (ww')^J$ (resp. $W^{J\cup\{\alpha\}} \to W^{J\cup\{\alpha\}}$, $w' \mapsto (ww')^{J\cup\{\alpha\}}$); here $(v)^{J'}$ for $v \in W$ and $J' \subset \Delta$ denotes the unique representative in $W^{J'}$ of the class of v in $W/W_{J'}$. Therefore we may assume from the beginning that $D \subset \Phi^+$. It suffices to see that the natural map $\mathfrak{M}_{J,D}(L) \to \mathfrak{M}_J(L)$ is injective. By (a) we know that the image of V^J, hence in particular the image of $V^J(D)$ in $\mathfrak{M}_J(L)$ is linearly independent. Together with the result of the first step this shows the wanted injectivity of $\mathfrak{M}_{J,D}(L) \to \mathfrak{M}_J(L)$. $\square$

For $w \in W$ let $(w)^J$ denote the unique element of W^J with $(w)^J W_J = wW_J$. Thus, $(.)^J$ is the projection from W onto the first factor in the direct product decomposition $W = W^J W_J$. Loosely speaking, applying $(.)^J$ means cutting off W_J-factors on the right.

We write $S = \{s_\alpha \,|\, \alpha \in \Delta\}$. Consider the following partial ordering $<_J$ on W^J. For $w, w' \in W^J$ we write $w <_J w'$ if there are $s_1, \ldots, s_r \in S$ such that, setting $w^{(i)} = (s_i \ldots s_1 w)^J$ for $0 \le i \le r$, we have $\ell(w^{(i)}) > \ell(w^{(i-1)})$ for all $i \ge 1$, and $w^{(r)} = w'$. We denote by $w_\Delta \in W$ resp. $w_J \in W_J$ the respective longest elements.

Lemma 1.3. *(a) For any $w \in W$ we have $l(w) \ge l((w)^J)$.*
(b) For $w_1 \in W^J$ and $w_2 \in W_J$ we have $l(w_1 w_2) = l(w_1) + l(w_2)$.
(c) For any $w \in W$ we have $l(w_\Delta w) = l(w w_\Delta) = l(w_\Delta) - l(w)$.

Proof: Any $v \in W^J$ is the unique element of minimal length in the set of representatives for the coset vW_J; this gives (a). For the easy statements (b) and (c) see [6] 1.8 and 1.10. $\square$

Lemma 1.4. *Let $w \in W^J$ and $s \in S$.*
(a) $w <_J (sw)^J$ implies $\ell(w) < \ell(sw)$.
(b) $\ell(w) < \ell(sw)$ and $w \neq (sw)^J$ together imply $sw \in W^J$, hence $w <_J (sw)^J = sw$.
(c) $(sw)^J <_J w$ if and only if $\ell(sw) < \ell(w)$.
(d) There exists a unique maximal element $z^J \in W^J$ for the ordering $<_J$; it lies in V^J. We have $z^J = w_\Delta w_J$. For any $u \in W$ such that $z^J \le_\emptyset u$ and for any $s \in S$ with $\ell(sz^J) < \ell(z^J)$ we have $\ell(su) < \ell(u)$.

Proof: (a) We have $l(w) < l((sw)^J) \le l(sw)$ where the first inequality follows from the definition of $<_J$ and the second one from Lemma 1.3 (a) (applied to sw).
To prove (b) assume $\ell(w) < \ell(sw)$ and $sw \notin W^J$. Then we find some $\alpha \in J$ with $\ell(sws_\alpha) =$

$\ell(sw) - 1 = \ell(w)$. Take a reduced expression $w = \sigma_1 \ldots \sigma_r$ with $\sigma_i \in S$. By the deletion condition for Weyl groups we get a reduced expression for sws_α by deleting some factors in the string $s\sigma_1 \ldots \sigma_r s_\alpha$. Namely, as $\ell(sws_\alpha) = \ell(w)$, exactly two factors must be deleted. If s remained this would mean $\ell(ws_\alpha) < \ell(w)$, contradicting $w \in W^J$. If s_α remained this would mean $\ell(sw) < \ell(w)$, contradicting our hypothesis. Thus $sws_\alpha = w$, i.e. $w = (sw)^J$.
(c) First assume $\ell(sw) < \ell(w)$. Then we get $l((sw)^J) < l(w)$ from Lemma 1.3 (a) (applied to sw). As $(s(sw)^J)^J = w^J = w$ we get $(sw)^J <_J w$ from the definition of $<_J$. Now assume $\ell(sw) > \ell(w)$ and $(sw)^J <_J w$. Then there are $\alpha_1, \alpha_2 \in J$ such that $\ell(sws_{\alpha_1}s_{\alpha_2}) < \ell(w)$. On the other hand, $w \in W^J$ implies $\ell(ws_{\alpha_1}) > \ell(w)$ and $\ell(ws_{\alpha_2}) > \ell(w)$. From $\ell(sw) > \ell(w)$ (or from $\ell(ws_{\alpha_1}) > \ell(w)$) together with $\ell(sws_{\alpha_1}s_{\alpha_2}) < \ell(w)$ it follows that $\ell(sws_{\alpha_1}) = \ell(w)$. As $\ell(sw) > \ell(w)$ and $\ell(ws_{\alpha_1}) > \ell(w)$ this implies $w = sws_\alpha$ as in the proof of (b). But then $\ell(ws_{\alpha_2}) > \ell(w)$ contradicts $\ell(sws_{\alpha_1}s_{\alpha_2}) < \ell(w)$.
(d) From Lemma 1.3 (c) it follows that $(w_\Delta)^J = w_\Delta w_J$. We claim that $z^J = (w_\Delta)^J = w_\Delta w_J$ is maximal in W^J with respect to $<_J$, and is uniquely determined by this property. To see this we need to show, by (b), that for any $w \in W^J - \{z^J\}$ there is some $s \in S$ with $\ell(sw) > \ell(w)$ and $w \neq (sw)^J$. As $w \neq z^J = w_\Delta w_J$ we find $s \in S$ with $\ell(sww_J) = \ell(ww_J) + 1$, hence

$$l(sw) \geq l(sww_J) - l(w_J) = l(ww_J) + 1 - l(w_J) > l(w)$$

where we used $l(ww_J) = l(w) + l(w_J)$ as recorded in Lemma 1.3 (b). If we had $w = (sw)^J$ this would mean $sw = wu$ for some $u \in W_J$, hence $l(sww_J) = l(wuw_J) \leq l(ww_J)$ by Lemma 1.3 (b): contradiction !
Finally, we have $z^J = w_\Delta w_J = w_{\check{J}} w_\Delta$ for

$$\check{J} = \{\beta \in \Delta \,|\, s_\beta = w_\Delta s_\alpha w_\Delta \text{ for some } \alpha \in J\}.$$

For $u \in W$ such that $z^J = w_{\check{J}} w_\Delta <_\emptyset u = (uw_\Delta)w_\Delta$ we get $uw_\Delta \in W_{\check{J}}$. Similarly, $z^J = w_{\check{J}} w_\Delta$ means that $\ell(sz^J) < \ell(z^J)$ for $s \in S$ can only happen if $s = s_\alpha$ for some $\alpha \in \Delta - \check{J}$. Therefore $l(suw_\Delta) > l(uw_\Delta)$ since $uw_\Delta \in W_{\check{J}}$. By Lemma 1.3 (c) this means $\ell(su) < \ell(u)$. $\square$

Lemma 1.5. *For each $w \in V^J - \{z^J\}$ there is some $w' \in V^J$ and some $s \in S$ with $w <_J w'$, with $\ell((sw)^J) < \ell(w)$ and with $\ell((sw')^J) \geq \ell(w')$.*

Proof: Consider the set

$$J' = \{\alpha \in \Delta \mid \ell(s_\alpha w) > \ell(w)\}$$

and let $w_{J'}$ denote the longest element of $W_{J'}$. For any given $\alpha \in \Delta$ we have $\alpha \notin J'$ if and only if $\ell((s_\alpha w)^J) < \ell(w)$, by Lemma 1.4.
Case (i): $z^J w^{-1} \notin W_{J'}$. Take a reduced expression $z^J w^{-1} = \sigma_1 \ldots \sigma_r$ with $\sigma_i \in S$. Let $1 \leq i \leq r$ be maximal such that $\sigma_r = s_\alpha$ for some $\alpha \in \Delta - J'$ (such an i exists since $z^J w^{-1} \notin W_{J'}$). By Lemma 1.4(c) we have $\ell(z^J) = r + \ell(w)$, by Lemma 1.4(b) we then see

$w' \in V^J$ for $w' = \sigma_{i+1} \dots \sigma_r w$. This w' together with $s = s_\alpha$ is fine.
Case (ii): $z^J w^{-1} \in W_{J'}$. Here we claim that $w' = z^J$ satisfies the wanted conclusion. Assume on the contrary that $\ell(s_\alpha z^J) < \ell(z^J)$ for all $\alpha \in \Delta - J'$. Then we also have $\ell(s_\alpha w_{J'} w) < \ell(w_{J'} w)$ for all $\alpha \in \Delta - J'$. This follows from Lemma 1.4(d) since $z^J w^{-1} \in W_{J'}$ implies $z^J \leq_\emptyset w_{J'} w$. On the other hand $\ell(s_\alpha w_{J'} w) < \ell(w_{J'} w)$ for all $\alpha \in J'$, too (because $\ell(w_{J'} w) = \ell(w_{J'}) + \ell(w)$ as follows from the definition of J'), hence for all $\alpha \in \Delta$. This means $w_{J'} w = w_\Delta$. But then $w = w_\Delta w_{\check{J}}$ for some $\check{J} \subset \Delta$ (as in the proof of Lemma 1.4(d)). As $V^J \cap V^{\check{J}} = \emptyset$ for $J \neq \check{J}$ this shows $J = \check{J}$ and $w = z^{\check{J}}$, contradicting our hypothesis $w \neq z^J$. □

The next result concerns the partial ordering $<_\emptyset$ of W (i.e. $<_J$ for $J = \emptyset$), called the weak ordering of W in [2].

Consider the following subgroup W_Ω of W. We write our set of simple roots as $\Delta = \{\alpha_1, \dots, \alpha_l\}$ and denote by $\alpha_0 \in \Phi$ the unique highest root. Then we define the elements $\epsilon_1, \dots, \epsilon_l$ in the $\mathbb{R}$-vector space dual to the one spanned by Φ by requiring $(\epsilon_i, \alpha_j) = \delta_{ij}$ for $1 \leq i, j \leq l$. For $1 \leq i \leq l$ we let $w_{\Delta^{(i)}} \in W$ denote the longest element of the subgroup of W generated by the set $\{s_{\alpha_j} \mid j \neq i\}$. Then

$$W_\Omega - \{1\} = \{w_{\Delta^{(i)}} w_\Delta \mid 1 \leq i \leq l,\ (\epsilon_i, \alpha_0) = 1\}.$$

The conjugation action of W_Ω on $\{s_{\alpha_0}, s_{\alpha_1}, \dots, s_{\alpha_l}\}$ identifies W_Ω with the automorphism group of the Dynkin diagram of the affine root system (see [8] pp. 18-20).

Proposition 1.6. *Suppose that the underlying root-system is of type A_l, B_l, C_l or D_l. There exists a sequence $w_\Delta = w_0, w_1, \dots, w_r = 1$ in W such that for all $i \geq 1$ we have $w_{i-1} <_\emptyset w_i$, or $w_i = u w_{i-1}$ for some $u \in W_\Omega$.*

Proof: We use the respective descriptions of W_Ω given in [8] pp. 18-20. We write $s_i = s_{\alpha_i}$.
Case A_l: Then W can be identified with the symmetric group in $\{1, \dots, l+1\}$. We write an element $w \in W$ as the tuple $[w(1), \dots, w(l+1)]$. As simple reflections we take the transpositions $s_i = [1, \dots, i-1, i+1, i, i+2, \dots, l] \in W$ for $i = 1, \dots, l$. Then W_Ω consists of the elements

$$w_{\Delta^{(i)}} w_\Delta = [i+1 \dots, l+1, 1, \dots, i] \qquad (0 \leq i \leq l).$$

The length $\ell(w)$ of $w \in W$ is the number of all pairs (i, j) with $i < j$ and $w(i) > w(j)$. We pass from w_Δ to 1 via the sequence

$$\begin{aligned} w_\Delta = [l+1, \dots, 1] &\overset{(*)}{\mapsto} [1, l+1, \dots, 2] <_\emptyset [l, l+1, l-1, \dots, 1] \\ &\overset{(*)}{\mapsto} [1, 2, l+1, l, \dots, 3] <_\emptyset [l-1, l, l+1, l-2, \dots, 1] \\ &\overset{(*)}{\mapsto} \dots <_\emptyset [2, \dots, l+1, 1] \overset{(*)}{\mapsto} [1, \dots, l+1] = 1. \end{aligned}$$

Here each step of type $(*)$ is obtained by left-multiplication with an element of W_Ω.
Case B_l: Here W can be identified with the group of signed permutations of $\{\pm 1, \dots, \pm l\}$, i.e.

with all bijections $w : \{\pm 1, \dots, \pm l\} \to \{\pm 1, \dots, \pm l\}$ satisfying $-w(a) = w(-a)$ for all $1 \leq a \leq l$. We write an element $w \in W$ as the tuple $[w(1), \dots, w(l)]$. As simple reflections we take the elements $s_i = [1, \dots, l-i-1, l-i+1, l-i, l-i+2, \dots, l]$ for $1 \leq i \leq l-1$, together with $s_l = [-1, 2, \dots, l]$. Then the length of $w \in W$ can be computed as

$$\ell(w) = |\{(ij)\,;\, i < j,\, w(i) > w(j)\}| - \sum_{\substack{j \\ w(j)<0}} w(j)$$

(for all this see [2] chapter 8.1). The group W_Ω consists of two elements, its non-trivial element is

$$w_{\Delta^{(1)}} w_\Delta = [1, \dots, l-1, -l].$$

For $1 \leq i \leq l$ let

$$a_i = [-i, \dots, -l, i-1, \dots, 1],$$

$$b_i = [-i, \dots, 1-l, l, i-1, \dots, 1].$$

We pass from w_Δ to 1 via the sequence

$$w_\Delta = [-1, \dots, -l] = a_1 \overset{(*)}{\mapsto} b_1 <_\emptyset a_2 \overset{(*)}{\mapsto} b_2 <_\emptyset a_3 \overset{(*)}{\mapsto} \dots$$

$$\dots <_\emptyset a_l \overset{(*)}{\mapsto} b_l = [l, \dots, 1] \overset{(**)}{\mapsto} b_l = [1, \dots, l] = 1.$$

Here the relations $<_\emptyset$ result from left-multiplications with $s_{l-1} \dots s_1$, increasing the length by $l-1$, as one easily checks. Each step of type $(*)$ is obtained by left-multiplication with $w_{\Delta^{(1)}} w_\Delta$. It remains to justify the step $(**)$. Observe that

$$w_{\Delta^{(1)}} w_\Delta s_1 \dots s_l = [l, 1, \dots, l-1].$$

Moreover, for each $w \in W$ satisfying $w(i) > 0$ for all $1 \leq i \leq l$ we have $w <_\emptyset s_1 \dots s_l w$. Together it follows that, to prove that the step $(**)$ is permissible, it suffices to show that $(**)$ decomposes into left-multiplications with (powers of) $[l, 1, \dots, l-1]$, and transpositions $s_1, \dots, s_{l-1}$. But this was shown in our analysis of case A_l.

Case C_l: Here W is the same as in case B_l and we take the same simple reflections. Again W_Ω consists of two elements, but this time its non-trivial element is

$$w_{\Delta^{(l)}} w_\Delta = [-l, \dots, -1].$$

We pass from w_Δ to 1 via the sequence

$$w_\Delta = [-1, \dots, -l] \overset{(*)}{\mapsto} [l, \dots, 1] \overset{(**)}{\mapsto} [1, \dots, l] = 1.$$

Here $(*)$ is obtained by left-multiplication with $w_{\Delta^{(l)}} w_\Delta$. To justify the step $(**)$ observe that

$$w_{\Delta^{(l)}} w_\Delta s_l w_{\Delta^{(l)}} w_\Delta s_1 \dots s_l = [l, 1, \dots, l-1].$$

Moreover, for each $w \in W$ satisfying $w(i) > 0$ for all $1 \le i \le l$ we have $w <_\emptyset s_1 \dots s_l w$ (as already noted above), and

$$w_{\Delta^{(l)}} w_\Delta s_1 \dots s_l w <_\emptyset s_l w_{\Delta^{(l)}} w_\Delta s_1 \dots s_l w.$$

Thus left-multiplication of $[l, 1, \dots, l-1]$ to such $w \in W$ is a permissible operation for our purposes. Therefore we may conclude as in the case B_l.

Case D_l: Here W can be identified with the group of signed permutations of $\{\pm 1, \dots, \pm l\}$ having an even number of negative entries, i.e. with all bijections $w : \{\pm 1, \dots, \pm l\} \to \{\pm 1, \dots, \pm l\}$ satisfying $-w(a) = w(-a)$ for all $1 \le a \le l$, and such that the number $|\{i \,|\, w(i) < 0\}|$ is even. We write an element $w \in W$ as the tuple $[w(1), \dots, w(l)]$. As simple reflections we take the elements s_i for $1 \le i \le l-1$ used in cases B_l and C_l, together with

$$s_l = [-2, -1, 3, \dots, l].$$

The length of $w \in W$ can be computed (see [2] chapter 8.2) as

$$\ell(w) = |\{(ij)\,;\, i < j,\, w(i) > w(j)\}| + |\{(ij)\,;\, w(i) + w(j) < 0\}|.$$

W_Ω consists of the four elements $1, w_{\Delta^{(1)}} w_\Delta, w_{\Delta^{(l-1)}} w_\Delta$ and $w_{\Delta^{(l)}} w_\Delta$. We have

$$w_{\Delta^{(1)}} w_\Delta = [-1, 2, \dots, l-1, -l]$$

and, according to the parity of l,

$$w_{\Delta^{(l)}} w_\Delta = [-l, \dots, -1] \qquad (l \text{ even})$$

$$w_{\Delta^{(l)}} w_\Delta = [l, 1-l, \dots, -1] \qquad (l \text{ odd})$$

(and $w_{\Delta^{(l-1)}} w_\Delta = [l, 1-l \dots, -2, 1]$ if l is even, $w_{\Delta^{(l-1)}} w_\Delta = [-l, \dots, -2, 1]$ is l is odd). We pass from w_Δ to 1 via the sequence

$$w_\Delta = [-1, \dots, -l] \overset{(*)}{\mapsto} [l, \dots, 1] \overset{(**)}{\mapsto} [1, \dots, l] = 1 \qquad (l \text{ even})$$

$$w_\Delta = [1, -2, \dots, -l] \overset{(*)}{\mapsto} [l, \dots, 1] \overset{(**)}{\mapsto} [1, \dots, l] = 1 \qquad (l \text{ odd}).$$

Here $(*)$ is obtained by left-multiplication with $w_{\Delta^{(l)}} w_\Delta$. To justify the step $(**)$ observe that

$$w_{\Delta^{(1)}} s_1 \dots s_{l-2} = [l, 1, \dots, l-1].$$

For each $w \in W$ with $w(i) > 0$ for all $1 \le i \le l-2$ we have $w <_\emptyset s_1 \dots s_{l-2} w$. Thus left-multiplication of $[l, 1, \dots, l-1]$ to such $w \in W$ is a permissible operation for our purposes and we may conclude as before. □

Corollary 1.7. *For each $w \in V^J$ there is a sequence $z^J = w_0, w_1, \dots, w_r = w$ in W such that for all $i \ge 1$ we have $w_i^J = u w_{i-1}^J$ for some $u \in W_\Omega$, or [$\ell(w_{i-1}^J) < \ell(w_i^J)$ and $w_i^J = s(w_{i-1}^J)$ for some $s \in S$].*

Proof: Recall that $z^J = (w_\Delta)^J$. Furthermore observe that $\ell(w') < \ell(w)$ and $w = sw'$ for some $s \in S$ implies that $[w^J = s(w')^J$ and $\ell((w')^J) < \ell(w^J)]$ or $w^J = (w')^J$. Thus the corollary follows from Proposition 1.6. $\square$

Remark: For the irreducible reduced root systems of type E_8, F_4 and G_2 we have $W_\Omega = \{1\}$ by [8]. Therefore the statement of Proposition 1.6 cannot hold true in these cases. We do not discuss the cases E_6, E_7.

2 Functions on the Iwahori subgroup

Let F be a non-Archimedean locally compact field, $\mathcal{O}_F$ its ring of integers, $p_F \in \mathcal{O}_F$ a fixed prime element and k_F its residue field. Let G be a split reductive group over F, connected and different from its center. (Here we commit the usual abuse of notation: what we really mean is that G is the group of F-rational points of such an algebraic F-group scheme, similarly for the subgroups considered below.) Let T be a split maximal torus, $N \subset G$ its normalizer in G and let $W = N/T$, the corresponding Weyl group. For any $w \in W$ we choose a representative (with the same name) $w \in N$. Let $P = TU$ be a Borel subgroup with unipotent radical U. Let $\Phi \subset X^*(T) = \mathrm{Hom}_{alg}(T, \mathbb{G}_m)$ be the set of roots, let $\Phi^+ \subset \Phi$ be the set of P-positive roots, let $\Delta \subset \Phi^+$ be the set of simple roots. Since T is split this root system is reduced.

For $\alpha \in \Phi$ let $U_\alpha \subset G$ be the associated root subgroup. Then $U = \prod_{\alpha \in \Phi^+} U_\alpha$ (direct product, for any ordering of Φ^+). We need the parabolic subgroups $P_J = PW_JP$ of G; each parabolic subgroup of G containing P is of this form (for a suitable J). For $w \in W$ let $P_{J,w} = wP_Jw^{-1}$ and let $P^-_{J,w}$ be the parabolic subgroup of G opposite to $P_{J,w}$. We then find

$$\Phi - \Phi_J(w) = \{\alpha \in \Phi \mid U_\alpha \subset P_{J,w}\}$$

or equivalently: $\prod_{\alpha \in \Phi_J(w)} U_\alpha$ is the unipotent radical of $P^-_{J,w}$. Note that $P_{J,w} = P_{J,w'}$ for any $w' \in wW_J$.

We choose an Iwahori subgroup I in G compatible with P, in the sense that we have the Iwahori decomposition

$$G = \bigcup_{w \in W} IwP$$

(disjoint union). For any subgroup H in G we write $H^0 = H \cap I$.

Lemma 2.1. *Let $D \subset \Phi$ be a J-quasi-parabolic subset. Then $\prod_{\alpha \in D} U^0_\alpha$ is a subgroup of G and is independent of the ordering of D. We denote it by U^0_D.*

Proof: Take any ordering of D. Then choose an ordering of Φ which restricts to this ordering on D and such that the product map

$$\prod_{\alpha \in \Phi} U_\alpha \longrightarrow G$$

is injective. Write $D=\cap_{w\in T}\Phi_J(w)$ (some $T\subset W$). Then of course

$$\prod_{\alpha\in D}U_\alpha^0=\cap_{w\in T}\prod_{\alpha\in\Phi_J(w)}U_\alpha^0$$

(all products w.r.t. the fixed ordering of Φ, and the intersection is taken inside G). Hence it is enough to see that $\prod_{\alpha\in\Phi_J(w)}U_\alpha^0$ is independent of the ordering of $\Phi_J(w)$ — but this is clear: $\prod_{\alpha\in\Phi_J(w)}U_\alpha^0$ is the intersection of I with the unipotent radical of $P_{J,w}^-$. $\square$

For a topological space $\mathcal{T}$ and an L-module M let $C^\infty(\mathcal{T},M)$ denote the L-module of locally constant M-valued functions on $\mathcal{T}$.

Applying the functor $C^\infty(I,.)$ to the exact sequence (4) we obtain an exact sequence

$$C^\infty(I,\bigoplus_{\alpha\in\Delta-J}L[W^{J\cup\{\alpha\}}])\longrightarrow C^\infty(I,L[W^J])\longrightarrow C^\infty(I,\mathfrak{M}_J(L))\longrightarrow 0. \tag{5}$$

Observe that we have natural embeddings, which we view as inclusions,

$$\bigoplus_{\substack{\alpha\in\Delta-J\\ w\in W^{J\cup\{\alpha\}}}}C^\infty(I/P^0_{J\cup\{\alpha\},w},L)\subset C^\infty(I,\bigoplus_{\alpha\in\Delta-J}L[W^{J\cup\{\alpha\}}]),$$

$$\bigoplus_{w\in W^J}C^\infty(I/P^0_{J,w},L)\subset C^\infty(I,L[W^J]),$$

by summing over the respective direct summands.

Proposition 2.2. *The sequence*

$$\bigoplus_{\substack{\alpha\in\Delta-J\\ w\in W^{J\cup\{\alpha\}}}}C^\infty(I/P^0_{J\cup\{\alpha\},w},L)\xrightarrow{\partial_C}\bigoplus_{w\in W^J}C^\infty(I/P^0_{J,w},L)\xrightarrow{\nabla_C}C^\infty(I,\mathfrak{M}_J(L))$$

obtained by restricting (5) is exact.

Proof: Choose an enumeration D_0, D_1, $D_2,\ldots$ of all J-quasi-parabolic subsets of Φ such that $n<m$ implies $|D_n|\le|D_m|$. Let $(f_w)_{w\in W^J}\in\operatorname{Ker}(\nabla_C)$. By induction on m we show: adding to f an element in the image of ∂_C if necessary, we may assume $f_w|_{U^0_{D_n}}=0$ for all $w\in W^J$, all $n\le m$.

Assume we have $f_w|_{U^0_{D_n}}=0$ for all $w\in W^J$, all $n<m$. Let us write $D=D_m$.

(i) We first claim $f_w|_{U^0_D}=0$ for all $w\in W^J-W^J(D)$. Indeed, for such w we have $|D\cap\Phi_J(w)|<|D|$, hence $D\cap\Phi_J(w)=D_n$ for some $n<m$. Thus

$$f_w(U^0_D)=f_w(U^0_{D_n}\prod_{\alpha\in D-D_n}U^0_\alpha)=f_w(U^0_{D_n})=0$$

where in the first equation we used that we may form U^0_D with respect to any ordering of D, where the second equation follows from $U^0_\alpha\subset P^0_{J,w}$ for $\alpha\notin\Phi_J(w)$ (and the invariance property of f_w), and where the last equation holds true by induction hypothesis.

(ii) Our sequence in question restricts to a sequence

(6) $$\bigoplus_{\substack{\alpha\in\Delta-J\\ w\in W^{J\cup\{\alpha\}}(D)}} C^\infty(I/P^0_{J\cup\{\alpha\},w},L)\xrightarrow{\partial_C^D}\bigoplus_{w\in W^J(D)} C^\infty(I/P^0_{J,w},L)\xrightarrow{\nabla_C^D} C^\infty(I,\mathfrak{M}_J(L)).$$

For any $x\in U_D^0$, evaluating functions at x transforms (6) into a sequence isomorphic with the one from Proposition 1.2 (b). Let us denote by $(\partial_C^D)_x$ resp. by $(\nabla_C^D)_x$ the differentials of this sequence, which by Proposition 1.2 (b) is exact. From (i) it follows that

$$f^D(x)=(f_w(x))_{w\in W^J(D)}\in\mathrm{Ker}((\nabla_C^D)_x),$$

hence this lies in the image of $(\partial_C^D)_x$. For all $x\in U_D^0$ choose preimages of $f^D(x)$ under $(\partial_C^D)_x$. Since the f_w are locally constant, these preimages can be arranged to vary locally constantly on U_D^0, and moreover, in view of (i) we may assume that for all $x\in U_D^0\cap\cup_{n<m}U^0_{D_n}$ these preimages are zero.

For any $\alpha\in\Delta-J$ and $w\in W^{J\cup\{\alpha\}}(D)$ the natural map $U_D^0\to I/P^0_{J\cup\{\alpha\},w}$ is injective. Thus we find an element

$$g^D=(g_{\alpha,w})_{\alpha,w}\in\bigoplus_{\substack{\alpha\in\Delta-J\\ w\in W^{J\cup\{\alpha\}}(D)}} C^\infty(I/P^0_{J\cup\{\alpha\},w},L)$$

which on U_D^0 assumes the preimages of the $f^D(x)$ just chosen, and which vanishes at all $x\in\cup_{n<m}U^0_{D_n}$ with $x\notin U_D^0$. We obtain

$$f^D(x)-\partial_C^D(g^D)(x)=0$$

for all $x\in\cup_{n\le m}U^0_{D_n}$: for $x\in U_D^0$ this follows from our definition of $g^D|_{U_D^0}$, for $x\in\cup_{n<m}U^0_{D_n}$ with $x\notin U_D^0$ this follows from the vanishing of g^D at such x together with the induction hypothesis. Now set $g_{\alpha,w}=0$ for all $\alpha\in\Delta-J$ and $w\in W^{J\cup\{\alpha\}}-W^{J\cup\{\alpha\}}(D)$. By (i) and what we just saw we find

$$((f_w)_w-\partial_C((g_{\alpha,w})_{\alpha,w}))(x)=0$$

for all $x\in\cup_{n\le m}U^0_{D_n}$.The induction is complete. In other words, we have shown that, adding to $(f_w)_w$ an element in the image of ∂_C if necessary, we may assume $f_w|_{U_D^0}=0$ for all $w\in W^J$, all J-quasi-parabolic subsets D. In particular we find $f_w|_{U^0_{\Phi_J(w)}}$ for all $w\in W^J$. But $U^0_{\Phi_J(w)}$ is a set of representatives for $I/P^0_{J,w}$, hence $f_w=0$. We are done. □

Definition: Let J be a subset of Δ. We define the G-representation $\mathrm{Sp}_J(G,L)$ by the exact sequence of G-representations

$$\bigoplus_{\alpha\in\Delta-J} C^\infty(G/P_{J\cup\{\alpha\}},L)\xrightarrow{\partial} C^\infty(G/P_J,L)\longrightarrow\mathrm{Sp}_J(G,L)\longrightarrow 0,$$

where ∂ is the sum of the canonical inclusions, and the G-action is by translation of functions on G. We call $\mathrm{Sp}_J(G,L)$ the J-special G-representation with coefficients in L.

Theorem 2.3. $\mathrm{Sp}_J(G,L)$ *is L-free. There exists an I-equivariant embedding*

$$\mathrm{Sp}_J(G,L) \stackrel{\lambda_L}{\hookrightarrow} C^\infty(I, \mathfrak{M}_J(L)).$$

Its formation commutes with base changes: for a ring morphism $L \to L'$ the composite

$$\mathrm{Sp}_J(G,L) \otimes_L L' \cong \mathrm{Sp}_J(G,L') \stackrel{\lambda_{L'}}{\hookrightarrow} C^\infty(I, \mathfrak{M}_J(L')) \cong C^\infty(I, \mathfrak{M}_J(L)) \otimes_L L'$$

is $\lambda_L \otimes_L L'$.

Proof: Recall that for $w \in W$ we defined $P^0_{J,w} = I \cap wP_Jw^{-1}$. Note that $P^0_{J,w}$ and wP_J depend only on the coset wW_J, not on the specific representative $w \in wW_J$. The same is true for the isomorphism

$$I/P^0_{J,w} \cong IwP_J/P_J,$$
$$i \mapsto iw.$$

It follows that for any inclusion of cosets $wW_J \subset wW_{J\cup\{\alpha\}}$ we have a commutative diagram

$$\begin{tikzcd}
I/P^0_{J,w} \arrow[r] \arrow[d, "\cong"] & I/P^0_{J\cup\{\alpha\},w} \arrow[d, "\cong"] \\
IwP_J/P_J \arrow[r] & IwP_{J\cup\{\alpha\}}/P_{J\cup\{\alpha\}}
\end{tikzcd}$$

where the horizontal arrows are the obvious projections and the vertical arrows are the above isomorphisms. Now recall the Iwahori decompositions

$$G/P_J = \cup_{w\in W^J} IwP_J/P_J, \qquad G/P_{J\cup\{\alpha\}} = \cup_{w\in W^{J\cup\{\alpha\}}} IwP_{J\cup\{\alpha\}}/P_{J\cup\{\alpha\}}$$

(disjoint unions). They give

$$C^\infty(G/P_J, L) = \bigoplus_{w\in W^J} C^\infty(IwP_J/P_J, L),$$

$$C^\infty(G/P_{J\cup\{\alpha\}}, L) = \bigoplus_{w\in W^{J\cup\{\alpha\}}} C^\infty(IwP_{J\cup\{\alpha\}}/P_{J\cup\{\alpha\}}, L).$$

With these identifications, the above commutative diagrams (for all $\alpha \in \Delta - J$) induce a commutative diagram

$$\begin{tikzcd}
\bigoplus_{\alpha\in\Delta-J} C^\infty(G/P_{J\cup\{\alpha\}}, L) \arrow[r] \arrow[d, "\cong"] & C^\infty(G/P_J, L) \arrow[r] \arrow[d, "\cong"] & \mathrm{Sp}_J(G,L) \arrow[r] & 0 \\
\bigoplus_{\substack{\alpha\in\Delta-J \\ w\in W^{J\cup\{\alpha\}}}} C^\infty(I/P^0_{J\cup\{\alpha\},w}, L) \arrow[r] & \bigoplus_{w\in W^J} C^\infty(I/P^0_{J,w}, L) \arrow[r] & C^\infty(I, \mathfrak{M}_J(L)) &
\end{tikzcd}$$

where the vertical arrows are isomorphisms. The top row is exact by the definition of $\mathrm{Sp}_J(G,L)$, the bottom row is exact by Proposition 2.2, and clearly all arrows are I-equivariant. Hence we get the wanted injection $\lambda_L : \mathrm{Sp}_J(G,L) \hookrightarrow C^\infty(I, \mathfrak{M}_J(L))$. We then derive the freeness of $\mathrm{Sp}_J(G,L)$: first for $L = \mathbb{Z}$ since $C^\infty(I, \mathfrak{M}_J(\mathbb{Z}))$ is $\mathbb{Z}$-free, then by base change $\mathbb{Z} \to L$ for any L. Similarly we get the stated base change property. $\square$

Corollary 2.4. *(Conjectured by Vignéras [15]) The submodule* $\mathrm{Sp}_J(G,L)^I$ *of* I*-invariants in* $\mathrm{Sp}_J(G,L)$ *is free of rank*

$$\mathrm{rk}_L(\mathrm{Sp}_J(G,L)^I) = \mathrm{rk}_L(\mathfrak{M}_J(L)) = |V^J|.$$

PROOF: The inequality $\mathrm{rk}_L(\mathrm{Sp}_J(G,L)^I) \leq \mathrm{rk}_L(\mathfrak{M}_J(L)) = |V^J|$ follows from Theorem 2.3. On the other hand, by the Iwahori decomposition again, $C^\infty(G/P_J,L)$ is free of rank $|W^J|$ ([15] Proposition 9). Now W^J is the disjoint union of all $V^{J'}$ with $J' \supset J$. Since $C^\infty(G/P_J,L)$ admits a G-equivariant filtration whose graded pieces are the $\mathrm{Sp}_{J'}(G,L)$, the inequalities $\mathrm{rk}_L(\mathrm{Sp}_{J'}(G,L)^I) \leq \mathrm{rk}_L(\mathfrak{M}_{J'}(L)) = |V^{J'}|$ for all $J' \supset J$ imply the inequality $\mathrm{rk}_L(\mathrm{Sp}_J(G,L)^I) \geq \mathrm{rk}_L(\mathfrak{M}_J(L)) = |V^J|$.

Alternatively, the bijectivity of

$$\mathrm{Sp}_J(G,L)^I \longrightarrow C^\infty(I,\mathfrak{M}_J(L))^I \cong \mathfrak{M}_J(L)$$

follows immediately from the proof of Theorem 2.3, namely from the surjectivity of

$$\bigoplus_{w\in W^J} C^\infty(I/P^0_{J,w},L)^I \longrightarrow C^\infty(I,\mathfrak{M}_J(L))^I$$

which we get from the very definition of $\mathfrak{M}_J(L)$. $\square$

Corollary 2.5. *Let* π *be a smooth irreducible (hence finite dimensional) representation of* I *on a* $\mathbb{C}$*-vector space. Then* π *occurs in* $\mathrm{Sp}_J(G,\mathbb{C})$ *with multiplicity at most* $|V_J|\dim_{\mathbb{C}}(\pi)$*.*

PROOF: π occurs in $C^\infty(I,\mathfrak{M}_J(\mathbb{C}))$ with multiplicity $|V_J|\dim_{\mathbb{C}}(\pi)$. $\square$

Remark: If L is a complete field extension of F we may replace all spaces of locally constant functions occuring here by the corresponding spaces of locally F-analytic functions. In particular we may define locally analytic G-representations $\mathrm{Sp}^{an}_J(G,L)$ and $C^{an}(I,\mathfrak{M}_J(L))$. Then Theorem 2.3 and Corollary 2.4 carry over, with the same proofs: there exists an I-equivariant embedding

$$\mathrm{Sp}^{an}_J(G,L) \hookrightarrow C^{an}(I,\mathfrak{M}_J(L))$$

and we have $\mathrm{rk}_L(\mathrm{Sp}^{an}_J(G,L)^I) = \mathrm{rk}_L(\mathfrak{M}_J(L)) = |V^J|$.

3 Special representations of finite reductive groups

Now we assume in addition that G is semisimple and that the root system Φ is irreducible. There is a unique chamber C in the standard apartment associated to T in the Bruhat-Tits-building of G which is fixed by our Iwahori subgroup I. Let x_0 be the special vertex of C corresponding to our Borel subgroup P (see below for what this means). Let $\mathcal{G}_{x_0}/\mathcal{O}_F$ denote the $\mathcal{O}_F$-group scheme with generic fibre the underlying F-group scheme $\mathbb{G}$ of $G = \mathbb{G}(F)$ and such that for each unramified Galois extension F' of F with ring of integers $\mathcal{O}_{F'}$ we have

$$\mathcal{G}_{x_0}(\mathcal{O}_{F'}) = \{g \in \mathbb{G}(F') \quad | \quad gx_0 = x_0\}$$

(see [13] 3.4). This $\mathcal{G}_{x_0}$ is a group scheme as constructed by Chevalley ([13] 3.4.1). Its special fibre $\mathcal{G}_{x_0} \otimes_{\mathcal{O}_F} k_F$ is a split connected reductive group over k_F with the same root datum as G ([13] 3.8.1; compare also [9], part II, 1.17, and for adjoint G see [8] p.30/31 where the Bruhat decomposition of $\overline{G} = (\mathcal{G}_{x_0} \otimes_{\mathcal{O}_F} k_F)(k_F)$ is discussed similarly to how we are going to use it here). Let $K_{x_0} = \mathcal{G}_{x_0}(\mathcal{O}_F)$ and

$$U_{x_0} = \mathrm{Ker} \quad [\quad K_{x_0} \longrightarrow \mathcal{G}_{x_0}(k_F) \quad].$$

For H any of the groups G, P_J, P, T, N, U, U_α let

$$\overline{H} = \frac{(H \cap K_{x_0}, U_{x_0})}{U_{x_0}} = \frac{H \cap K_{x_0}}{H \cap U_{x_0}}.$$

Our choice of x_0 above is characterized by the fact I is the preimage of $\overline{P}$ under the homomorphism $K_{x_0} \to \overline{G}$. On groups of k_F-rational points we have: $\overline{P}_J$ is a parabolic subgroup in $\overline{G}$, containing the Borel subgroup $\overline{P}$. This $\overline{P}$ has $\overline{U}$ as its unipotent radical and contains the maximal split torus $\overline{T}$, whose normalizer in $\overline{G}$ is $\overline{N}$. The quotient $\overline{N}/\overline{T}$ is canonically identified with the Weyl group $W = N/T$, and similarly as before we choose for any $w \in W$ a representative (with the same name) $w \in \overline{N}$. Let $\overline{P}^- = \overline{TU}^-$ denote the Borel subgroup opposite to P, with unipotent radical $\overline{U}^-$. For $w \in W$ let $\overline{U}^w = \overline{U} \cap w\overline{U}^- w^{-1}$. Then

$$\overline{U}^w = \prod_{\substack{\alpha \in \Phi^+ \\ w^{-1}(\alpha) \in \Phi^-}} \overline{U}_\alpha$$

and $\overline{U}^1 = \{1\}$. By transposition of [15] par. 4.2, Prop. 4 (b) we have

$$\overline{U}^w w \overline{P}_J = \overline{P} w \overline{P}_J \tag{7}$$

for any $w \in W_J$, and the left hand side product is direct.

Lemma 3.1. *Let $w \in W^J$ and $s \in S$.*
(a) If $(sw)^J = w$ then

$$us\overline{U}^w w\overline{P}_J = \overline{U}^w w\overline{P}_J$$

for each $u \in \overline{U}^s$, and these are direct products.
(b) If $\ell((sw)^J) > \ell(w)$ then

$$\overline{U}^s s\overline{U}^w w\overline{P}_J = \overline{U}^{sw} sw\overline{P}_J$$

and these are direct products.
(c) If $\ell((sw)^J) < \ell(w)$, then $w^{-1}(\beta) \in \Phi^-$, where $s = s_\beta$. The product

$$\overline{U}' = \prod_{\substack{\alpha \in \Phi^+ - \{\beta\} \\ w^{-1}(\alpha) \in \Phi^-}} \overline{U}_\alpha$$

(any ordering of the factors) is a subgroup of $\overline{U}^w$. *We have*

$$\overline{U}^s su\overline{U}'w\overline{P}_J=\overline{U}^w w\overline{P}_J \qquad \textit{for } u\in\overline{U}^s-\{1\},$$

$$us\overline{U}'w\overline{P}_J=\overline{U}^{sw}sw\overline{P}_J \qquad \textit{for } u\in\overline{U}^s$$

and all these are direct products.

PROOF: We use general facts on Bruhat decompositions.
(a)We have

$$s\overline{U}^w w\overline{P}_J=s\overline{P}w\overline{P}_J\subset\overline{P}w\overline{P}_J\cup\overline{P}sw\overline{P}_J=\overline{P}w\overline{P}_J=\overline{U}^w w\overline{P}_J$$

where at the inclusion sign we use $s\overline{P}w\subset\overline{P}w\overline{P}\cup\overline{P}sw\overline{P}$, and where in the equality following it we use the hypothesis $(sw)^J=w$, i.e. $swW_J=wW_J$. Applying s we see that this inclusion is an equality. Since $u\in\overline{P}$ and $\overline{U}^w w\overline{P}_J=\overline{P}w\overline{P}_J$ we get (a).
(b) $\ell((sw)^J)>\ell(w)$ implies $\ell(sw)>\ell(w)$ and again by general properties of Bruhat decompositions we find

$$\overline{U}^s s\overline{U}^w w\overline{P}_J=\overline{U}^s s\overline{P}w\overline{P}_J=\overline{P}s\overline{P}w\overline{P}_J=\cup_{v\in W_J}\overline{P}s\overline{P}w\overline{P}v\overline{P}$$

$$=\cup_{v\in W_J}\overline{P}sw\overline{P}v\overline{P}=\overline{P}sw\overline{P}_J=\overline{U}^{sw}sw\overline{P}_J$$

where the assumption $\ell(sw)>\ell(w)$ implied $\overline{P}s\overline{P}w\overline{P}=\overline{P}sw\overline{P}$, and where we made repeated use of (7) (in the first and last equation with this J, and in the second equation by setting $J=\emptyset$ in (7)).
(c) $\ell((sw)^J)<\ell(w)$ implies $\ell(sw)<\ell(w)$, hence $w^{-1}(\beta)\in\Phi^-$. One checks that $\overline{U}'=s\overline{U}^{sw}s$, hence this is a subgroup. Moreover, $s\overline{U}'=\overline{U}^{sw}s$ and since $\overline{U}^s\subset\overline{P}$ and $\overline{U}^{sw}sw\overline{P}_J=\overline{P}sw\overline{P}_J$ the last equality follows. Finally, again by general facts on Bruhat decompositions we have

$$s\overline{U}^w w\overline{P}_J \quad\subset\quad \overline{U}^w w\overline{P}_J\cup\overline{U}^{sw}sw\overline{P}_J$$

and the union on the right hand side is disjoint (since $swW_J\neq wW_J$). We just saw that $s\overline{U}'w\overline{P}_J=\overline{U}^{sw}sw\overline{P}_J$, hence $s(\overline{U}^w-\overline{U}')w\overline{P}_J\subset\overline{U}^w w\overline{P}_J$. It follows that

$$\overline{U}^s su\overline{U}'w\overline{P}_J\subset\overline{U}^w w\overline{P}_J$$

for $u\in\overline{U}^s-\{1\}$. To see the reverse inclusion it is enough to show $\overline{U}'w\overline{P}_J\subset\overline{U}^s su\overline{U}'w\overline{P}_J$. Since $\overline{U}'=s\overline{U}^{sw}s$ this boils down to showing $\overline{U}^{sw}sw\subset s\overline{U}^s sus\overline{U}^{sw}sw\overline{P}_J$, i.e. (by (7)) to $\overline{U}^{sw}sw\subset s\overline{U}^s sus\overline{P}sw\overline{P}_J$. A small computation in $\mathrm{SL}_2(k_F)$ shows that, because of $u\neq 1$, there is some $\tilde{u}\in\overline{U}^s$ with $s\tilde{u}sus\in\overline{P}$. This implies the wanted inclusion. □

Definition: Similarly as before, we define the J-special $\overline{G}$-representation $\mathrm{Sp}_J(\overline{G},L)$ with coefficients in L by the exact sequence of $\overline{G}$-representations

$$\bigoplus_{\alpha\in\Delta-J}C(\overline{G}/\overline{P}_{J\cup\{\alpha\}},L)\xrightarrow{\partial}C(\overline{G}/\overline{P}_J,L)\longrightarrow\mathrm{Sp}_J(\overline{G},L)\longrightarrow 0.$$

Consider the natural map

$$C(\overline{G}/\overline{P}_J, L) \longrightarrow C^\infty(G/P_J, L),$$

$$f \mapsto [g = ky \mapsto f(\overline{k})]$$

where we decompose a general element $g \in G$ as $g = ky$ with $k \in K_{x_0}$ and $y \in P_J$ (using the Iwasawa decomposition $G = K_{x_0}P_J$), and where $\overline{k}$ denotes the class of k in $\overline{G} = K_{x_0}/U_{x_0}$. We have similar maps for the various $P_{J\cup\{\alpha\}}$, hence an embedding

$$\mathrm{Sp}_J(\overline{G}, L) \hookrightarrow \mathrm{Sp}_J(G, L). \tag{8}$$

For $w \in W^J$ we write

$$g_w = \chi_{\overline{P}w\overline{P}_J} = \chi_{\overline{U}^w w\overline{P}_J},$$

the characteristic function of $\overline{P}w\overline{P}_J = \overline{U}^w w\overline{P}_J$ on $\overline{G}$. We also write g_w for the class of g_w in $\mathrm{Sp}_J(\overline{G}, L)$.

Proposition 3.2. *(a) The embedding (8) induces an isomorphism*

$$\mathrm{Sp}_J(\overline{G}, L)^{\overline{P}} \cong \mathrm{Sp}_J(G, L)^I.$$

(b) The set $\{g_w \mid w \in V^J\}$ is an L-basis of $\mathrm{Sp}_J(\overline{G}, L)^{\overline{P}}$.

Proof: For $\overline{G} = \mathrm{GL}_n(k_F)$ (some n) a proof of (b) is given in [12] par.6. For general $\overline{G}$ the proof carries over (this is then similar to [15] par.4). But of course, to compute $\mathrm{Sp}_J(\overline{G}, L)^{\overline{P}}$ (i.e. proving (b)) one may also proceed as in the proof of Corollary 2.4 above, and then (a) follows by comparing with the very statement of Corollary 2.4. $\square$

We define the Hecke-Algebra

$$\mathcal{H}(\overline{G}, \overline{P}; L) = \mathrm{End}_{L[\overline{G}]} L[\overline{P}\backslash\overline{G}].$$

For a $\overline{G}$-representation on a L-vector space V with subspace $V^{\overline{P}}$ of $\overline{P}$-invariants, Frobenius reciprocity tells us that there is an isomorphism

$$\mathrm{Hom}_{L[\overline{G}]}(L[\overline{P}\backslash\overline{G}], V) \cong \mathrm{Hom}_{L[\overline{P}]}(L, V) \cong V^{\overline{P}}$$

which sends $\psi \in \mathrm{Hom}_{L[\overline{G}]}(L[\overline{P}\backslash\overline{G}], V)$ to $\psi(\overline{P}) \in V^{\overline{P}}$. Hence $V^{\overline{P}}$ becomes a right $\mathcal{H}(\overline{G}, \overline{P}; L)$-module. For $g \in \overline{G}$ we define the Hecke operator $T_g \in \mathcal{H}(\overline{G}, \overline{P}; L)$ by setting

$$(T_g f)(\overline{P}h) = \sum_{\overline{P}h' \subset \overline{P}g^{-1}\overline{P}h} f(\overline{P}h')$$

for $f \in L[\overline{P}\backslash\overline{G}]$, where for the moment we identify $L[\overline{P}\backslash\overline{G}]$ with the L-module of functions $\overline{P}\backslash\overline{G} \to L$. For $n \in \overline{N}$ the Hecke operator T_n only depends on the class of n in $W = \overline{N}/\overline{T}$. It acts on $v \in V^{\overline{P}}$ as

$$vT_n = \sum_{u \in \overline{P}/(\overline{P}\cap n^{-1}\overline{P}n)} un^{-1}v. \tag{9}$$

Notice that for $s \in S$ we may identify $\overline{U}^s \cong \overline{P}/(\overline{P} \cap s\overline{P}s)$. Thus formula (9) for the Hecke operator T_s acting on $g_w \in \mathrm{Sp}_J(\overline{G}, L)^{\overline{P}}$ becomes

$$g_w T_s = \sum_{u \in \overline{U}^s} (\text{the class of } \chi_{us\overline{U}^w w\overline{P}_J}) \tag{10}$$

in $\mathrm{Sp}_J(\overline{G}, L)^{\overline{P}}$.

For the rest of this section we assume that L is a field with $\mathrm{char}(L) = \mathrm{char}(k_F)$.

Lemma 3.3. *Let $w \in W^J$ and $s \in S$.*
(a) If $(sw)^J = w$ then

$$g_w T_s = 0.$$

(b) If $\ell((sw)^J) > \ell(w)$ then

$$g_w T_s = g_{sw}.$$

(c) If $\ell((sw)^J) < \ell(w)$ then

$$g_w T_s = -g_w.$$

PROOF: This follows from Lemma 3.1 and from $|\overline{U}^s| = 0$ in L. For example, for (c) we compute, using the notations of Lemma 3.1 (c), in particular the direct product decomposition $\overline{U}^w = \overline{U}^s \overline{U}'$:

$$\begin{aligned} g_w T_s &= \sum_{u \in \overline{U}^s} [\chi_{us\overline{U}^w w\overline{P}_J}] = \sum_{u \in \overline{U}^s} \sum_{u' \in \overline{U}^s} [\chi_{usu'\overline{U}' w\overline{P}_J}] \\ &= \sum_{u \in \overline{U}^s} \sum_{u' \in \overline{U}^s - \{1\}} [\chi_{usu'\overline{U}' w\overline{P}_J}] + \sum_{u \in \overline{U}^s} [\chi_{us\overline{U}' w\overline{P}_J}]. \end{aligned}$$

Lemma 3.1 (c) together with $|\overline{U}^s| = 0$ in L shows that the second term vanishes and that the first term is $-[\chi_{\overline{U}w\overline{P}_J}]$. $\square$

Proposition 3.4. *Each non-zero $\mathcal{H}(\overline{G}, \overline{P}; L)$-submodule E of $\mathrm{Sp}_J(\overline{G}, L)^{\overline{P}}$ contains the element g_{z^J}. In particular, the $\mathcal{H}(\overline{G}, \overline{P}; L)$-module $\mathrm{Sp}_J(\overline{G}, L)^{\overline{P}}$ is indecomposable.*

PROOF: By Proposition 3.2 we find an element

$$h = \sum_{w \in V^J} \beta_w g_w$$

in E, with certain $\beta_w \in L$, not all of them zero. Choose an enumeration $z^J = w_0, w_1, w_2, \dots$ of V^J such that $w_j <_J w_i$ implies $i < j$. For $t \geq 0$ consider the property

$$\mathfrak{P}(t) \quad = \quad [\ \beta_{w_i} = 0 \text{ for all } i > t\ ].$$

By descending induction it is enough to show the following: If $\mathfrak{P}(t)$ holds true for some $t > 0$, then passing to another $h \neq 0$ if necessary, $\mathfrak{P}(t')$ holds true for some $t > t' \geq 0$. Notice that in view of the decreasing nature of our enumeration, Lemma 3.3 shows that the property $\mathfrak{P}(t)$ is

preserved under application of T_s to h, for any $s \in S$.
Let t be minimal such that $\mathfrak{P}(t)$ holds true (i.e. such that in addition $\beta_{w_t} \neq 0$), and assume $t > 0$ (otherwise we are done). By Lemma 1.5 we find $s, s_1, \ldots, s_r \in S$ such that, setting $w^{(i)} = (s_i \ldots s_1 w_t)^J$ for $0 \leq i \leq r$, we have the following: $\ell(w^{(i+1)}) > \ell(w^{(i)})$ for all $i \geq 0$, and $\ell((sw^{(i)})^J) < \ell(w^{(i)})$ for all $r > i \geq 0$, and $\ell((sw^{(r)})^J) \geq \ell(w^{(r)})$. By Lemma 3.3 we may replace h by hT_s to assume $\beta_{w^{(r)}} = 0$ [while keeping the other hypotheses on h: in particular, $\beta_{w_t} \neq 0$ also for the new h — this follows from our induction hypothesis which tells us that for the old h we have $\beta_{(sw_t)^J} = 0$ (if $(sw_t)^J \in V^J$), therefore this old $\beta_{(sw_t)^J}$ (if $(sw_t)^J \in V^J$) does not, by an instance of Lemma 3.3 (b), contribute to the new $\beta_{w_t} = \beta_{s(sw_t)^J}$.] By descending subinduction on $0 \leq g \leq r$ we show that, passing to another $h \neq 0$ if necessary, we may assume $\beta_{w_i} = 0$ for all $i > t$, and $\beta_{w^{(g)}} = 0$. For $g = 0$ this is what we want. For $g = r$ this was just shown. Now if for $0 \leq g < r$ we have $\beta_{w^{(g)}} \neq 0$ and $\beta_{w^{(g+1)}} = 0$, we replace h by $h + hT_{s_{g+1}}$: then, inspecting once more the formulae of Lemma 3.3, we find $\beta_{w^{(g)}} = 0$ for this new h, but $\beta_{w^{(g+1)}} \neq 0$, ensuring $h \neq 0$. $\square$

4 Irreducibility in the residual characteristic

Following our conventions we put $T^0 = I \cap T$ and then let $\widetilde{W} = N/T^0$ (sometimes referred to as the extended affine Weyl group). $\widetilde{W}$ acts on the apartment A and can be canonically identified with the semidirect product $(T/T^0).W$. It contains the affine Weyl-group W^a, the subgroup of $\widetilde{W}$ generated by the reflections in the walls of A. On the other hand, let Ω be the subgroup of $\widetilde{W}$ stabilizing the standard chamber in A (i.e. the one fixed by I). Then $\widetilde{W}$ is canonically identified with the semidirect product $\Omega.W^a$. If G is of adjoint type the canonical projection $\varphi : \widetilde{W} \to W$ is injective on Ω and its image $W_\Omega = \varphi(\Omega) \subset W$ coincides with the one defined in section 1.

We define the Iwahori Hecke algebra

$$\mathcal{H}(G, I; L) = \mathrm{End}_{L[G]} L[I\backslash G].$$

For a smooth G-representation on a L-vector space V with subspace V^I of I-invariants, Frobenius reciprocity tells us that there is an isomorphism

$$\mathrm{Hom}_{L[G]}(L[I\backslash G], V) \cong \mathrm{Hom}_{L[I]}(L, V) \cong V^I$$

which sends $\psi \in \mathrm{Hom}_{L[G]}(L[I\backslash G], V)$ to $\psi(I) \in V^I$. Hence V^I becomes a right $\mathcal{H}(G, I; L)$-module. For $g \in G$ we define the Hecke operator $T_g \in \mathcal{H}(G, I; L)$ by setting

$$(T_g f)(Ih) = \sum_{Ih' \subset Ig^- Ih} f(Ih')$$

for $f \in L[I\backslash G]$, where for the moment we identify $L[I\backslash G]$ with the L-module of compactly supported functions $I\backslash G \to L$. The Hecke operator T_n for $n \in N$ depends only on the class of

n in $\widetilde{W}$, and the T_n for n running through a system of representatives for $\widetilde{W}$ form an L-basis of $\mathcal{H}(G,I;L)$ ([14] section 1.3, example 1). They act on $v\in V^I$ as

$$vT_n=\sum_{u\in I/(I\cap n^{-1}In)}un^{-1}v.$$

By Proposition 3.2 we have an isomorphism

$$\mathrm{Sp}_J(\overline{G},L)^{\overline{P}}\cong \mathrm{Sp}_J(G,L)^I. \tag{11}$$

For $w\in W$ we had defined a Hecke operator T_w acting on the $\mathcal{H}(\overline{G},\overline{P};L)$-module $\mathrm{Sp}_J(\overline{G},L)^{\overline{P}}$. On the other hand, if we denote again by w a representative in N of the image of w in $\widetilde{W}$ (under the embedding $W\hookrightarrow(T/T^0).W\cong\widetilde{W}$), we get a Hecke operator T_w acting on the $\mathcal{H}(G,I;L)$-module $\mathrm{Sp}_J(G,L)^I$. (Note however that, for fixed Iwahori subgroup I, the embedding $W\to\widetilde{W}$ depends on the choice of x_0 (or equivalently, of P). Hence the $\mathcal{H}(G,I;L)$-elements T_w for $w\in W$ depend on this choice.) It is clear from our constructions that these actions coincide under our isomorphism (11). Recall that for $w\in W^J$ we wrote g_w for the class in $\mathrm{Sp}_J(\overline{G},L)^{\overline{P}}$ of the characteristic function of $\overline{P}w\overline{P}_J$ on $\overline{G}$. Now we also write g_w for its image in $\mathrm{Sp}_J(G,L)^I$ under (11), i.e. for the class in $\mathrm{Sp}_J(G,L)^I$ of the characteristic function of IwP_J on G.

For the rest of this section we assume that L is a field with $\mathrm{char}(L)=\mathrm{char}(k_F)$.

Lemma 4.1. *Assume that G is of adjoint type. For each $u\in W_\Omega$ there exists a lifting $\widetilde{u}\in N$ (under the canonical projections $N\to\widetilde{W}\to W$) which normalizes I and such that for all $w\in W^J$ we have $g_wT_{\widetilde{u}^{-1}}=g_{(uw)^J}$ in $\mathrm{Sp}_J(G,L)^I$.*

Proof: By [8] Proposition 2.10 we can lift $u\in W_\Omega$ to an element $\widetilde{u}\in N$ which normalizes I. Therefore $T_{\widetilde{u}^{-1}}$ acts on $\mathrm{Sp}_J(G,L)^I$ simply through the action of $\widetilde{u}\in N\subset G$ and for $w\in W^J$ we compute $\widetilde{u}IwP_J=I\widetilde{u}wP_J=I(uw)^JP_J$. The Lemma follows. (The hypothesis that G be of adjoint type should be superfluous here, but [8] assumes this.) $\square$

Theorem 4.2. *If the underlying root-system is of type A_l, B_l, C_l or D_l then the $\mathcal{H}(G,I;L)$-module $\mathrm{Sp}_J(G,L)^I$ is irreducible.*

Proof: By Proposition 3.4 we know that each non-zero $\mathcal{H}(G,I;L)$-submodule of $\mathrm{Sp}_J(G,L)^I$ contains the element g_{z^J}. Therefore it is enough to show that $\mathrm{Sp}_J(G,L)^I$ is generated as a $\mathcal{H}(G,I;L)$-module by the element g_{z^J}.

(a) We first asume that G is of adjoint type. We claim that for each subspace E of $\mathrm{Sp}_J(G,L)^I$ containing g_{z^J} and stable under all T_w for $w\in W$, and stable under all $T_{\widetilde{u}^{-1}}$ for $\widetilde{u}\in N$ normalizing I as in Lemma 4.1, we have $E=\mathrm{Sp}_J(G,L)^I$. Indeed, we know that $\mathrm{Sp}_J(G,L)^I$ is generated as an L-vector space by all g_w for $w\in V^J$. By Lemmata 4.1 and 3.3 it is therefore enough to find for each $w\in V^J$ a sequence $z^J=w_0,w_1,\ldots,w_r=w$ in W such that for all $i\geq 1$ we have $w_i^J=uw_{i-1}^J$ for some $u\in W_\Omega$, or $[\ell(w_{i-1}^J)<\ell(w_i^J)$ and $w_i^J=s(w_{i-1}^J)$ for some $s\in S]$. But this is the content of Corollary 1.7 which is available since we assume that G be of adjoint type.

(b) In the general case we find a central isogeny $\pi : G \to G'$ with G' split, connected, semisimple and of adjoint type, and with the same root system. We find a split maximal torus T' with normalizer N', a Borel subgroup P' and an Iwahori subgroup I' in G' such that $\pi^{-1}(T') = T$, $\pi^{-1}(P') = P$, $\pi^{-1}(I') = I$ and such that $W \cong N'/T'$. As $\ker(\pi) \subset T$ it is clear that π induces a G-equivariant isomorphism $\mathrm{Sp}_J(G', L) \cong \mathrm{Sp}_J(G, L)$ which restricts to an isomorphism of Iwahori invariant spaces $\mathrm{Sp}_J(G', L)^{I'} \cong \mathrm{Sp}_J(G, L)^I$ (both of dimension $|V^J|$, by Corollary 2.4).

We identify the Bruhat-Tits buildings of G and G'; then C is fixed by I', and x_0 corresponds to $P' \subset G'$ (just as it corresponds to $P \subset G$). Let $\widetilde{u} \in N'$ as in Lemma 4.1, in particular normalizing I'. For $n' \in N'$ we have

$$(12) \qquad T_{n'}T_{\widetilde{u}^{-1}} = T_{n'\widetilde{u}^{-1}} = T_{\widetilde{u}^{-1}}T_{\widetilde{u}n'\widetilde{u}^{-1}} \qquad \text{in } \mathcal{H}(G', I'; L)$$

by general facts on $\mathcal{H}(G', I'; L)$ (the 'braid relations'), or just by the definition of the T_g's. Now $\widetilde{u}\pi(N)\widetilde{u}^{-1} = \pi(N)$ and this is contained in N'. Since $\mathcal{H}(G, I; L)$ is generated by the T_n with $n \in N$ (see, e.g. [14] section 1.3, example 1), the relations (12) imply

$$(13) \qquad \mathcal{H}(G, I; L)T_{\widetilde{u}^{-1}} = T_{\widetilde{u}^{-1}}\mathcal{H}(G, I; L)$$

inside $\mathrm{End}_L\mathrm{Sp}_J(G, L)^I$ (here we keep the names of $\mathcal{H}(G, I; L)$ and $T_{\widetilde{u}^{-1}}$ also for their images in $\mathrm{End}_L\mathrm{Sp}_J(G, L)^I$). We get

$$(14) \qquad (g_{z^J}\mathcal{H}(G, I; L))T_{\widetilde{u}^{-1}} \subset (\widetilde{u}g_{z^J})\mathcal{H}(G, I; L)$$

inside $\mathrm{Sp}_J(G, L)^I$ (recall that $T_{\widetilde{u}^{-1}}$ acts from the right on $\mathrm{Sp}_J(G, L)^I$ by left multiplication with $\widetilde{u}$). By Proposition 3.4 we have $g_{z^J} \in (\widetilde{u}^{-1}g_{z^J})\mathcal{H}(G, I; L)$. We apply $T_{\widetilde{u}^{-1}}$, by equation (13) again this gives $\widetilde{u}g_{z^J} \in g_{z^J}\mathcal{H}(G, I; L)$, and together with (14) we get

$$(g_{z^J}\mathcal{H}(G, I; L))T_{\widetilde{u}^{-1}} \subset g_{z^J}\mathcal{H}(G, I; L).$$

By what we have seen in (a) this proves the Theorem. □

Remark: In conclusion, it turns out that, in case the root system is A_l, B_l, C_l or D_l (possibly also in case it is E_6, E_7), to prove the irreducibility of the $\mathcal{H}(G, I; L)$-module $\mathrm{Sp}_J(G, L)^I$ it is enough to use the action of $\mathcal{H}(\overline{G}, \overline{P}; L)$ together with the Hecke operators $T_{\widetilde{u}^{-1}}$ of Lemma 4.1. To deal with the remaining exceptional groups where the operators $T_{\widetilde{u}^{-1}}$ are not availabe one has to work out the action of sufficiently many other Hecke operators (besides those in $\mathcal{H}(\overline{G}, \overline{P}; L)$). We remark that Corollary 2.4 together with [15] Proposition 10 provides us with an isomorphism of $\mathcal{H}(G, I; L)$-modules

$$(15) \qquad \mathrm{Sp}_J(G, L)^I \cong \frac{C^\infty(G/P_J, L)^I}{\sum_{\alpha \in \Delta - J} C^\infty(G/P_{J\cup\{\alpha\}}, L)^I}.$$

In the case $G = \mathrm{SL}_n(F)$ (or $G = (\mathrm{P})\mathrm{GL}_n(F)$) Rachel Ollivier found an independent proof of the irreducibility of the right hand side of (15).

Corollary 4.3. *Suppose that the underlying root-system is of type A_l, B_l, C_l or D_l. The G-representation $\mathrm{Sp}_J(G,L)$ is irreducible.*

PROOF: Let $I_1 \subset I$ denote the pro-p-Iwahori subgroup in I, where $p = \mathrm{char}(k_F)$. Then I is generated by I_1 and $T^0 = T \cap I$. As T acts trivially on $\mathrm{Sp}_J(G,L)$, the spaces of invariants under I and I_1 are the same:

$$\mathrm{Sp}_J(G,L)^I = \mathrm{Sp}_J(G,L)^{I_1}.$$

Replacing I by I_1 in our definition of the Iwahori Hecke Algebra $\mathcal{H}(G,I;L)$ we obtain the algebra $\mathcal{H}(G,I_1;L)$. Similarly as before, $\mathrm{Sp}_J(G,L)^{I_1}$ is an $\mathcal{H}(G,I_1;L)$, and the irreducibility of $\mathrm{Sp}_J(G,L)^I$ as an $\mathcal{H}(G,I;L)$-module (Theorem 4.2) immediately implies the irreducibility of $\mathrm{Sp}_J(G,L)^{I_1} = \mathrm{Sp}_J(G,L)^I$ as an $\mathcal{H}(G,I_1;L)$ module. Now recall the well known fact that for every smooth representation of a pro-p-group — like I_1 — on a non-zero L-vector space E the subspace E^{I_1} of I_1-invariants is non-zero (since $\mathrm{char}(L) = p$). Applied to a non-zero G-subrepresentation E of $\mathrm{Sp}_J(G,L)$, the irreducibility of $\mathrm{Sp}_J(G,L)^{I_1}$ as a $\mathcal{H}(G,I_1;L)$ module implies $E^{I_1} = \mathrm{Sp}_J(G,L)^{I_1}$. But $\mathrm{Sp}_J(G,L)$ is generated as a $L[G]$-module by $\mathrm{Sp}_J(G,L)^{I_1}$; this follows from [15], Proposition 9, where it is shown that even the $L[G]$-module $C^\infty(G/P_J,L)$ is generated by its I_1-fixed vectors. Thus $E = \mathrm{Sp}_J(G,L)$ and we are done. □

Remark: For any J with $|V^J| = 1$, like $J = \emptyset$, we get the irreducibility of $\mathrm{Sp}_J(G,L)$ for *any* G (not necessarily of type A_l, B_l, C_l or D_l). The irreducibility of the Steinberg representation $\mathrm{Sp}_\emptyset(G,L)$ had been obtained earlier by Vignéras [15]. In fact she conjectures [15] the irreducibility of $\mathrm{Sp}_J(G,L)$ for any J, without any restrictions on Φ (like those imposed in Corollary 4.3).

Corollary 4.4. *(a) (Vignéras) The G-representations $\mathrm{Sp}_J(G,L)$ for the various subsets $J \subset \Delta$ are pairwise non-isomorphic.*
(b) Suppose that the underlying root-system is of type A_l, B_l, C_l or D_l. The G-representations $\mathrm{Sp}_J(G,L)$ with J running through all subsets $J \subset \Delta$ form the irreducible constituents of the G-representation $C^\infty(G/P,L)$, each one occuring with multiplicity one.

PROOF: The irreducibility of the $\mathrm{Sp}_J(G,L)$ in (b) is Theorem 4.3, everything else can be found in the paper [15]. Namely, there it is shown that each $\mathrm{Sp}_J(G,L)$ admits a P-equivariant filtration, with factors the natural P-representations $C_c^\infty(PwP/P,L)$ for $w \in V^J$. These factors are shown to be irreducible ([15] Proposition 1, Theorem 5). They are non-isomorphic for different $w \in W$. Indeed, let $R(w) = \{\alpha \in \Phi^+ \,|\, w^{-1}(\alpha) \in \Phi^+\}$. Let U^- denote the unipotent radical of the Borel subgroup P^- oppposite to P. For $w \in W$ let

$$U^w = U \cap wU^-w^{-1} = \prod_{\alpha \in \Phi^+ - R(w)} U_\alpha.$$

Similarly to (7) we have $U^w = PwP/P$. Therefore $R(w)$ is the set of all $\alpha \in \Phi^+$ for which U_α acts trivially on $C_c^\infty(PwP/P,L)$. But $R(w)$ uniquely determines w. □

Question: Is the theory of extensions between the various G-representations $\mathrm{Sp}_J(G,L)$ (for L a field with $\mathrm{char}(L) = \mathrm{char}(k_F)$) parallel to the theory of extensions between the various G-representations $\mathrm{Sp}_J(G,\mathbb{C})$ (as worked out in [11], [12]) ?

Corollary 4.5. *Suppose that the underlying root-system is of type A_l, B_l, C_l or D_l. Let $\mathcal{O}_K$ be a complete discrete valuation ring with fraction field K and residue field k_K. Suppose* $\mathrm{char}(k_K) = \mathrm{char}(k_F)$. *Up to $K^\times$-homothety,* $\mathrm{Sp}_J(G,\mathcal{O}_K)$ *is the unique G-stable $\mathcal{O}_K$-lattice inside* $\mathrm{Sp}_J(G,K)$.

PROOF: (I thank Marie-France Vignéras for completing my (originally incomplete) argument here.) Let N be another G-stable $\mathcal{O}_K$-lattice inside $\mathrm{Sp}_J(G,K)$. Let $p_K \in \mathcal{O}_K$ be a uniformizer. Since $\mathrm{Sp}_J(G,k_K)$ is irreducible by Corollary 4.3, the image of $p_K^n N \cap \mathrm{Sp}_J(G,\mathcal{O}_K)$ in $\mathrm{Sp}_J(G,\mathcal{O}_K) \otimes_{\mathcal{O}_K} k_K = \mathrm{Sp}_J(G,k_K)$ for $n \in \mathbb{Z}$ must be either (a) zero, or (b) all of $\mathrm{Sp}_J(G,k_K)$. Case (a) implies $p_K^{n-1}N \subset \mathrm{Sp}_J(G,\mathcal{O}_K)$. Case (b) implies

$$\mathrm{Sp}_J(G,\mathcal{O}_K) \subset p_K \mathrm{Sp}_J(G,\mathcal{O}_K) + p_K^n N. \tag{16}$$

Now $\mathrm{Sp}_J(G,\mathcal{O}_K)$ is finitely generated as an $\mathcal{O}_K[G]$-module (e.g. by $\mathcal{O}_K$-generators of $\mathrm{Sp}_J(G,\mathcal{O}_K)^I$, as was already used in the proof of Corollary 4.3), therefore there exists some $m >> 0$ with $p_K^m \mathrm{Sp}_J(G,\mathcal{O}_K) \subset N$. This means that (16) simplifies as $\mathrm{Sp}_J(G,\mathcal{O}_K) \subset p_K^n N$. In view of this dichotomy (a)/(b) for any $n \in \mathbb{Z}$ we get $p_K^n N = \mathrm{Sp}_J(G,\mathcal{O}_K)$ for some $n \in \mathbb{Z}$ since $\cap_n p_K^n N = 0$ and $\cup_n p_K^n N = \mathrm{Sp}_J(G,K)$. $\square$

5 Harmonic Chains

Here L is an arbitrary ring again and $G = \mathrm{GL}_{d+1}(F)$ (some $d \geq 1$). Let X denote the semisimple Bruhat-Tits building of G. Let X^0 denote the set of vertices of X. For $x \in X^0$ let

$$K_x = \{g \in G \quad | \quad gx = x \text{ and } \det(g) \in \mathcal{O}_F^\times\}$$

and let U_x be the unique maximal normal open subgroup of K_x. Let $P_{J,x} = K_x \cap P_J$. The group K_x acts on the set of simplices of X containing x. Let $\sigma_x = \sigma_x(J)$ denote the unique maximal such simplex which is fixed by $P_{J,x}$. It is k-dimensional, where $k = |\Delta - J| = d - |J|$. Inside the set of all k-dimensional simplices of X we define

$$X_x(J) = \{g\sigma_x \,|\, g \in K_x\}.$$

In each $\sigma \in X_x(J)$ we distinguish the vertex $x \in \sigma$, its pointing. K_x acts on $X_x(J)$. We let

$$X(J) = \coprod_{x \in X^0} X_x(J)$$

and call this the set of pointed J-simplices (so *by definition* this is a disjoint union, i.e. each element of $\sigma \in X(J)$ comes with a distinguished vertex $x \in \sigma$, its pointing). G acts on $X(J)$. Let $x \in X$ and $\alpha \in \Delta - \sigma$. For $\sigma \in X_x(J)$ and $\tau \in X_x(J \cup \{\alpha\})$ (i.e. pointed at the same vertex

x) we write $\tau < \sigma$ if $\tau \subset \sigma$. Now let $x, x' \in X^0$ such that $\{x, x'\} \in X^1$ (i.e. is a 1-simplex in X; we identify simplices in X with their sets of vertices). Let

$$U_{x,x'} = U_{x',x} = (U_x, U_{x'}),$$

the subgroup of G generated by U_x and $U_{x'}$. Then $U_{x,x'} \subset K_x$ and $U_{x,x'} \subset K_{x'}$. For $k \in K_x$ and $k' \in K_{x'}$ such that $k^{-1}k' \in P_J$ we say that the families of pointed J-simplices

$$\mathfrak{F} = U_{x,x'} k \sigma_x = \{\sigma \in X_x(J) \quad | \quad \sigma = uk\sigma_x \text{ for some } u \in U_{x,x'}\} \subset X_x(J)$$

and $\mathfrak{F}' = U_{x,x'} k' \sigma_{x'} \subset X_{x'}(J)$ are *adjacent* in $X(J)$.

Definition: $\mathfrak{har}_J(1)$ and $\mathfrak{har}_J(2)$ are the minimal L-submodules of $L[X(J)]$ satisfying:
(1) For each $\alpha \in \Delta - J$, each $\tau \in X(J \cup \{\alpha\})$, if we let $\mathcal{B}(\tau) = \{\sigma \in X(J) \,|\, \tau < \sigma\}$, then

$$\sum_{\sigma \in \mathcal{B}(\tau)} \sigma \in \mathfrak{har}_J(1).$$

(2) If $\mathfrak{F}$ and $\mathfrak{F}'$ are adjacent families in $X(J)$, then

$$\sum_{\sigma \in \mathfrak{F}} \sigma - \sum_{\sigma' \in \mathfrak{F}'} \sigma' \in \mathfrak{har}_J(2).$$

We let $\mathfrak{har}_J = \mathfrak{har}_J(1) + \mathfrak{har}_J(2)$ and define

$$\mathfrak{H}_J(L) = \frac{L[X(J)]}{\mathfrak{har}_J}.$$

We call $\mathfrak{H}_J(L)$ the L-module of L-valued J-chains on X. It carries an obvious G-action.

Theorem 5.1. *There exists a G-equivariant isomorphism*

$$\mathfrak{H}_J(L) \cong \mathrm{Sp}_J(G, L).$$

Proof: For $x \in X^0$ we have the K_x-equivariant isomorphism

$$L[X_x(J)] \cong C(U_x \backslash K_x / P_{J,x}, L), \tag{17}$$

$$g\sigma_x \mapsto \chi_{U_x g P_{J,x}}$$

($g \in K_x$) where $\chi_{U_x g P_{J,x}}$ denotes the characteristic function of $U_x g P_{J,x}$. The Iwasawa decomposition $G = K_x P_J$ (which holds since x, like all vertices in X, is a special vertex) provides a natural isomorphism

$$C(U_x \backslash K_x / P_{J,x}, L) \cong C(U_x \backslash G / P_J, L).$$

Together we obtain an isomorphism

$$\frac{L[X_x(J)]}{L[X_x(J)] \cap \mathfrak{har}_J(1)} \cong \frac{C(U_x \backslash G / P_J, L)}{\sum_{\alpha \in \Delta - J} C(U_x \backslash G / P_{J \cup \{\alpha\}}, L)}.$$

Furthermore, for $\{x,x'\}\in X^1$ the isomorphisms (17) for x and x' induce an isomorphism

$$\mathfrak{har}_J(2)\cap(L[X_x(J)]\oplus L[X_{x'}(J)])\cong C(U_{x,x'}\backslash G/P_J,L).$$

Together we deduce a G-equivariant exact sequence

$$\bigoplus_{\{x,x'\}\in X^1} C(U_{x,x'}\backslash G/P_J,L)\longrightarrow\bigoplus_{x\in X_0}\frac{C(U_x\backslash G/P_J,L)}{\sum_{\alpha\in\Delta-J}C(U_x\backslash G/P_{J\cup\{\alpha\}},L)}\longrightarrow\mathfrak{H}_J(L)\longrightarrow 0. \tag{18}$$

On the other hand we have according to [12] section 6, Theorem 8 a G-equivariant exact sequence

$$\bigoplus_{\{x,x'\}\in X^1}\mathrm{Sp}_J(G,\mathbb{Z})^{U_{x,x'}}\longrightarrow\bigoplus_{x\in X_0}\mathrm{Sp}_J(G,\mathbb{Z})^{U_x}\longrightarrow\mathrm{Sp}_J(G,\mathbb{Z})\longrightarrow 0.$$

Using [12] section 6 Proposition 15 we see that by base extension $\mathbb{Z}\to L$ we derive an exact sequence

$$\bigoplus_{\{x,x'\}\in X^1} C(U_{x,x'}\backslash G/P_J,L)\longrightarrow\bigoplus_{x\in X_0}\frac{C(U_x\backslash G/P_J,L)}{\sum_{\alpha\in\Delta-J}C(U_x\backslash G/P_{J\cup\{\alpha\}},L)}\longrightarrow\mathrm{Sp}_J(G,L)\longrightarrow 0. \tag{19}$$

Comparing the exact sequences (18) and (19) we conclude. □

Remarks: (a) We may identify X^0 with the set of homothety-classes $[\Lambda]=\{\lambda\Lambda\mid\lambda\in F^\times\}$ of free $\mathcal{O}_F$-submodules Λ of rank $d+1$ in a fixed $(d+1)$-dimensional F-vector space. A k-dimensional simplex in X is then given by the set of its $k+1$ vertices. This set carries a canonical cyclic ordering, namely the cyclic ordering $\ldots,[\Lambda_0],\ldots,[\Lambda_k],[\Lambda_0],\ldots$ if we can choose the representatives Λ_j such that

$$\Lambda_0\supsetneq\Lambda_1\supsetneq\ldots\supsetneq\Lambda_k\supsetneq p_F\Lambda_0.$$

Giving a pointing of the simplex amounts to fixing this cyclic ordering into a true total ordering $([\Lambda_0],\ldots,[\Lambda_k])$ (here $[\Lambda_0]$ is the pointing). For $\{x_0,x_0'\}\in X^1$ and pointed k-simplices $(x_0,\ldots,x_k)$ and $(x_0',\ldots,x_k')$ (represented as indicated) the families $U_{x_0,x_0'}(x_0,\ldots,x_k)$ and $U_{x_0,x_0'}(x_0',\ldots,x_k')$ are adjacent if and only if $\{x_i,x_i'\}\in X^1$ for all $0\le i\le k$.
(b) Let $\widehat{X}^k$ denote the set of *all* pointed k-dimensional simplices in X. One may define a G-stable submodule $\widehat{\mathfrak{har}}_J$ of $L[\widehat{X}^k]$ as the minimal submodule of $L[\widehat{X}^k]$ containing $\mathfrak{har}_J$ and all relations of the following kind. Let $\sigma=(\Lambda\supsetneq\Lambda_1\supsetneq\ldots\supsetneq\Lambda_k\supsetneq p_F\Lambda)\in\widehat{X}^k$ (pointed at $[\Lambda]$) and set

$$\mathcal{C}(\sigma)=\{\sigma'=(\Lambda\supsetneq\Lambda_1'\supsetneq\ldots\supsetneq\Lambda_k'\supsetneq p_F\Lambda)\in X(J)\mid$$

$$\text{for all } 1\le j\le k \text{ we have } \Lambda_j'\subset\Lambda_j \text{ or } \Lambda_j\subset\Lambda_j'\}.$$

Then

$$\sigma-\sum_{\sigma'\in\mathcal{C}(\sigma)}\sigma'\in\widehat{\mathfrak{har}}_J.$$

One may ask for which J the inclusion $L[X(J)] \subset L[\widehat{X}^k]$ induces an isomorphism

$$\mathfrak{H}_J(L) \cong \frac{L[\widehat{X}^k]}{\widehat{\mathfrak{har}}_J}.$$

In the case where J consists of the first $d-k$ simple roots (in the Dynkin diagram) this holds true: this follows from work of de Shalit [7] (he works with a different but equivalent definition of $\widehat{\mathfrak{har}}_J$ in this case). For these J Theorem 5.1 has been obtained by de Shalit in the case $\mathrm{char}(F)=0$, and by Aït Amrane (as the main result of [1]) for F of arbitrary characteristic.

Formula: Let J be arbitrary again (and $G=\mathrm{GL}_{d+1}(F)$). We conclude with an explicit description of the embedding $\lambda_L : \mathrm{Sp}_J(G,L) \hookrightarrow C^\infty(I, \mathfrak{M}_J(L))$ of Theorem 2.3 in terms of the isomorphism $\mathfrak{H}_J(L) \cong \mathrm{Sp}_J(G,L)$ of Theorem 5.1, without giving proofs. We identify W with the automorphism group of the set $\{0,\ldots,d\}$ and Δ with the set of transpositions $(s-1,s)$ for $1\le s\le d$. For $0\le i\le d$ let $e_i\in X_*(T)$ denote the cocharacter $e_i:\mathbb{G}_m\to T$ sending $y\in\mathbb{G}_m$ to the diagonal matrix $e_i(y)$ with $e_i(y)_{ii}=y$ and $e_i(y)_{jj}=1$ for $j\ne i$. Let $\{s_1<\ldots<s_k\}$ denote the set, in increasing enumeration, of all $s\in\{1,\ldots,d\}$ such that the transposition $(s-1,s)$ does *not* belong to J. In particular, $k=d-|J|$. For $1\le i\le k$ let

$$\xi_i^J = \sum_{0\le j\le s_i-1} e_j \in X_*(T).$$

For $w\in W^J$ let

$$\widetilde{Y}_A^0(J,w) = \{\sum_{i=1}^k m_i w(\xi_i^J)\mid m_i\in\mathbb{Z}_{\ge 0}\} \subset X_*(T).$$

Under the natural projection

$$X_*(T)\otimes\mathbb{R} \xrightarrow{\pi} X_*(T)\otimes\mathbb{R}/(e_0+\ldots+e_d) = A$$

the set $\widetilde{Y}_A^0(J,w)$ projects to a set $Y_A^0(J,w)$ of vertices in the standard apartment A of X. This $Y_A^0(J,w)$ is the set of vertices of a connected full simplicial subcomplex $Y_{J,w}$ of X all of whose maximal simplices are k-dimensional. We let $Y_A(J,w)$ denote the subset of $X(J)$ consisting of all pointed J-simplices in X having all their vertices in $Y_A^0(J,w)$. Thus the simplex underlying an element of $Y_A(J,w)$ is a chamber in $Y_{J,w}$. We may assume that I fixes the chamber in X whose set of vertices is $\{\pi(\xi_0^\emptyset),\pi(\xi_1^\emptyset),\ldots,\pi(\xi_d^\emptyset)\}$, where we set $\xi_0^\emptyset=0\in X_*(T)$. Let $I.Y_A(J,w)\subset X(J)$ denote the union of all I-orbits of elements of $Y_A(J,w)$ and then put

$$Y(J) = \bigcup_{w\in W^J} I.Y_A(J,w)$$

(this is a disjoint union inside $X(J)$). For $\sigma\in Y(J)$ there exists a unique $w\in W^J$, a unique $\sigma'\in Y_A(J,w)$ and some $g\in I$ such that $\sigma=g\sigma'$. Here g is not uniquely determined, but the coset $gV_{\sigma'}$ in I is independent of the choice of g, where $V_{\sigma'}\subset I$ denotes the stabilizer of σ' in I. There is a unique element $\sigma(w)\in Y_A(J,w)$ which is pointed at the central vertex (i.e. at

$\pi(0) \in A$). Let $m(\sigma(w), \sigma') \in \mathbb{Z}_{\geq 0}$ denote the gallery distance between $\sigma(w)$ and σ' (i.e between their underlying chambers in $Y_{J,w}$). Let

$$\widetilde{\lambda}_L(\sigma) = (-1)^{m(\sigma(w),\sigma')}\chi_{gV_{\sigma'}} \otimes \nabla(w)$$

(with ∇ as in the exact sequence (4)), an element of $C^\infty(I, L) \otimes \mathfrak{M}_J(L) = C^\infty(I, \mathfrak{M}_J(L))$. By L-linearity we obtain a map $\widetilde{\lambda}_L : L[Y(J)] \to C^\infty(I, \mathfrak{M}_J(L))$. One can show:

(i) The canonical map $L[Y(J)] \to \mathfrak{H}_J(L)$, induced by the inclusion $L[Y(J)] \subset L[X(J)]$, is surjective. (More precisely, for $w \in W^J$ the image of $L[I.Y_A(J, w)]$ in $\mathfrak{H}_J(L)$ corresponds, under the isomorphism $\mathfrak{H}_J(L) \cong \mathrm{Sp}_J(G, L)$, to the image of $C^\infty(I/P^0_{J,w}, L)$ in $\mathrm{Sp}_J(G, L)$, cf. the proof of Theorem 2.3.)

(ii) The composition

$$L[Y(J)] \longrightarrow \mathfrak{H}_J(L) \cong \mathrm{Sp}_J(G, L) \overset{\lambda_L}{\hookrightarrow} C^\infty(I, \mathfrak{M}_J(L))$$

is the map $\widetilde{\lambda}_L$ just described.

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